STUDENT STRUGGLE DURING COLLABORATIVE PROBLEM-SOLVING IN ONE MATHEMATICS CLASSROOM

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In solving mathematics problems in collaboration, students encounter a range of mathematical and social struggles. As teachers cannot possibly respond to every such struggle, they may need to respond to those with which students require most support. Yet, little is known about students' success in overcoming the various types of struggle they encounter. In this study, we examined the types of struggle students experienced as they worked together in solving a cognitively-demanding problem. We analyzed the relative proportions of the various types of struggle they encountered, their success in overcoming each type, and the resources they leveraged in doing so. While students overcame many mathematical struggles, they had less success overcoming struggles related to reaching consensus or having their questions answered by peers. We argue that teachers may merit from support in learning to attend to these latter, more social struggles.

Keywords: Problem Solving, Middle School Education, Teacher Educators

Conceptual Framework

Struggle is of crucial importance to student learning (Hiebert & Grouws, 2007; NCTM, 2014). Indeed, if students already know all they need to know to solve a problem, the problem is unlikely to result in much struggle or any new learning. Struggle is a sign that one's prior knowledge is inadequate for solving a problem and that something new needs to be learned. When struggling, students may identify gaps in their understanding, which can result in new learning if addressed (Loibl & Rummel, 2014). Research even shows that, when given the chance to struggle in solving a problem before a lesson on the underlying concepts, students develop richer understandings of these concepts than when given the lesson without first having had opportunities to struggle (Kapur, 2010).

Recent research has described the mathematical, or cognitive, struggles students encounter when solving challenging mathematics problems, as well as teachers' responses to such struggles (Warshauer, 2015). In today's mathematics classroom, however, students are likely to encounter not only individual, cognitive struggles, but also a host of other struggles related to the collaborative context in which they are increasingly being asked to solve problems (NCTM, 2014). When solving problems in collaboration with others, students encounter various social struggles, such as the struggle to have their questions answered or ideas taken up by their peers (Langer-Osuna, 2011). If not overcome, these social struggles may limit students' opportunities to engage substantively with the mathematics under consideration and to benefit in their learning as a result. In solving problems collaboratively, students may also encounter struggles that reside at the intersection of the mathematical and the social. For example, they may struggle to explain a solution strategy of theirs to a peer or to understand a peer's explanation (Franke et al., 2015). If granted the opportunity to work through such struggles, both the student providing an explanation and the student listening to one may develop important mathematical understandings (Ing et al., 2018; Webb et al., 2009).

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Prior research has examined teachers' responses to these various cognitive struggles (Warshauer, 2015), social struggles (Dunleavy, 2015), and struggles to engage with one another's mathematical thinking (Franke et al., 2015). Given the many demands on their time and attention when teaching, teachers cannot possibly attend and respond to every such struggle that students encounter. They may thus need to respond to those struggles students have the hardest time overcoming. And yet, knowledge of the relative success students have in overcoming the various types of struggle they encounter remains underspecified. Moreover, little is known about the resources students call upon when struggling or which of these resources prove most helpful in overcoming their struggles.

As a first step to addressing these perceived research needs, the current study involved an analysis of video portraying students' collaborative mathematical work. The study is part of a broader project seeking to unpack the nature of students' struggles, both productive and unproductive, in learning mathematics. We examined the types of struggle a group of four students encountered as they worked together to solve a cognitively-demanding problem (Stein et al., 1996) over the course of a lesson. We analyzed the number and proportion of each type of struggle encountered by the group as a whole and by each individual student in the group. We also examined the extent to which students overcame each type of struggle they encountered and the resources they drew upon in doing so. Specifically, we examined the following research questions:

1. What types of struggle do students in one classroom encounter when solving a cognitively-demanding mathematics problem in collaboration?
2. How successful are the students in overcoming the various types of struggle they encounter?
3. What resources do the students leverage to overcome these struggles?

Methods

Study Context

The data we examined in this study consisted of one classroom video portraying a group of four 7th-grade students solving a cognitively-demanding mathematics problem collaboratively. We chose this video from a large collection of videos collected from the same school district. This particular district is a large, metropolitan district serving students from diverse racial, linguistic, and cultural backgrounds. The district uses a task-based curriculum comprised of challenging tasks designed to be solved using multiple solution strategies and representations, and for which an existing solution strategy should not be immediately apparent (Stein et al., 1996). Moreover, the district has a strong focus on and commitment to the tenets of Complex Instruction (Cohen & Lotan, 2004). For example, in each of the district's middle-school mathematics classrooms, teachers strive to delegate authority to students. They also assign students group roles to ensure that they all contribute substantively to the group's problem-solving efforts and to disrupt a pattern whereby only the voices of high-status students are heard.

Clip Selection

We chose to analyze video as the analysis we describe below would not have been feasible to conduct in a classroom in real time. The video we analyzed for this study was chosen through an iterative process. To begin, the first author viewed 83 videos in their entirety, each about 45-minutes in length. These videos portrayed groups of 3-4 students solving challenging

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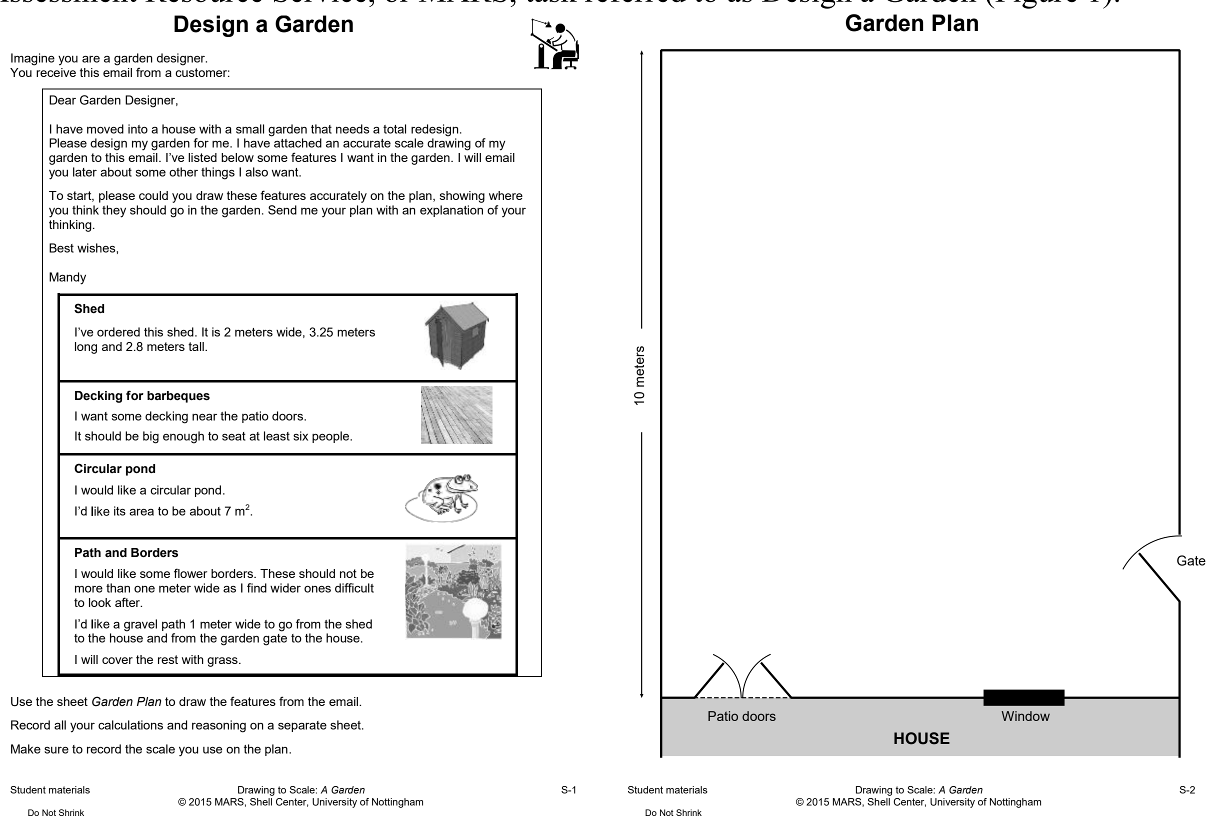
mathematics tasks and were collected as part of a separate research study. The videos were collected with an iPad placed on a tripod, while audio was collected using a table mic. Following data collection, audio from the table mic was synced with the video from the iPad. In viewing these 83 videos, the first author flagged videos in which students appeared to be struggling often, for sustained periods of time, and in varying ways. This struggle was evidenced by students' disagreements, questions, and expressions of confusion. Altogether, this process yielded a collection of 30 classroom videos. Next, the first author revisited notes he had written describing the audio- and video-quality of the videos, dropping those that were of inadequate quality (e.g., in which a student was out of view or students were hard to hear). This left 19 videos. Finally, he read through descriptions he had written for each of these 19 videos, choosing eight that portrayed a variety of different types and magnitudes of struggle. This collection of eight videos became the focus of a larger research program examining struggle.

Together, the two authors of the present study narrowed down this collection of videos from eight to three for detailed analysis. Our selections were guided by a set of criteria rooted in literature on productive struggle and the use of classroom video (Sherin et al., 2009; Warshauer, 2015). The criteria specified that the videos portray substantial student discussion of the mathematics, a range of resources being called upon, and various types of struggle.

In applying these criteria, we identified three videos, one that we analyzed in depth for this paper. We chose to analyze this video first as we had developed some familiarity with it through a separate analysis and believed this familiarity would facilitate the process of applying codes as part of the current analysis. This video portrays a group of four students solving the Mathematics Assessment Resource Service, or MARS, task referred to as Design a Garden (Figure 1).


Design a Garden

Imagine you are a garden designer.
You receive this email from a customer:

Dear Garden Designer,

I have moved into a house with a small garden that needs a total redesign. Please design my garden for me. I have attached an accurate scale drawing of my garden to this email. I've listed below some features I want in the garden. I will email you later about some other things I also want.

To start, please could you draw these features accurately on the plan, showing where you think they should go in the garden. Send me your plan with an explanation of your thinking.

Best wishes,

Mandy

Shed
I've ordered this shed. It is 2 meters wide, 3.25 meters long and 2.8 meters tall.

Decking for barbeques
I want some decking near the patio doors.
It should be big enough to seat at least six people.

Circular pond
I would like a circular pond.
I'd like its area to be about 7 m^2.

Path and Borders
I would like some flower borders. These should not be more than one meter wide as I find wider ones difficult to look after.
I'd like a gravel path 1 meter wide to go from the shed to the house and from the garden gate to the house.
I will cover the rest with grass.

Use the sheet *Garden Plan* to draw the features from the email.
Record all your calculations and reasoning on a separate sheet.
Make sure to record the scale you use on the plan.

Student materials Drawing to Scale: *A Garden* S-1
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Do Not Shrink

Figure 1. The Design a Garden Task

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For this task, students were asked to create a scale drawing of a garden containing the following items: a) a shed, b) a deck (i.e., a patio), c) a circular pond, and d) a path and some borders. We analyzed the 45 minutes and 14 seconds of video portraying students solving this particular problem, excluding a brief warm-up activity at the start of the lesson. This 45:14 of video begins with the teacher launching the Design a Garden task, during which time she asked students to identify all the different items they might find in a garden. Students then used the remaining time to work together to design their garden, ensuring that they included each of the items the task asked them to include. Although each student had their own copy of the task in front of them, they worked together throughout, cognizant that they needed to arrive at consensus regarding where they placed the various items in their design, as well as the dimensions of each item.

Data Analysis

We began our analysis of this video by applying codes to a 12-minute sample, which represented about 10% of the total duration of all three videos we ultimately selected. We did this to refine our coding procedure and establish inter-rater agreement before then splitting off to each code part of the video. Ultimately, however, we decided that we would both code the entire video together.

We coded the video in three phases. First, the two authors independently parsed the 12-minute sample of video into segments that captured any instance in which a student in the video encountered a "roadblock," which we defined as an impediment or obstacle that slowed students' progress. Although it was common for more than one student to be involved in particular roadblocks, we created roadblock segments one student at-a-time, as it was too challenging to create segments for multiple students simultaneously. We did not distinguish smaller roadblocks from more substantive ones, instead creating segments for any impediment students encountered, regardless of its magnitude. Next, we met to compare our segments and to arrive at consensus regarding the start and end times for each one. This resulted in a collection of segments for each of the four students in the 12-minute video-clip. In the second phase of coding, we independently applied the following codes to each segment: 1) type of struggle encountered (cognitive, socio-cognitive, social, or materials) and 2) struggle overcome (yes or no). *Cognitive* struggles consisted of individual, mathematical struggles like the struggle to understand the problem or implement a procedure for solving it. *Socio-cognitive* struggles consisted of students struggling to explain a strategy to a peer or to reach consensus regarding a particular approach for solving the problem. *Social* struggles consisted of struggles related to group dynamics, including the struggle to have one's questions or ideas taken up. Finally, the *materials* code captured students' struggles to access or use a material (e.g., a ruler, a calculator). We applied a series of rules for determining whether or not a struggle was overcome, which varied somewhat depending on the type of struggle under consideration. As an example, if a student repeatedly asked a question that was not answered, we determined that this struggle, a social struggle, was not overcome. As another example, the socio-cognitive struggle to reach consensus was overcome if students ultimately reached agreement regarding the idea over which they were in disagreement. If the conversation shifted to a different topic before such agreement was reached, we determined that the struggle was not overcome. In cases were a student acknowledged understanding something mathematical that had previously puzzled them, we determined that they had overcome a cognitive roadblock. After applying these codes in this second phase of coding, we came together to compare our codes and discuss, then resolve, any disagreements. For the vast majority of segments, we had applied the same codes independently. In the third phase of coding, we independently applied codes for the various resources students asked for or were offered

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related to each roadblock we identified previously. These resources consisted of: a) a peer, b) the teacher, c) a tool, d) the problem itself, e) students' multiple mathematical knowledge bases or MMKBs (e.g., linguistic resources) (Turner et al., 2012), and f) notes/the board. We then met to discuss and resolve any disagreements in these code applications.

We then repeated the steps described here with the remainder of the video.

Examining the coded data. After coding all the data, we each independently examined the coded data, then wrote and shared analytic memos documenting our observations. Our analysis of the coded data was guided by several conjectures. First, we conjectured that the students would encounter different types of roadblocks and that different students would encounter these roadblock types to varying degrees. To evaluate this conjecture, we examined the relative proportions of each type of roadblock encountered by the group, as well as the relative proportions of each type of roadblock each student encountered. Second, we conjectured that students would have more success overcoming certain types of roadblocks than others. To evaluate this conjecture, we examined the proportion of each type of roadblock that students overcame. Third, we anticipated that calling upon certain resources might prove more helpful to students in overcoming the roadblocks they encountered. To evaluate this final conjecture, we identified which resources were called upon most for those roadblocks that students overcame.

Findings

Overall, students encountered 107 roadblocks in the 45:14 of video we analyzed. Of these 107 roadblocks, 30 (28.0%) were *cognitive*, 48 (44.9%) were *socio-cognitive*, 16 (15.0%) were *social*, and 13 (12.1%) were related to students accessing or using *materials*. Hence, a full 59.9% of the struggles students encountered (i.e., socio-cognitive and social) were related to the collaborative nature of their work.

In terms of the number of roadblocks each individual student encountered, and for which we had evidence, Student 1 encountered 25, Student 2 encountered 32, Student 3 encountered 13, and Student 4 encountered 37. Table 1 portrays the number and proportion of each of the four types of roadblock each individual student encountered. The proportion of cognitive roadblocks each student encountered was fairly similar, although for Student 3, the total *number* of cognitive roadblocks he encountered was smaller than was the case for the other students. Moreover, while Students 1, 2, and 4 encountered a similar number and proportion of socio-cognitive roadblocks, Student 3 grappled with far fewer socio-cognitive roadblocks. Lastly, the number of social roadblocks Student 3 encountered was the same as the number encountered by Students 1, 2, and 4 combined.

Table 1: Number & Proportion of Each Type of Roadblock Encountered by Each Student

	Student 1	Student 2	Student 3	Student 4
Cognitive	10 (40.0%)	8 (25.0%)	3 (23.0%)	9 (24.3%)
Socio-cognitive	13 (52.0%)	17 (53.1%)	1 (7.7%)	17 (45.9%)
Social	1 (4.0%)	3 (9.4%)	8 (61.5%)	4 (10.8%)
Materials	1 (4.0%)	4 (12.5%)	1 (7.7%)	7 (18.9%)
Total	25	32	13	37

Note. Percentages show the proportion of the total number of roadblocks each student encountered for each roadblock type (e.g., 10/25 or 40% of Student 1's roadblocks were cognitive).

With regard to students' success in overcoming roadblocks, we found that students overcame more cognitive and materials roadblocks than either socio-cognitive or social roadblocks (Table 2).

Table 2: Number & Proportion of Each Type of Roadblock Overcome or Not Overcome

Type of Roadblock	Overcome	Not Overcome	Total
Cognitive	15 (50.0%)	15 (50.0%)	30
Socio-cognitive	2 (4.2%)	46 (95.8%)	48
Social	4 (25.0%)	12 (75.0%)	16
Materials	11 (84.6%)	2 (15.4%)	13

Specifically, students overcame half of the *cognitive* roadblocks they encountered and a full 84.6% of the *materials* roadblocks they encountered. This latter number is likely as high as it is in part because it includes certain materials roadblocks that were fairly easy to overcome (e.g., accessing an eraser). One such roadblock, however, involved students figuring out how to use a SAFE-T compass to draw the circular pond in their garden design. This roadblock was, in our view, harder to overcome than the roadblock of finding an eraser, yet was ultimately overcome. Unlike most of the materials roadblocks, the cognitive roadblocks were more challenging. One such roadblock involved Student 2 trying to find the radius of a circle with the area being given, something the student ultimately overcame with support from Student 1, who located a website that calculated the circle's radius when the area was entered. Regarding the *socio-cognitive* roadblocks, the vast majority of these involved three students – Students 1, 2, and 4 – trying to reach consensus regarding the inclusion, dimensions, and placement of various objects in the garden. For one such roadblock, Student 1 suggested placing the patio in a particular location in the garden, which Student 4 disagreed with. To convince Student 1 to place the patio in a different location, Student 4 pointed out that the existing garden plan included a pair of patio doors, and that the patio should be placed by these doors, something Student 1 immediately agreed with. This was one of only two socio-cognitive roadblocks that was overcome. Another such struggle regarded whether or not to add stairs to the patio. While Student 4 wanted to include stairs, the other students, especially Students 1 and 2, disagreed, pointing out that the problem did not say to include stairs and that they had seen patios before that did not have stairs. This particular struggle re-surfaced multiple times, yet consensus was never reached and the struggle remained unresolved. In terms of *social* roadblocks, students overcame one-fourth of this type of roadblock. However, further analysis revealed that Student 3, who encountered most of the social roadblocks, overcame only one-eighth of these. For Student 3, the social roadblock repeatedly encountered involved having their ideas heard and questions responded to.

Lastly, we examined the types of resources students called upon both when successful in overcoming a roadblock and when unsuccessful in doing so. As shown in Table 3, when students overcame a roadblock, the resource they most often called upon was one another (i.e., peers). At times, a peer's support was asked for, while at other times, a peer offered support that was unsolicited. As an example, for the socio-cognitive struggle mentioned above, in which Students 1 and 4 sought to reach consensus regarding the placement of the patio, Student 4 pointed out the patio doors to Student 1. In this example, Student 1 had asked Student 4 if they agreed with Student 1's idea, and as such, the peer resource was asked for. Although the peer resource was often called upon when roadblocks were overcome, it was also called upon often when students did not manage to overcome a roadblock.

Table 3: Resources Leveraged in Overcoming Each Roadblock

	Peer	Teacher	Tool	Problem	MMKBs	Notes/board
Overcome	18 (56.3%)	2 (6.3%)	15 (46.9%)	3 (9.4%)	0 (0.0%)	3 (9.4%)
Not overcome	63 (84.0%)	9 (12.0%)	11 (14.7%)	23 (30.7%)	9 (12.0%)	3 (4.0%)

Note. There were 32 roadblocks overcome and 75 not overcome; totals here exceed these numbers, as multiple resources were called upon for certain roadblocks. Percentages represent the proportion of roadblocks overcome, or not overcome, when a given resource was called upon (e.g., for 18/32 or 56.3% of the roadblocks overcome, students called upon the peer resource).

Noteworthy is that one resource – students' multiple mathematical knowledge bases (MMKBs) – was called upon only when students were engaged in a disagreement and striving to reach consensus. As an example, when discussing whether or not to add stairs to the patio, Student 2 referenced knowledge of their home patio, specifically, that this patio had no stairs and was so low from the ground that stairs were not needed. Lastly, we think it is worth noting that, for the 32 roadblocks students overcame, the teacher was accessed as a resource only twice.

Discussion

In this study, we examined three research questions: 1) What types of struggle do students in one classroom encounter when solving a cognitively-demanding mathematics problem in collaboration? 2) How successful are the students in overcoming the various types of struggle they encounter? 3) What resources do the students leverage to overcome these struggles? We anticipated that students would encounter a variety of different types of struggle, experience greater success in overcoming certain types of struggle, and call upon certain resources more than others when striving to overcome their struggles.

We found that the majority of the struggles, or "roadblocks," students encountered were related to the collaborative context in which they solved the Design a Garden problem. Specifically, 59.9% of their struggles were either socio-cognitive or social in nature. This suggests that, if teachers are asked to train their attention primarily on students' cognitive (i.e., individual, mathematical) struggles, as prior work has sought to do (Warshauer et al., 2021), teachers may miss a significant part of the struggle picture. These findings also suggest that teacher educators may find it beneficial to support teachers in attending to a greater range of different types of struggle that students encounter.

This seems particularly important given the potential association between social, socio-cognitive, and cognitive struggles. Of the roadblocks that Student 3 encountered, 61.5% were social in nature. Moreover, this student overcame only one of the eight social struggles they encountered. Unlike Students 1, 2, and 4, few of Student 3's struggles were mathematical in nature. While the majority of the roadblocks Students 1, 2, and 4 grappled with were cognitive or socio-cognitive, this was not the case for Student 3. This suggests that, unless a student overcomes the social struggles they encounter during collaborative problem-solving, they may lack opportunities to grapple with the mathematical struggles that seem likely to result in them arriving at important mathematical insights (Webb, 1991). Given that so few of Student 3's social struggles were overcome, it may be important for teachers to attend and respond to (Jacobs et al., 2010) social struggles like a student struggling to have their questions heard and ideas taken up, perhaps more so than the other types of struggle we describe here. Existing research provides guidance in this regard, describing practices (e.g., Shuffle Quizzes) teachers may enact to ensure all students are included in a group's mathematical conversations (Dunleavy, 2015).

Teachers may also wish to attend and respond to students' socio-cognitive struggles given how few of the struggles of this type students overcame. Prior work provides guidance regarding the sorts of moves teachers might make to help students engage with each other's strategies (Franke et al., 2015). Why so few socio-cognitive roadblocks were ultimately overcome is not immediately apparent. However, we think this may be related to the particular nature of the socio-cognitive roadblock with which these students engaged most: deciding whether or not to include stairs in the garden design. Although this was related to the problem, it was less mathematical than, say, the struggle to reach consensus regarding the solution to a problem, something that would likely be less open to debate. Moreover, we imagine that a discussion of the solution to a problem may do more to merit students' mathematical understandings than a discussion of whether or not to include stairs in the design of a garden. As such, some socio-cognitive roadblocks appear more worthy of students' time than others. Finally, we find it noteworthy that, despite referencing the problem itself and calling upon their out-of-school knowledge, students did not overcome many of their socio-cognitive struggles related to reaching consensus. This suggests that such roadblocks may be difficult to overcome even if students call upon the sorts of resources teachers might hope that they call upon.

Of the various types of roadblocks students encountered, they had most success overcoming cognitive and materials roadblocks. Indeed, students overcame half of their cognitive roadblocks, often with the support of their peers and rarely with the support of the teacher. In our view, this suggests that, rather than intervening right away when students appear to be grappling with a mathematical struggle, it may be best to leave the students to continue grappling, as there is a good chance that they will overcome the struggle on their own, without the teacher's support.

Limitations and Future Directions

The purpose of this analysis was to make visible the complexity of students' struggles in the course of one lesson. The patterns discernable in this video, however, cannot broadly predict what students' struggles may look like in other contexts. For instance, in the classroom observed in this video, like many others in this district, norms appeared to have been established whereby students understood they were to rely on each other for support. This may explain, in part, the degree to which students turned to each other as a resource and suggests that students in classrooms where such norms are not yet present may call upon their peers with lesser frequency.

Additionally, while certain resources (e.g., students' multiple mathematical knowledge bases) did not appear to be called upon much when students overcame a particular roadblock, we do not believe this suggests that these resources are not helpful with regards to overcoming struggle. It is possible that such resources would prove more helpful with problems involving contexts other than designing a garden. We think this is an area worthy of further examination.

Lastly, there are some limitations to our coding procedure. For instance, when students ultimately reached agreement, we determined that a socio-cognitive roadblock to reach consensus had been overcome. However, such agreement could have been reached as a result of one student leveraging their status or overpowering a peer. It is important to distinguish such instances from instances when agreement is reached as a result of a more equitable exchange, yet our coding procedure does not capture such distinctions. Future work could examine the role of students' status, power, and positioning in overcoming struggles like the struggle to reach consensus.

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