RANKING THE COGNITIVE DEMAND OF FRACTIONS TASKS

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We report on and validate a system for ranking the cognitive demand of mathematical tasks. In our framework, task rankings are determined by the sequences of units and unit transformations students might use to solve each task. Using this framework, we ranked a set of 10 fractions tasks. We then interviewed 12 pre-service teachers to assess the validity of the ranking system. Results validate the task ranking system by demonstrating that increases in task ranking predict increases in the cognitive demand experienced by the pre-service teachers, as evidenced by their responses to the tasks. These results hold implications for instruction that maintains appropriate cognitive demand and future research that models students' mathematics.

Keywords: Cognition; Learning Theory, Number Concepts and Operations; Problem Solving.

In mathematics education, the cognitive demand of mathematical tasks has been categorized in terms of qualitative distinctions, such as procedures without connections and doing mathematics (e.g. Stein, Grover, Henningsen., 1996). In cognitive psychology, cognitive demand is quantified in terms of the number of action schemes a student might need to hold in mind in order to solve the task (Pascual-Leone, 1970). Here, we present a framework that accounts for the cognitive demand of mathematical tasks in terms of the sequences of units and unit transformations students might use to solve fractions tasks. Our framework integrates the math-specific construct of units coordination with the general cognitive construct of working memory.

The purpose of this study is to test a task ranking system based on our integrated framework. This purpose addresses one the major goals of PME-NA, "to further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof." We created task rankings based on the hypothesis that longer sequences of units/transformations would induce higher cognitive demand. To test the hypothesis, we applied a simple statistical test from 12 pre-service teachers' (PSTs) responses to 10 ranked fractions tasks. Results confirm that the task's rank predicts the cognitive demand experienced by PSTs, as evidenced by their behavioral (including verbal) responses to the task. Thus, our results validate the task ranking system and its underlying framework.

Theoretical Framework

Piaget characterized mathematics as a coordination of mental actions (e.g. Beth & Piaget, 1966). Mathematics educators who have adopted Piagetian perspectives on mathematical learning have attempted to account for the actions students rely upon to construct mathematical concepts (Simon, Placa, Avitzur, & Kara, 2018; Tzur & Simon, 2004). We are particularly concerned with the mental actions students use to construct and transform units.

Steffe (1992) originally defined units coordination as the distribution of one composite unit (a unit containing units of 1) across each of the units in another composite unit. For example, a student might conceptualize the product 5 times 7 as the distribution of seven units of 1 within each of five units of 1, simultaneously producing five 7s and 35 1s. However, units coordination can be understood more broadly as any coordination of mental actions used to construct or transform units. For example, the unit fraction 1/5 might be constructed by partitioning a whole into five parts; conversely, iterating one of those parts five times reproduces the whole. The coordination of this

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partitioning action and the corresponding iterating action establish 1/5 as a one-to-five relationship between the unit fraction and the whole (Wilkins & Norton, 2011)

Research on students' mathematics has identified several mental actions that undergird their fractions knowledge (Hackenberg & Tillema, 2009; Steffe & Olive, 2010; Ron Tzur, 1999; Boyce & Norton, 2016; Wilkins & Norton, 2011). In addition to partitioning and iterating, these mental actions include unitizing, disembedding, and distributing, as summarized in Table 1.

Table 1: Mental actions for constructing and transforming

Mental Action	Description	Fractions Example
Unitizing (U_n)	Taking a collection of n items or units, or a continuous span of attention, as a whole unit	Treating a rectangular bar as a whole unit, of 1
Iterating (I_n)	Making *n* identical, connected copies of a unit to form a new unit	Iterating 1/7 of a whole three times to produce 3/7 of the bar
Partitioning (P_n)	Creating *n* equal parts within a whole	Partitioning a whole bar into 15 equal parts
Disembedding (D_n)	Taking *n* parts out of a whole while maintaining their inclusion as part of the whole	Taking one part from a whole that has been partitioned into 9 parts, to make 1/9
Distributing ($T_{m:n}$)	Inserting the *m* units of one composite unit into each of the *n* units in another composite unit to produce a unit of units of units	Inserting three parts within each of the nine parts in 9/9 to make 27 parts in the whole

Working memory is a limited cognitive resource with special relevance in solving mathematical tasks (Bull & Lee, 2014; Swanson & Beebe-Frankenberger, 2004). Here, we adopt Pascual-Leone's definition: "working memory involves the process of holding information in an active state and manipulating it until a goal is reached" (Agostino, Johnson, & Pascual-Leone, 2010, p. 62). In our framing, in the context of solving fractions tasks, working memory involves holding in mind sequences of mental actions used to construct and transform units.

Methods

Task Ranking

We chose to focus on fractions tasks because of the wealth of literature on students' development of fractions knowledge and the mental actions that undergird that knowledge (Boyce & Norton, 2016; Hackenberg & Tillema, 2009; Steffe & Olive, 2010). The literature identifies unitizing, partitioning, iterating, distributing and disembedding as mental actions potentially available to students in solving fraction tasks. We used these five actions along with three types of units (whole units, composite units, and fractional units) as the atoms of fractions knowledge. The fraction tasks we used were modified from Hackenberg and Tillema's (2009) work. We report on a subset of 10 fraction tasks we ranked, listed in Table 2.

In order to determine how cognitively demanding a task might be for a student, we examined results from the literature that reported on students' prior responses. The literature we chose, and Hackenberg & Tillema (2009) in particular, included detailed accounts for the schemes and mental actions students seemed to use in solving the tasks. However, in some cases, we had to break down schemes and advanced ways of operating into the aforementioned atoms—simpler units and actions that undergird students' fractions knowledge. That is, we hypothesized potential solution paths for mathematical tasks using one unit/action at a time, without chunking them into larger structures, such as schemes.

Table 2: Fractions tasks

Task	Rank	Description
3	5	Imagine a cake that is cut into 13 equal pieces. You take 4 pieces. So, how much of the whole cake do you have?
4	7	Imagine you have 1/7 of a whole candy bar. So, could you use that to figure out how long 3/7 of the whole candy bar would be?
5	8	Imagine this [drawing a rectangle] is 5/9 of a whole candy bar. So, how could you make 1/9 of the whole candy bar from what you have?
6	10	Imagine a rectangular cake that is cut into 15 equal pieces. You decide to share your piece of cake fairly with one other person. So, how much of the whole cake would that person get?
7	10	Imagine you share a sub sandwich fairly among 17 people. Now each person shares their piece with two other people (three people total share each piece). So, could you figure out how much one little piece is of the whole sandwich?
8	12	Imagine you are at a party and a cake is cut into nine equal pieces. Two people show up to the party late and you decide to share your piece of cake with them. So, what fraction of the whole cake do the latecomers get together?
9	12	Imagine cutting off 2/5 of 1/3 of a cake. So, how much is that of the whole cake?
10	14	Imagine cutting off 1/4 of 5/6 of a cake. So, how much is that of the whole cake?
11	16/17	Imagine, you need 1/3 of a pound of sugar and all you have are bags of sugar that are 1/7 of a pound. So, how many 1/7 bags do you need?
12	16	Imagine I have 7/9 of a yard of ribbon, but every ninth it changes colors. My friend needs 2/3 of what I have, and she wants all of the colors. So, tell how much of a yard she has.

For example, consider Task 8. To solve this task, a student might start with the whole cake (whole_1) and partition it (P_9) into nine pieces (9), then disembed one of those pieces (D_1) to make their piece of cake (whole_2). The student could then partition (P_3) that piece into three pieces (3) and disembed one (D_1) of those to make a new piece (whole_3). Knowing two pieces are needed, the student could iterate that piece twice (I_2). The student still needs to name the fractional size of the small piece. To generate the relationship between the original whole cake and the small piece, the student could iterate (I_{27}) the small piece to exhaust the original whole. This process is captured in the graph shown in Figure 1. We refer to such graphs as unit transformation graphs.

Since our theory assumes that units and actions count towards the cognitive demand a student experiences when solving a task, both were counted when determining the rank of a task. This total sum of units and actions enumerated the cognitive demand of the task. In the unit transformation graph for Task 8 (Figure 1) there are six units (denoted by circles) and six mental actions (denoted by arrows), together giving a task demand of 12. All tasks were ranked using this same process and the ranks are reported in Table 2. Once all the tasks were ranked, we tested our theoretical ranking system through empirical evidence.

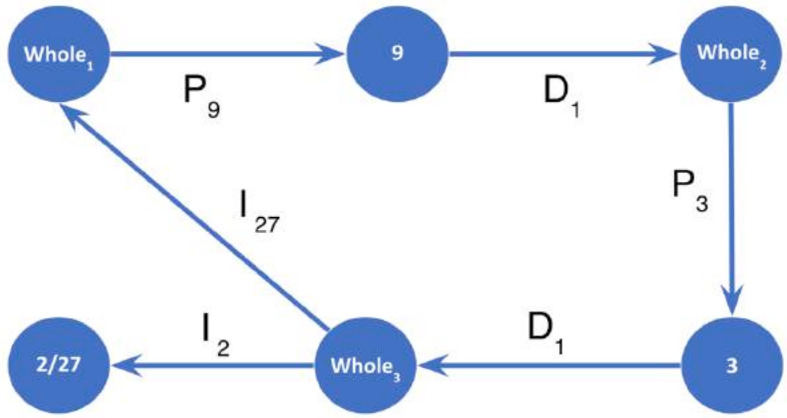

Figure 1: Graph of units and actions for Task 8

Data Collection

Participants were recruited from two sections of a mathematics for elementary school teachers content course taught by the same instructor at a large university in the mid-Atlantic United States. PSTs were selected for this study because they engage in metacognitive skills that enable them to express their thinking well; they commonly practice explaining their mathematics in the mathematics for elementary school teachers course. Moreover, they are mathematically mature enough to have constructed the mental actions required to productively engage with fraction tasks.

Twelve PSTs volunteered to participate in a 75-minute semi-structured clinical interview (Goldin, 2000). Each interview consisted of three parts: a units coordination assessment (Norton et. al, 2015), a working memory assessment (Morra, 1994), and a set of fraction tasks. This paper focuses on the final component of the interview. The PSTs were given the fraction tasks verbally one at a time and asked to solve them, initially without using figurative material. Sometimes follow-up questions were asked to probe a PST's thinking; sometimes PSTs were encouraged to use drawing to support their solution. A subset of tasks from Table 2 was selected for each PST, depending on our assessment of that PST's units coordinating ability and working memory. The tasks were always given in increasing order (top to bottom in Table 2), posing lower ranked tasks before higher ranked tasks. Each interview was video recorded with any written work collected using a Livescribe pen and notebook. The videos were selectively transcribed.

Data Analysis

Data analysis for this report consisted of two phases: coding for cognitive demand the PSTs experienced and a quantitative analysis of the task ranking system from the results of the coded cognitive demand.

Coding cognitive demand. Videos were analyzed for the purpose of coding the cognitive demand of each task, as experienced by each PST. We relied on video recorded behavioral data (including verbalizations) as indicators of this experienced demand.

Videos were analyzed one PST at a time with at least two of the three authors present. Experienced cognitive demand was coded as Low, High, or Over. The Low code was given when the PST was able to solve the task easily and confidently. Behavioral indicators included relaxed posture, providing an answer without verbal rehearsal or giving a fluent rehearsal of solution strategy. When a PST struggled but still had success engaging with the task, we assigned a High code. Behavioral indicators for a High code included asking for the question to be repeated during the solution process, expressing doubt throughout the task, unsure or repeated rehearsal needed to convince themselves of the task solution, and losing track of units during the solution process. The Over code meant that the

PST clearly was unable to assimilate the task or unable to resolve it without significant support from the interviewer or figurative material. This code often provided the easiest behavioral indicators with participants saying things like "Ah, it's just hard to do it in my head. Umm…" or "my brain is confused now."

The following pair of transcripts, from PST G provide an example of the difference between a Low code and a High code. The first transcript is for Task 6 (rank 10), and the second is for Task 8 (rank 12). In her response to Task 6, we see that PST G is quickly successful in solving the task with minimal rehearsal needed. Indeed, she seemed to have an answer ready (one-thirtieth) before saying anything, so that her verbalizations served as explanations to the researcher, rather than a necessary process in generating a solution to the task. The verbal run through of her solution process was succinct and confident.

Researcher: This is the next task [Task 6]. Imagine a rectangular cake that is cut into fifteen equal pieces. You decide to share your piece of cake fairly with one other person. So, how much of the whole cake would that person get?

PST G: [pauses for seven seconds and looks up.] You get one fifteenth of the cake and split that in half. My first thought was one-thirtieth…Of the cake, because…[makes splitting motion with hands in the air.] Splitting that in half, like if you were to split every piece of fifteen in half, then that would be like one thirtieth of the entire cake.

In comparison to Task 6, Task 8 seemed to induce additional cognitive demands for PST G, who required verbal rehearsal of her thought process to determine the answer to this new task. While she was successful in the end, throughout the solution processes there were several times she expressed doubt about a step or result. She would say things like, "Wait, that doesn't seem right," and "I don't know if that'd give you the same answer." She was eventually able to be successful on this task after attempting it twice. The fact that she was able to work through and solve the task despite some expressed doubt meant it did not qualify for an Over coding. However, Task 8 appeared to require substantial mental effort to produce a correct solution, indicating demand was High.

PST G: So, it's split up into nine equal pieces. So, then, you would split one ninth into…Two people come, but you still have a little bit? So, that… So, you would split that up into three. So, then I… Well, I guess you would do one ninth times two thirds to get how much they equal, like how much both their pieces would be. And then whatever that is, I guess it would be… two over… two eighteenths? Wait, that doesn't seem right. [pauses for five seconds] I feel like… I mean, I guess… You take those nine pieces, splitting that one ninth into thirds. But to find out how much two of those thirds are, you'd multiply one ninth by two thirds… Or no. You'd… you'd multiply the one ninth by one third, and then just do that twice? I don't know if that'd give you the same answer.

Researcher: Okay. Uh, let's… Maybe I can help you.

PST G: Okay.

Researcher: If you want me to be your calculator again, I'll do it.

PST G: [begins to draw on table with finger] So, you do one ninth, which divided by three, so you could times it by one third. So, then you'd have one over um… [pauses for five seconds.] Oh wait… [whispers to self] Three times nine, that's twenty-seven. Oh no, one over twenty-seven. And then you multiply that by two… to get two-thirds or to get two parts of the thing… So, then I guess… What's one over twenty-seven times two? Is that just two-twenty-sevenths? Okay.

Researcher: Nice, I like the way you reasoned through it. Yeah.

PST G: Okay. I was like, because I was thinking one over twenty-seven times two over one and I was like I guess that's just two, twenty-sevenths.

Quantitative analysis. The variable we measured was cognitive demand. This ordinal categorical data was coded as Low, High, and Over as described above. To test whether the task ranking system was valid, we only considered instances where a PST experienced a change in cognitive demand between two successive and differently ranked tasks. We excluded any cases where the same cognitive demand was experienced on successive tasks and instances where the demand changed within the same ranked task. For example, if Task 6 (rank 10) was coded as High but Task 7 (rank 10) was coded as Low, we did not count this as a trial. In fact, such instances were common and expected because PSTs might rely on their solution to the first task within a given rank to facilitate their solution to the second task of that rank. After excluding these cases, we were left with 21 trials. The trials are labeled in Table 3 with the number placed in the cell of the higher ranked task where the change occurred. Since our theory assumes an increase in task ranking predicts an increase in experienced cognitive demand, we consider a successful trial to be one where the change in cognitive demand increased for successively given tasks of increasing rank. There are 18 successful trials. The three unsuccessful trials occurring with PSTs F, G, and J, denoted in Table 3 with an asterisk (Trials 7, 11, and 17).

To test the validity of the task ranking system, we asked the following: What is the probability of 18 successes in a sample of 21 trails if the chance of changed cognitive demand is 50%. To answer this question, we used a binomial test with a p-value of 0.05. We assume the independence of observation needed for a binomial test holds across PSTs since each was interviewed separately and any discussion of the interview between PST outside of the interview setting was negligible. We also assume the independence of observation holds within a PTS's interview because of the novelty of the tasks and the exclusion of same ranked tasks from the trials. We analyzed the data using Microsoft Excel (version 15.33).

Results

Table 3 illustrates the cognitive demand of tasks as experienced by each PST. Green indicates Low demand, yellow High, and red Over. Two pairs of tasks, 6-7 and 8-9, have the same rank; if a PST was given both tasks of the same rank, we only consider the first of the same ranked task given to eliminate familiarity with the task as a confounding variable of cognitive demand. At a glance, we can see that the predicted trend of increasing rank with increased cognitive demand did occur. There are three PSTs (F, G, and J) for whom this pattern did not strictly hold outside of same ranked tasks. For PST F, the codes followed the pattern of Over, High, then Over again. PST G had a High code after two Over codes. Lastly, PST J had one High code in the middle of two Low codes before getting coded as Over. We attribute PST F's deviation from the predicted pattern to her initial assimilation of fractions in an unconventional manner (e.g., assimilating "three-sevenths" as one-third of 1/7) before adjusting this understanding in subsequent tasks as parts out of wholes. The switch from Over to High that PST G experienced was a case of persistence in trying to solve a task as she made use of new strategies used on previous tasks. For PST J, she experienced a perturbation with her scheme for "one-ninth" in Task 5 that led to an Over code but was resolved for subsequent tasks.

Table 3: Summary of cognitive demand by task and PST

Task	Rank	A	B	C	D	E	F	G	H	I	J	K	L
12	16							11*		15			
11	16/17												
10	14						8	10	13			20	
9	12												
8	12			3		6	7*	9	12	14	18	19	21
7	10	1											
6	10		2		4	5					17*		

5	8										16		
4	7												
3	5												

With 18 successes out of 21 trials, and a probability of success for a single trial of 50%, we obtained a p-value of 0.0006. This statistical result validates the task ranking system. In particular, it supports the hypothesis that increased length in sequences of units/actions required to solve fractions tasks predicts cognitive demand, as experienced by the PSTs and evidenced by their behavioral responses to the tasks.

Discussion

In validating the task ranking system with a simple statistical test, we have affirmed the hypothesis that informed it. In turn, the affirmation of this hypothesis demonstrates the utility of our framework—a framework that integrates the psychological construct of working memory (Pascual-Leone, 1970) with the mathematics education construct of units coordination (Steffe, 1992). Furthermore, it supports the Piagetian perspective of mathematics as a coordination of actions (Beth & Piaget, 1966), while recognizing students' mental actions as the source of their own mathematical power.

Other mathematics education studies have addressed cognitive demand (e.g., Stein, Grover, Henningsen, 1996) or have identified how students might rely on sequences of mental actions to solve mathematical tasks (Simon, Placa, Avitzur, & Kara, 2018; R. Tzur & Simon, 2004). We used unit transformation graphs to account for both: mental actions constitute the atoms of students' mathematical knowledge, as represented by the circles (unit constructions) and arrows (unit transformations) in our graphs. We enumerated cognitive demand by the number of circles and arrows in each graph. Although this characterization of cognitive demand aligns best with the psychological construct of working memory, it also relates to Stein and colleagues' (1996) categorization.

Stein and colleagues were especially concerned with maintaining high cognitive demand of instructional tasks, where high demand referred to aspirations of engaging their students in "procedures with connections" and "doing mathematics" (Boston & Smith, 2009; Stein et al., 1996). Unit transformation graphs might support such aspirational goals by informing teachers of ways they can help students manage the demands of mathematical tasks without reducing them to the lower categories of "memorization" or "procedures without connections." Within our framework, maintaining such demand would involve facilitating students' coordination of the mental actions involved in a task's solution by providing appropriate figurative supports, such as manipulatives and opportunities for student drawings. Such supports could allow students to offload demands on working memory, especially in long sequences of units/actions, without eliminating the demand for their coordination (cf., Costa et al., 2011).

Prior research has highlighted additional factors that contribute to cognitive demand. For example, Pajares (1994) demonstrated that self-efficacy and math anxiety can moderate the cognitive demands that students experience in response to mathematical tasks. Although we did not take such factors into account in our study, the complexities of teaching necessitate that teachers do. We recognize these complexities and intend unit transformation graphs as a tool teachers and researchers might use to manage them.

Ultimately, we see unit transformation graphs as a means of recognizing and empowering students' mathematics by explicitly accounting for their available mental actions and coordinations thereof. We might expand the program by relying on research that identifies students' mental actions in other domains of mathematics, such as algebra (e.g., Lee & Hackenberg, 2014) and covariation (e.g., Carlson et al., 2002).

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