RETHINKING HOW UNITS COORDINATION IS ASSESSED IN PRESERVICE TEACHER POPULATIONS

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Fraction proficiency continues to be a challenge for many learners of mathematics. Valid and reliable methods for assessing fraction understanding are critical tools in the pursuit of meeting this challenge. Written assessments have been widely used with K-12 students to assess fraction understanding, including units coordination. However, using these types of assessments with a preservice PreK-8 teacher population has proved difficult and inconclusive. Preservice PreK-8 teachers have a variety of algorithmic techniques at their disposal, which has resulted in the need to reexamine how units coordination is assessed in this population. This paper shares the subsequent reconceptualization of assessing preservice PreK-8 teachers' units coordination.

Keywords: Mathematical Knowledge for Teaching, Rational Numbers, Preservice Teacher Education

For decades, proficiency with fraction concepts and computations has been a bane to many students and teachers alike (e.g., Ball, 1990; Bentley & Bosse, 2018; Borko et al., 1992; Izsák et al., 2010; Menon, 2009; Olanoff et al., 2014; Rathouz, 2010; Rizvi & Lawson, 2007; Schneider & Siegler, 2010; Stafylidou & Vosniadou, 2004; Tirosh, 2000). In a previous study of preservice PreK-8 teachers' (PSTs') fraction knowledge (Busi et al., 2015; Lovin et al., 2018), we found evidence that many PSTs struggled with the more sophisticated reasoning needed for fluency with rational numbers. Subsequently, we investigated ways to improve PSTs' fraction content knowledge through changes in our pedagogy (Stevens et al., 2020).

The framework we have used to guide our work in assessing and making sense of PSTs' conceptions of fractions is based on a trajectory of fraction schemes and operations (Norton & Wilkins, 2012; Steffe & Olive, 2010; Wilkins & Norton, 2011). A key component of moving through this trajectory relies on the number of levels of units the learner can coordinate simultaneously. Specifically, to reach the higher levels of reasoning in the trajectory, the learner must be able to coordinate three levels of units simultaneously (3UC) – meaning they can anticipate the outcome of this coordination before they do it. Having this anticipation is known as interiorizing the ability to coordinate units. If someone is unable to anticipate the outcome of the coordination, they may either not have acquired this coordination or may solely coordinate units in action, in the midst of solving a fraction task.

Throughout our work, we have experienced a productive struggle with confidently assessing PSTs' ability to coordinate three levels of units. Our initial study identified 13 cases in which it appeared PSTs had developed a fraction scheme in the developmental trajectory beyond coordinating three levels of units before they had acquired 3UC. This is contradictory to the validated theory in which each step of the developmental trajectory requires the acquisition of the previous scheme or operation.

One confounding fact is that the written assessment used in this first study was initially developed for use with upper elementary and middle school students (Norton & Wilkins, 2012; 2013; Wilkins & Norton, 2011). When used with PSTs, PSTs' overreliance on procedures to find common denominators or to do fraction computations masked evidence of whether they had interiorized the operation of coordinating three units. Since this first study, we have been exploring alternative tasks and strategies to better assess PSTs' ability to coordinate three levels of units. Our ensuing productive struggle led us from solely written assessment tasks to observations of PSTs completing written tasks to structured interviews and has helped identify issues with our tasks that can be used to create improved assessments. Our goal is to share some observations from this process.

Theoretical Framework

An existing developmental trajectory of fraction schemes and operations serves as our framework. This trajectory was validated for upper elementary and middle school students (Norton & Wilkins, 2012; 2013; Wilkins & Norton, 2011) and later validated for PSTs (Busi et al., 2015; Lovin et al., 2018). These schemes and operations can be grouped into three bands of developmental knowledge of fractions with each subsequent band relying on an increasing number of levels of units the learner can simultaneously coordinate: fractions as solely part-whole concepts (only requires the coordination of one level of unit); fractions as measures (requires the coordination of two levels of units); and fractions as numbers "in their own right" (requires the coordination of three levels of units) (Hackenberg, 2007, p. 27; Hackenberg et al., 2016). Our previous work discovered a majority of PSTs were not proficient in being able to reason about fractions as numbers "in their own right" (Hackenberg, 2007, p. 27). This finding corroborates existing research (e.g., Chinnappan, 2000; Olanoff et al., 2016; Son & Crespo, 2009; Son & Lee, 2016). The catalyst for developing this reasoning is being able to simultaneously coordinate three levels of units (Steffe & Olive, 2010), which is the part of the trajectory we focus on in this paper. (For more information about this developmental trajectory of fraction schemes and operations, please see Norton & Wilkins (2009), Norton & Wilkins (2012), Norton et al. (2018), Steffe (2002), Steffe & Olive (2010), Wilkins & Norton (2011).)

Methods

Participants and Instrument

Participants in the study comprised seven undergraduates enrolled in one of three required mathematics content courses for PSTs at a southeastern university. The first in this sequence of courses focuses on number concepts and operations, with significant time dedicated to developing fraction understanding. Four of the participants were enrolled in the first course and participated in the study prior to fraction instruction. The other three participants were enrolled in one of the subsequent courses.

Because the motivation for the study was to investigate strategies that may impact or mask PSTs' ability to demonstrate the interiorization of 3UC, a written 3UC assessment was developed that paralleled assessments used in previous studies (Busi et al., 2015; Lovin et al., 2018). Previous studies showed some evidence of PSTs' reliance on algorithms such as dividing fractions or finding equivalent fractions as masking evidence of 3UC interiorization. This study sought to further investigate these potentially confounding algorithms by combining clinical interviews of the participants with the written assessment so PSTs' approaches and reasoning could be explored further.

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Data Collection

The participants completed a twelve-item assessment designed to determine whether or not they were able to coordinate three levels of units. PSTs who have interiorized the ability to do this have an immediate, productive plan to solve a 3UC fraction task and can anticipate the results irrespective of context, denominator choice, or representation; they do not rely on their written work to discover a productive strategy in action (Hackenberg et al., 2016).

The assessment began with four items with no accompanying representations or context. For example, PSTs were posed the following question: "Envision 2/3 of a whole. Now consider 1/12 of the same whole. How many 1/12s are in the 2/3 you originally envisioned?" The remaining eight items were written within a specific context (e.g., an amount of pizza or the length of a jump rope) and provided a specific representation (e.g., a portion of a circle or a line). The fractions used in both sections were varied in structure; the denominators either allowed for halving strategies (e.g., relating 3/¾nd 1/8) or required strategies other than halving (e.g., relating 3/5 and 1/15).

For each item, the PSTs were asked to provide both a solution and a demonstration of their reasoning. For the first four items, the PSTs provided no written documentation of their thinking; their explanations were verbal. For the remaining eight items, the PSTs were asked to use the provided representation to diagram their thinking, and the researchers asked them clarifying questions about their diagrams. The PSTs were observed and video-recorded while completing the entire assessment. The observations and clinical interviews were an essential portion of the study because they enabled the researchers to watch the participants' approaches in action, rather than solely evaluating written evidence of their strategies after they submitted the assessment.

Data Analysis

All four researchers independently rated the written responses for each item and then compared the documentation to the video recordings of the verbal explanations to evaluate each participant's interiorization of 3UC. The researchers then discussed their ratings and came to a consensus based on the evidence provided by the comparisons.

The video recordings allowed the researchers to look for discrepancies in the written documentation, the participants' observed approaches, and the participants' verbal descriptions of their strategies. Participants may show evidence for 3UC in their written work, but then describe their reasoning in a manner that indicates otherwise. In this case, looking solely at written work would result in a *false positive*. Participants may also show counterevidence for 3UC in their written work, but then describe their reasoning in a manner that indicates otherwise. In this case, looking solely at written work would result in a *false negative*. Based on previous findings, the researchers hypothesized these discrepancies would exist between some participants' written evidence and their verbal descriptions of their approaches.

Results

We will share one illustrative example of a *false positive* assessment of 3UC and one illustrative example of a *false negative* assessment of 3UC.

False Positive Example

The PST was given this written question: "The candy bar shown below (represented by a rectangle) is 5/6 of a whole candy bar. If each person wants 1/24 of a whole candy bar, how many people can share the amount shown below?" In her written work (Figure 1), the PST seemed to create the whole by partitioning the given diagram into five 1/6 pieces and adding on one additional 1/6 piece to the given diagram to create six 1/6 pieces. From there, it seemed like

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she further partitioned each 1/6 piece into four smaller pieces, essentially cutting each 1/6 piece into fourths. There are now 24 pieces within the whole candy bar. Each group of four 1/24 pieces (contained within each of the original 1/6 pieces) is marked with four symbols to show they make a group. For example, there are four x's above four 1/24 pieces in one 1/6 piece and four +'s above four 1/24 pieces in another 1/6 piece. (Note: the symbols above the four 1/24 pieces in the added-on 1/6 piece are difficult to decipher.) The PST seemed to be coordinating the 1/24 pieces within each of the 1/6 pieces within the whole and showing she has five groups of four 1/24 pieces in the given amount, indicating she was coordinating three levels of units.

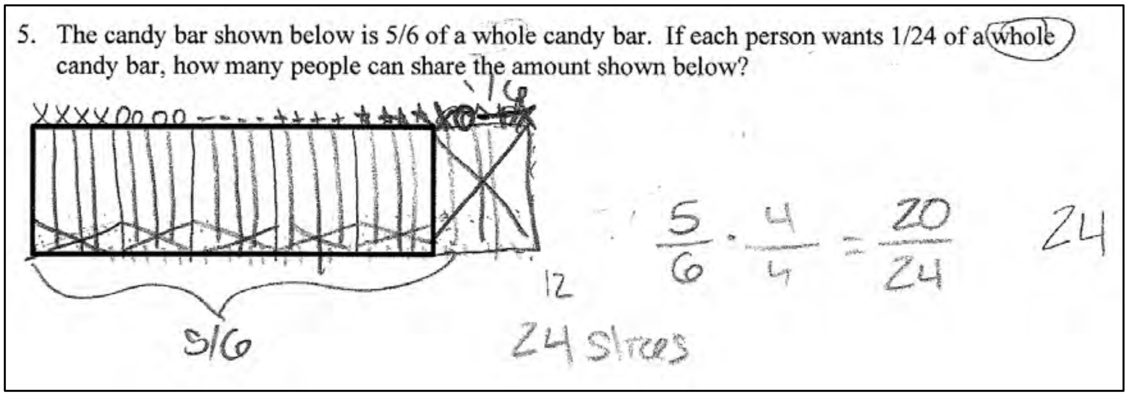

Figure 1: PST Solves Candy Bar Problem

The same PST was given this written question: "The length of rope shown below (represented by a line) is 3/5 of a whole length of jump rope. If each jump rope requires 1/10 of the whole length of rope, how many jump ropes can you make from the length of rope shown below?" In her written work (Figure 2), the PST again seemed to create the whole by partitioning the given diagram into three 1/5 pieces and adding on two additional 1/5 pieces to the given diagram to create five 1/5 pieces. This can be seen with the dotted lines. From there, it seemed like she further partitioned each 1/5 piece into two smaller pieces, essentially cutting each 1/5 piece in half. This can be seen with the solid lines. She then labeled each piece as 1/10 in size and circled the six 1/10 pieces that were in the given diagram. The PST seemed to be coordinating the two 1/10 pieces within each 1/5 piece, five of which make the whole, again indicating she was coordinating three levels of units.

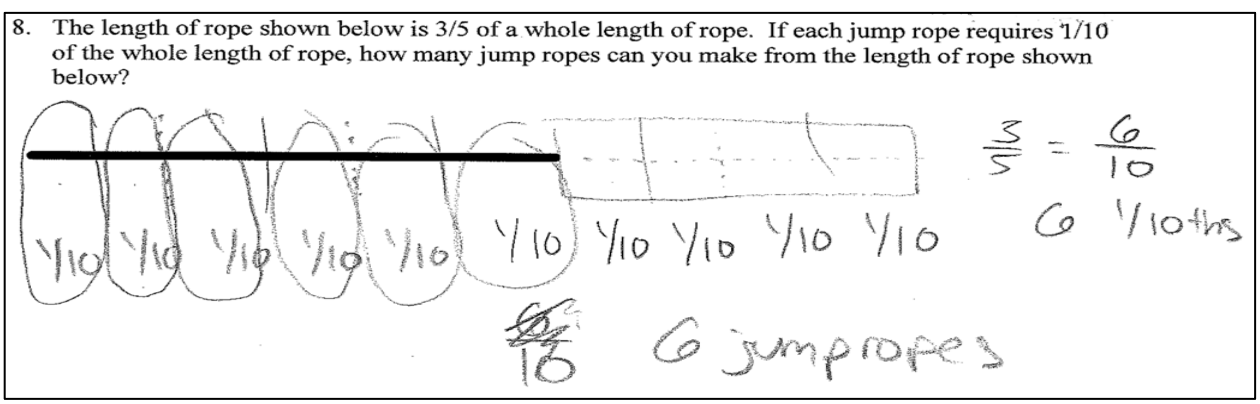

Figure 2: PST Solves Jump Rope Problem

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Based solely on her written work, it would seem this PST was coordinating three levels of units. She identified four 1/24 pieces within each 1/6 piece, five of which were given and six of which make the whole. She also identified two 1/10 pieces within each 1/5 piece, three of which were given and five of which make the whole. However, listening to this PST answer the first four questions of the interview protocol, it was clear this PST is not coordinating units and is instead using a generalized procedure to create equivalent fractions.

For example, she was asked: "Suppose you have 3/¾f a whole, can you explain to me how many 1/8 pieces of the whole you have?" She almost immediately answered correctly that she would have six 1/8 pieces of the whole because she "converted [3/4] into eighths." She described a procedure she named the "giant one" (see Figure 3 for a visual representation of the "giant one") which tells her she has to multiply the numerator and the denominator by the same factor so she is "not changing the value… just changing the representation of it." In this example, she explained she multiplied the four by two (in the denominator) so she must multiply the three by two (in the numerator) to get six 1/8s.

In another example, she was asked: "Suppose you have 2/3 of a whole, can you explain to me how many 1/12 pieces of the whole you have?" The PST used the same process of "multiplying by the giant one, or four-fourths" to know there would be two times four or eight 1/12 pieces of the whole. With this strategy, this PST claimed, "I don't change the value of the original fraction, I'm just changing the way it looks." When posed with a third, similar question, the PST asked the interviewer, "Is it okay if I use the same explanation?" indicating the "giant one" strategy is what she is most comfortable with and most confident in.

This PST's strategy for each conceptual problem at the beginning of the interview protocol was to create an equivalent fraction. She could clearly articulate her strategy of multiplying the numerator and the denominator of the given fraction by the same number. She could also clearly articulate that by doing this, she is not changing the value of the fraction, she is just changing "the way it looks" or the "representation" of it. However, this strategy does not give any evidence of coordinating three levels of units; there is no indication she sees two 1/8 pieces in each 1/¼iece or four 1/12 pieces in each 1/3 piece.

When listening to this PST explain her thinking about her written work, there was further evidence she is not actually coordinating three levels of units. The PST was given this written question: "The pizza shown below is 2/3 of a whole pizza (represented by 2/3 of a whole circle). If each person wants 1/9 of a whole pizza, how many people can share the amount shown here?" In her written work (Figure 3), the PST initially performed the "giant one" procedure to get an answer of 6/9, which she correctly interpreted as six people eating pizza. Then she moved to the diagram. She split the given amount (2/3 of a whole pizza) into thirds and then split each of those into three smaller equal pieces, making 1/9-sized pieces relative to the given amount (2/3 of a whole pizza). When she did this, it seemed like she might be coordinating three 1/9-sized pieces within each 1/3 piece, even though she is ignoring the size of the whole pizza, giving some indication of coordinating units. However, when she verbally described her thinking, she explained, "I have three parts of a pizza and if each person wants 1/9, I must split up the thirds…I must multiply by something to get nine and I know three times three is nine. So, I split it up again into three equal pieces…so then it was a total of nine pieces and when I multiplied by the numerator it was six, so I know that six people could eat pizza." When asked to identify the six 1/9 pieces in the diagram, she could not find them. She went on to say, "I was just thinking about it numerically. I was thinking about whatever I multiply by the denominator, I must multiply by the numerator." This PST did not seem to be coordinating units and was instead attempting to use the diagram to explain the "giant one" procedure. In reality, conceptualizing equivalent fractions requires one to see there are two groups of three 1/9 pieces within the given amount and three groups of three 1/9 pieces within the whole, which does require the coordination of units.

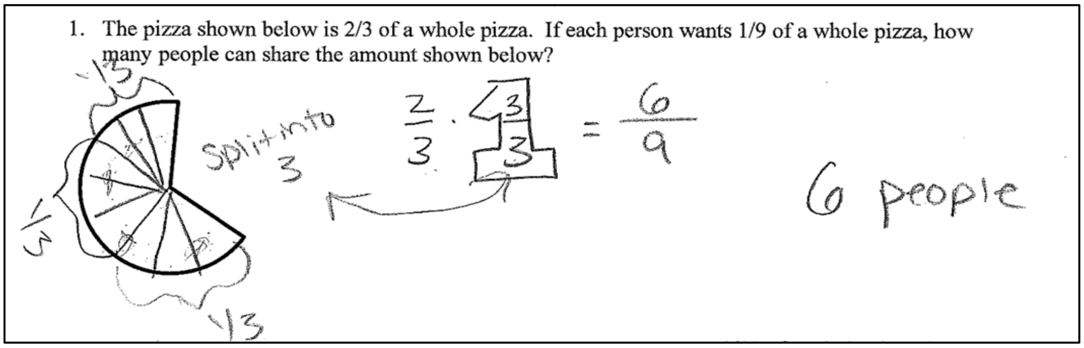

Figure 3: PST Solves Pizza Problem Using a "Giant One"

This situation was thus labeled as a false positive. After the clinical interview, it was determined the PST was in fact completing and trusting the multiplication algorithm to find an equivalent fraction and then translating that number onto the diagram. As static work, it appeared multiple levels of units had been coordinated, but after hearing from the PST, it became clear she was not seeing units within units. Rather, she was retroactively placing the units onto the diagram without any coordination of unit size. It was ultimately concluded this PST has not interiorized the operation of coordinating three levels of units, even though her written work seemed to provide evidence that she had.

False Negative Example

The PST was given this written question: "The pizza shown below is 2/3 of a whole pizza (represented by 2/3 of a whole circle). If each person wants 1/9 of a whole pizza, how many people can share the amount shown here?" In her written work (Figure 4), it seemed as though the PST is trying to figure out how to partition a whole circle into nine relatively equal pieces. She made a few attempts, including a familiar cut-in-half, cut-in-half method, before achieving her goal. But that was where she stopped; she did not provide an answer to the question. There is no evidence of coordinating units in this written work.

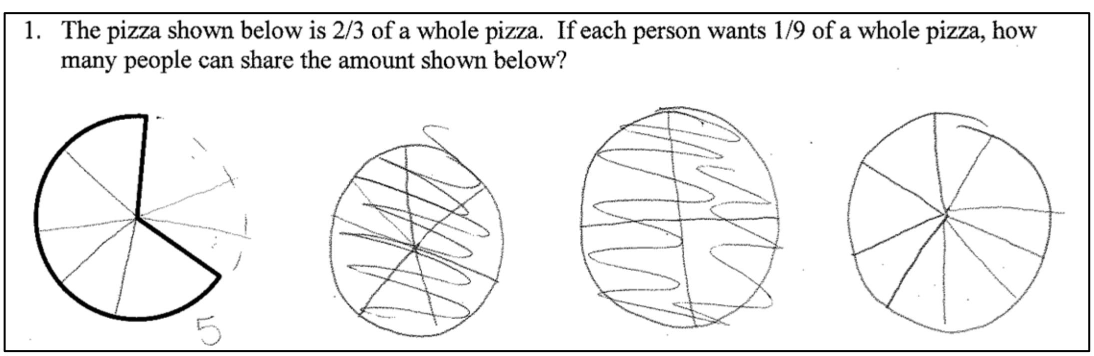

Figure 4: PST Attempts to Partition a Circle into Nine Pieces

The same PST was given this written question: "The length of rope shown below (represented by a line) is 3/5 of a whole length of jump rope. If each jump rope requires 1/10 of the whole length of rope, how many jump ropes can you make from the length of rope shown below?" In her written work (Figure 5), the PST initially labeled the given amount as 3/5, extended the line to represent the whole length of rope, and drew four larger pieces, creating 1/¼ieces, with six smaller pieces within each of the larger pieces, creating 1/24 pieces. She labeled each set of two of the larger pieces as 1/10, making a total of 2/10. The PST abandoned this attempt and started again below it. She drew a second line that has five clear larger pieces, creating 1/5 pieces, with three smaller pieces within each of the larger pieces, creating 1/15 pieces. She labeled three of the larger 1/5 pieces as 3/5. Like the previous question, the PST stopped and did not provide an answer to the question. Considering both attempts on this question, there is no evidence this PST was coordinating multiple levels of units.

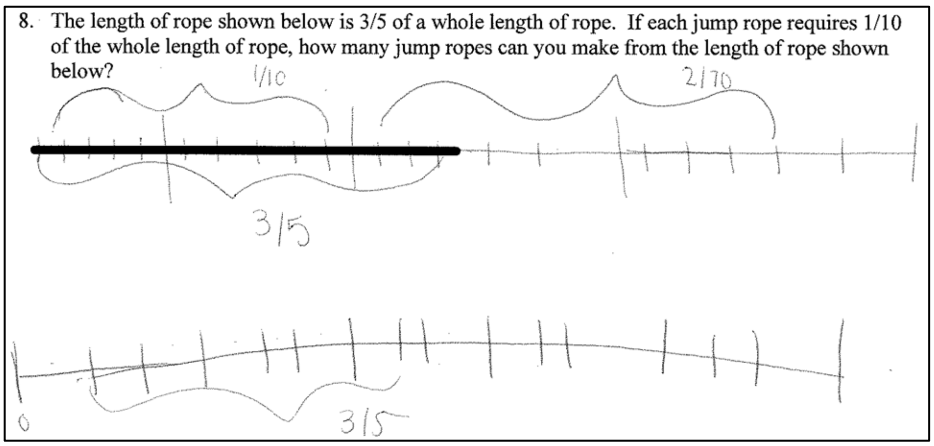

Figure 5: PST Makes Two Attempts at Jump Rope Problem

Based solely on her written work, it would seem this PST was not coordinating three levels of units. She struggled to make connections between the relative sizes of the pieces of the pizza, which was represented with a circular area model, or the pieces of the jump rope, which was represented with a linear model. However, listening to this PST answer the first four conceptual questions of the interview protocol, it did seem like she is able to coordinate units.

She was first asked: "Suppose you have 3/¾f a whole, can you conceptually explain to me how many 1/8 pieces of the whole you have?" She responded with, "If you have a pizza and you cut it into four slices and you shade in three of them, then you can divide all of the fourths into half again and that will give you eighths. So then the section of the three-fourths that isn't shaded, you would have two-eighths not shaded and the rest…you would have six-eighths shaded." The PST was able to confidently describe cutting each 1/¼iece in half to create eight 1/8 pieces within the whole pizza.

Next, she was asked: "Suppose you have 2/3 of a whole, can you conceptually explain how many 1/12 pieces you would have in the whole?" Her answer was, "You would take a pizza and divide it into thirds and then shade two of those thirds. And then you could divide each slice into fourths, each third into four additional sections. And then you would have the section not shaded; there would be four pieces not shaded of the one-twelfths, so four-twelfths not shaded. And then you would have the remaining part of the pizza would be the shaded twelfths…so you would have eight-twelfths shaded." The PST was able to confidently identify the number of 1/12 pieces within each 1/3 piece of the whole.

This PST gave a very similar response to the remaining two questions included in this portion of the interview protocol. In each response, this PST gave a clear articulation of the number of fractional pieces within another. Furthermore, she was able to describe the pieces within the pieces of the shaded part of her diagrams as well as the pieces within the pieces of the unshaded part of her diagrams. For example, she was able to coordinate the number of 1/12s not shaded (four 1/12s) as well as the number of 1/12s shaded (eight 1/12s). This shows a coordination of units within units within the whole (i.e., 3UC).

It is interesting this PST initially asked if she could draw a visual representation to help her solve these four problems. Even though the interviewer asked her to share her thinking without drawing a visual representation, the PST still visualized and described exactly what she would have drawn. This PST clearly favors visualization, but struggled with the visual representations of any type in the written work portion of the interview.

This situation was thus labeled as a false negative. After the clinical interview, it was clear this PST was in fact able to confidently coordinate units even though she struggled to show it in writing. Her static work appeared void of unit coordination. But, when given the opportunity to talk about the problems, it became clear she was very capable of this coordination. It was ultimately concluded this PST has interiorized the operation of coordinating three levels of units.

Discussion

In our previous work with written assessments, many PSTs used computational procedures to solve 3UC tasks, masking evidence of coordinating three levels of units (Busi et al., 2015; Lovin et al., 2018). Through this previous work, it became evident that intentionally designed assessments were necessary to help unpack the masking issue. Originally, the new written assessments aimed to vary contexts, (e.g., candy bars), denominator choices (e.g., allowing for halving strategies) and representations (e.g., rectangular area) to further explore PSTs' true ability to coordinate three levels of units. However, it quickly became apparent PSTs were still exhibiting inconsistencies with how they solved these written problems. We again noticed algorithm use and incomplete diagrams caused us to be inconclusive in our attempts to determine if 3UC was evidenced in the work.

To help guard against the inconclusive nature of the written work, a clinical interview protocol was also created. Striking observations were made in terms of the differences between looking at a PST's static work and hearing a PST talk about her reasoning. As described in the results section above, there were some PSTs whose written work showed evidence of coordinating three levels of units, but when listening to their reasoning during the clinical interview, it became clear that seeing units within units within the whole was not occurring. This indicated they had in fact not interiorized 3UC. On the other hand, there were some PSTs whose written work indicated they could not coordinate three levels of units. But when they described their thinking about the problems during the clinical interviews, they could clearly and confidently talk about units within units within the whole. This showed evidence that they in fact had interiorized 3UC.

The additional interview data is providing evidence that PSTs' written work as a single artifact of evidence is not sufficient to determine the presence of the interiorization of 3UC. This is a significant finding given that many previous studies (e.g., Busi et al., 2015; Caglayan & Olive, 2011; Lovin et al., 2018; Son & Lee, 2016; Ubah & Bansilal, 2018) have relied on written assessments to determine PSTs' ability to coordinate units. This begs the question: how do we best assess 3UC in PSTs? The clinical interviews we conducted seem to be effective. By

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listening to a PST reason about 3UC problems conceptually and by listening to a PST describe her thinking about a specific problem in context, we felt confident about our assessment of whether or not that PST had interiorized 3UC. Although clinical interviews are time consuming, our findings indicate they must be conducted to develop and validate interventions for developing PSTs' 3UC.

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