SUPPORTING MIDDLE-SCHOOL STUDENTS' DEVELOPMENT OF EMERGENT GRAPHICAL SHAPE THINKING

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Emergent graphical shape thinking (Moore & Thompson, 2015) is a way of reasoning that is critical across numerous STEM fields. However, evidence indicates that the underlying component ideas for emergent thinking are underdeveloped in school mathematics education (e.g., Thompson & Carlson, 2017), and few studies directly report on students' development of this way of thinking. We present the results of a teaching experiment conducted with eighth-grade students to support stable meanings for emergent graphical shape thinking. We focus on the in-the-moment meanings expressed by a pair of students as they engaged in a sequence of tasks that we conjecture could support stable meanings for constructing and interpreting graphs.

Keywords: Algebra and Algebraic Thinking, Middle School Education, Learning Trajectories and Progressions

Across STEM fields, constructing and interpreting graphs is a crucial skill (e.g., Glazer, 2011; Potgieter et al., 2008). For instance, in a study looking at the use of graphical representations across numerous science textbooks and practitioner journals, Paoletti et al. (2020) determined that, at least implicitly, an individual must engage in *emergent graphical shape thinking* (hereafter emergent thinking) to interpret most graphs in these sources. Moore and Thompson (2015) defined emergent thinking as conceiving a graph simultaneously in terms of "what is made (a trace) and how it is made (covariation)" (2015, p. 785). Specifically, with a conception of a point as a multiplicative object, a student can conceive of a graph in terms of an emergent, progressive trace generated by the point's movement and dictated by the covarying quantities' magnitudes represented on the axes. The resulting graph represents the tracking of the two quantities' simultaneous covariation. Although there is some evidence that students in grades 6-12 can engage in emergent thinking in-the-moment (e.g., Ellis et al., 2015; Johnson, 2015), other research suggests that pre-service (e.g., Moore & Thompson, 2015; Moore et al., 2019) and in-service (e.g., Thompson et al., 2017) mathematics teachers in the United States often do not reason emergently in tasks designed to elicit such reasoning. Therefore, there is a need to examine how to productively support students in developing emergent thinking.

In this report, we address the research questions: *How do two eighth-grade students develop meanings for graphs that entail emergent thinking?* To investigate this question, we conducted a teaching experiment (Steffe & Thompson, 2000). In this report, we examine the work of two eighth-grade students as they completed the *Faucet Task* (Paoletti, 2019). Prior to this, we define components of emergent thinking to help readers understand how the task could support students' developing meanings for graphs. We then describe the in-the-moment meanings (Thompson, 2016) the two students developed as they engaged in the task. Finally, we share the results of a task developed by Thompson et al. (2017) that the students completed after the instructional sequence to determine whether such meanings may have become part of the students' stable meanings for constructing and interpreting graphs.

Components of Emergent Thinking

Covariational Reasoning and Multiplicative Objects

Several researchers (see Thompson & Carlson, 2017, for a review) have explored ways in which students' covariational reasoning can support them in developing productive meanings for various mathematical ideas. Researchers have contended covariational reasoning is developmental (Carlson et al., 2002; Saldanha & Thompson, 1998). Initially, a student is likely to coordinate two quantities by thinking "of one, then the other, then the first, then the second, and so on" (Saldanha & Thompson, 1998, p. 299) until the student has developed an operative image of covariation that entails a relationship between quantities that results from imagining both quantities being tracked for some duration. Saldanha and Thompson (1998) elaborated:

> [Covariational reasoning] entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two. As a multiplicative object, one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value. (p. 299)

Saldanha and Thompson's use of *multiplicative object* stems from Piaget's notion of 'and' as a multiplicative operator (the Cartesian product). Thompson et al. (2017) noted, "A person forms a multiplicative object from two quantities when she mentally unites their attributes to make a new attribute that is, simultaneously, one and the other" (p. 98). Hence, covariational reasoning entails understanding the simultaneity of two quantities' values in relation to each other.

Reasoning in a Coordinate System

To represent and coordinate two conceived quantities, students can construct a coordinate system (Lee, 2016; Lee et al., 2020). In the Cartesian coordinate system, once a student has conceived that quantities' magnitudes can be represented via line segments, the student can consider changes in the lengths of these segments, oriented orthogonally on horizontal and vertical axes, as the situational quantities covary. With such a coordinate system in mind, a student can then conceive of a point as a multiplicative object (Lee, 2016; Lee et al., 2020; Thompson, 2011) that simultaneously represents the two covarying quantities via the two segments' magnitudes. Such a meaning is a prerequisite for reasoning about (or imagining) a graph as representing an emergent trace of a point representing covarying quantities.

Setting and Methods

The middle school where the study took place serves a diverse student population (over 75% students of color) in the northeastern United States. We conducted the teaching experiment in an accelerated eighth grade math class with eight students who had completed high school level Algebra I and Geometry courses. The experiment occurred over five days in June after administration of the Geometry end-of-course assessment. The first author, who was not the students' normal teacher, served as the classroom teacher-researcher (TR).

All portions of the teaching experiment were video- and audio-recorded. The two focus students for this study, Kendis (female, African American) and Camila (female, Hispanic), were a pre-established group in the class. During the instruction, Kendis and Camila used a Chromebook computer to view and manipulate interactive applets and recorded their work on paper worksheets and a dry-erase board. To analyze this data, we watched the videos to identify occurrences providing insights into each student's in-the-moment meanings for constructing, interpreting, or representing quantities and relationships between quantities (Thompson, 2008).

Additionally, we collected data from the *uv*-Task described in Thompson et al. (2017) one day after the instruction concluded. In the *uv*-Task, a coordinate system is shown, and bolded segments representing quantities *v* and *u* (on the horizontal and vertical axes, respectively) vary as the animation plays. (The animated task can be seen at http://bit.ly/CovaryMagnitudes.) Consistent with Thompson et al.'s (2017) methods, we gave participants a paper with a set of axes and the initial segments representing *v* and *u* shown, and the animation was played six times. The students were asked to sketch a graph that depicted the value of *u* relative to the value of *v* (see Figure 1 (left) for an accurate graph). Using the rubric from Thompson et al. (2017) shown in Figure 1 (right), we independently coded the student responses on the shape of the sketched graph (92% interrater reliability). Although the *uv*-Task was not explicitly designed to measure emergent thinking, we contend imagining the graph as the trace of the (imagined) point corresponding to the endpoints of the two segments as they covary is required to produce a more accurate graph shape; we infer scoring a 2 or higher is likely indicative of a person engaging in emergent thinking.

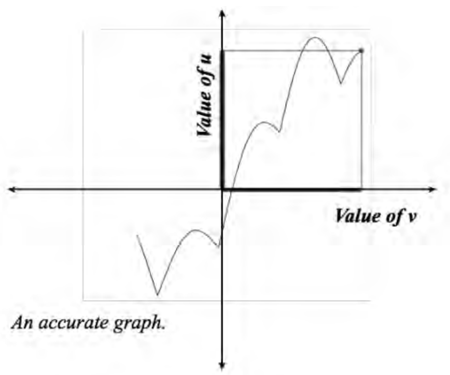

Table 4. Scoring rubric for Dimension B, shape of teachers' sketched graphs (SSG).

Level B4a:	The graph has four local minima in ascending order and three local maxima in ascending order*
Level B4b:	The graph begins decreasing, is generally increasing, has at least 2 local extrema, and has these points of inflection: – 6 if graph starts concave up and ends concave up – 7 if graph starts concave up (down) and ends concave down (up) – 8 if graph starts concave down and ends concave down
Level B3a:	The graph has 6 or 8 local extrema with minima in ascending order and maxima in ascending order.
Level B3b:	Same as B4b except that the graph has one too few or one too many points of inflection given the way the graph starts and ends.
Level B2:	The graph is generally increasing and has 2-5 or 9-12 local extrema, ignoring ascending order.
Level B1:	The graph has no more than 1 local minimum and is otherwise monotonically increasing.
Level B0:	The graph does not fit any of the above levels.

* By ascending order, we mean that from left to right each local minimum's *y*-coordinate was greater than the previous one, and likewise for each local maximum's *y*-coordinate.

Figure 1: (left) The accurate graph and (right) the scoring rubric for shape of the sketched graph on the *uv*-Task (Thompson et al., 2017).

Kendis and Camila's Development of Emergent Thinking

In the sections that follow, we present evidence that Kendis and Camila developed in-the-moment meanings for graphs that entailed emergent thinking. We first present evidence of their construction of component meanings to highlight how this thinking developed.

Constructing Quantities and Reasoning Covariationally

Critical to thinking emergently is conceiving of two covarying quantities. To help students construct quantities situationally, the TR presented the class with a GeoGebra applet (https://www.geogebra.org/m/rdxkrwek) intended to represent a faucet with hot and cold knobs (Figure 2). The TR directed students to use sliders to represent turning each knob on or off; changing the sliders changes the representations of amount of water (width of the rectangle below the faucet) and temperature (color of the rectangle). We intended for students to reason about the changing amount of water and temperature as two quantities to coordinate.

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Figure 2: Screenshots of part one of the *Faucet Task* (turning cold water on).

The TR asked, “What are some of the things that this applet is trying to represent?” Camila stated, “the more to the right you dragged [the slider], the wider it [the rectangle below the faucet] got.” Kendis added that the width of the rectangle represents “how much water comes out of the faucet.” When the TR drew students’ attention to the changing colors of the rectangle, Kendis volunteered that the color represented “temperature,” and she explained that turning each knob outwards (cold on and hot off) would result in lowering the water temperature. Kendis’s responses demonstrated that she conceptualized two situational quantities.

Next, to support the students in coordinating two covarying quantities, the TR asked students to make predictions for what would happen to the amount and temperature of the water in four scenarios, assuming that both the hot and cold knobs start halfway on. Each scenario consisted of turning one knob either all the way on or off. By making predictions for changes in both amount and temperature of water in each scenario, we provided students opportunities to coordinate simultaneous changes in two quantities and thereby understand the simultaneity of the two quantities changing as a multiplicative object. As evidence of such reasoning, when asked what would happen if the hot knob were turned all the way off, Kendis responded, “the water is going to get colder, and it’ll be less [water].” Kendis’s response explicitly described changes in the magnitudes of both quantities, indicating her meaning that the quantities simultaneously covary.

Constructing and Using a Coordinate System

The next prompts were designed to develop two ideas related to constructing and using a coordinate system: using line segments to represent quantities and understanding a point in a coordinate system as a multiplicative object. These components support emergent thinking.

Using line segments to represent quantities. In the next prompt, students accessed a revised applet that included (a) a vertical (graduated, but unlabeled) thermometer (colored red) to represent the water temperature and (b) a horizontal pink line segment that corresponded to the width of the rectangle that represented the water stream (Figure 3, left). The positioning of the segments as vertical and horizontal was designed to foreshadow the creation of a coordinate system using segments to represent quantities’ values on the vertical and horizontal axes.

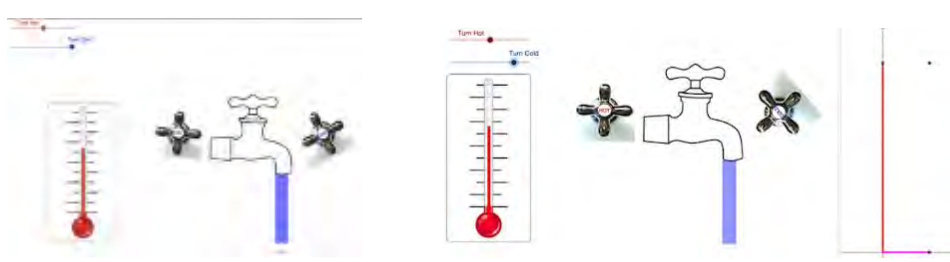

Figure 3: Screenshots of (left) part two and (right) part three of the *Faucet Task*.

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The TR then asked students to describe how the segments varied for the same four knob-turning scenarios as before. Our goal was to provide opportunities to connect the lengths of the segments to the previously established quantities. To exemplify the productivity of such opportunities, consider a dialogue about turning the cold knob off between Camila and the TR:

Camila: Um, you turn the cold to the left, and then the temperature will increase, and the red line will get longer because of that. And the pink line will be shorter. [*TR asks Camila to repeat.*] The red line is going to get longer.
TR: It's going to get longer? Why?
Camila: Because you're eliminating the cold water, so the hot is left, and the hot water increases the temperature.
TR: [*TR restates what Camila said.*]…[A]nd then the pink segment's going to go?
Camila: It's going to get shorter.
TR: It's going to get shorter because there's going to be?
Camila: Less water overall.

In this dialogue, Camila connects changes in the quantities in the situation (temperature and amount of water) to changes in the lengths of the corresponding line segments, indicating that she understood the segments as representing the situational quantities. Further, we note Camila readily transferred this reasoning when presented with the segments on the coordinate system.

Understanding a point in a coordinate system as a multiplicative object. Shortly after the previous exchange, the TR showed students a new applet. This applet included a coordinate system with the pink segment (representing amount of water) positioned along the horizontal axis, the red segment (representing temperature) positioned along the vertical axis, and a point with position corresponding to the endpoints of both segments (Figure 3, right). The TR directed students to describe the motion of the point as they explored the applet to provide an opportunity to conceive of relationships between the point's movement and variations in both segments.

While working as a pair, Camila and Kendis had the following conversation with the TR:

TR: So how is this point moving around the screen?
Camila: In accordance with the…
Kendis: [*moves fists horizontally back and forth*]
TR: In accordance with what?
Kendis: The, the temperature… [crosstalk]
Camila: [crosstalk] Temperature.
Kendis: …and the, and how much water was coming out.
TR: With both?
Kendis: [*nods and gestures a vertical line with hand*] It stays in line with both of them.

We interpreted Kendis's reference to "both of them" as the segments representing amount of water and temperature. We inferred that Kendis's horizontal gesture was intended to show that the top endpoint of the temperature segment and the point on the coordinate system formed a horizontal line (and similar for the vertical gesture and the amount of water segment). We infer Kendis understood that the point's movement was dictated by the two quantities' magnitudes represented by segments; the point served as a multiplicative object in the coordinate system.

Using the same applet, the TR told students to investigate a point's movement in several scenarios. Responding to a scenario starting with both knobs turned halfway on, and asked to

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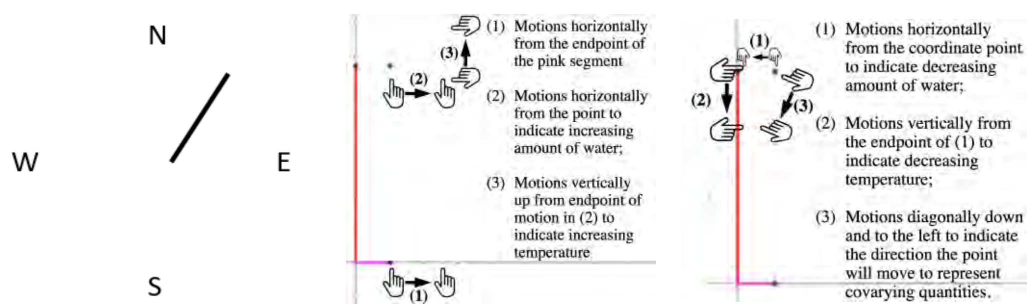
predict how the quantities will change when they turn hot all the way on, both Kendis and Camila related segment lengths to the situational quantities. Kendis stated:

> Yeah, this is going to go up [*traces finger along the vertical axis from the origin upward beyond the length of the red segment]*… more temperature…. [*traces finger along the horizontal axis from the origin to the right beyond the length of the pink segment*] It's going to move to the right and up.

In response to Kendis's reasoning, Camila used the Chromebook to do a Google Image search for a compass and produced a drawing (reproduced in Figure 4a). We infer that Camila interpreted the described action ("move to the right and up") as occurring simultaneously, and the diagonal line segment represented her understanding of the point's movement.

Indicative of not yet explicitly connecting the coordinate point to the situational multiplicative object she had constructed earlier, Kendis initially disagreed with Camila's representation, stating:

> [I]t's going this way [*traces right along the horizontal axis, as in (1) in Figure 4b*] and, look, it's going to stay in a line with [the red segment], so it's just going to move over and up [*traces from the point to the right a short distance (2) and then up (3) in Figure 4b*].



(a) (b) (c)

Figure 4: (a) Recreation of Camila's drawing. (b, c) Recreations of Kendis' hand motions.

Although both Camila and Kendis understood where the point would end up relative to its starting position, they conceptualized the point's movement differently. Consistent with the developmental nature of covariational reasoning (Saldanha & Thompson, 1998), Kendis initially conceived of the changes in the underlying segments as sequential (the point would move to the right, then up) as opposed to the simultaneous movement Camila had described.

As the pair continued to discuss the scenarios, evidence emerged that Kendis also began to explicitly connect the motion of the point with the simultaneously covarying situational quantities. For instance, when predicting the point's movement when the two knobs start halfway on and the hot knob is turned off, Kendis described "[the red segment]'s gonna go down, and then [*points to the horizontal axis*] it's less water also so it's gonna go diagonal [*making a diagonal cutting motion with her hand*]." Immediately after this, Kendis silently engaged in a series of movements. She first motioned horizontally to the left from the point as if indicating a decreasing amount of water (indicated by (1) in Figure 4c), then motioned down as if indicating a decreasing temperature of water ((2) in Figure 4c). Critically, and differing from her earlier activity, after these two motions, Kendis lastly motioned diagonally down-and-to-the-left ((3) in Figure 4c) to indicate that the point would move in such a way to reflect the simultaneous variations of the two segment and quantities magnitudes. We took this as evidence of Kendis's formation of a multiplicative object.

Reasoning Emergently and Interpreting Graphs in Multiple Ways

For the final activity in the *Faucet Task*, the TR provided students with several graphs that we told them resulted from turning the knobs. The TR asked students to determine what position each knob was in initially and what action(s) occurred. It is important to note that the graphs were undirected (i.e., no starting or ending point was identified). Thus, each graph had at least two possible interpretations. Through this activity, we intended to provide students the opportunity to reason emergently by interpreting (at least) one possible trace of the graph.

As the pair discussed the actions that would produce a graph (Figure 5 (left)) that was moving "down and to the right," Camila reasoned that the action was turning the cold knob on. She stated, "It's going down in temperature and to the right, so it means you're increasing water, and it's going down, so it means you have to be adding cold water." Camila's reasoning moved between imagining the tracing of a point on the graph, the underlying quantities and how they covary, and the action in the situation. We infer she was reasoning emergently.

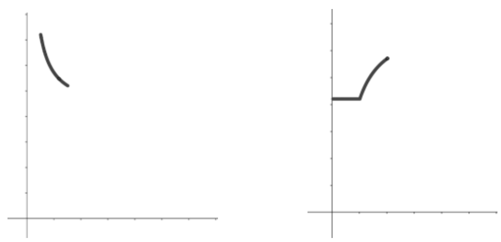


Figure 5: Two trace graphs the TR asked students to interpret.

Kendis and Camila did not independently consider that more than one action could produce the same graph. However, during the class discussion of Figure 5 (left), another group described an interpretation of the graph as turning the cold knob off (reading the graph from right to left). Once the discussion revealed that reading the graph as a trace from right to left could be produced by a different action that would result in the same final graph, Camila was able to apply this idea to describe two different possible productions of the graph in Figure 5 (right):

Camila: First step is to turn the cold on, then turn the hot one on.
TR: [T]hey're both starting completely off, turning cold on then turning hot on…. [S]o in terms of the two quantities, how did you know that was [*trails off*]?
Camila: Well, it continued to go to the right, so it means [the amount of water]'s increasing in quantity, and then, after the second transition, it's going up in temperature, which means you're going to be adding hot water. So, the first one we started off as cold adding it, and then we had to add more of hotter temperature.
TR: … Could there be another way this plays out?
Camila: Hot water off.
TR: Hot water, so you start with both of them on, turn hot water off get to here…
Camila: And then the cold is at halfway and then you could also turn it off.

We take Camila's independent description of two different action sequences that would produce the graph as strong evidence that she was engaging in in-the-moment emergent thinking.

uv-Task Results

As shown above, the work on the *Faucet Task* provided evidence of Kendis and Camila developing in-the-moment emergent thinking. We hypothesized that repeated experiences with such thinking in different contexts would allow emergent thinking to become part of the students' stable meanings for constructing or interpreting graphs. The remaining sessions of the teaching experiment provided students with seven additional opportunities to construct and five additional opportunities to interpret graphs in different tasks and contexts. We use the results from the *uv*-Task (Table 1) to provide some evidence that such opportunities were productive for both Kendis and Camila, as well as their classmates, in developing stable meanings (Thompson, 2016) for constructing and interpreting graphs that entail emergent shape thinking.

Table 1 presents the results of US secondary mathematics teachers as reported in Thompson et al. (2017) and our participants' results on the *uv*-Task. We interpreted a score of 0/IDK ("I don't know") on this task as no evidence of employing covariational reasoning, a score of 1 as evidence of employing gross covariational reasoning (Thompson & Carlson, 2017), and a score of 2 or greater as evidence of employing some level of emergent reasoning. Kendis and Camila each received a score of 2 for the shape of their sketched graphs (see Figure 6). These scores, which exceeded the performance of over 70% of US mathematics teachers in the Thompson et al. (2017) study, indicated to us that Kendis and Camila may have developed emergent thinking as a component of their stable meanings for constructing or interpreting graphs, as evidenced by their ability to apply such reasoning in an unfamiliar, decontextualized situation.

Table 1: Scores on the Shape of Sketched Graph Rubric for the *uv*-Task

	0/IDK	*1*	*2*	*3*	*4*
US teachers (n = 121)	65 (53.7%)	22 (18.2%)	11 (9.1%)	14 (11.6%)	9 (7.4%)
8th graders (n = 8)	2 (25.0%)	0 (0.0%)	4 (50.0%)	1 (12.5%)	1 (12.5%)

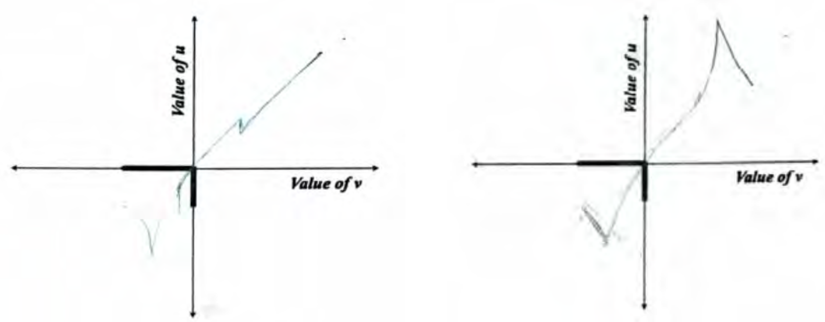


Figure 6: (left) Camila's graph and (right) Kendis's graph in response to the *uv*-Task.

Conclusion

Addressing our research question, we described two students' activity as they engaged in an instructional sequence that emphasized aspects of covariational reasoning (Thompson & Carlson, 2017) and reasoning within a coordinate system (Lee, 2016; Lee et al., 2020) to support them in developing emergent thinking. We highlight that despite individual differences in students' in-the-moment meanings during instruction, each student demonstrated evidence of stable meanings that entailed emergent thinking by the end of the study; each student conceived graphs as "what is made (a trace) and how it is made (covariation)" (Moore & Thompson, 2015, p. 785). We add

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to the literature by providing a proof-of-concept that at least some middle school students can develop emergent thinking as part of their stable meanings for graphs.

Although our quantitative results present students' performance on a task relative to a sample of U.S. mathematics teachers (Thompson et al., 2017), we do not intend for comparisons to be drawn between these populations. Rather, we intend to provide a frame of reference that conveys the non-trivial nature of constructing meanings that entail emergent thinking. Our study demonstrates that a purposeful learning progression can develop eighth-grade students' stable meanings for graphs via emergent reasoning; we conjecture other populations (e.g., teachers) could develop comparable meanings if provided similar opportunities.

We acknowledge that the small sample size and the use of an accelerated math class limit the generalizability of our findings. Given the importance of emergent thinking as a way of interpreting graphs across STEM fields (Paoletti et al., 2020), it is critical to continue to investigate ways to develop such meanings throughout school mathematics education.

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