WHEN IS A GUESS MORE THAN JUST A GUESS? MIDDLE-GRADES STUDENTS' GUESS AND CHECK STRATEGIES

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The appropriateness of guess and check as a problem-solving strategy has been debated. This qualitative study examines the use of guess and check by middle-grades students to solve linear systems of equations. Students' reasoning is examined within the number sequences framework, which is based in part on students' units coordination. Only students at the fourth and fifth stages (out of five) correctly solved systems of equations algebraically; this is attributed to their operations on two- and three-level unit structures, and to a disembedding operation. Students at the third stage applied strategic guess and check methods, which is attributed to assimilating with composite units (i.e., units of units), but these students could not correctly use an algebraic method. For students at the second stage, guess and check was non-strategic, which is attributed to their construction of composite units in activity. Implications for instruction are discussed.

Keywords: Algebra and Algebraic Thinking; Middle School Education; Number Concepts and Operations

Literature Review

Guess and check is a common strategy for students to apply in problem solving situations (Johanning, 2004). *Systematic guess and check* is form of reasoning in which a student "works with the situational context and applies relational reasoning to solve the problem" (Johanning, 2010, p. 123). Thus, students operate within the problem-solving context while simultaneously reasoning about the quantitative relationships to arrive at increasingly better approximations of the solution. More general definitions of guess and check range from trial-and-error (Gallagher et al., 2000), which may or may not be systematic, to "random guess and try" (Capraro et al., 2012, p. 112).

Guess and check is particularly relevant to solving algebra problems. While Knuth and colleagues (2006) define guess and check strategies as pre-algebraic, Kieran (1996) describes global meta-level activity as an algebraic activity that aligns closely with systematic guess and check. Therefore, it is unclear to what extent guess and check, and particularly systematic guess and check, is a productive algebraic strategy.

Johanning (2010) asked middle-grades students to solve linear systems of equations word problems and found that systematic guess and check was the most common method applied. She argued that guess and check is algebraic in nature and supports students in developing more sophisticated algebraic methods. By this rationale, systematic guess and check is a worthwhile skill with the potential to improve students' reasoning about systems of equations. In contrast, Malloy and Jones (1998) found that eighth-grade students who applied guess and check to linear systems of equations problems often failed to find a solution and did not initiate the use of alternative methods when guess and check failed. As these studies demonstrate, the conclusions surrounding the productive nature of guess and check are inconsistent. Furthermore, the research does not offer a theoretical rationale for students' widespread dependence on guess and check. This study asks, in what ways do the number sequences account for students' guess and check solutions to linear systems of equations? And, are students' strategies for solving systems of

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equations more closely tied to their number sequence or course enrollment?

Theoretical Framework

Olive and Çaglayan (2008) framed middle-grades students' algebraic solutions to linear systems of equations within their units coordination. The number sequences are based on units coordination (Steffe, 2010; Ulrich, 2015; 2016a), but also take into account mental operations such as iterating and disembedding (Steffe, 2010). This allows the number sequences to distinguish among three groups of students with varying stages of fluency operating with composite units (Ulrich, 2016b). Students who have constructed the tacitly nested number sequence (TNS) construct composite units in activity (Steffe, 2010); students who have constructed an advanced tacitly nested number sequence (aTNS) assimilate with composite units (Ulrich, 2016b); and students who have constructed an explicitly nested number sequence (ENS) assimilate with composite units, iterate units of one, and disembed (Steffe, 2010). Zwanch (2019, in review) demonstrated that the distinction among these three stages can be used to model their representations of multiplicative algebraic relationships. As such, the number sequences will be used to analyze students' use of guess and check to solve linear systems of equations.

Tacitly Nested Number Sequence (TNS)

TNS students assimilate with one level of units and construct a second level, or *composite unit*, in mental activity (Steffe, 2010; Ulrich, 2015). The operations of a TNS support double counting because TNS students can monitor the number of times that they count on. Consider the problem asking, what is seven more than 24? To a TNS student, the number word "seven" stands for a counting sequence from one through seven, but in mental activity can be chunked into one composite unit containing a counting sequence of seven units. Thus, TNS students can transpose the counting sequence to monitor their counting beginning at 24 and increasing to 31.

TNS students' algebraic reasoning. TNS students do not disembed, but Hackenberg (2013) found that disembedding is critical to algebraic reasoning. *Disembedding* is a mental operation that allows students to think about removing one unit from another without destroying either unit, and to reflect on the relationship between the two units (Steffe, 2010). For instance, to abstract the relationship between quantities such as 10 and 8 or 6 and 4 as x and $x-2$, requires the student to disembed the smaller quantity from the larger and reflect on the relationship (Hackenberg, 2013). This reflection supports the algebraic representation of the two related quantities. As TNS students do not disembed, Hackenberg's (2013) findings suggest that TNS students will be limited in their symbolic representations of related unknowns.

Advanced Tacitly Nested Number Sequence (aTNS)

aTNS students assimilate with composite units and construct or coordinate a third level of units in activity, but do not disembed (Ulrich, 2016b). To *assimilate* with composite units implies that aTNS students can immediately perceive of a number word, like "seven," as one unit containing seven units of one. This allows aTNS students to reason strategically by operating on embedded composite units (Ulrich, 2016b). For example, an aTNS student may find the difference between 39 and 62 is 23 by reasoning that 40 plus 22 is 62, and 39 is one less than 40, so the difference is one more than 22. aTNS students are only tacitly aware of the nesting of the subsequences, 39 and 23, within 62. This makes explaining their thought process challenging (Ulrich, 2016b).

aTNS students' algebraic reasoning. Zwanch (2019, in review) found that aTNS students can write algebraic equations to represent additive and multiplicative relationships, but they do so inconsistently. Their algebraic reasoning is supported by assimilatory composite units, which

Hackenberg et al. (2017) find support operations on unknowns. However, aTNS students' algebraic representations are inconsistent due to not disembedding (Zwanch, in review). This research demonstrates that aTNS students can write symbolic equations representing one-step additive and multiplicative relationships because they can operate on composite units in activity, thereby forming a third level of units. Following mental activity, however, the third level of units decays. As aTNS students cannot disembed one quantity from the other to reflect on the relationship following this mental decay, they have no material for reflection (Zwanch, 2019, in review).

Explicitly Nested Number Sequence (ENS)

ENS students also assimilate with composite units, but in addition can disembed and iterate units of one (Steffe, 2010; Ulrich, 2016a). Iterable units of one and disembedding support multiplicative reasoning (Steffe, 2010) because ENS students can, for instance, think about removing a unit of one from a composite unit of seven and repeating the unit seven times to fill the whole – seven is seven times the size of one.

ENS students' algebraic reasoning. Olive and Çaglayan (2008) utilized the coin problem (Figure 1) to examine how units coordination was related to students' algebraic solutions to a linear system of equations problem. One participant, Ben, who assimilated with composite units, wrote the equations $(.05N) + (.1D) + (.25Q) = \$5.40$, $D = N + 3$, and $Q = N - 2$. Although Ben explained that N, D, and Q represented the numbers of nickels, dimes, and quarters, respectively, he struggled to substitute $n + 3$ and $n - 2$ in place of D and Q. When he was pressed to do so, he conflated the numbers of dimes and nickels with their values. This was a limitation of his units coordination because he could not operate on the initial equation, which represents a three-level unit structure (i.e., the value of a single coin, within the number of a type of coin, within the total value, $5.40; Olive & Çaglayan, 2008).

Generalized Number Sequence (GNS)

A GNS is the most sophisticated number sequence, and GNS students assimilate with three levels of units and can construct four or even five in activity (Steffe, 2010; Ulrich, 2016a). One mental operation of a GNS is iterable composite units. This implies that GNS students can "collapse" a composite unit to form a "singleton unit" (Steffe, 2010, p. 42), and conceive of composite units as identical, which allows them to be iterated to solve problems (Steffe, 2010).

GNS students' algebraic reasoning. In response to the coin problem (Figure 1), Maria, who assimilated with three levels of units wrote the equation $.05N + .1(N + 3) + .25(N - 2) = 5.40$ with "ease" (Olive & Çaglayan, 2008, p. 280). Assimilating with three levels of units allowed Maria to operate on the initial equation, a three-level unit structure, by substituting expressions for D and Q without the same difficulty as Ben.

Research Questions

The literature demonstrates that students' algebraic reasoning can be modeled by their number sequences. Additionally, differences in students' fluency with composite units and the construction of a disembedding operation are critical to their algebraic reasoning. Therefore, this study asks, in what ways do the number sequences account for students' guess and check solutions to linear systems of equations? Furthermore, the literature is unclear as to the appropriateness of guess and check strategies. This study will also ask, are middle-grades students' solution methods for linear systems of equations more closely related to their number sequence or course enrollment?

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Methods

This study included 18 students in grades six through nine at a rural middle and high school in the southeastern United States. Students are listed in Table 1 by math class and number sequence. The first letter of each pseudonym matches the first letter of their number sequence attribution. According to the state standards and the teachers of these students, Math 6, Math 7, and Pre-Algebra did not include any instruction on solving systems of equations. Algebra 1, Algebra 1 Parts, and Algebra 2 did include instruction on algebraic methods for solving systems of equations. Students with an asterisk are students who had received instruction on algebraic methods for solving systems of equations in their math class. Students' number sequence was determined by a survey (Ulrich & Wilkins, 2017) and confirmed by screening questions during semi-structured clinical interviews. Clinical interviews were conducted with each student on two occasions, for approximately 45 minutes each, and in addition to confirming their number sequence attribution also included algebra tasks. The tasks reported here are the coin problem and the modified coin problem (Figure 1). Students were given time to solve each problem with any method they chose but were prompted to try an algebraic method if they did not do so independently.

Table 1: Participants by Math Course and Number Sequence

	Math 6	Math 7	Pre-Alg	Alg1	Alg1 Parts	Alg2
TNS	Tabitha				Travis*	
aTNS	Aaron	Alyssa		Amanda*	Alex*	
	Abby	Andy				
	Ann	Ava				
ENS	Elle		Emily	Erin*	Elizabeth*	Emma*
	Evan					
GNS			Greg	Gavin*		

*Denotes students who received instruction on algebraic methods for solving systems

<u>The Coin Problem (Problem 1; from Olive & Çaglayan, 2008)</u>: Ms. Speedy keeps coins for paying the toll crossing on her commute to and from work. She presently has 3 more dimes than nickels and 2 fewer quarters than nickels. The total value of the coins is $5.40. Assuming that she does not have any pennies, find the number of each type of coin she has.
<u>The Modified Coin Problem (Problem 2)</u>: I have 17 coins – some quarters, some dimes, and some nickels. I have 6 more dimes than nickels and 1 fewer quarter than nickels. Find the number of each type of coin that I have.

Figure 1: The Coin Problem and Modified Coin Problem

Results and Analysis

This study asked whether students' methods for solving systems of equations were more closely tied to their math class or number sequence. Table 2 shows that the two TNS students correctly solved problem 2 using guess and check, although one had taken algebra and the other had not. All six aTNS students who attempted problem 1 used guess and check, and seven of eight aTNS students used guess and check on problem 2. This was also regardless of whether they had taken an algebra course. Thus, students who had constructed only a TNS or an aTNS tended to use guess and check, regardless of whether they had received instruction on algebraic methods to solve systems of equations. GNS students always used algebraic methods on problems 1 and 2, regardless of whether they had received algebra instruction. ENS students' methods varied. On the coin problem (1), all ENS students attempted an algebraic method, although only one ENS student was successful. The other five ENS students did not arrive at an answer algebraically and did not guess and check when the interviewer suggested it, presumably due to the quantitative complexity of problem 1. In contrast, on the modified coin problem (2), which involves less quantitative complexity, all three ENS students who had not received algebra instruction used guess and check, and all three ENS students who had received algebra instruction solved problem 2 using an algebraic method. Middle-grades students' solution methods to linear systems of equations were more closely tied to their number sequence than their course enrollment, with the exception of ENS students. ENS students' solutions were more closely tied to their course enrollment and the quantitative complexity of the problem. This pattern is indicated in Table 2 by the cluster of grayed cells among all TNS and aTNS students, and those ENS students who had not received algebra instruction, as well as the second cluster of grayed cells among all GNS students and the ENS students who had received algebra instruction.

Table 2: Results of the Coin and Modified Coin Problems by Number Sequence, Solution Method, and Math Course

	Coin Problem (Problem 1)				Modified Coin Problem (Problem 2)			
Method	Guess and Check		Algebraic Method		Guess and Check		Algebraic Method	
Course	<Alg	Alg	<Alg	Alg	<Alg	Alg	<Alg	Alg
TNS					1/1	1/1		
aTNS	¼	½			5/5	2/2	0/1	
ENS			0/3	1/3	3/3			3/3
GNS			1/1	1/1			1/1	1/1

Each numerator represents the number of students who correctly solved the problem with that method in that number sequence stage, compared to the number who attempted it (denominator). Grayed cells indicate 50% or more of solutions were correct. Neither TNS student attempted problem 1 due to their perceived frustration level. Two aTNS students did not complete problem 1 due to time. <Alg indicates a math class that did not offer algebra instruction. Alg indicates a math class that did offer algebra instruction (see Table 1).

This study also asked to what extent students' number sequences could be used to model their guess and check solutions to linear systems of equations. For brevity, this analysis is limited to the modified coin problem, and GNS students' solutions are not presented, as they did not guess and check. One response from each number sequence was selected to be representative.

TNS Students' Solutions

Travis guessed on the modified coin problem by saying, "I'm trying to get a number … [of] dimes that have six more than nickels so that I can see how many quarters…" This shows that he was thinking about each type of coin sequentially. His first guess was 13 dimes, 7 nickels, and 6 quarters. He was satisfied that this was the answer until the interviewer asked if there were 17 coins total. This is evidence that Travis did not keep track of the dual goals of utilizing all 17 coins and maintaining the relationships between the numbers of coins. Once he finished the problem, he summarized his solution: "I would pick a number [for dimes] and … see what would be 6 less than that, and one less than that. Then I would add them all up and see if they would equal 17." His summary shows the sequential nature of Travis's guess and check process.

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Building on Olive and Çaglayan's (2008) analysis, Travis's sequential determination of the numbers of dimes, nickels, and quarters is due to a limitation of the units coordination defined by his TNS. Travis could assimilate the task with one level of units (e.g., a number of dimes) and construct a second level in activity (e.g., a number of nickels in relation to a number of dimes). This facilitated his double counting and supported the sequential determination of the numbers of each type of coin. Following mental activity, the relationship between the numbers of coins decayed and Travis could only reflect on his answer. Travis's need to construct composite units in activity also limited his reflection on the relationship between his guess and the total. His first guess of 13 dimes, 7 nickels, and 6 quarters was, from the interviewer's perspective, implausibly large. To Travis, the guess was not concerning because he could not conceptualize the number of each type of coin embedded within the total number, so he worked through the problem by sequentially calculating the numbers of coins, and retrospectively checking the relationship to the total.

aTNS Students' Solutions

Abby solved the modified coin problem by drawing 17 circles to represent the coins. This shows that she anticipated the need to exhaust all 17 coins. Then she filled one circle with an N to represent one nickel, seven circles with ds to represent seven dimes, and no circles with Qs. Because she did not fill all of the circles, she knew her answer was not correct (Figure 3). Next, she made incremental adjustments to the coins by adding one nickel, dime, and quarter to her drawing, and then two nickels, dimes, and quarters. These incremental adjustments are evidence that Abby understood adding the same number of each type of coin would maintain the necessary relationships. Finally, she concluded that the solution was 4 nickels, 3 quarters, and 10 dimes.

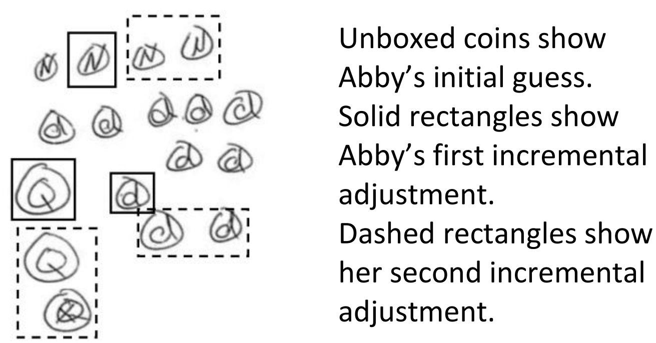

Figure 3: Abby's Representation of the Modified Coin Problem

Abby's guess and check included two key components – dual awareness of the goals of finding 17 coins total and of the relationships between the numbers of coins. This was supported by an assimilatory composite unit. Prior to activity, Abby could conceive of the situation holistically as a composite unit of 17 coins, containing 3 tacitly embedded composite units representing the numbers of dimes, nickels, and quarters. As aTNS students cannot disembed, Abby relied on figurative materials to support her reasoning. However, her operations on embedded composite units supported her understanding that she could make incremental adjustments to each guess; this is a form of strategic reasoning. In total, seven aTNS students

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solved the modified coin problem using guess and check, six made incremental adjustments to their guesses, and they all used figurative material to support their reasoning.

ENS Students' Solutions

Emily guessed the correct solution on the first try. She drew 4 nickels, then directly above the nickels she drew 10 dimes, and below the nickels drew 3 quarters (Figure 4). While drawing, she said, "Well, if there's 4 nickels then that would be 6 more dimes is 10, and 1 less quarter is 3. So that works." When asked how she generated that guess, she responded, "I just thought if it's 4 nickels, then 6 more is 10 dimes so that's 14 right there, and so the 3 [quarters] just worked out." Emily's explanation does not indicate how she arrived at four nickels as an initial guess, but there is no indication that she guessed other combinations of coins prior. Similarly, Evan guessed the correct solution on the first try "by accident."

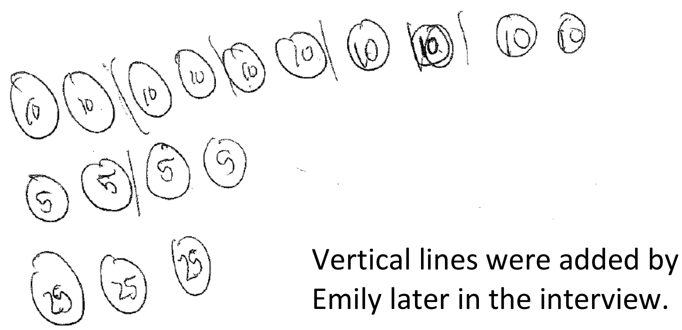

Figure 4: Emily's Representation of the Modified Coin Problem

ENS students' guesses were supported by an assimilatory composite unit, similar to aTNS students, which allowed them to conceive of the situation as a composite unit of 17 coins containing three embedded composite units representing the numbers of dimes, nickels, and quarters. However, the ease with which Emily and Evan guessed the solution indicates, however, that their reasoning was more sophisticated than that of the aTNS students. Based on the limited evidence provided by these two ENS students, it is difficult to attribute this sophistication to any particular mental operation. Elle's solution will be presented next because her work provides more clear evidence that a disembedding operation supported the accuracy of their guesses.

Elle used an unwinding strategy to solve the modified coin problem. This is another pre-algebraic strategy in which students solve a problem "by working backward through the constraints provided in the problem… by inverting operations and performing arithmetic operations rather than using algebraic manipulation" (Knuth et al., 2006, pp. 301–302). Elle applied an unwinding strategy when she said,

> I'm subtracting the amount of dimes from that that we already have [writes 17 minus 6 equals 11]. And I'm just trying to figure out, like, how many nickels and dimes. … So that [subtracting six] sort of equalizes the number of dimes and nickels, doesn't it, but we have one fewer quarter than nickels. … Well, we already have one less quarter than nickels, so that's one more [writes 11 plus 1 equals 12]. [Adding one] sort of balances it, quarters with the nickels.

Elle divided 12 by 3 to find that the solution was 4 nickels, 10 dimes, and 3 quarters.

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Elle's unwinding strategy is evidence that she applied both an assimilatory composite unit and a disembedding operation. As with the ENS students who guessed and checked, Elle assimilated the situation as a composite unit of 17 containing 3 embedded composite units. Disembedding allowed Elle to simultaneously conceive of the relationship between the numbers of dimes and nickels to the total, which supported her reasoning that equating the number of dimes and nickels would reduce the total to 11. She then applied the same disembedding operation to consider the relationship between the numbers of nickels and quarters to 11. Thus, Elle leveraged her reflection on the relationships between the numbers of each type of coin to simply the problem context. Although neither Emily nor Evan seemed overtly aware of this process, it is possible that similar reasoning supported their "accidental" guesses.

Discussion

Studying algebraic solutions to linear systems of equations is typical of middle- and high-school Algebra 1 curricula (e.g., CCSSI, 2010), but guess and check remains prevalent (Johanning, 2004). This study offers a lens to interpret these difficulties and implications for students' preparedness to receive instruction on solving linear systems of equations. TNS students did not solve systems of equations algebraically, regardless of their course enrollment. However, they did correctly solve a linear system of equations with limited quantitative complexity using guess and check. This was supported by their construction of composite units in activity. Although it is unlikely that these students are prepared to accept instruction on algebraic methods to solve systems of equations, they may benefit from instruction that promotes systematic guess and check. Knuth et al. (2006) maintain that guess and check is pre-algebraic. While this may be so, supporting TNS students' use of systematic guess and check may be more productive than attempting to teach them to apply algebraic methods without understanding.

aTNS students also did not use algebraic methods, regardless of their course enrollment, which implies that they are also unlikely prepared to accept instruction on algebraic methods for solving systems of equations and apply those methods in novel problem-solving situations. However, aTNS students had access to more sophisticated solutions than TNS students, including systematic guess and check. Thus, aTNS students may benefit from instruction that includes active reflection on the relationships between the unknown quantities in systems of equations to support that readiness for instruction. Additional longitudinal research is necessary to assess an instructional trajectory that may engender aTNS students' construction of disembedding. However, Zwanch and Wilkins (2021) found that students who have constructed an aTNS by sixth or seventh grade are more likely to construct an ENS and a disembedding operation by the time they enter eighth grade, when compared to students in sixth and seventh grades who have not yet constructed an aTNS. This implies that the early middle grades are a critical time in students' construction of number, particularly in supporting the construction of an ENS. In combination with the present study, these findings suggest that supporting aTNS students' construction of disembedding to support their preparedness to learn algebraic methods of solving systems of equations is likely more productive in the early middle-grades.

ENS students capitalized on the algebraic methods taught in their algebra classes, if they had taken one. This indicates that their assimilatory composite unit and disembedding operation prepare ENS students to accept instruction on algebraic methods for solving linear systems of equations. However, their success with these methods was limited to situations such as the modified coin problem, which had limited quantitative complexity. Longitudinal research should consider how to support ENS students' solutions to systems of equations with greater

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quantitative complexity, such as the coin problem, in novel problem-solving situations. Overall, these results demonstrate that providing instruction on linear systems of equations in a middle- or high-school Algebra 1 course is most likely to be productive if instruction is differentiated to support solution strategies that students are prepared to accept.

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