ANALYTIC EQUATION SENSE: A CONCEPTUAL MODEL TO INVESTIGATE STUDENTS' ALGEBRAIC MANIPULATION

Yufeng Ying
University of Georgia
yufengsmart@gmail.com

Kevin Moore
University of Georgia
kvcmoore@uga.edu

In this paper, I propose a new construct named analytic equation sense *to conceptually model a desired way of reasoning that involves students' algebraic manipulations and use of equivalent expressions. Building from the analysis of two existing models in the field, I argue for the need for a new model and use empirical evidence to explain the new model.*

Keywords: Algebra and Algebraic Thinking, Cognition

Students' success in learning algebra has concerned educators for decades, and researchers have stressed algebra's importance to students' learning and growth extensively and repeatedly (Kaput, 2000; Usiskin, 1995; Wu, 2001). A central difficulty to students' algebra learning, as Behr et al. (1980) captured, is a sense of "extreme rigidity about written sentences," which includes "an insistence that statements be written in a particular form" and "a tendency to perform actions (e.g., add) rather than to reflect, make judgments, and infer meaning" (p. 16). Such a sense of rigidity in doing algebra appears in scenarios such as students interpreting an equal sign as "calculate the left side" (e.g., Knuth et al., 2006), students meeting difficulties in using the substitution method in solving equations (e.g., Jones, 2008), and students hesitating to transform an expression into its equivalent expressions when beneficial (e.g., Ying, 2020).

The field has conducted many studies regarding a sense of rigidity that appears in students' conception of the equal sign and students' use of the substitution method (e.g., Alibali et al., 2007; Baroody, 1983; Knuth et al., 2008; McNeil., 2006, Jones et al., 2012). Comparatively, less research has focused on the sense of rigidity in students' symbol manipulation and use of equivalent expressions for problem-solving (such as given $x+y=2$, $xy=2$, students should be able to evaluate x^2+y^2 without solving for x and y but realize $x^2+y^2= (x+y)^2-2xy$). On the one hand, such an area that is challenging to research as 1) the idea of using equivalent expressions or symbol manipulation is so general that converting $2+x=5$ to $x=5-2$ can also be argued as using equivalent expression; 2) a flexible use of symbol manipulation and equivalent expressions may be influenced by a complicated set of mathematical knowledge and is hard to list out clearly (Hoch, 2006); 3) it is doubtful whether some algebraic manipulations are just symbol playing which carry little educational value (Booth, 2018). However, on the other hand, studies have reported an important connection between students' flexibility in using algebraic manipulation and their success in mathematics (e.g., Novotona & Hoch, 2008; Vincent et al., 2017; Kieran, 2006) and how some delicate algebraic manipulations echo the essential aesthetic nature of mathematics and are accompanied by deep mathematical thinking (e.g., Arcavi, 1994; Dreyfus & Eisenberg, 1986).

The aforementioned difficulties and affordances of reasoning flexibly with algebraic equations collectively suggest the need to construct a conceptual framework in studying students' algebraic manipulation and the use of equivalent expressions. Accordingly, the paper reports a result from an ongoing research effort in constructing such a conceptual framework. Specifically, the paper begins by discussing the affordances and limitations of two existing constructs. Building from this analysis, the paper proposes a new construct named *analytic*

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

equation sense with empirical evidence illustrating a) how the construct was conceptualized from analyzing students' algebraic manipulations and b) three factors that the construct captured as the core elements in supporting students' algebraic manipulation and use of equivalent expressions.

Existing Constructs in Studying Algebraic Manipulations and Use of Equivalent Expressions

One of the early studies that drew attention to algebraic manipulation and equivalent expressions was Arcavi's (1994) work on symbol sense. Arcavi established the construct of symbol sense as an analog to number sense in the context of algebra. The definition of symbol sense included all sense-making activities relevant to symbols, which includes students' algebraic manipulations and use of equivalent expressions but extends much further.

Specifically, the idea of symbol sense addresses algebraic manipulations that are complemented with what Arcavi calls *reading through symbols*. As Arcavi (2005) indicated, the detachment of meaning in symbol manipulation helps with efficiency, but reading through symbols adds a layer of meaning and connectedness to performed manipulations. One example Arcavi (1994) provided was the problem of finding the numerical property of the result n^3-n when n is an integer. Arcavi suggested the problem could be solved by converting the expression n^3-n to the expression n(n-1)(n+1) and realizing that the latter term was the product of three consecutive integers, which further implied that n^3-n can be divided by 6. Arcavi argued in such a solution, one had to both apply manipulations (convert n^3-n to n(n-1)(n+1)) and read through symbols (conceive n(n-1)(n+1) as representing three consecutive integers) to fully solve the problem. Consequently, Arcavi argued that algebra manipulations and reading through algebra symbols are complementary to each other.

In the case of using equivalent expressions, Arcavi suggested that equivalent expressions can be conceptualized with non-equivalent meanings. Using the same example above, Arcavi believed the expression n(n-1)(n+1) helps students to interpret the term as the sum of three consecutive integers, which is an observation that the original expression n^3-n may not afford directly. Similarly, many expressions, when transformed into different equivalent expressions, can generate a richer set of implications and meanings. Therefore, Arcavi suggested an important aspect of symbol sense is to treat the result of manipulations not only as results but also as "potential sources of new meaning" (p.28).

Collectively, Arcavi's idea of symbol sense stresses the importance of incorporating a search for meaning while performing algebraic manipulation and using equivalent expressions. In other words, educators should attend more to symbol manipulations that are accompanied with meanings.

Nevertheless, since symbol sense also contains many other aspects, researchers have adopted the term in a broad range of areas. For instance, symbol sense was also used in studying students' conception of the minus sign (Lamb et al., 2012), students' understanding of the quantitative relationship between different expressions (Pope & Sharma, 2001), students' calculus performance (Thompson et al., 2010), and students' function graphing skills (Kop et al., 2020). As a result, the versatile use of the construct symbol sense has the risk to obscure researchers' real interest when working with such a construct. As Pierice & Stacy (2004) categorized, the applicability of symbol sense contains almost everything involving symbols. Furthermore, as Bokhove & Drijvers (2010) stated, "observing symbol sense is not a straightforward affair," as students "exhibit both symbol sense behaviors and behavior lacking", and it was hard to decide whether students "are relying on standard algebraic procedures or are actually showing insight into the equation of expression" (p.48). In summary, it is questionable whether the idea of symbol sense is at an appropriate grain size for studying algebraic manipulations and the use of equivalent expressions, and the definition of symbol sense might be too general to provide sufficient details to pinpoint students' cognitive difficulties in algebraic manipulations.

Another widely adopted framework in the field was Hoch's (2003) idea of structure sense. Building on Arcavi's symbol sense, Hoch narrowed her scope of interest to manipulations that leveraged algebraic structures. Hoch & Dreyfus (2005) defined algebraic structures as combinations of external appearances (the way an expression is written) and internal orders of an expression (the potential implications of an expression). As an oversimplification, one might interpret Hoch's definition of algebraic structures as almost all possible information that one can derive from algebraic expressions, and the conception of structure sense is then a collective set of skills in leveraging the derived information to make manipulations and use equivalent expressions. In short, Hoch's idea of structure sense is a trimmed version of symbol sense that focuses on students' flexible algebraic manipulations.

Many researchers have used Hoch's idea of structure sense and studied relevant students' algebraic manipulations. Hoch and Dreyfus (2005) primarily investigated students' structure sense by asking students to solve problems that contain some "cancelable" parts on both sides of the equation, such as solving for x knowing $(x^3 + 2x) - x = 5 + (x^3 + 2x)$. During the study, Hoch found only 6.3% of the students recognized the cancellation without a bracket, 13.6% with one bracket, and 17.7% with two brackets. Hoch and Dreyfus (2006) found that structure sense increases students' accuracy in solving algebra problems, but even high performers lack structure sense. Jupri and Sispivati (2017) reported that experts (mathematics lecturers in college) would solve some challenging problems in a consistent way with Hoch's picture of structure sense. An interesting observation by the authors was that sometimes experts started the problem by following procedural solutions without exploiting algebraic structures, and then these experts came back to leverage structures when they met difficulties. Researchers have also used the idea of structure sense in a broader setting, including college algebra and basic arithmetic (e.g., Novotna et al., 2006; Novotna & Hoch, 2008; Meyer, 2017; Bishop, 2018), and the lack of structure sense among teachers and students was a common theme across many findings (e.g., Musgrave et al., 2015; Vincent et al., 2017).

The idea of structure sense has a much smaller grain size than the idea of symbol sense, and researchers have applied the term with more coherence in studying students' algebraic manipulations. However, the construct still suffers from salient constraints: namely, if one carefully reviews the mathematical tasks that researchers have used in studying structure sense, one may find a lack of clarity in the mathematical understanding that the idea structure sense tries to capture. Consider the following three questions as examples:

Q1: $\frac{1}{4} - \frac{x}{x-1} - x = 5 + (\frac{1}{4} - \frac{x}{x-1})$; Q2: (x-3)4-(x+3)4 ; Q3: 10012-9992;

All three questions are taken from Hoch and Dreyfus's (2005, 2006) research. Hoch and Dreyfus believed that students' abilities in solving these questions elegantly measured their structure sense. In the appearance, all four questions do measure students' abilities in performing certain algebraic manipulations, but it is doubtful whether the intellectual capacities required in each task are well-connected or consistent. For instance, Q1 requires students to be sensitive toward a potential cancellation on both sides of an equation, Q2 expects students to view a compound expression (x-3) as a single entity, and Q3 asks students to apply the property of (a^2-

b^2)=(a+b)(a-b) into a numerical expression. Cognitively, each task seems to demand a different set of mathematical knowledge. In relating those tasks to the broader field of mathematics education, one may find Q1, Q2, and Q3 all indeed align with different existing research topics. For instance, Q1 overlaps with Carpenter's (2005) idea of relational thinking, which models students' coordination between both sides of an equation (e.g., realizing cancellation). Q2 touches on the broader topic of transitioning between arithmetic and algebra, and many researchers have studied students' difficulties in forming an algebraic way of thinking and mastering algebraic rules (e.g., Carraher et al. 2006; Filloy & Rojano, 1989; Herscoviss & Linchevski, 1994; Kirshner, 2004). Q3 brings the theme of creativity in mathematics problem solving and number sense. As a result, the cognitive commonalities between these tasks and between the thinking required in these tasks are unclear. Consequently, without explicating students' thinking behind all those tasks, to group those tasks under the same quilting of structure sense might be counter-productive in helping teachers to locate students' real struggles with the learning of algebra.

The lack of cognitive explanation on the thinking behind the construct structure sense is most salient for the question that asks the student to prove $(x+y)^4=(x-y)^4+8xy(x^2+y^2)$ (Hoch & Dreyfus, 2006). In Hoch and Dreyfus's writing, this question should be solved with certain manipulation tricks. However, why students should not just expand the polynomial on both sides? As Jupri and Sispivati (2017) illustrated, experts also attempt problems by procedural solutions, and it is psychologically natural for students to take an approach that is less cognitively demanding. Therefore, we remain cautious in believing all manipulation problems share equal values. Moreover, to help differentiate between random symbol playing and desired manipulations, I believe a cognitive explanation to the thinking behind algebraic manipulations is needed.

In summary, both the constructs of symbol sense and structure sense have helped researchers studying students' symbol manipulations. However, both constructs lack specificity and cognitive explanatory power in a) identifying beneficial and preventive factors that are relevant to students' algebraic manipulation; b) explicating a way of reasoning that teachers and students can adopt in engaging algebra manipulations. Some studies also touch on such an area, such as Harel & Soto's (2017) work on structural reasoning, Hausberger's (2015) work on structuralist thinking, and Schoenfeld's (2014) work on problem-solving. Similarly, their works situate in different grain sizes and lack specialized cognitive analysis of the thinking behind desired algebraic manipulations. Still, all aforementioned works are indispensable, and they are the giants' shoulders the paper stands on.

Method and Methodology

The ongoing research project aims to design a conceptual framework in studying students' flexible and meaningful algebraic manipulation and use of equivalent expressions in problem-solving. The term conceptual framework follows Thompson's (2008) writing on conceptual analysis. Epistemologically, we share many premises with general constructivism (e.g., Glaserfeld, 1995) and believe that students construct their own mathematics. Accordingly, the building of a conceptual framework creates a hypothetical thinking model through observing and analyzing students' thinking so that such a framework becomes a viable way of assessing students' mathematical knowledge and provides a viable way of thinking that students and teachers can adopt in relevant tasks (Thompson, 2008, 2013).

Up to the date when the manuscript was written, we conducted four semi-structured clinical interviews (Barriball & While, 1994) separately with four pre-service high school teachers, and the length of each interview varied between two to three hours. We have only recruited pre-service teachers so far as 1) pre-service teachers' knowledge is an important factor that influences the general teaching quality (e.g., U.S Department of Education, 2000); 2) pre-service teachers share similar mathematical understandings with high school students on K-12 math content (Carlson, Oehrtman, & Engelke, 2010). All participants have completed several college-level math courses (e.g., multi-variable calculus) but not high-level analysis courses (e.g., complex analysis). In each session, we asked the participant to go through 6-8 sequenced algebra problem, and a talk-aloud approach was adopted. Such approach asks interviewers to encourage the participant to share their thinking at every step verbally, while staying cautious in not intervening participants' own thought process (Carlson & Bloom, 2005). Most of the problems were challenging algebraic questions with multiple solutions, and we do not expect nor push participants to solve all of them. Instead, we encourage each participant to try as much as possible, and view both their successful attempt and unsuccessful attempts as valuable data in indicating their thought process. We transcribed all recordings and used open coding (Khandkar, 2019) to find emergent themes that assisted the modeling of students' thought process.

Analytic Equation Sense along with empirical supports

Based on the empirical findings, the paper proposes a conceptual model named analytic equation sense (AES). We define AES as a positive cyclic reasoning process with three important aspects:

1. Equation aspect: Students should conceptualize an equation as generative to further equivalences.
2. Analytic aspect: Students should analytically navigate between different equivalences in a given problem beyond solely relying on visual clues.
3. Sense aspect: Students should reflect on the encountered problem to gain more knowledge about the potential affordances and limitations of different manipulations and equivalent forms. The reflection, in return, strengths students' awareness that an equation have multiple equivalent forms and helps students to develop stronger skills in navigating between various equivalent forms.

Equation Aspect

We chose the term equation as we found that students' conceptualization of an equation plays an important role in performing algebraic manipulation and using equivalent expressions. In specific, we build off Ying's (2020) research on differentiating between two different conceptions of the equation: Students with a type A conception conceive an equation as representing one equivalent relationship, and that students will be able to substitute quantities that are shown in the relationship. For instance, when given the equation $x^2-x+1=0$, students with type A conception can substitute the term x^2 with the term x-1 when needed. Students with type B conception will further conceive an equation as also representing a family of equivalent relationships, and that students will be able to transform the equation to generate substitutions for new quantities. For instance, when given the same equation $x^2-x+1=0$, students with type B conception can also generate a substitution for unappeared terms, such as $\frac{1}{x}$. The student may

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

realize x^2-x+1=0 implies $x - 1 + \frac{1}{x} = 0$ and then aware that $\frac{1}{x}$ can be substituted with the term 1-x when needed. We argue students' flexible algebraic manipulations require students to develop the type B conception.

We use students' work on the following task as an illustration: "Given a^2-3a+1=0, find the value of $3a^3 - 8a^2 + a - 1 + \frac{3}{a^2+1}$". A challenge in solving this problem is tackling the term $3a^3$ and $\frac{3}{a^2+1}$ (using the fact a^2-3a+1=0, the term $\frac{3}{a^2+1}$ equals $\frac{1}{a}$). One way to substitute both terms is to transform the given equation a^2-3a+1=0 into a^3-3a^2+a=0 (multiply a on both sides) and $a - 3 + \frac{1}{a} = 0$ (divide a on both sides) and use those two new equations for substitution. All participants displayed a sense of struggle with this problem, especially with tackling the term $3a^3$ and $\frac{3}{a^2+1}$. We believe many of the observed struggles related to their lack of type B conception, which is the conception that an equation represents a family of equivalent relationships, consider: In dealing with the term a^3, students did not attempt substitute a^3 directly (which can be accomplished through converting a^2-3a+1=0 to a^3-3a^2+a=0). Rather, students rewrote a^3 as a(a^2) and substitute a^2. Such manipulation displays a sense of preference to operate only with the term that was shown in the given equation a^2-3a+1=0. Similarly, in dealing with the term $\frac{3}{a^2+1}$, all participants deduced that a^2-3a+1=0 implies $a^2 + 1$=3a and rewrote $\frac{3}{a^2+1}$ $as \frac{3}{3a}$. However, when tackling the term $\frac{3}{3a}$ or $\frac{1}{a}$, all participants were puzzled and confused. When we asked participants whether they could infer anything about 1/a from the given equation a^2-3a+1=0, they suggested no. Since all participants performed substitution, we believe students have developed the type A conception of an equation. However, their inabilities to deal with the term 1/x and their preference to only operate with the term that was shown in the original equation indicated their potential lack of the type B conception.

After showing the solutions to the students and asking for their feedback, all of the participants expressed a sense of shock regarding the possibility of transforming the given equation to generate new equations. Their feedback reaffirmed our hypothesis that students may not conceptualize an equation as representing a family of equivalent relationships. In specific, one participant said, "I automatically think of modifying what's already there as opposed to changing the equation itself before we begin to solve, before we begin to work and solve actual problem." He also elaborated, "you are given these two equations, so the major response was to, ok, what can we do with these two, by themselves, to get the answer. Rather than what can we change about these two, you know like multiplying by a on both sides and dividing a on both sides before we begin actually go about solving." Similarly, another student stated, "I was thinking a lot of it like taking things like this (circling the original equation a^2-3a+1=0) as it was instead of moving terms around." In another problem, one participant also shared a similar sense of reluctance in transforming the given equation and stated that "these numbers are kind of sets, and usually I guess, these are usually presented in the way that is easiest to solve." Based on those responses, we infer that many students do not conceptualize an equation as a potential source to generate new equations, and such thinking thwarts students' flexibility in performing algebraic manipulations and their use of equivalent expressions.

In short, we use the term equation to highlight the need for students to understand that an equation can be transformed and leveraged in various equivalent forms, and educators should be aware of some unproductive beliefs, such as that equations "are usually presented in the way that is easiest to solve."

Analytic

We chose the term analytic as we found that an analytic way of reasoning plays an important role in doing algebraic manipulation and use of equivalent expressions. In specific, we followed Stylianou's (2006) research on differentiating between three different types of proof schemes, which are external (random guess), empirical (based on old memory or visual similarity), and analytic (with mathematical rationale). We believe a student displays an analytic way of reasoning in algebra manipulations if the student can provide a mathematical rationale or justification for the manipulation he or she wants to perform and performed. In contrast, we believe that a student does not display an analytic way of reasoning if he or she performs an operation solely based on random guesses or visual similarities. We argue that an analytic way of reasoning helps students with flexible algebraic manipulations. Consider the following two scenarios:

The first scenario is the case where the student adopted an analytic way of reasoning. The problem is "given a=2003, b=2007, c=1997, Evaluate $a^2+b^2+c^2-ab-bc-ac$". One way to solve the problem is realizing that the targeted expression equals $\frac{(a-b)^2+(a-c)^2+(b-c)^2}{2}$.

The student started the problem by writing down the expression $(a-b)^2$. Interestingly, he did not remember the exact formula but quickly calculated (a-b)(a-b) on paper to derive the expansion. He then wrote out the expansions for $(b-c)^2$ and $(c-a)^2$. And he said that he was going to try to use these three perfect squares expressions to get the answer. Finally, he realized that $(a-b)^2+(a-c)^2+(b-c)^2$ is $2(a^2+b^2+c^2-ab-bc-ac)$ and solved the problem. When the interviewer asked about his thought process in deciding such an approach, he replied, "the way the question is framed, with the squares, and also the subtraction of ab, bc, and ac. That makes me think of this formula how a different of squares will get you… get you there…Also I am seeing, after I saw this that, it will be easier to get a square if I can subtract out some of the larger number from each other". Later, he also explained that he wrote out all three expressions because he believed all three perfects squares were needed to substitute the terms "ab," "bc" and "ac that were shown in the expression.

Such a process displayed a desirable analytic way of reasoning. The student started the problem by trying to establish associations between the expression that he needed to evaluate $(a^2+b^2+c^2-ab-bc-ac)$ and the expression that he was acquainted with $((a-b)^2)$. After making such an association, he reaffirmed those associations' usefulness by realizing their potential in simplifying calculation (notice the difference between a,b, and c are relatively small). He further noticed that since the three middle terms were "ab," "bc," and "ac", if he wanted to rewrite the entire expression based on those perfect squares, he would also need $(b-c)^2$ and $(a-c)^2$. In such a thought process, his final success in finding the solution was accompanied by mathematical rationales, and those rationales guided and reaffirmed his choices of manipulation.

The second scenario is where the student adopted a non-analytic way of reasoning. When solving one problem, the student needed to evaluate $x^2-1+\frac{1}{x^2}$ from given equation $\frac{1}{x}+x=1$. One possible approach was to take squares on both sides of the equation $\frac{1}{x}+x=1$. Facing the problem, the student stated, "this expression (referring to $x^2-1+\frac{1}{x^2}$) was kind of similar to the one we were given (referring to $\frac{1}{x}+x=1$) , but I need to substitute something to replace the

x^2". After some thought, the student decided to substitute x^2 by x-1 (which derived from timing x on both sides of the equation $\frac{1}{x} + x = 1$) and transformed $x^2 - 1 + \frac{1}{x^2}$ to $(x-1) - 1 + \frac{1}{x-1}$, and then she was puzzled and stuck. We asked why she performed such a substitution, and she explained, "you wanna have similar terms on each of these, so just thinking about how a manipulation will help you give you something similar to whatever the expression is you are trying to find the value of."

From her response and writing, we infer the mathematical operation that she performed was largely motivated by pursuing visual similarities, and she might regard $\frac{1}{x} + x = 1$ and $(x-1) - 1 + \frac{1}{x-1}$ as similar since those two expressions visually appear so. Nevertheless, visual similarities between equations do not always translate into the similarities between equations' mathematical meanings. In this case, the student's pursuing of visual similarities thwarted her chance to find the solution, since converting between $\frac{1}{x} + x = 1$ and $(x-1) - 1 + \frac{1}{x-1}$ takes much more effort than converting between $\frac{1}{x} + x = 1$ and $x^2 - 1 + \frac{1}{x^2}$. Indeed, during our study, many students displayed non-analytic ways of reasoning and chose to perform some manipulations for reasons such as "this is what I did in the previous problem" or "I want to make this look like that". Frequently, those non-analytic ways of thinking lead students in an unproductive direction. More importantly, without analytic ways of reasoning, students frequently meet difficulties in evaluating whether a particular approach is worth continuing.

Based on the contrast between these two scenarios, we argue that students who are guided solely by visual features of expressions without analytically considering their mathematical meanings will have a more challenging time performing appropriate algebraic manipulations and use appropriate equivalent expressions. In summary, we chose the term analytic to highlight the necessity of helping students to generate a mathematical rationale regarding every manipulation that students made in solving algebra problems.

Sense

We inherit the word sense from Arcavi and Hoch's writings as we believe students' algebraic manipulation and use of equivalent expression is essentially a sense-making process in solving algebra-related problems. Since we do not believe one may develop his or her sense-making ability all in a sudden. We believe students' skills in algebraic manipulation and use of equivalent expression, as one's skill in sense-making, requires continuous effort in practicing and reflecting.

In the example provided above where the student decided to substitute x^2 by x-1 for the expression $x^2 - 1 + \frac{1}{x^2}$, it is worth noticing that such substitution was derived from actively transforming the given condition $\frac{1}{x} + x = 1$ to $x + x^2 = x$. But that student, in the earlier problem which is "Given a^2-3a+1=0 and evaluate $3a^3 - 8a^2 + a - 1 + \frac{3}{a^2+1}$", did not attempt to change the given equation. Similarly, after solving one problem which required taking the reciprocal, that same student actively started to try to take reciprocals in the next problem. Unfortunately, we cannot prove that she gained these insights by reflecting on the earlier problem. However, her performance does raise the possibility that one's intuitions and skills for algebraic manipulations can be gained through practice and reflecting on encountered problems. Those practices and reflections, in return, can strength students' awareness that an equation have multiple equivalent forms and helps students to develop stronger skills in navigating between various equivalent forms. Therefore, we use the term sense to indicate our belief that the development of algebraic manipulation skills is a constant learning process that requires continuous effort in practicing and reflecting.

Conclusion

In a nutshell, the conceptualization of AES represents a sincere effort to capture the potential sense-making process in which students can engage in algebraic manipulation and use of equivalent expressions. AES can be used both as a way of reasoning that students can adopt in solving algebra problems or as a research framework in understanding relevant students' difficulties. In a broader context, AES speaks directly to the general theme of rigidity that educators try to tackle, and the construct encourages students to engage in algebra problems flexibly, analytically, and creatively.

References

Alibali, M. W., Knuth, E. J., Hattikudur, S., McNeil, N. M., & Stephens, A. C. (2007). A longitudinal examination of middle school students' understanding of the equal sign and equivalent equations. *Mathematical Thinking and learning*, *9*(3), 221-247.

Arcavi, A. (1994). Symbol sense: informal sense-making in formal mathematics. *For the Learning of Mathematics, 14*(*3*), 24–35.

Arcavi, A. (2005). Developing and using symbol sense in mathematics. *For the learning of mathematics*, *25*(2), 42-47.

Baroody, A. J., & Ginsburg, H. P. (1983). The effects of instruction on children's understanding of the" equals" sign. *The Elementary School Journal*, *84*(2), 199-212.

Behr, M., Erlwanger, S., & Nichols, E. (1980). How children view the equals sign. *Mathematics teaching*, *92*(1), 13-15.

Bishop, J. P., Lamb, L. L., Philipp, R. A., Whitacre, I., & Schappelle, B. P. (2016). Leveraging structure: Logical necessity in the context of integer arithmetic. *Mathematical Thinking and Learning, 18*(3), 209-232.

Bokhove, C., & Drijvers, P. (2010). Symbol sense behavior in digital activities. *For the learning of mathematics*, *30*(3), 43-49.

Booth, L. (1989). A question of structure to action: Early learning of algebra: A structural perspective. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp 20-32). Virginia, VA: Lawrence Erlbaum.

Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational studies in Mathematics*, *58*(1), 45-75.

Carlson, M., Oehrtman, M., & Engelke, N. (2010). The precalculus concept assessment: A tool for assessing students' reasoning abilities and understandings. *Cognition and Instruction*, *28*(2), 113-145.

Carpenter, T. P., Levi, L., Franke, M. L., & Zeringue, J. K. (2005). Algebra in elementary school: Developing relational thinking. *Zentralblatt für Didaktik der Mathematik*, *37*(1), 53-59.

Carraher, D. W., Schliemann, A. D., Brizuela, B. M., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics education*, 87-115.

Dreyfus, T., & Eisenberg, T. (1986). On the aesthetics of mathematical thought. *For the learning of mathematics*, *6*(1), 2-10.

Filloy, E., & Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. *For the learning of mathematics*, *9*(2), 19-25.

Glasersfeld, E. Von. (1995). *Radical constructivism: A way of knowing and learning*. London: Falmer.

Ginsburg, H. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. Cambridge University Press.

Hausberger, T. (2015, February). Abstract algebra, mathematical structuralism and semiotics. In *CERME 9-Ninth Congress of the European Society for Research in Mathematics Education* (pp. 2145-2151).

Harel, G., & Soto, O. (2017). Structural reasoning. *International Journal of Research in Undergraduate Mathematics Education*, *3*(1), 225-242.

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational studies in mathematics*, *27*(1), 59-78.
Hoch, M. (2003). Structure sense. In M. A. Mariotti (Ed.), *Proceedings of the 3rd Conference for European Research in Mathematics Education* (CD). Bellaria, Italy: CERME.
Hoch, M., & Dreyfus, T. (2005). Structure sense in high school algebra: The effect of brackets. In M. J. Hoines & A. B. Fuglestad (Eds.), Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education (Vol. 3, pp. 49–56). Bergen, Norway: PME
Hoch, M., & Dreyfus, T. (2006). Structure sense versus manipulation skills: An unexpected result. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 305–312). Prague, Czech Republic: PME.
Jones, I., Inglis, M., Gilmore, C., & Dowens, M. (2012). Substitution and sameness: Two components of a relational conception of the equals sign. *Journal of experimental child psychology*, *113*(1), 166-176.
Jupri, A., & Sispiyati, R. (2017). Expert strategies in solving algebraic structure sense problems: The case of quadratic equations. In *Journal of Physics (Conference Series)* (Vol. 812, No. 1).
Kaput, J. (1999). Teaching and learning a new algebra with understanding. In E. Fennema, & T. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 133-155). Mahwah, NJ: Erlbaum
Khandkar, S. H. (2009). Open coding. *University of Calgary*, *23*, 2009.
Kirshner, D., & Awtry, T. (2004). Visual salience of algebraic transformations. *Journal for research in mathematics Education*, *20*, 224-257.
Kieran, C. (2006). Research on the learning and teaching of algebra: A broadening of sources of meaning. In *Handbook of research on the psychology of mathematics education* (pp. 11-49). Brill Sense.
Knuth, E. J., Alibali, M. W., Hattikudur, S., McNeil, N. M., & Stephens, A. C. (2008). The importance of equal sign understanding in the middle grades. *Mathematics teaching in the Middle School*, *13*(9), 514-519.
Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education*, *37*, 297-312.
Lamb, L. L., Bishop, J. P., Philipp, R. A., Schappelle, B. P., Whitacre, I., & Lewis, M. (2012). Informing Practice: Developing Symbol Sense for the Minus Sign: research matters for teachers. *Mathematics Teaching in the Middle School*, *18*(1), 5-9.
Kop, P. M., Janssen, F. J., Drijvers, P. H., & van Driel, J. H. (2020). The relation between graphing formulas by hand and students' symbol sense. *Educational Studies in Mathematics*, *105*(2), 137-161.
McNeil, N. M., Grandau, L., Knuth, E. J., Alibali, M. W., Stephens, A. C., Hattikudur, S., & Krill, D. E. (2006). Middle-school students' understanding of the equal sign: The books they read can't help. *Cognition and instruction*, *24*(3), 367-385.
Meyer, A. (2017). Students' development of structure sense for the distributive law. *Educational Studies in Mathematics, 96*(1), 17-32.
Musgrave, S., Hatfield, N., & Thompson, P. (2015). Teachers' meanings for the substitution principle. In T. FukawaConnelly, N. E. Infante, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 18th meeting of the MAA special interest group on research in undergraduate mathematics education* (pp. 801–808). Pittsburgh, PA: RUME.
National Commission on Mathematics, Science Teaching for the 21st Century (US), & United States. Dept. of Education. (2000). *Before it's too late: A report to the nation from the National Commission on Mathematics and Science Teaching for the 21st Century*. Diane Publishing Company.
Novotná, J., & Hoch, M. (2008). How structure sense for algebraic expressions or equations is related to structure sense for abstract algebra. *Mathematics Education Research Journal*, *20*(2), 93-104.
Pope, S., & Sharma, R. (2001). Symbol sense: Teacher's and student's understanding. *Proceedings of the British Society for Research into Learning Mathematics*, *21*(3), 64-69.
Pierce, R., & Stacey, K. (2004). Monitoring Progress in Algebra in a CAS Active Context: Symbol Sense, Algebraic Insight and Algebraic Expectation. *International Journal for Technology in Mathematics Education*, *11*(1).
Schoenfeld, A. H. (2014). *Mathematical problem solving*. Elsevier.
Stylianou, D., Chae, N., & Blanton, M. (2006). Students' proof schemes: A closer look at what characterizes students' proof conceptions. In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez (Eds.), *Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, (Vol 2, pp. 54-60). Mérida, México: Universidad Pedagógica Nacional.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In *Proceedings of the annual meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 31-49). PME Morelia, Mexico.

Thompson, P. W. (2013). In the absence of meaning…. In *Vital directions for mathematics education research* (pp. 57-93). Springer, New York, NY.

Thompson, P. W., Cheepurupalli, R., Hardin, B., Lienert, C., & Selden, A. (2010, March). *Cultivating Symbol Sense in Your Calculus Class*. Paper presented at the IM&E Workshop, San Diego, CA. Retrieved from http://ime.math.arizona.edu/2009-10/Pamphlets/Symbols.pdf

Usiskin, Z. (1995). Why is algebra important to learn. *American Educator*, *19*(1), 30-37.

Vincent, J., Pierce, R., & Bardini, C. (2017). Structure Sense: A Precursor to Competency in Undergraduate Mathematics. *Australian Senior Mathematics Journal*, *31*(1), 38-47.

Wu, H. (2001). How to prepare students for algebra. *American Educator*, *25*(2), 10-17.

Ying, Y. (2020). A conceptual analysis of the equal sign and equation -the transformative component. *Proceedings of the 42nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp.246-254).