CHARACTERIZING PROSPECTIVE SECONDARY TEACHERS' FOUNDATION AND CONTINGENCY KNOWLEDGE FOR DEFINITIONS OF TRANSFORMATIONS

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One promising approach for connecting undergraduate content coursework to secondary teaching is using teacher-created representations of practice. Using these representations effectively requires seeing teachers' use of mathematical knowledge in and for teaching (MKT). We argue that Rowland's (2013) Knowledge Quartet for MKT, in particular, the dimensions of Foundation and Contingency, is a fruitful conceptual framework for this purpose. We showcase an analytic framework derived from Rowland's work and our analysis of 85 representations of practice. These representations all featured geometry. We illustrate examples of combinations of "high" and "low" Foundation and Contingency, and show results of coding juxtaposed with performance on an instrument previously validated to measure MKT. We describe the potential for generalizing this framework to other domains, such as algebra and mathematical modeling.

Keywords: Mathematical Knowledge for Teaching, Undergraduate Education, Preservice Teacher Education

Many secondary teachers find their undergraduate mathematical preparation disconnected from their teaching (Goulding et al., 2003; Ticknor, 2013; Wasserman et al., 2018; Zazkis & Leikin, 2010). The mathematicians who teach these teachers may want to connect content coursework to secondary teaching (e.g., Lai, 2019; Lischka et al., 2020; Ticknor, 2013). At the same time, mathematics faculty may lack the resources to say precisely what connections may be there, and how to give feedback to teachers regarding the connections (Lai, 2019).

In recent years, several groups have addressed this problem by developing (a) tasks for content courses where teachers create representations of practice, and (b) design principles for such tasks (e.g., lvarez et al., 2020; Wasserman et al., 2019). Mathematics faculty can now use these principles to create such tasks, but not necessarily to provide constructive feedback to teachers about their responses. To our knowledge, the field lacks frameworks for characterizing the mathematical knowledge in and for teaching (MKT) observable in representations of practice created by teachers, in ways that would support feedback. Such frameworks could position mathematics faculty to bridge mathematics and teaching more powerfully.

Our purpose is to characterize dimensions of MKT visible in teacher-created representations of practice, and to do so in a way that can potentially inform feedback to prospective teachers about their mathematical understanding and its use in teaching. In our work, representations of practice are snapshots of discourse used in responding to student contributions. Using such representations, created by prospective teachers, we asked: *What MKT is observable in teacher-created representations of practice?* We argue that Rowland's Knowledge Quartet framework for MKT is a productive analytic framework for analyzing representations of practice. We contribute a framework for observing "high" and "low" levels of knowledge in two dimensions of the Knowledge Quartet, namely, the Foundation and Contingency dimensions. Our corpus consists of teacher-created representations of practice featuring geometry from a transformation

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perspective. We conclude by considering how our work can generalize to other mathematical domains.

Conceptual Perspective

Mathematical Knowledge in and for Teaching (MKT)

Across the various literatures on MKT (e.g., Ball et al., 2008; Davis & Simmt, 2006; Heid et al., 2015; Rowland, 2013; Thompson & Thompson, 1994) and on mathematics learning (e.g., Daro et al., 2011; National Academies of Sciences, Engineering, and Medicine, 2018; Simon, 2006; Thompson, 2000), we have found ideas of Rowland and colleagues (2013, 2016), Simon (2006), and Thompson (2000) most generative for our work.

Among the four dimensions that compose Rowland's (2013) Knowledge Quartet framework for MKT, we focus on two: *Foundation* (knowledge and understanding of mathematical ideas, the nature of mathematics, as well as principles of mathematical pedagogy) and *Contingency* (the ability to respond to unanticipated events ranging from network outages to learners' alternative strategies). The remaining dimensions are *Transformation* (presenting ideas to learners) and *Connection* (cohering ideas over time). Foundation knowledge is observed through actions associated with the other dimensions, as well as in teacher talk outside of teaching (e.g., in a debrief of student teaching). Rowland, Thwaites, and Jared (2016) validated the use of this framework for identifying instances in student teaching at the secondary level where an observing teacher educator can infer the use of content or pedagogical content knowledge.

Rowland and colleagues used videos of teaching across multiple topics in multiple schools. We examined teacher-created representations of practice responding to a limited set of prompts. Hence, we found it useful to delimit and elaborate on Foundation and Contingency as follows. First, we delimited the Foundation dimension to knowledge of mathematics, because of our interest in content coursework. Second, the dependence of Foundation on *mathematical understanding* suggested that we be theoretically clear about a conception of mathematical understanding. We used Simon's (2006) characterization: mathematical understanding is the "learned anticipation of the logical necessity of a particular pattern or relationship" (p. 364). For instance, we consider understanding mathematical procedures to include relating that procedure to its underlying definitions or concepts, and to anticipate doing so when explaining procedures or troubleshooting a use of a procedure. Then, we delimited Contingency to the ability to integrate given student thinking into teacher talk. Finally, we used Thompson's (2000) notion of interacting reflectively to elaborate on integrating student thinking (cf. Ader & Carlson, 2018). When teachers interact reflectively, they interpret and leverage student thinking. When teachers interact unreflectively, they do not adopt the student's perspective.

Rowland and colleagues' work results in a framework for identifying instances where MKT may be used, but it does not result in a framework for characterizing levels of use such as would be useful for guiding an instructor to provide feedback to a teacher.

Teacher-Created Representations of Practice

In all assignments we analyzed, prospective secondary teachers created representations of practice based on a description of a teaching situation provided to them, where the teaching situation included various samples of student work. These representations of practice may be considered an approximation of practice in Grossman et al.'s (2009) framework, meaning that they are "opportunities for novices to engage in practices that are more or less proximal to the practice of a profession" (p. 2058).

Mathematical Context for Analysis

A transformation perspective is characterized by defining congruence and similarity via transformations (Usiskin & Coxford, 1972). The transformations critical to congruence and similarity are reflections, rotations, translations, and dilations. Across the two units in the materials used in this study, prospective teachers developed community definitions for reflection, rotation, translation, and dilation. Then, prospective teachers used these definitions to construct images of these transformations, as well as to determine whether two proposed figures can be connected by one of these transformations.

There were four prompts for creating representations of practice in the modules. Two prompts asked teachers to video-record themselves, and two asked teachers to write narratives. All prompts provided images of secondary student work and asked prospective teachers to respond in a way that would move students toward understanding connections between definitions and constructions of images of relevant transformations. All prompts provided the secondary level task that the sample secondary student work was responding to. Figure 1 shows images of student work from some of the prompts.

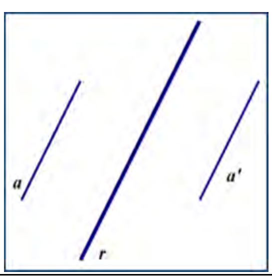
(Sample for prompt focused on rotation) As students are working on rotations of a flag, you observe two students with the following work completed.

Student 1:

Student 2:

(Sample for prompt focused on reflection) As students are working, you observe two students with the following work completed.

Student 1:



Student 2:

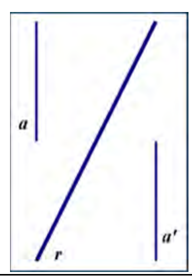


Figure 1: Images of secondary student work from two prompts

Data & Method

Overview

To develop a framework for characterizing Foundation and Contingency knowledge in teacher-created representations of practice, we analyzed teacher-created representations of practice in two rounds of coding. The first round aimed to characterize three levels of Foundation knowledge using all representations of practice from the first year of data collection. The second round used a purposive sample from three years of data collection, as detailed below. In this

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round, we sought to streamline coding for Foundation and develop coding levels of Contingency. We focused on streamlining because we sought a framework that could be ultimately usable by mathematics faculty who may not be education researchers, and that could potentially generalize across domains. In the second round, our process for coding and reconciling for two levels each of Foundation and Contingency knowledge ("High", "Low") took an average of 10 minutes per coder, per representation of practice.

Sampling

Data were drawn from the *Mathematics of Doing, Understanding, Learning, and Educating for Secondary School*s (*MODULES*[2]) project, which has developed curriculum materials for content courses for prospective teachers in four content strands (algebra, geometry, statistics, and modeling). We analyzed prospective teachers' responses to tasks in the geometry materials. Data included more than 300 teacher-created representations of practice from 93 prospective teachers in different regions of the US. The first round of coding used 54 teacher-created representations (2 representations x 27 teachers) from the first year of data collection. The second round of coding used a purposive sample of 31 teacher-created representations of practice (4 representations x 7 teachers + 3 representations x 1 teacher). The purposive sample was selected to document the range of potential MKT. Among the 93 teachers, 61 had completed pre- and post-semester forms of GAST, an instrument validated to measure knowledge for teaching geometry at the secondary level (Mohr-Schroeder et al., 2017). After assigning "high-GAST" and "low-GAST" thresholds for each item, we narrowed the pool to 20 teachers, consisting of the top 10 teachers ranked by proportion of "high-GAST"-"high-GAST" pre-post item scores, and the top 10 teachers ranked by "low-GAST"-"low-GAST" pre-post item scores. Only 8 of these teachers had submitted all 4 representations of practice assigned in the modules. During analysis, we realized that one assignment was scanned incompletely. This resulted in our sample of 31 teacher-created representations of practice.

Analysis

To develop a framework for characterizing foundation and contingency knowledge, we first considered Rowland's (2013) descriptions and Weston and Rowland's (n.d.) elaborations of Foundation and Contingency dimensions. We then reflected on how these considerations may apply to the specific teacher-created representations of practice analyzed and, at the same time, how they may apply to other domains. To do this, we involved researchers with expertise in a variety of mathematical domains – such as mathematical modeling, algebra, and geometry – in our discussion. In both rounds, we were blind to pre/post-test scores; no coders scored the pre/post-tests. We created lists of characteristics of representations of practice demonstrating "High" and "Low" Foundation and Contingency knowledge. We used these lists of characteristics to classify representations of practice first individually, then reconciling differences collaboratively, following a constant comparison method (Strauss & Corbin, 1998).

Results

Characterizing Levels of Knowledge in Teacher-Created Representations of Practice

Our main result is a framework for characterizing levels of Foundation and Contingency knowledge in teacher-created representations of practice. This framework is shown in Figure 2. Codes from the second round of analysis, using the framework, are shown in Figure 3, along with those teachers' post-test scores.

In our full presentation, we illustrate characterizations of all four combinations of Foundation/Contingency (High/High, Low/High, High/Low, Low/Low) with teacher-created representations of practice in response to multiple prompts and discuss contrasts in knowledge observed in video and written representations of practice.

	FOUNDATION	CONTINGENCY
HIGH	Recognizes the logical necessity connecting the definition of a transformation to ways of constructing a transformation image *Examples:* ● Explains a method of construction by marking points on a preimage and then "applying the definition to each of the marked points" to obtain the image ● Reasons that an attempted image is incorrect by showing that it does not satisfy the transformation definition	Frames questions or explanations about connection between construction and definition in terms of students' thinking *Examples:* ● Directs attention to student work to understand the idea that all properties of the definition must be followed to produce a correct construction. ● Engages students in selecting locations in sample student work, and reasoning whether the definition is satisfied
LOW	Explicitly or implicitly treats the definition of a transformation as separate from constructing images, and/or demonstrates lack of understanding of definition *Examples:* ● Describes a method for constructing, and never mentions any definition. ● Provides incorrect definition	Does not integrate student thinking into explanation of connection between construction and definition *Examples:* ● Evaluates student work as "right" or "wrong"; does not cite work otherwise ● Provides a correct explanation that does not reference student work

Figure 2. Framework for Characterizing Foundation and Contingency Knowledge

		FOUNDATION			
PSMT	post	📹	✏	📹	✏
GMM 205	12	H	H	H	H
GMM 202	11	H	H	H	L
GTA 218	10	L	L	L	L
GMM 302	9	L	H	L	L
GMM 308	9	H	L	H	L
GMM 201	9	L	L	L	L
GTA 206	6	L	H	L	--
GTA 208	3	L	L	L	L

		CONTINGENCY			
PSMT	post	📹	✏	📹	✏
GMM 205	12	L	H	L	H
GMM 202	11	H	H	H	L
GTA 218	10	L	L	L	L
GMM 302	9	L	H	L	H
GMM 308	9	L	H	H	H
GMM 201	9	L	L	L	L
GTA 206	6	L	L	L	--
GTA 208	3	L	L	L	L

PSMT = prospective secondary mathematics teacher, listed by pseudonym
post = score on post-test administration of GAST
- = video representation practice, ✏ = written representation of practice
- Representations of practice are listed in order they were assigned

H = High, L = Low

Figure 3. Characterizations of knowledge observed in purposive sample, with GAST scores

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In this paper, we illustrate two combinations of Foundation and Contingency knowledge (High/High, Low/High), using responses from one prompt. In this prompt, teachers were asked to write a narrative describing how they would "elicit student thinking about these reflections, with specific use of the two example students work, and move the class toward understanding connections between methods of reflection and the definition of reflection." The sample student work was shown previously in Figure 1. A class definition for reflection is given in Figure 4.

A reflection across a line L is a transformation that, for every point P in the plane: • $P' = P$ if P is on L • L is the perpendicular bisector of segment PP' if P is not on L.
Note: These materials teach prospective teachers the convention that P' refers to the image of a preimage P under the transformation discussed.

Figure 4. Version of class definition of reflection

Illustration 1: High Foundation-High Contingency

In our framework, the quality of Foundation knowledge is characterized by linking constructions to the definition, and the quality of Contingency knowledge is characterized by integrating student work into the work of connecting constructions and definitions. GMM302 was a representative case to illustrate High Foundation - High Contingency knowledge. GMM302 began their representation of practice:

> "To start, I would draw the student responses and our definition of Reflection on the board. [...] Pointing to the first response, [I would ask,] if we were to draw a line between the points *P* and the corresponding *P'*s, what can we tell about the line segments made by *P* and *P'*? As students respond, I draw and make the corresponding changes to the figure on the board." (see Figure 5a).

Then, after describing some potential responses from students, GMM302 prompted students to link construction and definition: "What is it we know about our line of reflection in regard to our definition of reflection?" GMM302 then marked the angles (see Figure 5b), asked students questions to review the two defining properties of perpendicular bisectors (bisecting, and with perpendicular angles), and posed the question, "Since our main problem here is the angles, how might we approach this in a way that results in right angles instead?" Finally, after drawing a correct reflection (see Figure 5c) but without evaluating it as such to the students, GMM302 asked, "Looking at our new figure, does this hold true to the definition of a Reflection?" GMM302 concluded, after describing potential responses, "Yes, it does hold true. So, we know [segment] *a'* is the reflection of [segment] *a* across the given line." We characterized this response as High/High because GMM302 created tight connections from incorrect and correct images to the definition, positioned students to engage with these links, and did so while centering student work.

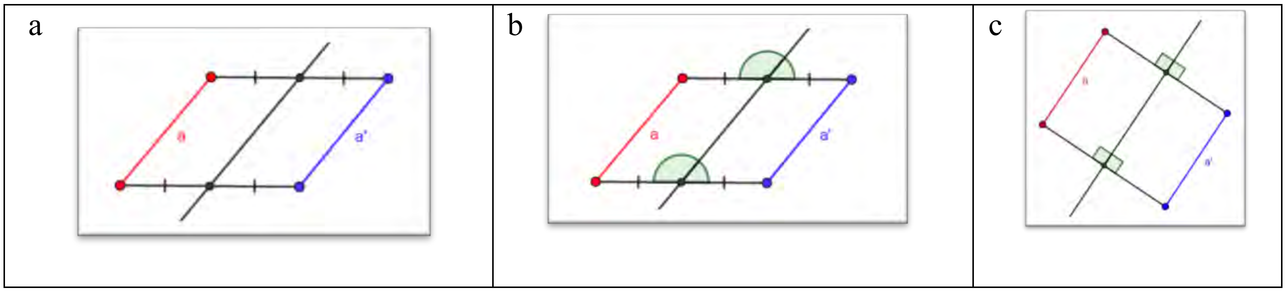

Figure 5. Board drawings proposed by GMM302 to link construction to definition

Illustration 2: Low Foundation-High Contingency

We chose participant GMM308 as a representative case to illustrate Low Foundation - High Contingency knowledge. GMM308 began their representation by analyzing the student work, and suggesting what may have been going on in the students' mind that led to these constructions.

> It looks as though they have drawn lines across the line of reflection from each point to the reflected point. I believe that they have thought that since it is reflected that the distance from the line of reflection is now opposite for each point (the point on top of the reflected image is the same distance as the point on the bottom of the pre-image and vice versa).

In this way, GMM308 exemplified the notion of interacting reflectively with student thinking (Thompson, 2000). GMM308 then described how the student work could be linked to the definition:

> I would use [Student 2's work] to discuss with students how this attending to some points of the definition, but not quite (sic). They have used the idea of the same distance from the line of reflection, but it was utilized incorrectly. I would use this to be able to discuss with students how this doesn't fully fit the definition of a reflection and how we can fix that. We would work as a class to improve the original reflection and make sure it fits all of the necessary components of the definition needed.

This representation of practice exemplified High Contingency knowledge because GMM308 identified how specific student work could be connected to the definition of reflection, especially the role of perpendicular bisectors. However, GMM308 did not articulate the reasoning about perpendicular bisectors precisely, and so we characterized the Foundation knowledge as Low. GMM308's analysis of Student 2's work was similar to their analysis of Student 1's work in that it did not articulate how precisely students might be able to determine whether an image and preimage could satisfy the definition of reflection.

Discussion

We analyzed teacher-created representations of practice in two rounds, resulting in a framework for observing Foundation and Contingency knowledge, characterizations of combinations of Foundation/Contingency levels, and the potential for comparing these characterizations to performance on an instrument previously validated to measure MKT. Previous research has identified dimensions of MKT (e.g., Rowland, 2013), conceptualized mathematical understanding (e.g., Simon, 2006), and unpacked teachers' actions to understand

and act on student thinking (e.g., Ader & Carlson, 2018; Thompson, 2000; Weston, 2013). These scholars grounded their work in videos of teaching and interviews. We synthesized previous research to contribute a framework for observing varying levels of different dimensions of MKT, applied to a pedagogy of teacher education, that of teacher-created representations of practice.

In evaluating the robustness of our framework, we consider the limitation of our data to four prompts for representations of practice in geometry, and the potential for our framework to generalize across domains. Our framework, as reported, is tailored to the use of definition to a particular concept of geometry, and derived from the analysis of a limited number of prompts. However, we see our framework as generalizable to mathematics more generally because of its underpinnings in the Knowledge Quartet (Rowland, 2013), mathematical understanding (Simon, 2006) and interacting reflectively (Thompson, 2000), all of which are intended to apply broadly to mathematics teaching and learning. Moreover, the centrality of definition to mathematics, as well as reasoning with definition or other assumptions (Kitcher, 1984), suggests the potential for adapting this framework to domains of mathematics with strongly structured logical systems, such as algebra. For instance, in place of linking definitions with construction methods, the framework could emphasize connecting definitions with common procedures or tests (e.g., ways to solve equations, vertical line test), and to what extent student thinking is centered in engaging with these procedures or tests. For domains such as mathematical modeling, which apply mathematics in phases of distinctive practices (e.g., Blum & Leiß, 2007), the framework could emphasize the rationale for each phase as well as anticipation of movement across phases, for instance, knowing that the proposal of a mathematical model can be followed by considering the real world or the results of the model, that that these phases can work together to refine one's model (e.g., Czocher, 2018).

We would be remiss to not issue caveats about the use of "levels" of knowledge. Most importantly, these characterizations, like other hierarchical characterizations in the literature (e.g., Ader & Carlson, 2018; Serbin et al., 2018), are not intended to be characterizations of teachers or their ultimate potential. Rather, we present these levels as descriptions of observable features of representations of practice that may be ultimately usable by teacher educators to guide formative feedback for prospective teachers. When the teacher educator is a mathematics faculty member, characterizing only an ideal may not be sufficient for helping that teacher educator articulate, for example, where a teacher might have involved student thinking more. A teacher educator could use the framework as a way to begin a dialogue with prospective teachers to support their growth. We believe that the risk of characterizing "levels" is outweighed by the potential benefit of supporting mathematics faculty members in seeing how to connect mathematics and teaching.

When we began this work, we had in mind conversations with mathematics faculty members as well as the research that indicates that mathematics faculty may want to connect mathematics and teaching, but do not know how. We also had in mind the mathematics faculty members that pilot our materials, which come in four domains: geometry, algebra, mathematical modeling, and statistics. We argue that attention to the dimensions of Foundation and Contingency are a fruitful framework for characterizing knowledge in teacher-created representations of practice. GMM308's representation of practice, and other examples of High/Low and Low/High combinations, illustrate that the dimensions of Foundation and Contingency can be viewed as separable, and therefore be distinct categories for feedback to teachers. Whereas Foundation knowledge might be a dimension that mathematics faculty find familiar, the Contingency dimension may be more foreign. We hypothesize that narrowing the scope of the unfamiliar to

Contingency, in the way that we have delimited it, may make it more accessible to mathematics faculty. If our hypothesis holds, then we will have a framework that can shape instruction and curriculum for mathematics content courses in many domains. Our future work involves testing the promise of this framework for building stronger connections between mathematics and teaching.

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