REPRESENTING PROPORTIONAL REASONING ALGEBRAICALLY TO PROBLEM SOLVE

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Proportional reasoning problems can be solved using algebraic reasoning. Therefore, making connections between proportional reasoning and algebraic thinking is important for solving problems. This study examined K-8 teachers' problem-solving strategies as they worked out a real world multi-step problem that involved proportional reasoning and algebraic thinking. The findings revealed that many teachers found this problem challenging. Particularly, some teachers had difficulty figuring out how to translate the variables into an algebraic equation. Some teachers who used variables as labels tended to engage in additive reasoning. They had difficulty representing the proportional problem context algebraically and solving the problem for the unknown quantity. Implications for further research are discussed.

Keywords: Teacher Knowledge, Proportional Reasoning, Algebraic Thinking

Proportional reasoning and algebraic thinking are often taught independently of each other. Therefore, when encountering a real-world problem that involves proportional reasoning and algebraic thinking, students who only developed procedural knowledge find such problems difficult to solve. This is because teachers tend to focus on aspects of problems that require only procedural knowledge, with a singular solution, strategy, and representation (Glassmeyer & Edwards, 2015). Teachers need to develop a deep understanding of the interrelationship between the conceptual and procedural knowledge to support their students to engage in problem solving and reasoning (Ma, 1999; Rittle-Johnson, Siegler & Alibali, 2001). Researchers suggest that many teachers struggle with understanding proportional reasoning (Riley, 2010; Cohen, Templin & Labato, 2010; Weiland, Orrill, Brown & Nagar, 2019). Particularly, this is the case with distinguishing proportional and non-proportional situations. Furthermore, teachers tend to focus on additive reasoning as opposed to proportional reasoning. There is very little research on teacher knowledge on proportional reasoning (Weiland, Orrill, Brown & Nagar, 2019).

This study investigated teacher's ability to translate a proportional relationship into an algebraic equation. More specially, pre and post test data on how teachers solved a multi-step problem involving proportional reasoning and algebraic thinking after participating in content based professional development was analyzed.

This study focused on answering the following research questions:

1) Did the professional development improve in-service teachers' overall conceptual understanding involving proportional reasoning and algebraic thinking?
2) Did the professional development improve in-service teachers' conceptual understanding in algebraic thinking?

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3) Did the professional development improve in-service teachers' conceptual understanding in proportional reasoning?

Proportional Reasoning

A proportional situation is one that has "structural relationships among four quantities, (say *a*, *b*, *c*, *d*) where there is a covariance of quantities and an invariance of ratios, where a ratio is a comparison of two quantities" (Weiland et al, 2019. P. 233). The components involved in reasoning with proportions are unitizing, rational numbers, ratio sense, partitioning, quantities and change, and relative thinking (Lamon, 1999). Uniting is a cognitive process that assigns a unit of measurement to a specific quantity (Lamon, 1996). The ability to form and operate within complex unit structures allows for higher ordered and flexible thinking. An example of uniting would be referring to an hour as 1 unit of 60 minutes or 2 units of 30 minutes, or 6 of 10 minutes, or 12 of 5 minutes depending on the context. Partitioning refers to the ability to break down a unit into equal parts (Lamon, 1999).

When proportions are represented in fraction notation (ie. $\frac{a}{b}, \frac{c}{d}$, not *a*:*b*, *c*:*d*), the information is structurally represented where it can be manipulated algebraically in any given calculation process. It is important to remember that all of the four quantities (*a, b, c, d*) can each be equal to one (1) in a given situation, which is simply the multiplicative identity property as a proportion. This models a transfer and flexibility of thinking which is necessary to manipulate the proportion. Critical thinking is foundational to proportional reasoning, as it involves abstracting then possibly manipulating that information (depends on the situation).

Proportional reasoning situations can also be represented algebraically. Representing proportional reasoning situations algebraically involves a flexible understanding of the meaning variables such as representing a category, a known value, an unknown value, or a changing value (Moss & Lamberg, 2019) to represent a problem situation. Representing proportional reasoning problems algebraically involves the ability to model real world situations which is considered a main objective of algebra (Izsak, 2003; Kaput, 1999; Schoenfeld, 1992).

Algebraic Reasoning

Algebraic thinking involves engaging in reasoning and sense making (Kaput and Blanton, 2005, Swafford and Langrall, 2000). It is the ability to model quantitative situations by being able to represent relationships quantitatively (Driscoll, 2001). According to Driscoll (2001), algebraic thinking involves developing habits of mind to think about quantitative relationships such as the ability to organize information by discovering patterns, relationships, and rules. As these 'habits' are listed as a structure of steps, in essence, this too then is a procedural skill. When this is practiced in order to become a habit, it becomes a behaviorist model with the incentive of possibly developing a conceptual understanding at any given point in this habitual practice. However, algebraic reasoning is a practice with the distributive, commutative, associative, and identity properties of addition and multiplication, and its abstracted symbolic representation is used to denote the calculations which deliver the final analysis and result.

The cognitive process in algebraic thinking includes encoding information, then retrieving and manipulating it to produce a final representation—a function in the brain also known as working memory (Gluck et al., 2016). Reasoning algebraically with known, unknown, and changing values is an evident example of the working memory function in the brain, and multi-step reasoning problems require unitizing and managing the transitions as they are modeled,

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which are then abstractly denoted symbolically (or vice versa). Algebraic reasoning models the various stages which demonstrate one's depth of knowledge—a conceptual understanding of the mathematics in any given situation.

Method

Twenty-three K-8 teachers participated in four-week content based professional development in a western state and the data presented here is from a larger study. A pre and post test was administered at the beginning and end of the week-long institute. The week-long institute focused on developing teachers' understanding of fractions and proportional reasoning content and pedagogical knowledge. The following problem was analyzed in this study.

> *Jeff had one-fourth as much money as Peggy. Ed had twice as much money as Peggy, they counted their money and then gave $20 to one of their friends. If they now have a total of $84, how much money did they initially have. Write an equation for this problem and solve it.*

The data was coded based on strategies that teachers used in three categories, *evolving, emerging* and *effective* in proportional reasoning and algebraic thinking, as illustrated in Figure 1.

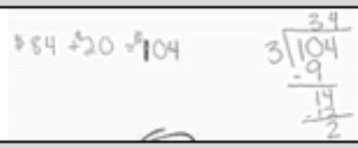
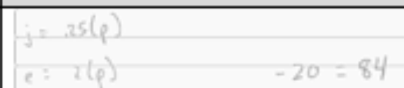
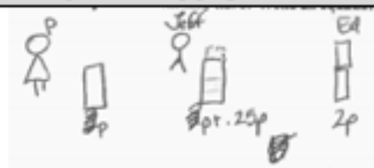
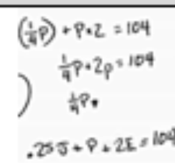
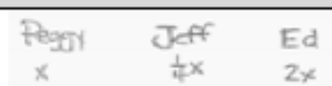
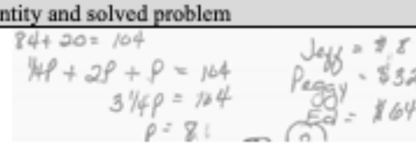
Score	Proportional Reasoning	Algebraic Thinking
1 *Evolving*	No Proportions, or no attempt	Variables used as labels, unable to model problem context problem and set up algebraic equation
		
2 *Emerging*	Attempted to set up proportions, but has errors	Variable used to represent an unknown quantity and modeled problem but did not solve original problem
		
3 *Effective*	Identified proper proportions correctly	Modeled problem using one variable as an unknown quantity and solved problem
		

Figure 1: Variables Rubric: Proportional Reasoning, Algebraic Thinking

In proportional reasoning, *effective* scores were able to identify the correct proportions. In this stage, participants were able to identify the unit (Peggy) creating a one-to-one relationship. They were also successful in demonstrating a correct ratio and partitioning understanding in forming the relationship between Jeff and Ed. For *emerging* scores, Peggy was correctly identified as the unit. However, for there was an incorrect partitioning or ratio relating Jeff or Ed to Peggy. For *evolving* scores, there was a lack of proportional understanding, in that there was

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no attempt to utilize a proportional relationship between Peggy, Jeff, or Ed. These strategies use a guess and check, and there was no clear demonstration of unitizing, partitioning or ratio concepts.

In algebraic thinking, *effective* scores were able to create an equation using one variable and correctly solve for all three values using algebraic properties of equality. For *emerging scores,* an equation was created using one variable, but was either not correctly solved or contained incorrect or missing proportions for either Jeff or Ed. For *evolving* scores an equation was used but contained multiple variables representing Peggy, Jeff and Ed or variables for Peggy, Jeff and Ed were identified but no equation was developed.

In full effect, an *effective* score in algebraic thinking includes correct referent units when presenting the solution. The presentation of referent units exhibits a thorough analysis and conceptual understanding of a problem, and demonstrate a focus on the unit that was manipulated in the problem. In this case a dollar symbol ($) was used to denote the referent unit in the rubric example. The role of referent units is for tracking information and changes between the known and unknown values.

A McNemar test (McNemar, 1947) was used to determine whether there was difference in proportion of participants classified as *non-effective (evolving & emerging)* and *effective* in both proportional reasoning algebraic thinking between the pre and post tests.

Results

The data revealed there was growth between the pre and post test in relation to teachers' ability to engage in proportional reasoning and algebraic thinking. In a paired t-test there was a significant difference in overall scores between the Pre ($M = 1.35$) and Post ($M=2.07$) scores, $p <.001$. This shows there was growth between the pre and post test (see Figures 2 & 3). This shows that overall, the professional development did have a positive impact on teachers' overall skill in a problem involving proportional reasoning and algebraic thinking.

The McNemar test (McNemar, 1947) revealed the following: In algebraic thinking the results were statistically significant *($p=.016$)* meaning there was a significant difference in the proportion of participants found to be *effective.* In proportional reasoning the results were statistically insignificant (*$p=.063$)* meaning that there was no significant difference in the proportion of participants considered to be *effective* in proportional reasoning between pre & post tests.

In an analysis of the pre and post tests, most teachers were able to set up the proportional reasoning aspect of the problem. However, they struggled modeling the problem algebraically using variables to solve the problem. More teachers initially struggled setting up an algebraic equation and solving the problem. While this was not the initial focus of this study, possible gaps of the teachers understanding in proportional reasoning are discussed.

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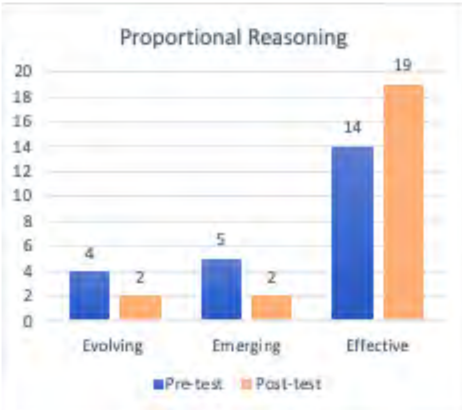

Figure 2: Proportional Reasoning Results

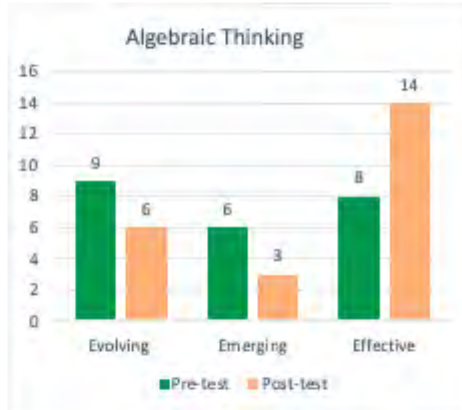

Figure 3: Algebraic Thinking Results

Discussion

The findings reveal that while there was improvement in the proportion of *effectiveness* in both proportional reasoning and algebraic thinking, the greatest gains and statistically significant results were in algebraic thinking. It was noted that some teachers initially struggled with proportional reasoning, and their ability to represent the multi-step proportional problem algebraically. These findings are consistent with other research findings that teachers struggle with conceptual understanding of proportional reasoning (Riley, 2010; Cohen, Templin & Labato, 2010; Weiland, Orrill, Brown & Nagar, 2019). The teachers that struggled with proportional reasoning were likely engaging in additive reasoning. Initially, many teachers had difficulty meaningfully modeling the proportional reasoning problem context using variables and equations. The ability to model and engage in algebraic reasoning is critical for understanding algebra (Izsak, 2003; Kaput, 1999; Schoenfeld, 1992). Algebraic reasoning involves being able to model the problem using expressions, equations, and variables. Specifically, it is helpful to distinguish between how variables are used such as labels or unknowns (Moss & Lamberg, 2019). Teachers who struggled with variables used them as labels to keep track of their thinking but were unable to set up an algebraic equation that involved proportional thinking. The post test results revealed that teachers became more proficient at solving a similar problem after they had engaged in professional development aimed at conceptual understanding of proportional reasoning and algebraic thinking.

More specifically, when setting up the initial proportions, teachers were able to effectively represent the proportions in one variable, which is foundational to the task to write an equation, and then solve it to answer the question. In further analysis, it was noted that all the participants in the *evolving* and *emerging categories* for algebraic thinking had a conceptual barrier in both unitizing and equality properties for proportional reasoning, whereby a one-to-one relationship allows for substitution in a problem, which then changes a multi-variable equation into a one-variable equation through properties of equality that is slated to be solved for first. Specifically, these participants were unable to identify Peggy as the unit and create a proportion or relationship between Jeff and Ed based on the unit (Peggy). In the *effective* category, it was noted that participants were successful in unitizing, creating a one-to-one relationship for Peggy, allowing for the multi-variable to be translated into a single-variable problem, using the correct proportions. See Figure 1.

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This suggests that identifying Peggy as the unit algebraically may have been a barrier in setting up the algebraic equation in terms of one variable. An example of this gap in algebraic thinking can be found in the *evolving* score as in Figure 1. This shows there is an algebraic idea developing, with denoting the three people as three different variables are to be added. However, no further steps are taken to solve the equation (the final task) represented in three different variables.

In the *emerging score,* the teacher starts building an equation in one variable (p), there is a complete mid-stop and disconnect to a final presentation of an equation, which thereby demonstrates there is an incomplete understanding in relating the unit or one-to-one proportional representation that is necessary for the algebraic representation in the equation, ultimately limiting their ability to solve the problem. This is an example of when Driscoll's habits of mind used as a practice set of steps to foster algebraic thinking (2001) represents a classical conditioning model (a behavioral process) that may lead to habituation over time (response to a stimulus declines). However, either does not necessarily ascertain the development of the necessary conceptual understandings of the algebraic content and reasoning being presented.

Classical conditioning and the working memory in the brain (processing new and incoming information) are not directly related and have different neural underpinnings in the brain (Gluck et al., 2016). The latter has been found to predict the learning of underlying conceptual structures when connecting multiple pieces of information (Banas & Sanchez, 2012). Flexibility and transfer of thinking when connecting information are developed in the function of the working memory. The only flexibility in thinking shown *emerging* teachers' written response in levels, was in changing fractions into decimal representations. The process of identifying the unit in a problem that used a proportion and algebraic thinking was not a specific concept covered in the professional development. More research should be done to document the cognitive processes and relationship between the proportional reasoning unit and algebraic translation.

Given that this study was limited to a single math problem, more research is needed in teacher's knowledge and ability to translate a proportional reasoning problem into an algebraic equation to model and solve the problem. Furthermore, more research is needed to understand and develop best practices to support instruction and student thinking when encountering problems that involve both proportional reasoning and algebraic thinking. Additionally, in future professional developments, more attention needs to be spent in the cognitive process of unitizing in the proportional reasoning to determine whether it enhances teachers' ability to improve their algebraic thinking.

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