TEACHERS' REPRESENTATIONAL AND CONTEXTUAL JUSTIFICATIONS FOR SELECTING PEDAGOGICAL REPRESENTATIONS

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There have been many efforts to measure pedagogical content knowledge with multiple-choice survey instruments, but little is known about how different types of items contribute. In this study, we examined interviews with 9 Grade 4 teachers to develop a deeper understanding of how teachers select pedagogical representations in the context of a survey assessment. Our analysis revealed two broad themes: representational justification (focused on how teachers interpreted features of the representation) and contextual justification (focused on how teachers considered their students and their own perspectives and experiences). These results indicated that content and pedagogical knowledge were highly intertwined in teachers' work on these tasks. However, the results also identify limitations for using this item type to measure teachers' pedagogical content knowledge in mathematics. Implications are discussed.

Keywords: Assessment, Elementary School Education, Mathematical Knowledge for Teaching, Rational Numbers.

In the time since Shulman (1986, 1987) first described pedagogical content knowledge (PCK) as an important part of the teacher knowledge base, the term has gain wide currency and accumulated a large body of scholarship. In mathematics education, perhaps even more than in other areas, concerted effort over the last two decades has been made to measure PCK and the related domain of mathematical knowledge for teaching (MKT) with survey instruments comprising multiple-choice items (Hill et al., 2005; Saderholm, et al., 2010).

Multiple choice teacher knowledge items are often written to measure specific categories within the domain of PCK or MKT, but the responses to a variety of different types of items all contribute to the same overall score on these instruments. Conceptualizations of teacher knowledge and psychometric results from several independent projects suggest that PCK and MKT are multidimensional constructs even though they have been measured primarily with unidimensional scales (Jacobson, 2017). Qualitative studies of teachers' written responses (Fauskanger, 2015) and teacher interviews (Lai & Jacobson, 2018) have revealed that teachers often draw on more than one category of knowledge when answering multiple-choice survey items. Even so, such items are still widely used because they provide an efficient means to assess teacher knowledge at scale.

MKT and PCK items often pair a specific mathematical topic with a pedagogical decision about that topic. Ball et al., (2008, p. 400) list 16 different "mathematical tasks of teaching" around which assessment items could be written such as, "evaluating mathematical explanations" and "modifying tasks to be easier or harder." Items on existing MKT and PCK instruments include a wide selection of these tasks of teaching, but little is known how well each type of task

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reflects teachers' knowledge. To investigate this issue, we designed a set of PCK items that share a single task of teaching: selecting pedagogical representations.

Theoretical Framework

In this study, we describe our efforts to use multiple choice survey items to efficiently assess teachers' knowledge of pedagogical representations, one of the two kinds of PCK described by Shulman (1986). Rather than framing our analysis in terms of PCK categories (and several different categorizations have been used in prior research), we ground our work with a foundational idea in one of Shulman's original articulations: PCK is an "amalgam of content and pedagogy", "the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners" (1987, p. 8). This study is part of a larger project to design assessments of teachers' PCK and to link teachers' scores on these assessments to classroom teaching and student learning.

We focus on pedagogical representations because of their central importance in mathematics instruction. Mathematics is inaccessible by direct experience, hence, teaching and learning are necessarily mediated by representations (Bruner, 1966; Duval, 2006). Following Cai and Lester (2005), we make a distinction between solution representations used by problem solvers to make sense of a problem and communicate their solution to others and pedagogical representations which are "the representations teachers and students use in their classroom as carriers of knowledge and thinking tools" (p. 223). These two kinds of representations can overlap, but the critical difference is that teachers are uniquely responsible for curating the pedagogical representations available in their classrooms to ensure they are both mathematically accurate and comprehensible to learners (Cai & Lester, 2005; Leinhardt, 2001).

The dual nature of teachers' responsibility vis a vis pedagogical representation is aligned with the dual constituents of PCK understood as an amalgam of content and pedagogy. To select a mathematically accurate representation, teachers need to attend to the way features of the representation map to features of the problem situation and the underlying mathematical ideas. Content knowledge is implicated in this attention: for example, teachers must know to check that a representation which purports to show a fractional quantity is equally partitioned. Mathematical accuracy is necessary but insufficient. Teachers must apply their pedagogical knowledge to recognize which representations are unfamiliar or confusing for students and thus provide more hindrance than help. Teachers who coordinate their knowledge of content and pedagogy to select accurate, comprehensible pedagogical representations demonstrate PCK.

The research question which guided this study was, *To what extent do teachers' rationales for selecting a pedagogical representation reflect PCK versus other factors unrelated to PCK?*

Method

Item Design

We designed eight items (denoted Q1 – Q8) to assess Grade 4 teachers' proficiency in selecting pedagogical representations for fraction and decimal instruction. Each item consisted of a mathematics problem and two diagrams illustrating different pedagogical representations for the mathematics problem (see Figure 2). The mathematics problems were aligned with state standards for Grade 4 fraction and decimal topics. The pedagogical representations were based on interviews with teachers and teacher educators, a review of fraction and decimal literature in practitioner and research journals, and a review of student thinking and misconceptions with

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fraction and decimal topics. Teachers responded by indicating whether or not the representation in each diagram was useful for students who were learning to solve the problem (A, B, both, or neither).

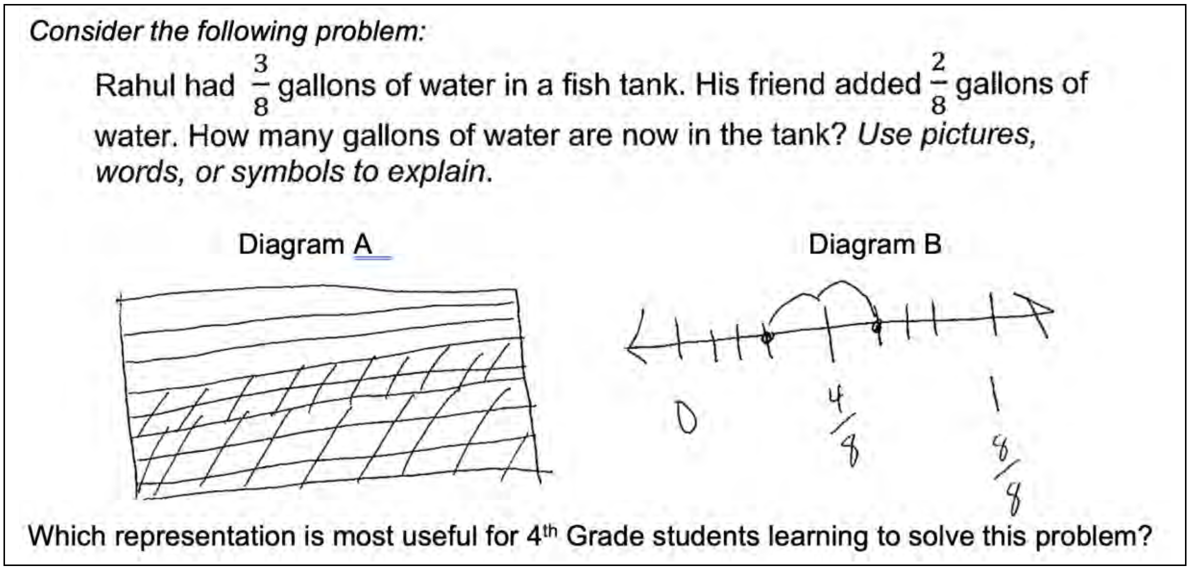

Figure 1. An example item for selecting pedagogical representations. Teachers could select A, B, both, or neither diagram.

The representations were designed with key features for teachers to notice that had either mathematical or pedagogical implications. All diagrams included representations of numbers in the problem, the solution, or both. The diagrams we intended teachers to endorse had representations that were both pedagogically warranted and mathematically accurate. Other diagrams that we intended teachers would not endorse included features that made them inaccurate or unclear.

We included one or more key features to make representations inaccurate. One inaccurate feature was using a representation that had a different sized whole for two fractions in a problem that were supposed to be commensurate. Another inaccurate feature had to do with the problem-representation alignment and focused on teachers' ability to distinguish between semantic equivalence of a problem and representation which can influence children's comprehension and the mathematical equivalence that an adult (but not a child) might use to solve a problem. For example, students would likely struggle to recognize a word problem describing 36 copies of 1/8 in a representation show 36 shared equally into eight groups, although an adult might use 36/6 as a way to compute 36 x 18. The third inaccurate feature was mathematical errors. For example, in one diagram we made a representation with a circle partitioned into non-congruent sections.

We also included key features to make diagrams unclear. (Although inaccurate representations are also unclear, here we describe features that are accurate but unclear.) One feature of unclear diagrams were representations that displayed the result instead of showing the process. For example, one unclear representation for a fraction comparison was an open number line with correctly labeled points but without any benchmarks or regular partitions by which order could be found. Another unclear feature was the alignment between the representation and common solution strategies students might use. For example, students often use a think addition strategy to solve change unknown subtraction problems. Teachers selecting a representation for a

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subtraction problem who were unaware of this strategy might inappropriately reject a representation that apparently showed addition. The third unclear feature was whether the representation illuminated or obscured a primary learning goal of the problem. Considering students who are learning how to add fractions with different denominators, a representation that already shows common denominators is less clear than a representation that also shows the original fractions.

Data Collection and Sample

Nine Grade 4 teachers (all White females), were selected from an economically and geographically diverse set of schools in a Midwestern State to complete a think-aloud item response interview. Teachers were in 7 different elementary schools across 5 different counties. Student eligibility for free or reduced lunch in these schools ranged from 8% to 85%, and enrollments of White students ranged from 64% to 98% across schools. Rates of students scoring at or above "proficient" levels on state mathematics tests ranged from 40% to 79% across the schools, with state school rankings ranging from the bottom 50% to the top 10%. This sample provided an opportunity to study the scope of teachers' thinking across a wide range of schooling contexts. Pseudonyms were used to report results.

Teachers answered the items and justified their representation selection. All interviews were video-recorded and transcribed. Because we limited interviews to 90 minutes and teachers varied in the time it took to discuss each item, not every teacher answered every item. Two teachers answered all 8 items, four answered 7 items, one answered 6 items, and two answered 5 items. In all, there were 60 item responses across all teachers. We used the selection (non/endorsement) and justification for each diagram (two diagrams per item per teacher) as the unit of analysis for this study. Thus, the size of the analytic sample was 120 diagram responses and the corresponding justifications.

Data Analysis

We first summarized teachers' responses and justifications with direct quotes (e.g., Vicky did not endorse Q1, Diagram 2 (a number line) because, "they [students] don't think about physical placement on the [number] line."). Then we conducted an inductive thematic analysis to analyze patterns in teachers' justifications, by reading the summaries and developing an initial codebook (Boyatzis, 1998; Rice & Ezzy, 1999). We then modified the codebook to remove ambiguities and overlapping codes. Two authors used a new codebook to iterate the code-reconcile-modify cycle until they arrived at a consensus on all codes. Once the codes were finalized, we reviewed each group separately considering to what extent the responses illustrated content knowledge, pedagogical knowledge, or an amalgam of the two.

Findings

Our analysis revealed two broad themes in teachers' response justifications: (1) some teachers based their justifications on the key features of the representation (as we had intended) and (2) some teachers based their justifications on contextual factors that teachers supplied from their own knowledge of students, the curriculum, and from their teaching experience. Among the 120 responses, responses were as likely to be justified based on features of the representation (*representational justification*; 56/120) as they were to be justified by non-diagrammatic concerns (*contextual justification*; 55/120). The themes were not strictly exclusive: 9 responses included both representational and contextual justifications. On some responses (18/120), teachers did not provide justification.

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Representational Justification

Three subthemes emerged from the analysis of teachers' representational justifications. Teachers *interpreted key features*, *attended to some key features* of the diagram (while overlooking others), or *misinterpreted key features*.

Interpreted key features. In this subtheme, teachers noticed the key features of the representation and they endorsed (or did not endorse) a representation as pedagogically useful because of the mathematical or the pedagogical affordance of the representation in light of these key features ($n = 41$).

We considered teachers' interpretation reasonable if they noticed the key features and provided an explicit, coherent justification to support their non/endorsement of a specific diagram. For example, Sarah justified why Diagram 2 in Q1 was not useful by observing that:
Although the number line is a good measurement, you have no benchmark. The number line is very abstract for them [students] to say this is 3/8 without benchmarks such as one and a half. When denominators are different, the number line with benchmarks is still hard.
In this example, Sarah noticed that the number line did not have benchmarks, something we left out intentionally to make the representation unclear. (Note that this teacher also comments on student difficulty in her response, an example of contextual justification which is discussed in the next section.)

Similarly, Amy asserted that Diagram 2 in Q4 was not useful "because some of the kids are going to count the number of coins, not the value of coins". Another teacher, Vicky, said "Well, some of them might see so many pennies and say that [pennies] must be more than that [dimes]." We chose to use coins because values in this kind of representation are based on convention instead of an observable quantity like count, length, or area. This key feature makes the representation unclear for some students because it obscures a primary learning goal of the problem: the relationship between tenths and hundredths. These teachers closely attended to the key features of the representations and used these features to justify their non/endorsement of the representations.

Attended to some key features. In this subtheme, teachers paid attention to some key features of the representations while overlooking others ($n = 12$). For example, in Q1, Rose noticed that both diagrams (bar model and number line model) showed how large a fraction is compared to one whole: "We can see that 3/¾s getting closer to one whole on the number line..Ind of like a bar graph"; however, this teacher overlooked the absence of benchmarks on the number line. This situation also occurred when teachers endorsed mathematically inaccurate diagrams. For example, a set model representation in Q5 incorrectly added denominators, and both Brooke and Molly apparently overlooked this feature while asserting that this diagram would be useful for students.

Misinterpreted key features. This subtheme captured responses of teachers who noticed the key feature of the diagram but interpreted them incorrectly ($n = 3$). This code was prevalent for Q2, Q3, and Q5 because one of the options was designed to be mathematically inaccurate. For example, three teachers (Megan, Vicky, and Cathy) asserted the usefulness of a non-equipartitioned circle model (Q3, Diagram 1). Vicky said students "would be able to figure it out pretty easily and tell." Megan reasoned: "With the circle graph, that might give a little trouble…but I think they could see that this is a fourth out and this is a fourth out." Teachers noticed the representational feature but still considered the diagram to be useful for their students despite its inaccuracy.

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Overall, responses in this category illustrated the potential for this kind of teacher knowledge task (selecting pedagogical representations) to assess teacher knowledge for using representations. Teachers who interpreted key features of the representations demonstrated pedagogical content knowledge whereas teachers of overlooked or misinterpreted key features did not.

Contextual Justification

The theme of contextual justification includes reasoning that influenced teachers' responses but had less to do with the features of the representations than with the teachers' own context and professional experience. Four subthemes emerged within the theme of contextual justifications: students' familiarity, students' competence, teachers' competence, and teachers' preference.

Students' familiarity. These non/endorsement responses were justified based on teachers' knowledge of students' familiarity with the representation (*n* = 30). Teachers used phrased like "used to", "have seen", or are "familiar with" while stating that their students' had a high degree or lack of familiarity with a representation. They used phrases like "never used" or "[they] haven'tcused those much" to indicate students' low degree of familiarity with a particular representation. In general, this category was not correlated with identifying inaccurate or unclear representations because key features were used to modify both frequently used (e.g., hundred grid) and less-frequently used representations (e.g., abacus) when we designed the items.

Students' Competence. Some teachers justified their diagram selection by anticipating students' competence (*n* = 13). Teachers justified their non-endorsement with statements, like, "They [students] don't think about physical placement on the line"," "number lines are very, very difficult for my fourth graders"," "even though Diagram 2 showed the whole, kids might not understand the pictures unless an explanation was given" and "the diagrams had too many things going on, which might be confusing to some students." sometimes, teachers selected a diagram over another by anticipating which one would be easier for their students (e.g., Laura chose Diagram 1 in Q7, stating "it might be easier to use [than Diagram 2]".)

Sometimes, teachers who were focused on students' competence seemed to neglect the pedagogical value of some representations that might be challenging to use, such as number lines. At other times, the anticipation of student struggle was associated with an unclear or inaccurate representation, and we could not determine whether these teachers may have had some tacit understanding of the problematic features of these representations. On the other hand, from our position as teacher educators, we found this stance towards selecting pedagogical representations concerning because it may lead to a classroom experience with only a very limited set of mathematical representations. Endorsements based on students' competence were weakly correlated with identifying inaccurate or unclear representations, and therefore can be taken as an indication of pedagogical content knowledge.

Teachers' competence. Teachers sometimes used their own anticipated competence with the representation to justify their selection (*n* = 7). In a typical example, Cathy explained why the abacus in Q6 would not be useful by stating "That one confuses me so I wouldn't even know how to explain it to my kids." Teachers also used phrases like "a little harder [for me] to understand" to explain why they thought the representation would not help their students.

Teachers' preference. A small number of teachers justified their selection based on their personal preference (*n* = 5). Teachers made statements such as "I like number line(s)" (Brooke, Q1) or "I am a fan of rectangle diagram(s)" (Sarah, Q1), usually without offering a rationale based on specific features of the representation. Only one teacher (Rose) justified their preferences based on the features of the diagram. Rose preferred Diagram 1 on Q2 because the

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zoomed-in portion of the number line representation allowed students to "see not only are tenths smaller than one but then each tenth is also broken down into smaller and smaller pieces." She continued "I haven't used this [number line with zoomed-in portion], but I like this." The teachers who made choices based on either their sense of competence or their preference often overlooked problematic key features, suggesting that both of the last two subthemes illustrate reasoning which was associated with a lack of pedagogical content knowledge.

Discussion

In this study, we examined interviews with nine Grade 4 teachers to understand how teachers select pedagogical representations in the context of a survey assessment. Our unit of analysis—a teacher's endorsement and justification of one diagram for one mathematical problem—provided a fine-grained tool to examine patterns in the variability across teachers and items. We present results that summarize 120 justification responses into two main themes with seven subthemes. These themes emerged from our analysis provide a comprehensive description of how teachers justified their decisions about the usefulness of specific pedagogical representations for particular mathematical problems.

Early in the analysis, we expected that responses involving representational justifications would provide opportunities for us to consider how content knowledge was used to select pedagogical representations, and that responses involving contextual justifications would help us understand how pedagogical knowledge was used. However, content and pedagogical knowledge were more intertwined in our data. As the example of dimes and pennies illustrates (Q4, Diagram 2), the largest subtheme—interpreted key features—involved some teachers who had made a sound, pedagogically-informed justifications for key features of representations which we had designed to be problematic. Similarly, contextual justifications sometimes provided information about teachers' content knowledge; the teacher who liked the zoomed-in number line representation did so because she saw how features of the representation mapped to important ideas about the content. These findings suggest that selecting pedagogical representations is an item type is well suited to engage teachers in reasoning that draws on pedagogical content knowledge.

This study offers researchers useful insights for developing and validating assessments of teachers' 'CK for selecting representations. For the most part, the interviews provided information about teachers' PCK that was aligned with what we would have inferred from their representation selection. However, the responses also illustrate the variety of reasoning that teachers used to justify their selection of pedagogical representations, including some justifications that were based on inaccurate mathematical thinking and inadequate pedagogical knowledge. An even larger number of responses were based on the teachers' personal experience and context, factors that are not related directly to the pedagogical content knowledge we aimed to assess.

The prevalence of justifications based on teachers' context and professional experience when selecting pedagogical representations is an important discovery because it highlights factors that are consequential to teachers' responses but are beyond the scope of survey item design. Every teacher sees the same text and inscriptions of an item, but these data reveal how the different experiences they have had with students, curriculum and their professional training provide a lens through which they interpret the task and their own response.

In practice, the best instructional decision when selecting a pedagogical representation will always depends on the context. A teacher with high knowledge may be able to use a new representation effectively, whereas a teacher with low knowledge may not be able to use a new representation without training. In such cases, it would be better for students to experience a familiar representation with mathematical accuracy than have the content misrepresented because of an unfamiliar representation. Even knowledgeable teachers' must balance the trade-off between the time and effort of investing in a representation that is unfamiliar to students and whatever pedagogical gains they anticipate once students have adopted the new representation. If the focus is on a new mathematical idea, then a familiar representation will help support student understanding, whereas an unfamiliar representation may impede students understanding by putting students in the position of needing to learn both the new mathematical idea and a new kind of literacy required to write and read with the new representation.

Can multiple choice survey items be designed that support valid inferences about teachers' PCK for selecting representations even though teachers' personal experience and context are equally salient as the key features of the representations in question? The responses that were justified by teacher's preference do not present a problem for this goal because they were generally aligned with low PCK. However, the remaining subthemes of contextual justification are all problematic to some degree. Responses based on the student competence subtheme are the least problematic, because these responses sometimes reflect PCK. On the other hand, these responses seem to come from a stance toward teaching and learning we find concerning because it might limit students' education experiences. Responses based on student familiarity—by far the most frequent contextual justification—are the most problematic, because these responses would reduce the accuracy of each item by adding noise to the signal. Noisy instruments are inefficient and must have more items to reach an acceptable level of reliability. If the noise could be mitigated through careful item design, it might be possible to build a trustworthy instrument to measure PCK for selecting representations. Addressing the important question of whether such a difficult task is possible is beyond the scope of this paper and will certainly require additional empirical work which we have begun to undertake.

The contribution of this study is to illuminate teachers' reasoning across a set of multiple choice teacher knowledge items of the same type: selecting pedagogical representations. Without a set of items with the same design focused on the same pedagogical decision and varying only in content (i.e., the mathematics problem, the representations, the key features), we could not have drawn generalizations about how teachers' reason about this type of item. In prior work, a PCK or MKT instrument might have had one or two items of this type among 20 to 30 items on an instrument, and interviews with teachers (e.g., Fauskanger, 2015) across a diverse set of items would not have supported the kind of discoveries we report about the range of teacher reasoning on a single type of question. We expect that there are important affordances and limitations of many other types of MKT and PCK items commonly in use that are not known because they have not been adequately investigated.

The data we have presented is drawn from a small qualitative study. The present study is focused on better understanding how teachers select pedagogical representations, and whether inferences about teacher knowledge can be drawn from multiple choice surveys about representation selection. The larger project engages in similar work for three additional kinds of pedagogical decisions. Further work is ongoing to examine whether these themes generalize to an independent sample of 40 preservice and inservice teachers.

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