AN INVITATION TO CONVERSATION: ADDRESSING THE LIMITATIONS OF GRAPHICAL TASKS FOR ASSESSING COVARIATIONAL REASONING

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We reflect on the limitations of our research group's prior methods for assessing covariational reasoning which primarily used graphical tasks found in extant literature. Graphical tasks dominate the literature on covariational reasoning, and through our use of these tasks we came to question the heavy reliance on them. Our concerns led us to ask the following: (1) What are the limitations of using tasks with graphs to assess covariational reasoning? (2) How can we improve assessment of covariational reasoning to accommodate students with nonnormative graphing schemes? We offer this piece as the beginning of a conversation to develop improved methodologies that attend to the ubiquity of students' nonnormative graphing schemes.

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The role of quantitative and covariational reasoning in the teaching and learning of mathematics has received increased attention in recent decades. Researchers have adopted numerous methodologies to investigate said reasoning, and, relatedly, they have developed a number of research-based tasks and instructional settings that afford such investigations. In our previous work, we drew connections between covariational reasoning and units coordination (Boyce et al., 2019). In this paper, we reflect on the relationships between our prior methods for assessing covariational reasoning, including how this influenced our data and claims regarding participants' maximum capacity to reason covariationally across settings. By sharing the analysis of our methods, we hope to spark a larger conversation that is important for a number of reasons.

First, reflection and review of research processes can refine our research and the quality of research in the field as a whole. Second, our research team consists of many newcomers to covariational reasoning research, including three first-year doctoral students—two of whom are the leading authors of this piece. This affords us the opportunity to start a dialogue between both novice and expert researchers concerning the interpretation and use of covariational reasoning frameworks in research. To further help facilitate such a conversation, we invited an expert on quantitative and covariational reasoning (the last author) to participate in crafting this paper. Third, in the sphere of covariational reasoning, there is a new generation of researchers who have started using the covariational reasoning frameworks that Thompson and Carlson (2017) developed over the past 30 years (Gonzalez, 2018; Stevens, 2019). By examining the evolution and contemporary usage of these frameworks, we can adopt a lens by which to identify previous works' contributions and limitations. Fourth, and most relevant to the present work, after reflecting on our methods we contend that extant

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studies have often relied too heavily on graphical items to assess covariational reasoning. We offer specific suggestions on how to improve interview protocols to address this issue.

Previous Study Design and Current Research Questions

Our initial interview protocols for assessing a students' overall ability to reason covariationally were guided by Carlson and Thompson's work (Carlson et al., 2002; Thompson & Carlson, 2017), the Project Aspire covariation tasks (Thompson, 2016; Thompson et al., 2017), and the classic Bottle Problem (Swan, 1985). Project Aspire, led by Thompson, created a validated diagnostic assessment of secondary teachers' mathematical meanings. Their work included drafting a number of covariation items and then piloting them with teachers to determine the best items for reliably assessing covariational reasoning. While the initial covariation item pool included both graphical and non-graphical items, all non-graphical items were discarded for a variety of reasons. Because we drew inspiration from the Aspire instrument and Carlson et al.'s (2002) tasks, and the covariational reasoning literature sparingly highlights the importance of non-graphical tasks, it did not occur to us to include such tasks in our interview protocol.

While retrospectively analyzing our methods and resulting data, we hypothesized that assessing overall covariational reasoning with mostly graphical tasks limited our ability to model students' thinking. This hypothesis stemmed from our observation that some students correctly described relationships between two covarying quantities using words and gestures, but these same students failed to graphically convey their described relationships because of their nonnormative graphing schemes. In hindsight, it makes sense that graphical tasks have limitations given the growing body of evidence that a number of successful undergraduate students use nonnormative graphing schemes (Frank, 2017; Lee et al., 2019; Moore et al., 2019), which often involve meanings for graphs that do not entail covariational reasoning. Consequently, such student graphs do not provide data for a researcher to assess their covariational reasoning beyond an *in-the-moment* absence of it. Indeed, Saldanha and Thompson (1998) conveyed a similar sentiment when stating: "The results of this study lead us to believe that understanding graphs as representing a continuum of states of covarying quantities is nontrivial and should not be taken for granted" (p. 303). Building off Saldanha and Thompson's sentiment, recent researchers' characterizations of students' nonnormative graphing schemes, and our inference of a possible overreliance on graphing tasks to assess covariational reasoning both within and outside of our own work, we ask: (1) What are the limitations of using tasks with graphs to assess covariational reasoning? and (2) How can we improve assessment of covariational reasoning to accommodate students with nonnormative graphing schemes?

Theoretical Framework

Covariational Reasoning

Thompson's research in covariational reasoning stemmed from his interest in how "students conceive situations as composed of quantities and relationships among quantities whose values vary" (Thompson and Carlson, 2017, pp. 424–425). In Thompson's (2011) view, "Quantification is the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship (linear, bi-linear, multi-linear) with its unit" (p. 37). Quantitative reasoning, then, is a person's conception of quantities and the relationships between those quantities. Building off prior quantitative approaches to covariational reasoning (Thompson, 1988, 1993, 1994; Thompson & Thompson, 1992), Carlson et al. (2002) defined covariational reasoning "to be the cognitive activities involved in coordinating two varying quantities while attending to the way they change in relation to each other" (p. 354). For instance, a person can conceive of the volume and height of water in a bottle as varying simultaneously as the bottle is emptied by evaporation.

Carlson et al. (2002) associated mental actions (MAs) with indicative behaviors (both graphical and verbal) in their covariational reasoning framework (see Table 1). Levels of covariational reasoning were then defined in terms of these mental actions. We originally interpreted this framework to mean that someone who exhibits MA2 verbal indicators would also exhibit the corresponding MA2 graphical indicators. Carlson et al. (2002) clarified that a student can exhibit behavior associated with a specific mental action without engaging in the mental action itself, so we suspected someone might be able to perform a graphical behavior indicative of covariational reasoning but be unable to verbalize some covariational relationship. However, it did not occur to us that someone might verbally describe a covariational relationship at a much higher level than their graphical behavior would suggest. We will return to these ideas in our results section.

Table 1: Portion of Carlson et al.'s (2002) Covariational Reasoning Framework

Mental action (MA)	Indicative behaviors
MA1 - Coordinating the value of one variable with changes in the other.	Labeling the axes with verbal indications of coordinating the two variables.
MA2 - Coordinating the direction of change of one variable with changes in the other variable.	Constructing an increasing straight line. Verbalizing an awareness of the direction of change of the output while considering changes in input.
MA3 - Coordinating the amount of change of one variable with changes in the other variable.	Plotting points/constructing secant lines. Verbalizing an awareness of the amount of change of the output while considering changes in the input.

Thompson and Carlson (2017) refined the Carlson et al. (2002) framework to craft a more broadly applicable covariational reasoning framework. Instead of listing indicative behaviors for each level of covariational reasoning, their descriptions focused on students' abilities to envision and anticipate rather than construct, plot, or verbalize an awareness of something. In particular, their descriptions attend to the type of variational reasoning a student uses to envision how each quantity changes as well as the nature of the multiplicative object a student constructs between each quantity's values. Thompson and Carlson followed Saldanha and Thompson's (1998) usage of Piaget's notion of "and" as a multiplicative operator to derive the notion of a multiplicative object. An individual forms a multiplicative object by uniting attributes of two objects to form a new third object. For example, Frank (2017) described how an individual can consider the attributes 'red' (perhaps from an apple) and 'circular' (from a ring) independently, then unite them to construct a single red circle as a multiplicative object. The resulting object is multiplicative because it is simultaneously red and circular. In regard to variational reasoning, Thompson and Carlson built off the variational reasoning research done by Castillo-Garsow (2010, 2012) and adapted the distinctions made by Castillo-Garsow et al. (2013) between discrete, chunky continuous, and smooth continuous variational reasoning.

Thompson and Carlson (2017) outlined two different ways researchers could use their covariational framework. First, the framework levels could be used to characterize a person's covariational reasoning in a specific instance. And second, the framework levels could describe a person's capacity or ceiling for covariational reasoning across settings. For example, a graduate student in mathematics may display a gross coordination of values in describing how the miles driven in their car and the gallons of gas used by the car increased as their road trip continued. Nevertheless, the student may also possess the ability to envision smooth continuous covariation. In this report, we are primarily concerned with the second way of using the framework levels.

Nonnormative Graphing Schemes

Research has recently begun to detail the extent to which students' meanings for graphs diverge from the normative meanings privileged by the mathematical community (Moore et al., 2019). Roughly, there are two key components of a graph: a curve itself and the coordinate system in which the curve is plotted. To attend to the nuanced, nonnormative graphing schemes students use, we consider one construct for each component: (a) the type of *shape thinking* (Moore and Thompson, 2015) students engage in while reasoning about a curve and (b) the quantitative *frame(s) of reference* (Joshua et al., 2015; Lee et al., 2019) students use to construct an underlying coordinate system.

Shape thinking. In order to focus on the meaning an individual has for a graph, Moore and Thompson (2015) introduced the constructs of static shape thinking and emergent shape thinking. Static shape thinking involves attending primarily to the perceptual shape of a graph and inferring associations based on shape, rather than the underlying covariational relationship. On the other hand, emergent shape thinking involves conceiving of a graph as a trace or record representative of the underlying covariational relationship. Thompson (2016) reported that 29 of 111 written responses high school teachers provided to a version of the bouncy ball task were coded as representing static shape thinking (see pp. 449–450). If a nontrivial proportion of teachers are reasoning with static shape thinking, some students likely are as well.

Frames of reference. Joshua et al. (2015) defined a frame of reference as "a set of mental actions through which an individual might organize processes and products of quantitative reasoning" (p. 32). In particular,

> An individual conceives of measures as existing within a frame of reference if the act of measuring entails: 1) *committing to a unit* so that all measures are multiplicative comparisons to it, 2) *committing to a reference point* that gives meaning to a zero measure and all non-zero measures, and 3) *committing to a directionality of measure comparison* additively, multiplicatively, or both. (Joshua et al., 2015, p. 32, emphasis added)

Constructing a complete quantitative structure for a two-dimensional Cartesian coordinate system requires measuring two quantities at once, which is accomplished by simultaneously combining two frames of reference. Ultimately, then, a coordinate system is the result of the many mental actions involved in constructing and combining frames of reference (Joshua et al., 2015). As Lee et al. (2019) found, it is a nontrivial task to coordinate across multiple frames of reference/coordinate systems at once. This suggests that students who do not construct canonical coordinate systems may struggle to interpret the conventional meanings a graph is meant to convey.

Methods

After publishing preliminary results on the relationship between students' units coordination and covariational reasoning (Boyce et al., 2019), our group reflected on the tasks and methods we used to assess students' capacity for covariational reasoning. During this reflective process, we analyzed artifacts from our previous research—including videos of interviews, individual/group coding notes, and meeting memos. For example, we looked at spreadsheets in which two members of the team independently commented on each interview and assigned each student a covariational reasoning level. Over five months, we engaged in sustained group discussion for an hour and a half each week about the theory of covariational reasoning, the literature on graphing and quantitative/covariational reasoning, and how we determined the levels we reported in Boyce et al. (2019).

Prior to each weekly meeting, we spent extensive time considering these issues individually and in small groups. This helped ensure that all group members had a voice and that we were generating several distinct suggestions for improving our covariational reasoning assessment protocol and analysis. As we revised our analysis procedures, we looked back at previous student interviews to determine how these new procedures would affect our past assessments of students' capacity for covariational reasoning.

Results

Limitations of Graphical Tasks

In this section, we discuss our results through an analysis of the actions of a college calculus student named Shania. Specifically, we respond to our first research question by highlighting the limitations of using a graphical task to assess her capacity to reason covariationally. As previously noted, Carlson et al. (2002) remarked that a student might exhibit a behavior associated with a particular mental action or level of covariational reasoning but not reason in a way consistent with that covariation level. They wrote, "Some students have been observed exhibiting behaviors that gave the appearance of engaging in [advanced mental actions] . . . When asked to provide a rationale for their construction, however, they indicated that they had relied on memorized facts to guide their construction" (pp. 361–362). Put more succinctly, graphical activity can lead a researcher to overestimate a student's overall ability to reason covariationally. Shania, as we will demonstrate, is representative of the opposite: her graphing activity may have led us to underestimate her capacity to reason covariationally.

Shania attempted a variation of Thompson's (2016) Bouncy Ball task. The task scenario (with provided graphs shown in Figure 1) follows below:

> A ball is hanging by a 10-foot rubber cord, from a board that is 20 feet above the ground. The ball is given a sharp push downward and is left free to bob up and down. The graph on the left represents the ball's *displacement from its resting point* in relation to the *time elapsed* since the ball was pushed. The graph on the right represents the ball's *total distance* traveled in relation to the *time elapsed*. The information given is in the first second after being pushed. The final graph represents the ball's *displacement from its resting point* in relation to its *total distance traveled* since being pushed.

After Shania read the task, the interviewer asked, "What's happening to the ball based on that description?" The interviewer also provided Shania with a coffee cup to physically illustrate the ball's motion. Following some dialogue to clarify the initial dangling position of the ball relative to the board, Shania gestured up and down with the cup to demonstrate how she believed the bouncy ball would bob up and down after being first pushed down. Already, via gestures, Shania demonstrated signs of reasoning covariationally at the gross coordination of values level. Her movements suggest that she understood how the ball's displacement and the number of seconds elapsed changed together.

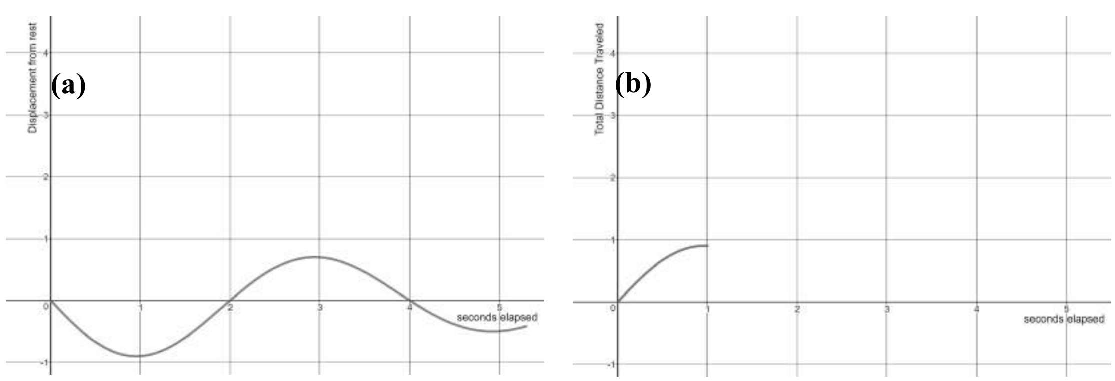

Figure 1: Graphs Provided for the Bouncy Ball Problem

Shania was then prompted to describe the *y*-axis label for the incomplete graph of total distance traveled with respect to time (See Figure 1b). After a brief conversation, the interviewer described how total distance traveled will always be positive. Shania agreed with this description and subsequently stated multiple times without prompting throughout the interview that, "The distance is always increasing." The interviewer then asked Shania to produce the remainder of the incomplete graph. In response, she drew a graph (Figure 2b) that seems to reflect the provided graph over the x-axis on the intervals where the displacement was negative. When describing her graph, Shania said that she thought of it as the derivative of the graph of displacement with respect to time (Figure 1a). But, when she described the relationship between different segments of her graph, she talked about the total distance traveled, rather than the displacement.

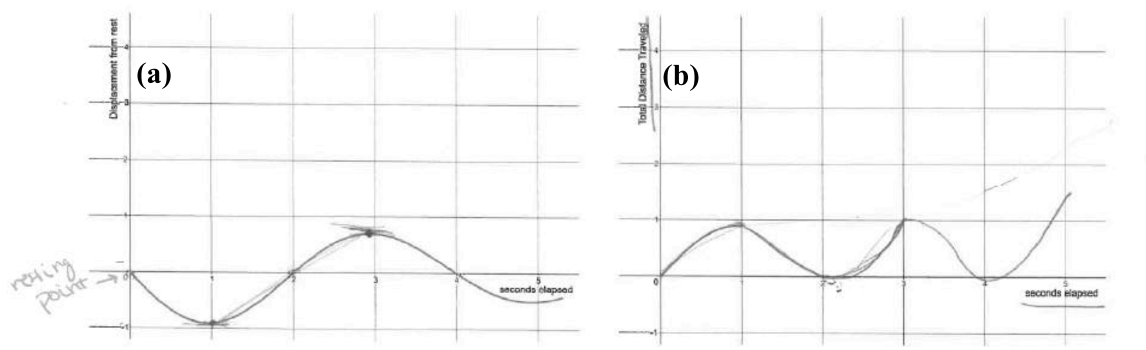

Figure 2: Shania's Work on the Bouncy Ball Task

The interviewer asked Shania why she thought the graph she was asked to produce was the derivative of the first graph. Shania responded, "I kind of saw, like, when we were doing the derivatives in class with sine and cosine and everything and then how sine is the opposite of cosine." The interviewer followed up and asked, "So it wasn't anything about the wording of the question that made you think derivative. Like, it was the memory of an oscillating shape?" To this, Shania responded, "yeah."

The summation of the interactions suggest Shania conflated three different sources to produce her graph: the graphs presented in her calculus course, the shape of the provided graph of displacement with respect to time, and her understanding of how total distance traveled must always be positive. Instead of attending to how the two quantities (total distance traveled and time) simultaneously vary, Shania intended her graph to represent a variety of concepts. We interpret this as significant evidence that Shania created her graph as a record that is intended to communicate a number of specific facts about the ball's movements. In other words, Shania was likely engaged in static shape thinking.

After the interviewer and Shania came to a joint conclusion that the task did not ask for a graph related to the derivative, Shania was given a second opportunity to produce a graph of total distance traveled with respect to time on a new piece of graph paper. Before producing her second graph, the interviewer asked, "What about that graph [Figure 1b] shows us the distance?" Shania responded by saying, "The distance is from here to here [draws an arrow going up parallel to y-axis on the new graph] since it's going up, increasing. And then our seconds were from zero to five." Here, Shania's language indicates that she was engaged in gross coordination of values due to her describing how the total distance traveled increases and how the time increases. Shania's original graph suggests that she did not envision how the two quantities were varying together. Instead, according to Thompson and Carlson's (2017) covariational framework, Shania would have most likely been classified at the precoordination level (a level beneath gross coordination of values).

As evidenced, nonnormative graphing schemes can cause underestimates of a student's capacity to reason covariationally. Frank (2017) demonstrated that in order to construct a graph as a record of quantities covarying, a student must conceive of a multiplicative object. Gross covariational reasoning, however, does not require a student to conceive of a multiplicative object. Thus, a gross covariational reasoner's graphing activity does not provide insight into their maximal capacity for reasoning covariationally. Subsequently, graphing tasks inhibit researchers' ability to assess some students' covariational reasoning capacity.

Improving Covariational Reasoning Assessments

We now transition to our second research question: *How can we improve assessment of covariational reasoning to accommodate students with nonnormative graphing schemes?* A reflective analysis of our own methodologies paired with a review of the literature has led us to one possible improvement that can be made: researchers could provide non-graphical tasks before—or possibly in place of—graphical ones. Although all interviewers in our prior study adhered to a general protocol, we noted in our analysis that one interviewer—the researcher who interviewed Shania—would often ask brief introductory questions to ensure interviewees fully understood the prompt before proceeding to any graphical tasks. Only upon retrospective analysis did the benefits of this practice become apparent. As detailed previously, the language and gestures Shania used to model the trajectory of the bouncy ball with a coffee cup provided strong evidence she was engaged in gross coordination of values. However, when she proceeded to the graphical component of the task, her tendency to engage in static shape thinking obscured evidence of this form of covariational reasoning. Further analysis and group discussion allowed us to see past her static shape thinking in our original study, but without data from the non-graphical introductory questions Shania's interviewer asked, we may have incorrectly assessed her covariational reasoning level.

The case of Shania highlights the importance of providing non-graphical means by which interviewees can engage in covariational reasoning. In line with this suggestion, Moore and Carlson (2012) noted that undergraduate precalculus students' graphical and computational solutions tended to match their emergent image of the problem's context as well as the quantitative structure they constructed for the problem. For example, Shania's gestures provided no evidence that her image of the problem incorporated the damping of the bouncy ball's displacement from rest as time passed. So, in line with Moore and Carlson's findings, it should come as no surprise that her graph of total distance with respect to time did not reflect this damping (see Figure 2b). Moore and Carlson (2012) contended that "It is critical that students first engage in mental activity to visualize a situation and construct relevant quantitative relationships prior to determining formulas or graphs" (p. 48). A simple way of supporting such mental activity could be to have a conversation with an interviewee about the situation and any relevant quantities. Even a short conversation has the added benefit of providing valuable data about the quantities an interviewee is constructing (as we have illustrated in the case of Shania). Another means for supporting the interviewee in constructing mental images of the quantities is to have them model the situation using gestures, like how Shania was instructed to model the trajectory of the bouncy ball using a cup. Of course, there are many other ways to aid interviewees in constructing an image and quantitative structure for the situation. We encourage researchers to be creative and design their pre-graphical tasks in ways that best align with their planned trajectory of tasks and intended methods of analysis.

The Invitation to Conversation

By reflecting on our prior research methods, we identified significant limitations of using predominantly graphical tasks to assess one's capacity for covariational reasoning across settings. Namely, students with nonnormative graphing schemes may be able to reason covariationally at higher levels in non-graphical settings. We suspect the primary difficulty for these students is modeling how two quantities covary in the *normative* Cartesian coordinate system privileged by the mathematical community. After all, even constructing a coordinate system—a prerequisite for creating or reasoning about a graph—requires constructing two frames of reference and then combining them. Not to mention, if the coordinate system a student constructs to model a covariational situation does not align with the normative Cartesian one, they may spend more time resolving perceived contradictions than demonstrating their capacity to reason covariationally. In sum, our analysis has revealed that graphical tasks can add considerable noise to a researcher's data for assessing students' covariational reasoning. If a researcher hopes to use graphs to assess covariational reasoning, we urge them to devise explicit methodologies for reducing or, at the least, acknowledging this noise.

Although we suspect experienced researchers have already begun to develop these types of methodologies (e.g., Johnson, 2015; Stevens & Moore, 2017), they are not yet explicitly outlined in the literature. We believe the community of covariational reasoning researchers, particularly novices, would benefit if experienced researchers shared the fine-grained details of their task design and analysis techniques. After all, such accounts are vital to communicating important considerations for conducting covariational reasoning research that may not yet be salient in the literature, such as the limitations of using graphical tasks. We call on researchers (ourselves included) to consider the role that non-graphical tasks should play in assessing covariational reasoning. To that end, we offer this piece as the beginning of what we hope becomes a larger conversation to develop improved methodologies that accommodate the ubiquity of students' nonnormative graphing schemes.

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