ENACTED TASK CHARACTERISTICS: SETTING AN INFRASTRUCTURE FOR STUDENTS' QUANTITATIVE REASONING

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In this study, we explored enacted task characteristics (ETCs) that supported students' quantitative reasoning (QR). We employed a design-based methodology; we conducted a teaching experiment with eight secondary school students. Through ongoing and retrospective analyses, we identified ETCs which supported students' quantitative reasoning. The ETCs can set the infrastructure for students' QR when students are: (a) identifying changing attributes of the tasks or situations, (b) coordinating the change among quantities, and (c) making generalizations about quantitative relationships. ETCs play an important role in development of students' meaningful understanding when tasks are designed with focus on quantitative reasoning and representational fluency.

Keywords: Enacted Task Characteristics, Quadratic Functions, Representational Fluency, Functional Thinking, Quantitative Reasoning

Rational and Research Aim

This research aims to identify sets of enacted task characteristics that support students' co-development of representational fluency and functional thinking in learning about quadratic functions within a quantitative context. Historically, quadratic functions have been identified as one function family students develop less sophisticated reasoning. Scholars reported that students often develop an unsophisticated understanding of quadratic functions, such as (a) conceiving a graph as an object (a pictorial entailment) (Ellis & Grinstead, 2008; Zaslavsky, 1997); (b) only articulating the parameters of quadratic functions with an unsophisticated understanding (Ellis & Grinstead, 2008; Even, 1998); (c) providing inappropriate generalization (Ellis & Grinstead, 2008); (d) conceiving of quadratic growth as exponential (Altindis & Fonger, 2018; Altindis & Fonger; 2019; Fonger & Altindis, 2019); and (e) depending heavily on algebraic representations, which limits the development of a robust understanding of quadratic functions (Ellis & Grinstead, 2008).

Developing greater sophisticated reasoning in learning about quadratic functions requires co-development of representational fluency and functional thinking. *Representational fluency* (RF) is defined as "the ability to create, interpret, translate between, and connect multiple representations—is a key to a meaningful understanding of mathematics" (Fonger, 2019, p. 1). *Functional thinking* (FT) is a creative thinking style about functions, creating patterns, and generalizing the functional relationships within concrete representations of functions (Blanton & Kaput, 2011; Stephens et al., 2017). In this study, FT included two types of reasoning about functions: correspondence and covariational reasoning. *Correspondence reasoning* is understanding the relationship between the x and y values by looking at the x and the y as corresponding dependent and independent values or quantities (Confrey & Smith, 1991; 1994; 1995). According to Thompson and his colleagues, covariational reasoning is being able to think about "two quantities' values varying" and the two quantities "varying simultaneously" (Thompson & Carlson, 2017, p. 425).

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Although we learn about the meaningful understanding grounds within the co-development of RF and FT from the literature (Even, 1998, Altindis & Fonger, 2019), we are still left with an inquiry on what type of enacted task characteristics may support the co-development of RF and FT. This study is guided by the research question: How can secondary school students be supported to develop a sophisticated understanding of quadratic functions?

Theoretical Framework and Background Literature

We networked the theory of quantitative reasoning (QR) (Thompson, 1994) with a theory of representations (Kaput, 1987a; 1987b) to support students' sophisticated understanding of quadratic functions. Quantitative reasoning sets a foundation for students' algebraic and covariational reasoning. Thompson's quantitative reasoning theory is based on Piaget's work on the mental images that learners create or *mental constructions* (Thompson, 1994). The creation of mental constructions is a demanding process for students to conceptualize quantities, quantification, and relationships among quantities (Thompson, 2011). According to Piaget (1967), *images* are conceptualizations that people must create, not something that already exists in their understanding of functions or the world. Piaget (1967) theorizes that a given subject's mental operation of a function and their mental image are connected and that the subject makes sense of an object by interacting with it. Following this logic, students might form an image of a function through reasoning about quantities that covary (Thompson, 1994). According to Thompson, when students try to grasp the concept of functions as equations that vary, they often focus on one variable as the source of the variation, usually the dependent variable. According to Thompson (1994), students' ability to build an image of changing quantities involves several layers: first, perceiving a change in one quantity; second, shifting into conceiving the two quantities as coordinated; and, finally, constructing an image of the two changing quantities as they covary simultaneously. These categories are based on Piaget's constructivist theory of learning.

Representations have been a focus of the mathematics education research community for decades. Scholars have explored students' understanding of mathematics regarding their representational activity, particularly their translations between and among representations—creating, interpreting, and transforming representations (e.g., Adu-Gyamfi & Bosse, 2014; Janvier, 1987a;1987b). In general, the relationship between mathematics and representations is understood as cause and effect—as long as teaching and learning mathematics exists, representations and their role will exist within it. In this study, we will focus on external (concrete) representations. Throughout this study, the word representation refers to the concrete representations of functions: graphs, tables, symbolic equations, and diagrams.

In the current study, we intend to network the theory of QR and the theory of representations to support students' sophisticated understanding of quadratic functions. We set the design principles and instructional supports, by the affordances of QR and representations, as follows: (a) creating opportunities for students to construct mental images of covarying quantities; (b) getting students to focus on quantitative operations rather than numerical operations; (c) emphasizing the role of concrete representations in quantitative processes; (d) grounding students' RF within the meaning of quantities; and (e) getting students to present the models of quantities in their minds via concrete representations.

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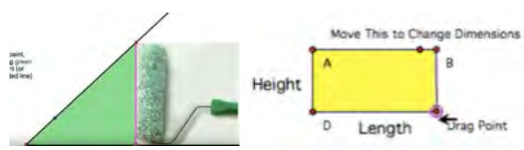
Methodology

In the present study, we employed a design-based research methodology (Cobb et al., 2017). We conducted a teaching experiment with eight Turkish American middle and high school students in the 8th, 9th, and 10th grades from urban and suburban school districts. The teaching experiment consisted of eight instructional lessons for two weeks lasting 45 to 60 minutes. We networked theories of quantitative reasoning (Thompson, 1994; 2011) and representations (Kaput, 1987a;1987b; Dreyfus, 2002) in designing a well-crafted learning ecology framework: enacted task characteristics, small and whole group dynamics, and teacher' pedagogical moves. In this research report, we will be focusing on enacted task characteristics. The data sources are enhanced transcription of small and whole group interactions. We are both the teacher researcher (TR) in this teaching experiment.

Tasks

In the present study, we used two tasks: the paint roller task and the growing rectangle task (see Figure 1). Both these tasks were created by Amy Ellis and her colleagues (2011; 2015). The context of these tasks may help students construct a much more profound understanding of quadratic functions as "a conception of two quantities varying simultaneously" (Thompson & Carlson, 2017, p. 444). With these tasks, students may notice attributes of the situation, such as seeing the paint roller's length and the size of the area being painted. Students may conceive of the triangle's height increasing and note that the area is also growing continually. These tasks include dynamic situations, diagrams, and videos that can help students see how a change in length affects a change in the area using color-coding that might help make the change in variables more visible to students (Johnson et al., 2018).



(a) (b)

Figure 1: (a) The Paint Roller Task (b) Growing Rectangle

Analyses

We used Cobb and Whitenack's (1996) techniques, which drew from Corbin and Strauss's (2008) constant comparison method. In the initial analysis, using phase one, we identified regularities in participants' interactions in small- and whole-group settings by creating enhanced transcriptions, structured and extended memos, and researcher journals. In the episode-by-episode analysis, we created the initial coding schema by coding the enhanced transcriptions of day 1 to day 8 using phase two. Then we re-coded to refute or agree with the codes or form the top-level codes—an emergent coding schema—using phase three. We then formed a developed coding schema—a learning-ecology framework—using phase four. In the analysis of analyses, we coded using the predetermined analytical frameworks of RF and FT—using phase five. Then we identified shifts in students' understanding of quadratic functions concerning the supports students received during the teaching experiment and verified the learning-ecology framework by coding 25% of the data using phase six.

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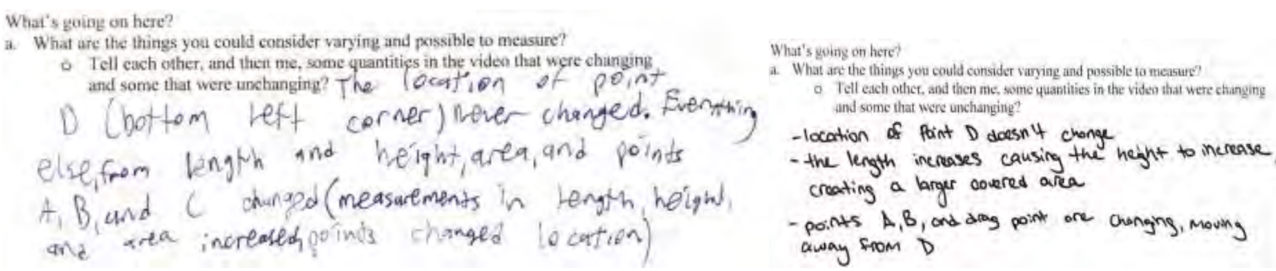
Result

Enacted Task Characteristics

We define ETCs as the instances in which students are given opportunities to articulate, talk about, answer, and discuss quantitative relationships within tables, graphs, and symbolic equations during small- and whole-group interactions (King, 2011; Stein et al., 2007). In other words, ETCs are statements and questions about a problem or a set of problems that encourage students to articulate, talk about, discuss, and create representations to present quantitative relationships. ETCs are a form of instructional support; the characteristics cluster around promoting students' QR and RF. ETCs can set the infrastructure for students' QR when students are: (a) identifying changing attributes of the tasks or situations, (b) coordinating the change among quantities, and (c) making generalizations about quantitative relationships.

Identifying changing attributes of tasks. One of the enacted task characteristics is asking students to identify attributes of a situation or their tasks—identifying relevant quantities and units to measure the quantities. Students were asked or prompted to identify quantities by looking at the task's attributes and identifying relevant quantities. After tracing appropriate quantities within the task context, they were prompted to think about a unit to measure the quantities.

Asli and Yener watched a video in the following vignette —featuring a growing rectangle being sketched via dynamic geometry software. Student handouts were structured so that students were asked to think and talk to each other about varying quantities and possible ways to measure those quantities. The task was structured to ask students to identify varying quantities; for example, the question in Figure 2: "What are the things you could consider varying and possible to measure?"



(a) (b)

Figure 2: (a) Yener's and(b) Asli's Varying Quantities of the Growing Rectangle

See the vignette below, which is the conversation students had in responding to the question on the task: "What are the things you could consider varying and possible to measure?"

6 Asli: Location of point D does not change.
7 Yener: Yeah. [Figure 2 (a) shows Yener's written answer: The location of point D (bottom left corner) never changed. Everything else, from the length and the height, area and the points A, B, and C changed (measurements in length, height, and area increased, points changed location)]
8 Researcher: Can you talk to each other?
9 Asli: We just wrote down when we talked about before we got the paper. [Figure 2 (b).]

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Asli and Yener identified the rectangle's corner; D (D is a point on the rectangle) was not changing (line 6–7). Asli referred to it as point D's location; Yener stated that D is at the ″bottom left corner," not changing (Figure 2). They agreed that everything else is changing on the task. Asli noticed that "the length increases causing the height to increase, creating a larger covered area″ (see Figure 2 (b)). Asli also recognized that the corners of the rectangle are changing, so she wrote, ″Points A, B, and drag points are changing, moving away from D." Yener agreed with Asli that A, B, and C changed. Length, height, and area changed as well. Yener recognized that the change in height, length, and area increases when the locations of A, B, and C (corners of the rectangle) change (line 7). Hence, we concluded that creating a foundation for students' QR might involve getting students to determine what is changing or varying in a dynamic task context. The tasks' structure, along with necessary tools, supports students in identifying varying relevant quantities. Students begin to recognize constant and variable quantities and how to measure them. This is evidence to suggest that enacted task characteristics should include questions or prompts that direct students' attention toward identifying relevant varying quantities on a task and noticing that the quantities are changing together.

Coordinating change among quantities. Another ETC is the coordination of change among quantities: probing, asking, or reinforcing students to coordinate changes among quantities. The tasks were structured to ask students how a change in one quantity affects the change in another in order to get students to coordinate the change between quantities. For example, one of the ETCs asks students: "How does the change in height affect change in area?" In the following vignette, Asli and Yener investigated the relationship between the height, length, and area of the growing rectangle task.

10 Yener: How does change in height is affect the change in area? If the height changes, the length changes.
11 Asli: The change in height increases the area covered. Because it contributes to the formula to get the area.
12 Yener: When the height changes, the area changes. Here is the area changes too.
13 Researcher: Can you be more specific? About how the height changes, the length changes. This also be an area.
14 Asli: When the length is increasing, the heights increase.
15 Yener: Increase Uhm. I think they might increase by the same amount. Yeah, they probably started over different, and then they increased amount each time the height and length.
16 Yener: Oh, I found this when height changes by 2, length changes by 3. That means that is constant.
17 Asli: Okay. So, what I wrote is the change in height increases the area covered because it contributes to the formula necessary to calculate the area [Figure 3 (a)].
18 Yener: Mine is same thing with height is affecting the change. [Figure 3 (b); he wrote: "The change in height is affecting the change in area by contributing to the formula for area therefore affecting the area."]

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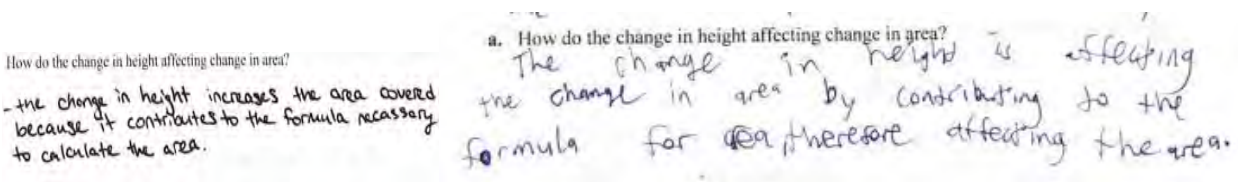

Figure 3: (a) Asli and Yener's (b) Response to "How Do the Change in Height Affecting Change in Area?

For this type of ETC, students are asked to see how the change in one quantity affects the change in another quantity (Figure 3). These questions (e.g., how does change in height affect the change in area?) form a foundation upon which students can coordinate change in quantities. For instance, Yener read the question (line 10): "How does change in height is affect the change in area?" Then he coordinated height with the length such that if the height changes (line 12), the length changes. Asli built on Yener's reasoning by stating (line 11), "The change in height increases the area covered."

Yener and Asli engaged in the task jointly; Yener agreed with Asli's statement, which encouraged Asli to justify her statement (line 11). She said, "Because it contributes to the formula to get the area." Asli's justification is about the corresponding reasoning. Yener said: "Increase, Uhmm. I think they might increase by the same amount, Yeah, they probably started over different, and then they increased amount each time the height and length." Yener noticed that the growing rectangle's height and length started with a different amount that changed in magnitude or amount each time (line 15). Then Yener said: "Oh, I found this when height changes by 2, length changes by 3. That means that is constant." Asli read her written responses: "Okay. So, what I wrote is the change in height increases the area covered because it contributes to the formula necessary to calculate the area" (line 17).

In responding to the task characteristics, students not only respond to questions on the tasks, but they also attempt to justify their responses. As we saw from Asli, she read her answer and even explained it (line 17). Furthermore, Yener read his response by comparing and contrasting his answer for the same question with Asli's (line 18).

Observing the results of this student exchange, we can infer that this student's ability to reason about relevant quantities and coordinate changes in quantities develops when prompted to consider how a change in one quantity affected change in another quantity. In other words, asking students about how a change in one quantity may affect the change in another can be an effective way to support healthy peer deliberation and the development of more advanced reasoning.

Generalization. Lastly, ETC involved structuring tasks to ask students to generalize the relationship between quantities. In terms of this study, a generalization is a form of support that pushes students to think about a pattern representing the relationship between quantities (e.g., the length of the paint-roller and its area). With ETC, students were asked to answer the same focus questions[1] in small- and whole-group settings in their handouts and had individual writing time for answering the same problem in their journal. The below vignette is taken from a whole-group interaction when students explored the relationship between the paint roller's length and the area covered by the paint roller. ETCs were structured with a focus question to allow the students to look for a pattern about the quantitative relationships.

And in the vignette below, the students were exploring the focus question: "What is the relationship between the length of the paint roller and the amount of the area being covered?"

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The focus question is designed to prompt students to coordinate a change in the paint roller's length and a change in the area it is covered. In other words, the question itself states that there is a relationship between the length of the paint roller and the area covered, which pushes students to generalize about the relationship.

Consider the vignette below:

31 Researcher: So, we will present the focus question ["What is the relationship between the length of the paint roller and the amount of the area being covered?"]. I will ask this group to present first. Yener. Ready.
32 Yener: I did not finish everything. But I have my answer.
33 Researcher: Okay. So, when someone is presenting, we want to ask questions, and we want to compare their thinking with ours—what they have on there. All right?
34 Yener: Wait. So, I just answer the focus question?
35 Researcher: Okay. Yeah. We are just answering the focus questions. But we are providing some evidence for our thinking.
36 Asli: Do you want to start first?
37 Yener: Okay, I'll do it first.
38 Yener: So, the focus question is, what's the relation between the length of the paint roller and the amount of area covered? And my answer is that every time the length increases by one centimeter, the amount the area changes by or the change in the change of area, it increases by 1 centimeter.

The focus Question: What is the relationship between the length of the paint roller and amount of the area covered?

Figure 4: A Focus Question for the Paint Roller Task

As we see with the above vignette, the TR stated that as a classroom community, students were trying to answer the focus question, which was about generalizing the relationship between quantities (line 31). Subsequently, the student's attention was directed to the relationship between the growing triangles' length and area (line 34). The paint roller task creates a growing triangle; the students' attention is directed to how the growing area is related to its length. As we see, the TR asked Asli and Yener if they could present, and when they agreed to present, she restated that as a community, they were trying to answer the focus question (line 31–33). Yener confirms that they were just answering the focus question by saying, "Wait. So, I just answer the focused question" (line 34). The TR oriented Yener toward answering the focus question and providing evidence to the claim they made in answering the focus question (line 33). Yener read the question (Figure 4): "What is the relationship between the length of the paint roller and amount of the area being covered?" and answered it by saying, "And my answer is that every time the length increases by one centimeter, the amount the area changes by or the change in the change of area, it increases by 1 centimeter" (line 38).

We concluded that having students answer the same focus questions about covarying quantities in social (small- and whole-group settings) and individual contexts (journals and individual handouts during writing time) might provide students with opportunities to articulate their thinking a more sophisticated understanding of their reasoning.

To use this ETC, the students' handouts and journals center on a focus question. For example, "What is the relationship between the length of the paint roller and the amount of the area being

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covered?" Students' handouts are designed to aid students in answering the focus question. Additionally, the TR's prompts in whole- and small-group settings, along with students' journals, center on answering the same focus questions. ETCs are a form of support in small- and whole-group settings where students are encouraged to generalize quantitative relationships.

In this example, we see that ETCs ask students to generalize the relationship by getting students to answer the focus question in small- and whole-group settings centered around identifying a pattern between quantities. Thus, ETCs are pushing students to generalize a relationship between quantities. The study suggested that enacted task characteristics that help support students' learning can include setting a focus question about quantitative relationships, which provides opportunities for students to generalize the quantitative relationships. Setting a focus question (see Figure 4) that asks students to explore the relationship among quantities is a form of support that may reinforce advanced reasoning about quantitative relationships. The findings suggest that the focus question provides students opportunities to articulate quantitative relationships in individual, small, and whole-group settings. Students benefited from the focus questions about quantitative relationships because students could answer the questions on their own, then discuss the same quantitative relationship in small- and whole-group settings where everyone articulated their thinking about the situation.

Discussion and Conclusion

We analyzed ETCs in the context of setting infrastructure for students' QR. Enacted task characteristics are purposefully designed elements that contribute to students' meaningful understanding of quadratic functions. Such characteristics allow students to talk, articulate and discuss quantitatively rich tasks while learning about quadratic functions. This study's findings parallel prior literature that posits that enacted tasks' design characteristics are a form of instructional support in learning and teaching about mathematics (King et al., 2011; Stein et al., 2007). In particular, the findings indicated that enacted task characteristics could effectively support student learning by setting infrastructure for students' QR. Thus, the present study's significance is in showing that the task characteristics should be designed with an emphasis on QR and RF and that these kinds of task characteristics can support students in co-developing RF and FT.

The findings showed that enacted task characteristics supported students' learning when ETCs enabled students to notice changing quantities and identify these quantities' attributes when learning about quadratic functions. The task characteristics made quantities and quantitative relationships visible to students. They provided opportunities for students to measure the magnitude of the quantities in the tasks, which effectively aligns with the prior research (e.g., Johnson et al., 2018). As the findings corroborate previous research on making quantities and their relationships visible to students, they further the literature by showing how task characteristics should be emphasized when focusing on RF and FT. Specifically, we found three salient task characteristics that enabled students to form foundations for QR. These characteristics include, typically, stating, probing, or asking students about (a) identifying changing attributes of the tasks, (b) coordinating change among quantities, and (c) generalizing the quantitative relationships.

First, we found that task design characteristics that direct students' attention to covarying quantities support students' meaningful understanding. Furthermore, such features support students' development of robust quantitative reasoning. Second, the current findings also focus on purposefully designing tasks to allow students to coordinate the change in one quantity with

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the change in another quantity. Lastly, the present results demonstrated that creating tasks with features, such as focus questions, that allow students to explore quantitative relationships is an effective form of support for students that further helps them form generalizations about quantitative relationships. While prior literature focused on making quantities visible to students (e.g., Johnson et al., 2018), this study builds on previous literature by suggesting that designing tasks with prompts, statements, or questions that redirect students' attention towards recognizing coordination among quantities can provide effective support for students' meaningful learning. This study also suggests that designing tasks with focus questions that require students to articulate or seek a generalized pattern about a quantitative relationship is beneficial to students' to develop quantitative reasoning skills.

Note

[1] For example, "What is the relationship between the length of the paint roller and the amount of the area being covered?"

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