TASK DEVELOPMENT TO ADDRESS ERROR PATTERNS IN PROSPECTIVE ELEMENTARY TEACHERS' POSING OF MULTI-STEP WORD PROBLEMS

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National and state standards in the US have emphasized the importance of solving and posing word problems in students' mathematics learning for decades. Therefore, it is essential for prospective teachers (PTs) to have the mathematical knowledge necessary to teach these skills to their future students. Unfortunately, little research has investigated how PTs develop problem-posing skills. By employing thematic qualitative text analysis, the researchers identified nine distinct patterns in errors identified in K-8 PTs' posing of two-step addition and subtraction word problems, in the context of a collegiate teacher education course. These results were used to inform the initial design of an interventional task to bring awareness of common errors to PTs.

Keywords: Preservice Teacher Education, Number Concepts and Operations, Elementary School Education, Instructional Activities and Practices

Introduction

The Standards for Preparing Teachers of Mathematics put forth by the Association of Mathematics Teacher Educators (AMTE; 2017) recognize that well-prepared beginning teachers of mathematics "regard doing mathematics as a sense-making activity that promotes perseverance, problem posing, and problem solving. In short, they exemplify the mathematical thinking that will be expected of their students," (p. 9). Research focuses heavily on teaching prospective elementary teachers (PTs) to persevere and problem solve (Alibali, et al., 2014; Bruun, 2013; Crespo, 2003; Green & Emerson, 2010; Jitendra et al., 2013; Polya, 1945) but, until recent years, has put little emphasis on the problem-posing skills PTs are expected to develop.

The Common Core State Standards (CCSS) Initiative in the United States (NGA & CCSSO, 2010, Table 1; NGA & CCSSO, 2010, Table 2) has presented taxonomies of common addition and subtraction and common multiplication and division situations that can be used to introduce PTs to the intricacies of the operations. These taxonomies can function as a steppingstone for helping PTs develop single-step problem-posing skills. However, the CCSS for Mathematics suggest that children "solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions" (NGA & CCSSO, 2010, p. 19) already starting in the second grade. Thus, we argue that once an introductory understanding of problem posing has been developed in teacher education coursework, attention should be drawn to developing PTs' skills for posing multi-step word problems.

When it comes to solving multi-step word problems, Heffernan and Koedinger (1997) found that solvers may experience what they called a "composition effect," meaning that "a two operator problem is harder than both of the parts that make it up put together" (p. 310). Alibali et al. (2014) extended this idea to posing, noticing that middle school students who were able to pose single-step problems also showed increased difficulty in posing two-step problems. As such, the researchers investigated two research questions:

1. What patterns emerge in the errors that arise when PTs write two-step addition and subtraction word problems?
2. What instructional interventions may help PTs to notice and potentially avoid making common errors when posing multi-step problems?

The researchers evaluated a corpus of 282 two-step addition and subtraction word problems posed by PTs enrolled in a problem-solving course for undergraduate education majors to identify common error patterns. With the results of this analysis, the researchers then created an error-analysis task for K-8 PTs to help draw attention to common two-step problem-posing errors with the hope that it might help them prevent making such errors in their own problem-posing experiences. This paper reports on the results of the thematic qualitative text analysis completed to answer the first research question and shows how these findings were used to develop an interventional task. The task was implemented in this course the semester following the word problem analysis. We will share our preliminary findings from this first implementation and discuss how they informed modifications. A second version of the task is currently being implemented and data is being collected to analyze the effectiveness of this task.

Literature Review

While problem posing research has been growing in popularity over the past several decades (Cai, Hwang, Jiang, & Silber, 2015; Einstein & Infeld, 1938; English, 1998; Kilpatrick, 1987; Silver, 1994; Silver & Cai, 1996; Singer et al., 2013), only over the past 15 years have researchers turned their attention to the preparation of PTs as problem posers (Cai et al., 2020; Crespo & Harper, 2020; Crespo & Sinclair, 2008; Ellerton 2013; Lavy & Shriki, 2007; Leikin & Elgrably, 2020). U.S. national and state standards have been encouraging teachers to implement problem-posing activities in their K-12 classrooms for over three decades (NCTM, 1989, 1991, 2000; NGA & CCSSO, 2010; TEA, 2012), but in order for those problem-posing tasks to be implemented well, PTs must be well-prepared to "manage the complexities of such contexts" (Singer et al., 2013). The Standards for Preparing Teachers of Mathematics put forth by AMTE (2017) indicate that effective mathematics education programs "develop positive dispositions toward mathematics, including persistence and a desire to engage in posing and solving problems," (p. 70). Our attention as mathematics educators then turns toward how we can efficiently prepare PTs to pose a variety of word problems and how the skills they learn can be applied in their future classrooms. For the sake of this study, the researchers will be referring to free problem posing, which occurs before problem solving, where problems are generated using a "contrived or naturalistic situation" (Stoyanova & Ellerton, 1996, p. 519).

Problem posing has shown to illuminate areas of conceptual misunderstanding in students (Alibali, 2014; Sharp & Welder, 2014), so it is important that introductory problem-posing activities involve mathematical content with which PTs are familiar. The CCSS Initiative has broken down addition and subtraction scenarios into 15 clearly distinguished categorical structures (NGA & CCSSO, 2010, Table 1). It is important that children are exposed to all 15 types of addition/subtraction problems or they may develop limited conceptions of the meanings of the operations (Van de Walle et al., 2019) or limited solution strategies (Carpenter et al., 2015). Therefore, PTs must not only be aware of the 15 different structure types of additive word

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problems, they must also be skilled in solving and posing every type. Once familiar with the various one-step scenarios, teacher educators can discuss how to connect multiple scenarios together to form multi-step problems. At this point, PTs can be engaged in developing their skills in posing two-step problems by connecting a variety of problem types.

In mathematics, the literature has recommended the use of error analysis as a means for gaining a deeper understanding of student knowledge (Fleishchner & Manheimer, 1997; Luneta & Makonye, 2010; Raghubar et al., 2009; Seng, 2010). By analyzing the types of errors made by PTs when posing two-step problems, we can design intentional instruction to support PTs in recognizing common errors and understanding why such errors occur. We utilized the skill of error analysis to form the basis of our task design, which asks PTs to analyze a set of flawed, or negative, problems. Research shows that the use of negative examples, in addition to positive examples, can be helpful in teaching good writing habits (Grow, 1987). Instructional psychologists also recommend using negative examples to "prevent certain classification behavior errors" (Ali, 1981). As such, the researchers of this study used the results of our analysis of PTs error to create an interventional task containing a set of negative examples for PTs to analyze, prior to posing their own multi-step problems.

Methods

A team of researchers at a tier one research institution in the southern United States worked over several years to create and incorporate a variety of instructional activities that could effectively support PTs in developing problem posing skills. This work was in the context of an undergraduate mathematics problem-solving course with the purpose of helping PTs understand how to teach mathematics through problem solving (Alwarsh, 2018; Bostic et al., 2016; Chapman, 2017; Fi & Denger, 2012), which puts word problems at the forefront of the lesson and introduces new content through the problems themselves (Alwarsh, 2018). Therefore, problem-posing activities were integrated into this course to support PTs in developing the skills they will need for posing problems for their students – specifically single- and multi-step. The writing and analysis of these word problems creates a productive dialogue between the PTs and the course instructors that allows PTs to develop a deeper understanding of the meanings for the mathematical operations and prepares them for posing problems for their future students to solve.

In this course, PTs are originally introduced to the previously mentioned taxonomy of common addition and subtraction structures as identified by the CCSS (NGA & CCSSO, 2010) as a basis for discussing structural differences between addition and subtraction word problems. The PTs then work on posing one-step word problems to match each of 14 possible problem structures (the fifteenth structure, part-part-whole – both parts unknown, involves multiple unknowns and, for the sake of one-step solvability, is left out of posing instruction). Class activities then turn to focus on categorizing and solving two-step addition and subtraction word problems. PTs are then given a series of assignments in which they practice posing a variety of two-step word problems . Increasingly more complex pairs of structures are assigned to guide PTs' posing to push them beyond using the same or easiest structure types (e.g., PTs tended towards posing change – add to – result unknown and part-part-whole – whole unknown problems, two structures introduced in Kindergarten (NGA & CCSSO, 2010)). Table 1 shows the culminating task for this set of lessons, in which PTs are assigned to pose four two-step addition/subtraction problems to match the four provided pairs of structures.

The data included in this report was collected from a single instructor's course across two semesters (including 74 total PTs). All PTs were enrolled in teacher certification programs in the areas of EC-6 (Generalist) or grades 4-8 mathematics and science or English and history. The assignment in Table 1 resulted in 282 PT-posed, two-step word problems that were analyzed in this study.

Table 1: Two-Step Addition and Subtraction Word Problem Prompts

Prompt	Assigned Pairs of Structures
1	change – subtract from – change unknown; compare – fewer – bigger unknown
2	change – add to – start unknown; compare – more – bigger unknown
3	part-part-whole – addend unknown; compare – more – difference unknown
4	change – subtract from – start unknown; part-part-whole – addend unknown

The researchers used thematic qualitative text analysis to analyze the submitted word problems resulting in 124 correct problems that matched the assigned structures and 158 problems that showed at least one error. As each error was identified, a temporary category was created based on the number of steps required to solve each incorrect problem and whether the structures used matched the assigned prompt. Two researchers independently coded the corpus of 282 word problems and discussed any disagreements until 100% of the analyzed word problems had been assigned an agreed upon code.

Results

Phase 1: Error Analysis

As previously mentioned, 124 of the submitted word problems correctly posed a question that required a two-step calculation utilizing two scenarios that both matched the assigned structures. The remaining 158 PT-submitted problems were coded as having one or more errors. The analysis of these errors led to the identification of nine distinct categories of error patterns (see Table 2), dependent upon the number of steps required to solve the problem, the appropriateness of the question(s) asked, and the use of the assigned structures. Below we will introduce each category of error pattern by providing examples of PTs' work and highlighting where the error occurred.

Two-step – Incorrect structure(s). Fifty-six of the problems correctly posed a question that required two connected steps that required addition or subtraction and were only deemed to be flawed because they did not entirely match the assigned prompts. For example, one such problem submitted in response to the first prompt read:

> Lauren has five apples on the table and Henry has two apples on the table. Lauren gives some of her apples to Henry. Lauren now has three apples. If Lauren now has less apples than Henry, how many apples does Henry have now?

Although the first step correctly matches the first assigned structure (change – subtract from – change unknown), the second step is a change – add to – result unknown scenario (instead of compare – fewer – bigger unknown). The question requires the solver to take the two apples that Lauren lost and add them to Henry's original two apples. There is no comparative structure included in this problem.

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Table 2: Frequency of Error Type Exhibited by PTs

Category	Sub-Category	Frequency
Two-step	Incorrect structure(s)	56
One-step	One question: Two correct structures	31
	One question: Incorrect structure(s)	32
	Two questions	5
More than two steps	Including two correct structures	8
	Incorrect structures	4
Zero-step		3
Algebra-style		4
Unanswerable		15
	Total	158

One-step problems. With 43% of the total errors, the most common error group resulted from PTs who could build up two addition/subtraction structural situations but could not properly combine the unknown values from each scenario into a question that required a two-step calculation. These problems can be separated into three subgroups: problems that used scenarios matching both of the assigned prompts , problems that used at least one incorrect scenario, and problems that asked two independent questions.

About 50% of the 63 one-step, one-question problems built contextual scenarios that correctly matched both of the requested problem structures. The remaining 50% of the one-step one-question problems were flawed in the sense that at least one scenario did not fully match the requested structures. However, due to a lack of connection between the two scenarios, in both of these groups, the question posed only required one calculation to be solved. This tended to result from a known from the first scenario being used in the second calculation. One example of such a problem was submitted to Prompt 2. The PT wrote:

> Sarah had some pieces of candy. Four more pieces were given to her, so she had ten pieces of candy total. Amanda had five more pieces of candy than the amount of candy Sarah was given. How many pieces of candy does Amanda have? (Problem 1)

In order to answer the question that was posed, the solver would use the four pieces of candy that Sarah was given and add the comparative difference of five to reach Amanda's total of nine. This only requires one calculation, i.e. step. The PT did correctly set up the structures from the prompt but did not connect the unknown from the first scenario to the second.

A second example of a one-step problem was submitted to the first prompt. It read, "Some balloons were blown up. Two of them were popped. Only three remained. Two of them were red and the rest were blue. How many were blue?" (Problem 2). Both scenarios match the prompts, but notice that with the three remaining balloons, "two of them were red and the rest were blue," only requires one calculation to solve (3-2=1). This PT developed a scenario where the two unknowns could have been connected by referring back to the original set of balloons, but instead posed a question that didn't require the solver to utilize the unknown information from the first step (the original number of balloons) in the second step. The use of a known from one scenario to set up the structure of a second scenario was not specified as an error in the table but was by far the most common issue in one-step problems. PTs could create scenarios matching the given prompt structures independently but experienced the "composition effect" (Heffernan & Koedinger, 1997) of not being able to merge two unknowns into a single question.

In the remaining five of these 68 problems, the poser wrote two separate questions, each only requiring a single step to solve. Four of these problems included the correct assigned structures, one did not. The PTs who wrote these problems knew that two steps were necessary to satisfy the given task but showed difficulty in connecting their unknowns into a single question. One such example is, "Tommy had seven envelopes, then he lost some of them. Now he has four envelopes. Tommy has five fewer envelopes than his friend Milton. How many envelopes did Tommy lose? How many envelopes does Milton have?" (Problem 3). The two, single-step questions here are clear, but a well-developed, two-step problem must require the solver to complete two connected calculations by only asking one question.

More than two steps. Twelve of the remaining problems successfully posed a valid multi-step addition/subtraction problem but asked a question that required three or more steps to answer. Eight of these problems correctly included both of the assigned structures, but the PTs were unable to formulate a question that only used the information found in those two steps. An example of this that was submitted to Prompt 1:

> Lucy has 3 fewer cookies than Julie. Lucy has two cookies. If Lucy and Julie both put their cookies in a jar but someone takes some cookies and leaves only 3 cookies in the jar, how many cookies were taken? (Problem 4)

The first step in solving this problem is a compare-fewer-bigger unknown (which matches the second assigned structure) where the solver calculates Julie's amount. The second step is a part-part-whole where Lucy and Julie merge their cookies together. Once the solver finds the joint number of cookies, some of those cookies are taken out. A third step then asks the solver to find how many cookies were removed, a change – subtract from – change unknown scenario (which matches the first assigned structure).

Zero-step problems. Three problems posed by PTs provided the answer to the posed question within the problem statement. These were labeled at zero-step problems since no computation was necessary to answer the posed question. In one example of a zero-step problem, the PT wrote, "I had some cookies. I got seven more cookies. Now I have 13 cookies. Courtney has some cookies. I now have seven less cookies than Courtney. How many cookies do I have?" (Problem 5). There are no steps required to solve this problem as the answer lies in the statement ("now I have 13 cookies"). This was the least frequent of the error patterns.

Algebra-style problems. A similarly small group of four problems offered scenarios for two unknowns that were not able to be solved by simply computing two sequential one-step arithmetic calculations. Instead, these problems provided information that connected the two unknowns in two different ways making the question posed answerable but requiring the solver to apply more complex algebraic thinking. For example, the following problem was submitted to the Prompt 2:

> Sally had some pencils. Then her mom gave her three more pencils to take to school. When she got to school Ruby had 3 more pencils than Sally. Altogether they had 13 pencils. How many pencils did Sally have when she got to school?

The given information could be represented with two numerical equations, but, in each equation, both quantities are unknown (i.e., $S + 3 = R$ and $S + R = 13$).

Unsolvable problems. The remaining 15 problems posed were deemed unsolvable as there was no clear way to actually answer the question that was asked. However, this error occurred due to multiple reasons three examples are given below. The first problem was unsolvable because no question was asked. This scenario was submitted for the second prompt, "Santa wrapped some presents. His wife helped him wrap 6 more, now there are 10 presents. Santa's elf helped too, he wrapped 2 more presents than Santa did initially" (Problem 6). The context was set up perfectly for the prompt, a question just needed to be asked.

The second example of an unsolvable problem was submitted for Prompt 4. The PT wrote, "Some oranges were on the plate, 3 were big and the rest were small. I ate 3 oranges. Then there were 2 oranges left on the plate. How many small oranges are left?" (Problem 7). This problem is missing contextual clarification. The poser chose to distinguish small oranges from big oranges, but then went back to discussing "oranges" in general. In order to solve the problem, the reader would need to know what size of oranges were eaten and what size of oranges remained at the end.

The final unsolvable problem example was submitted to Prompt 2. The PT wrote, "Nine bananas were on the table. Five were green and the rest were yellow. Then my mom came and took 2 of the yellow bananas. How many more bananas do I have than my mom?" (Problem 8). In order to confidently solve this problem, the solver needs more information about who each set of bananas belongs to. It is not clear whether all nine of the beginning bananas belong to the problem poser or the family in general.

Phase 2: Task Design

The results of the error analysis provided the researchers with valuable information regarding the challenges faced by PTs when posing two-step addition and subtraction problems. The researchers utilized this information to develop an interventional task for use with future PTs. The task was designed to bring awareness of common errors made when posing two-step problems to PTs prior to having them pose their own two-step problems. The task offered a set of flawed, or negative, problems that PTs would be asked to analyze. To create authenticity, we used example problems from our error analysis instead of constructing our own. The PTs were presented with a set of word problems and the following instructions:

> Imagine that you asked your students to write a 2-step addition/subtraction word problem and below are some of the word problems they wrote. For each word problem, explain to the student why their problem is incorrect or incomplete and briefly discuss how it could be fixed.

To create authenticity, we used the results from our Phase 1 analysis to identify key examples of PT-written word problems exhibiting the identified errors and used these on the task instead of constructing our own flawed problems. In developing this task, the researchers chose to focus their attention on PTs' ability to create well-written, two-step problems rather than on their ability to match a specified set of structures. For this reason, none of the 56 PT-submitted questions that correctly required two steps to solve (but were deemed flawed because the structures were not entirely matched) were included in this error analysis task. Additionally, in this initial version of the task, the researchers chose to focus only on problems that were written arithmetically. At that time, algebra-style problems were not being covered in the course, so those types of errors were excluded from the error analysis task as well. The task included eight questions previously coded as having a structural error (labeled as Problems 1-8 above).

Discussion

Implementation of Task

The error-analysis task created as a result of this analysis was implemented in the following semester. The task was strategically placed between two lessons that ensured PTs had already been exposed to analyzing two-step addition and subtraction word problems but had not yet attempted posing their own two-step word problems. Leading up to the error-analysis task, PTs had spent time learning about the 15 categories of addition and subtraction word problem types, practiced posing one-step addition and subtraction word problems, and categorized and written number sentences to solve two-step addition and subtraction word problems. The error-analysis task was assigned for PTs to complete individually outside of class time. Each problem on this task and PTs' responses to this task were discussed in the following class meeting. After this work, PTs began posing their own two-step addition and subtraction problems. As they practiced posing two-step word problems, first with addition/subtraction and later with multiplication/division and mixed operations, the errors described in this task continued to be identified and discussed.

Preliminary Task Results

Initial analysis of the task's first implementation showed promising results in terms of PTs being able to accurately identify most of the intended errors on the task and to identify common errors later when reflected in their own and others' posed problems. However, the researchers found that four of the problems were not demonstrating the errors strongly enough. During a preliminary qualitative analysis of the results, researchers aimed to identify the cause for PTs' difficulties in identifying the intended errors in the word problems on the error-analysis task. The researchers observed that Problem 2 was missing contextual information which distracted PTs from the intended error of it being a one-step problem. On Problem 4, a three-step problem, PTs focused on the tense of the verbs in the word problem and the order of which the numbers appeared; for example, some thought the result would be a negative amount of cookies. Similarly, on Problem 1, PTs seemed confused by the verb tense used throughout the problem and this took focus away from the intended issue that it was only a one-step problem. Finally, Problem 3, an example with two individual questions, also has the error of one of the knowns from the first step being used in the second step, making it a one-step problem. Although it was useful for PTs to point out all of these issues, these four problems were not adequately focusing PTs' attention on the type of error we intended the problem to highlight. Therefore, all four of the problems were traded for other PT-submitted problems that were coded under the same error category during the original data analysis. The instructions were also adjusted to reflect the fact that PTs' responses tended to offer fixed versions of the problems instead of identifying the error. The PTs were more explicitly asked to identify the error made, without fixing it, and to provide an explanation that would help a peer understand why the problem is not a two-step arithmetic addition and subtraction problem.

Future Research

The modified version of this task is being implemented in current sections of the course. Data is being collected and the researchers will analyze results from the implementation of the updated error-analysis task for future use and improvement. Additionally, researchers will perform a comparative analysis of PTs' posed word problems from the semesters in which the interventional task was implemented to previous semesters to investigate the possible implications the task may have had towards supporting PTs' problem-posing abilities through raising their awareness of common error patterns.

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