USING PROPORTIONAL TASKS TO EXPLORE TEACHERS' ABILITY TO MAKE SENSE OF STUDENT THINKING

Rachael Eriksen Brown
Pennsylvania State University Abington
reb37@psu.edu

Chandra Hawley Orrill
University of Massachusetts Dartmouth
corrill@umassd.edu

In this paper, we extend our previous work on challenges teachers face when engaging with proportional reasoning contexts to investigate two contexts that included four problems for middle grades teachers to solve as well as eight student solutions. Analysis included coding for correct solving of the problem as well as making sense of and determining reasonableness of the associated student work. Results indicate that making sense of student work was not dependent on correctly solving the problem. Determining reasonableness of student work was more challenging for our 32 participants. The think aloud interview, we argue, mimics responding to student thinking in a live setting. Implications for teacher knowledge as well as professional development and teaching will be discussed.

Keywords: Mathematical Knowledge for Teaching, Professional Development, Rational Numbers & Proportional Reasoning, Teacher Knowledge

Purpose

Proportional reasoning is an important mathematical concept for succeeding in K-12 math. However, not only students, but teachers, struggle with proportions (e.g., Akar 2010; Harel and Behr 1995; Post et al. 1988; Riley 2010). Teachers are often challenged to reason conceptually about proportions. Likely teachers, like their students, have an over-reliance on algorithms, like cross-multiplication, that leads to correct answers while not attending to multiplicative structures (e.g., Berk et al. 2009; Lobato et al. 2011; Modestou and Gagatsis 2010; Siegler et al. 2010).

Using Lamon's (2007) description of proportional reasoning as, "supplying reasons in support of claims made about the structural relationships among four quantities, (say a, b, c, d) in a context simultaneously involving covariance of quantities and invariance of ratios or products" (p.637-638) suggests the importance of teachers to identify what stays constant and recognize what varies in proportional relationships. One key idea is recognizing that a proportion is a multiplicative comparison and not an additive one (Lamon, 2007). In addition, teachers need to understand representations that highlight various components of proportional relationships, such as ratio tables and double number lines (Lobato & Ellis, 2010). Teachers need to understand how these representations support reasoning about the proportional structures.

One noted area of struggle is correctly identifying proportional reasoning situations and the tendency for students and teachers alike to use proportional reasoning in non-proportional situations (e.g., De Bock et al, 2002; Izsák & Jacobson 2017; Modestou & Gagatsis 2007). For example, De Bock and colleagues (2002) investigated students' persistent use of proportional thinking in a task focused on an area relationship. Of the 40 high school student participants in the study, 32 could not determine a correct answer even after being prompted with five scaffolds designed to highlight the area relationship. In our own work, we used a similar task, the Santa Task (see Figure 1), with middle grades math teachers and found several teachers were misled by the task. By the end of the task and our three scaffolds, only 13 of the 32 teachers correctly applied an area interpretation. Thus, this topic is challenging for teachers as well.

Santa Task

A painter painted a 56 cm high Santa on the door of a bakery. He needed 6ml of paint. Now he is making an enlarged version of the same painting on a supermarket window using the same paint. This copy should be 168 cm high. How much paint will Bart need to do this?

| Scaffold 1: Compare favorite answers of 18ml and 54 ml | Scaffold 2: A student drew rectangles around both images | Scaffold 3: A student used easier numbers. For smaller picture used 1 tube of red paint and figured out larger would use 9 tubes of red paint. |

Figure 1: Santa Task and student work

Given teachers need to not only work math problems correctly, but also make sense of students' work, we were interested in the relationship between teachers' abilities to solve the task and to make sense of sample student work on that task. We were also interested in their ability to determine whether the students' work was reasonable. We previously shared results of the Santa task analysis (Brown & Orrill, 2019). Our hypothesis was if teachers cannot solve a problem correctly, they are less likely to make sense of student thinking and less likely to determine if the solution is reasonable. Our Santa data suggest many participants could make sense of student thinking with or without solving the problem correctly; however, determining reasonableness was much more challenging when these participants had not solved the problem themselves correctly. Given this finding, we wanted to expand our focus to find out whether these trends stayed consistent across other items. Thus, we expanded our analysis to include the Milkshake task (see Figures 2-4). In this paper, we provide our analysis of both the Santa Task and the Milkshake Task. For each problem included in the tasks, we considered: (a) whether teachers engaged with students' reasoning and (b) whether they could determine the reasonableness of a student's approach. Our intent was to explore whether there was a connection between teachers' demonstrated content knowledge and their ability to make sense of students' reasoning in terms of how the student worked the task and whether the student's approach was a reasonable one. In our work, we define reasonableness to include determining whether the approach was mathematically viable or identifying the usefulness of a representation.

Perspective

Teachers facilitate students' interactions with mathematics in ways that allow them to develop meaning. Kilpatrick, Swafford, and Findell (2001) argue teachers decide when to allow students to struggle, ask questions, and provide guidance. Teachers also facilitate classroom discussions around key mathematical ideas. To do this well, teachers must engage with students' mathematical reasoning. *Principles to Actions* (NCTM, 2014) suggested teachers "elicit and use evidence of student thinking" (p. 10), including being able to assess student understanding in order to make instructional decisions.

While many have defined knowledge of teachers to include making sense of students' understanding (e.g., Shulman, 1986; Ball, Thames, & Phelps, 2008), little research has been done connecting teachers' understandings of mathematics to their understandings of students' ideas about mathematics. Rowland and colleagues' (e.g., Rowland, 2013; Turner & Rowland, 2011) Knowledge Quartet framework explores the connection between teacher content knowledge and how that knowledge is visible in teaching practice. Rowland (2013) wrote of the differences between the quartet and the Mathematical Knowledge for Teaching framework (Ball et al.,

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2008), "In the Knowledge Quartet, however, the distinction between different *kinds* of mathematical knowledge is of lesser significance than the classification of the situations in which mathematical knowledge surfaces in teaching." (p. 22). The Knowledge Quartet includes four dimensions: Foundation, Transformation, Connection, and Contingency. Foundation is teacher knowledge learned through schooling and professional development. The Transformation dimension is around using knowledge to support student learning in instruction. Connection addresses the coherent planning and teaching of mathematics. The fourth dimension is Contingency, a teacher's response to events in the classroom. This study addresses Contingency by investigating how a teacher reacts to unplanned student ideas about a task and how, if at all, that reaction relates to the teacher's ability to solve the same task.

Methods

In this study, we analyzed the data from the same 32 middle school teachers as in our previous work. The participants ranged from one to 26 years of experience. They were a convenience sample of middle school teachers from four states. Eight participants identified as male.

Milkshake Problem 2

Katrina wanted to make 3 cups of the recipe. How much of each ingredient did she need?

Student C:

30 cups
12 milk
18 ice cream

I drew out one batch then realized that it would be hard to make 3 cups from it. So, I found 30 cups because I could figure out 1/10 of that. Once I figured out there were 12 cups of milk and 18 of ice cream, I just multiplied by 0.10 to find out how much I would need. My answers were 1.2 c milk and 1.8 cups of ice cream.

Figure 2: Milkshake Problem 2 and student work

Each participant completed a think-aloud interview that included the Santa Task and the Milkshake Task. The Santa Task was around the middle of the protocol and was inspired by De Bock et al's (2002) study. The Milkshake Task was inspired by the Orange Juice Task (National Research Council, 2001) and was at the end of the protocol. Teachers were asked to solve a mathematical problem and then respond to student work on the same task. Both tasks prompted the teachers to explain what the student was doing and whether it was reasonable. Figures 1-4 provide details about the two tasks. The Santa Task included one task with three different student solutions (we refer to these as Scaffolds 1-3). The Milkshake Task included three problems. Part 1 included two student solutions (Students A and B; Figure 3); Part 2 included one student solution (Student C; Figure 2); Part 3 included two student solutions (Team A and B; Figure 4). We considered in the verbatim transcript of each participant's interview (a) whether the participant's answer to the problem was correct; (b) whether the participant was able to make sense of each student approach; and (c) whether the participant identified the reasonableness of

the approach. For participants who changed their initial solution, we analyzed only those responses that were given after the switch to correct reasoning. Transcripts were coded independently by each author and then discussed to reach 100% agreement.

Findings

Our intent was to determine whether there were relationships between participants' own mathematical thinking and their engagement with making sense of the students' thinking. Table 1 details the number of participants who solved each part correctly, were able to make sense of the student solutions, and were able to determine the reasonableness of the student solution process. The Santa problem was the more challenging problem for our participants to solve with only 13 solving it correctly. Of the three milkshake problems (Students A & B, Student C, and Teams A & B), the first was solved correctly by the majority of participants (97%). The second was solved correctly by 72% of participants, and the third was solved correctly by 88% of the participants.

Table 1: Coding for Santa and Milkshake Tasks

	Number of participants	Solved correctly	Able to Make Sense	Able to Determine Reasonableness
Santa Task				
Scaffold 1	32	12 (38%)	23 (72%)	12 (38%)
Scaffold 2	32	13 (41%)	26 (81%)	14 (44%)
Scaffold 3	32	13 (41%)	19 (59%)	10 (31%)
Milkshake Task				
Student A	32	31 (97%)	14 (44%)	16 (50%)
Student B	30*	29 (97%)	26 (87%)	27 (90%)
Student C	32**	23 (72%)	27 (84%)	23 (74%)
Team A	32	28 (88%)	27 (84%)	25 (78%)
Team B	30*	26 (87%)	17 (57%)	17 (57%)

*Data were missing for two participants for Student B and for two different participants for Team B.
**Only 31 people responded to the reasonableness question for Student C.

Overall, eight participants were able to correctly solve all four of the problems. Only three of those eight were able to both make sense of the student work and determine the reasonableness for all shared student work. Thus 9% of the participants (8 out of 32) were able to solve the problems correctly, make sense of and determine the reasonableness of all approaches.

Another eight participants were able to solve one or two of the four problems correctly. Seven of these participants were able to make sense of student work on problems they did not solve correctly, but these same seven could not always make sense of the student work associated with the problem they did solve correctly. Thus, making sense of student work does not seem related to being able to solve the problems correctly. With respect to determining reasonableness, five of these eight participants had a harder time determining the reasonableness of the student work (the number of times they could determine reasonableness was less than the number of times they made sense of student work).

Looking only at the Milkshakes problems we can see that many participants solved the problems correctly and many were able to make sense of the student work and correctly

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determine reasonableness. However, we noticed that the responses for Student A and for Team B were markedly more difficult for these participants. Closer examination revealed that both of these student responses involved non-standard uses of common representations. For example, Figure 3 shows Student A using a common, discrete representation of the ratio two to three. In Student A's explanation, the reasoning applied to that representation used variable parts reasoning. That is, the student describes the idea that a ratio can be thought of as a fixed number of parts that can vary in size (see Beckmann & Izsák, 2015). Teachers who correctly reasoned about the student work often praised the approach as being "clever" (Charlotte), "wonderful" (Felicia) or "nice" (Greg). As Felicia so simply articulated, "I don't know why I didn't think about it, but basically, as long as she keeps the ratio to milk to ice cream as 2 to 3, it should work." For reasonableness, many teachers commented on the compatibility of the numbers. For example, Greg responded "This example was easier because you were given that she has three-quarter cups of ice cream. Therefore, each of those could equal one-fourth, and that was easy. If she was given two cups of ice cream, then you would have to figure out how to make the three circles equal to two, which would be a harder question for most students." These teachers not only solved this problem correctly but could make sense of Student A's work and reasoning. In correctly determining reasonableness they recognized the importance of the ratio remaining constant and often articulated when this strategy would be more challenging to use.

Milkshake Problem 1

Katrina wants to follow the milkshake recipe of 2 c milk and 3 c ice cream, but she only has ¾ c ice cream. How much milk will she need?

Student A:

She'll need 3/4 cup. The ratio is 2 parts to 3 parts so it doesn't matter what those parts are as long as they are 2 to 3. In this recipe, it's 2/4 to 3/4.

Student B:

I used a ratio table to figure it out.

| milk	2	1		1/3	1/6	3/6
Ice cream	3	1 1/2		1/2	1/4	3/4

Once I did that, it was easy to see that I'd need half a cu

Figure 3: Milkshake Problem 1 and student work

Team B, in Figure 4, used strips to demonstrate the ratios in the four recipes from Milkshakes Part 3. In this student work, we can easily see the ratios where each rectangle in the strip represents one cup. This representation is common, especially when students think more additively. The constant of proportionality is much harder to attend to in this representation. As seen in Table 1, only 17 out of 30 (57%) teachers could make sense of this work and determine whether it was reasonable. For example, Diana, in response to the question if this approach will always work, said no "Because I feel, I really feel like these students are simply counting their boxes and aren't taking into consideration the ratio." Participants who were able to determine reasonableness were able to articulate the potential pitfalls with this work and many suggested they would want to ask students a follow up question, such as when Ella said, "let's take the fractions five eighths and two thirds like those are one twenty-fourth apart. So if we were to draw those, I don't think students really could see which one is more chocolaty."

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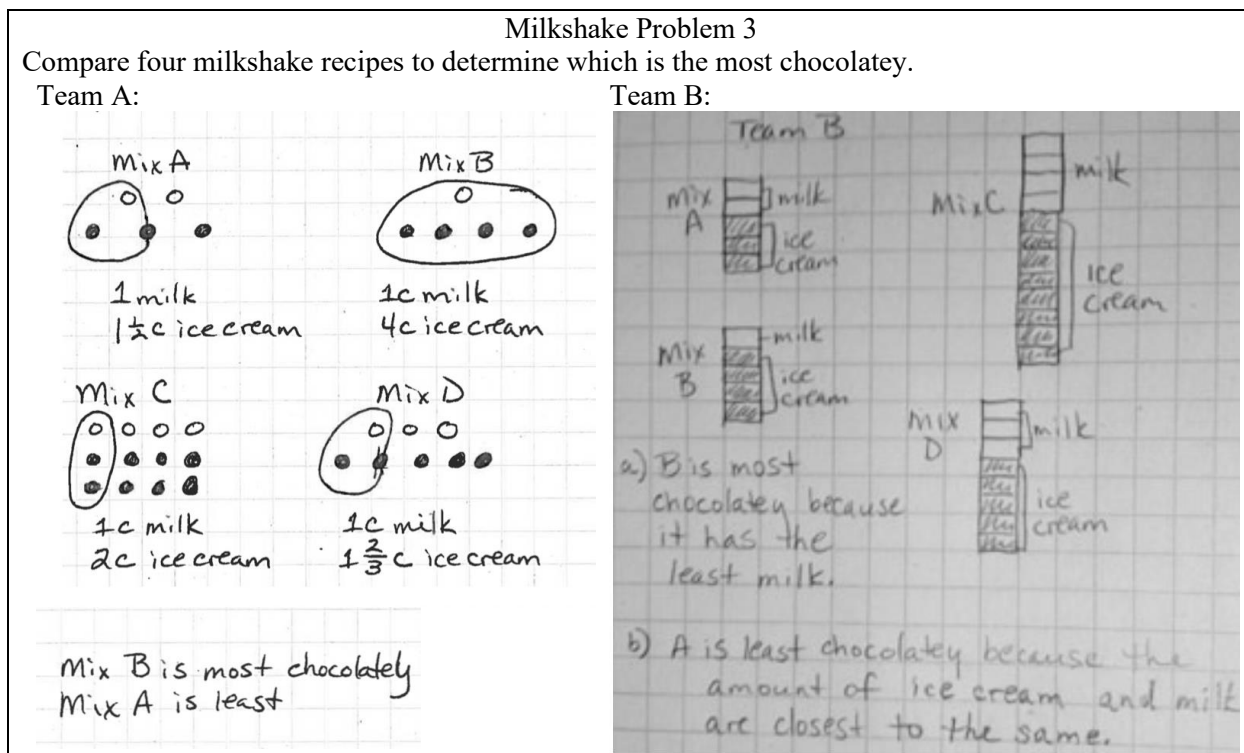
Figure 4: Milkshake Problem 3 and student work

Significance

These results contribute to the fields understanding of teacher knowledge, particularly using the Knowledge Quartet framework (Rowland, 2013) with practicing teachers (the Knowledge Quartet framework resulted from studies of preservice teachers). Our intent in this inquiry was to understand how teachers solve the problems and how they make sense of students' work, because those are both fundamental aspects of the work of teachers. We assert that evaluating seeing student work in an interview protocol is similar to seeing student work during an active lesson, only with the time constraints inherent in the classroom removed. Teachers need to make decisions about what a student is doing and whether it is a reasonable approach in order to respond in productive ways to students' work. The results from our participants indicate teachers, regardless of being able to solve a proportional reasoning problem themselves, can often make sense of what a student is doing to solve that same problem. Being able to determine the reasonableness of the approach appears to be more aligned with a teacher's ability to solve the problem correctly. Thus, Foundation and Contingency (2 of the four dimensions) may not result in the same teacher understandings used. The field should consider what teacher knowledge we are actually measuring.

In addition, these results suggest teachers are familiar with common representations, such as ratio tables. However, when common representations are used in unusual ways, such as our Student A and Team B examples, teachers have a harder time making sense of the work and determining whether it is reasonable. Thus, professional development providers and teacher educators should consider not only engaging teachers with these representations, but also engage them in considering unusual ways these representations could be used in productive and

unproductive ways. This likely means, engaging teachers with the structures of the mathematics (e.g., attending to invariance and what remains constant) rather than using the representations.

Acknowledgments

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