EXPLORING SECONDARY STUDENTS' PROVING COMPETENCIES THROUGH CLINICAL INTERVIEWS WITH SMARTPENS

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Twenty students who earned A or B course-grades in the proof unit(s) of a secondary course that addressed proof in geometry were asked to work on two proof tasks while sharing their thinking aloud and using smartpens. Students were classified into two categories: those who were successful with both proofs and those who were unsuccessful with both proofs. Large differences were observed in how often students in the two groups exhibited certain competencies. The largest gaps occurred in the ways that students: attended to the proof assumptions; used warrants in their proofs; and demonstrated logical reasoning.

Keywords: reasoning and proof, geometry and geometrical and spatial thinking

Introduction

Proving is an important aspect of mathematical competence, and geometry is the site which has historically been considered to be a good starting point to teach and learn mathematical argumentation and proving in secondary mathematics (Reiss, Hellmich, & Reiss, 2002). Hoyles (2002) suggested that proving involves a range of non-trivial "habits of mind" such as looking for structures and invariants; identifying assumptions; and organizing logical arguments. These processes, she noted, must be coordinated with visual or empirical evidence and mathematical results and facts. The need to coordinate the linguistic and visual registers is a particularly distinguishable feature of proof in geometry (Sinclair, Cirillo, & de Villiers, 2017). Yet, a number of studies conducted in the context of secondary geometry provide evidence that the teaching and learning of proof in secondary geometry is a challenging endeavor (Balacheff, 1988; Cirillo, 2011; Healy & Hoyles, 1998; Senk, 1985). To make progress on this challenge, we sought to gain insight into the competencies and behaviors displayed by students who were successful and unsuccessful with two proof tasks.

Theoretical Framework

A number of studies have documented students' difficulties with proof in geometry. For example, Senk (1985) conducted a study wherein she administered six proof-writing tasks to 1520 students in the U.S. After scoring the tasks on a scale of 0-4 where scoring >3 deemed a student to be "successful" on an item, Senk concluded the following: only about 30% of students in a full-year geometry course reached a 75% mastery of proof, and approximately 29% of students could not write even one valid proof. In another study, in this case, a study of high-attaining 14-15 year-old students in England and Wales, Healy and Hoyles (1998) found that only 19% of students were able to construct a proof of a familiar geometry statement, and fewer than 5% of students could construct a proof of an unfamiliar geometry statement. Last, in their study of 81 German upper secondary students, Reiss, Klieme, and Heinze (2001) found that only 20% of students were able to construct correct Euclidean geometry proofs. Various recommendations and student difficulties were noted in these studies.

Based on study results, Senk recommended that we must look for more effective ways to teach proof in geometry, noting, for example, that we must find ways to support students to start a chain of reasoning. In the studies conducted by Healy and Hoyles (1998) and Reiss and colleagues (2001), the

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researchers found that students were better at judging others' proofs than they were at constructing their own proofs. Also, in Reiss et al.'s (2001) study, only 10% of students were able to provide a definition of the central concept of "congruence" and name a mathematical theorem related to congruence (i.e., a triangle congruence theorem). Furthermore, this study provided evidence that geometric competence with respect to proving is dependent on a combination of metacognition, spatial reasoning, methodological knowledge, and declarative knowledge (e.g., the notion of congruence).

Some researchers have elaborated on why proof is so challenging for students. For example, Healy and Hoyles (1998) explained that the process of building a valid proof is complex in that it involves sorting out what is "Given" from what can be deduced, and then organizing the conclusions that can be drawn from the "Given" into a coherent and complete argument that meets the proof goal. When considering the large number of possible inferences that could be made from the "Given(s)" in a typical school geometry proof problem, Koedinger and Anderson (1990) noted: "Geometry proof problem solving is hard" (p. 512). To better understand this "hard" situation, they observed geometry "experts" and found that, prior to writing up the details of their proofs, experts tended to quickly and accurately develop an abstract proof plan that skips many of the steps required in the full proof. In other words, they first applied global thinking (i.e., considered the "big picture") rather than local thinking (i.e., worked on one step at a time) at the start of the process. This conclusion is consistent with Cai's (1994) finding in a study of problem solving in geometry that the more-experienced participants spent the majority of their time on orientation and organization, while less-experienced students spent the majority of their time on execution (i.e., doing rather than thinking or planning).

In Battista's (2007) review of school geometry research, he posed several unanswered questions related to students' learning of proof, including: Why do students have so much difficulty with proof? What components of proof are difficult for students and why? and How can proof skills best be developed in students? (pp. 887-888). Ten years later, speaking back to these questions in their research review, Sinclair et al. (2017) concluded that while some researchers have attempted to address these questions, the reported studies tended to focus on only one or a few teachers or did not provide evidence of effectiveness at scale. They also suggested that more research is needed on students' development of geometric proof skills and their understanding of the nature of proof. The lack of research in this area led us to pursue this topic. Thus, this study addresses the following research questions: (1) What proving competencies and behaviors are observed in students who were successful with solving geometric proofs? and (2) How do these competencies and behaviors compare to those of students who were unsuccessful with the proofs?

Methods

The study reported on in this paper is part of a larger research project titled: *Proof in Secondary Classrooms: Decomposing a Central Mathematical Practice* (*PISC;* PI: Cirillo).The goal of the *PISC* project is to better understand the difficulties involved in the teaching and learning proof in secondary geometry and to develop a new and improved intervention to address these challenges. Students who earned high marks (grades of A or B) in the geometry proof unit(s) were selected for individual clinical interviews for this sub-study. The rationale for interviewing students with high marks was to understand what high-performing students were taking away from the proof unit(s). Because past studies have shown that even high-attaining students struggle with non-routine as well as routine proof tasks in geometry (see, e.g., Healy & Hoyles, 1998; Cirillo, 2018), two proof tasks that, in theory, should have been somewhat familiar to the students, were selected for data collection and analysis. We chose triangle congruence proof as a topic for exploration because it is considered to be a central concept in school geometry and because limited time to conduct the interviews in the school setting did not permit us to take up large amounts of students' time with longer problems.

The criteria for participant selection for this sub-study were as follows: (1) students were enrolled in a secondary course that addressed proof in geometry; (2) students earned an A or a B in the proof unit(s); (3) students were identified by their teacher as students who would be willing to share their thinking aloud during the interview; and (4) students completed the full interview protocol in the allotted time; (5) there were no technology glitches during the data collection, and (6) students' success results on each of the two proof tasks were the same (i.e., successful or unsuccessful on both proofs). This selection process reduced the sample size from 31 students interviewed to 20 selected. Participating students spanned Grades 8-11 (ages 13-17).

The first author conducted all student interviews. The goal was to spend about 35 minutes with each student; the mean interview length was about 32 minutes. The full interview protocol consisted of seven items. The first item was a simple "warm-up" task about geometric notation. The next four tasks were selected or adapted from Cirillo and Herbst (2011). The last two tasks, which were the ones selected for this analysis, were full-proof tasks (see Figure 1). Students spent an average time of 7.28 minutes on Task 6 and 5.95 minutes on Task 7. Students were asked to read each task aloud to ease them into talking through the task and guarantee that they had read the "Given" information for each task. Smartpen technology (i.e., Livescribe pens) was used to audio-record the students' explanations of their thinking and capture their pen strokes as they worked through the proofs. This methodology allowed us to capture students' thinking in the form of verbal explanations and simultaneous diagram markings and other written work.

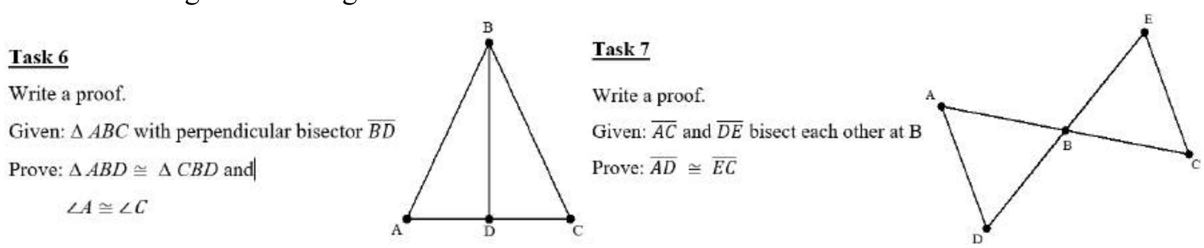

Figure 1: The two full-proof interview tasks analyzed for this study

Smartpen data were digitized to create a "pencast," or video, that simultaneously replays each student's handwriting and audio-recording (Livescribe, 2012). Prior to analyzing the smartpen data, students' final proofs were quantitatively scored from the paper hardcopies in ways that followed Senk's (1985) methods. Specifically, we adapted the rubrics for Senk's full-proof tasks so that every proof was scored on a scale of 0-4. Following Senk's approach, if students scored a 3 or a 4 on a proofs, they were considered to be Successful with the Proof tasks (abbreviated as SP; n=7). Students who scored less than 3 were considered Not successful with the Proof tasks (abbreviated as NP; n=13). The resulting data set consisted of two Proof Task Interviews (PTIs) from 20 students. The units of analysis are the individual proof tasks, resulting in 40 units of analysis. Tables 1 and 2 include age, grade level, course grade (A or B), and task scores (0-4) for each participant in the NP and SP groups.

We used constant comparative analysis (Boeije, 2002) to develop a coding dictionary. To develop the codes, the research team watched the PTI pencasts for six participants for both Tasks 6 and 7. Codes were developed for observed problem-solving behaviors and competencies that were exhibited through spoken and written work. This iterative process resulted in 45 possible codes for both NPs and SPs. After garnering an 86.59% interrater reliability and reconciling incongruent decisions, the second author coded the remaining data. The final phase of analysis involved looking across the coding results for patterns and themes.

Table 1: Study participants who were Not successful with both Proof tasks (NPs)

Participant Number	Age	Grade Level	Course Grade	Task 6 Score	Task 7 Score
P1	14	8	A	0	0
P2	16	10	B	0	0
P3	15	10	B	0	0
P4	15	10	B	0	0
P5	14	8	A	0	1
P6	14	8	A	1	1
P7	17	9	B	1	1
P8	17	10	B	1	1
P9	16	10	B	1	1
P10	15	9	A	1	1
P11	13	8	A	2	1
P12	17	11	B	2	2
P13	17	11	A	2	2

Table 2: Study participants who were Successful with both Proof tasks (NPs)

Participant Number	Age	Grade Level	Course Grade	Task 6 Score	Task 7 Score
P14	14	8	A	3	3
P15	14	8	A	3	4
P16	13	8	A	4	4
P17	13	8	A	4	4
P18	14	8	A	4	4
P19	14	8	A	4	4
P20	13	8	A	4	4

Findings

In reporting the findings, we first describe the most prevalent competencies and behaviors among the students who were **S**uccessful with both **P**roof tasks (SPs; n = 7). Because there were multiple competencies and behaviors that were exhibited by a high percentage of the SPs, we chose a threshold of 60% for the occurrences that would be reported. That is, we report on behaviors and competencies observed at least 60% of time for the SPs. This decision yielded five findings across the data set and includes behaviors related to working with the "Givens," marking the diagram, demonstrating logical thinking, and so forth.

Regarding the second research question about the competencies and behaviors of the students who were **N**ot successful with either **P**roof task (NPs; n = 13), the data were more inconsistent. We first compare NPs with SPs by reporting on the frequencies (as a percentage of the total occurrences for each PTI) with respect to the six findings from the SP group. Table 3 provides a summary of these behaviors and the frequency of occurrence. We then share three additional findings that relate to behaviors observed at least 20% of the time in the NP group.

Competencies and Behaviors of Students Who Were Successful with the Proofs

All seven SPs made productive and explicit use of the "Given" information for both tasks (i.e., 100% of the time). They did so either as they were planning their proofs or as they began to work on a proof. They explicitly identified the relevant mathematical objects from the assumptions (i.e., the "Given"). Here is an example of P14 thinking aloud about Task 7 (P14-T7; from here forward, we will use the notation PX-TY to denote each participant and task): "First thing we know is that ABC and DE bisect each other at B. Why? Because it's the Given. Next. Well you know that B, B is the midpoint. Why? Because definition of line segment bisector…"

Table 3: Frequencies of observed competencies and behaviors for both groups

Observed Competencies & Behaviors (rounded to nearest whole percentage)	**SPs (%)**	**NPs (%)**
***Students productively attended to the "Given" information**	**100**	**23**
Students correctly identified bisectors	100	12
Students indicated what object was being bisected	93	12
Students used the diagram as a resource	**100**	**47**
Students marked the diagram	100	65
Students used the diagram as a check list or planning tool	100	46
Students made valid claims supported by assumptions about the diagram	100	31
***Students identified warrants as postulates, axioms, definitions, or theorems**	**86**	**8**
Students clearly connected claims to definitions	86	12
Students stated or explained a definition	79	4
Students articulated a definition of congruent triangles	100	8
***Students demonstrated that they were thinking in a logical manner**	**93**	**8**
Students attended to important details while working through their proofs	**68**	**13**
Students articulated a plan for the proof prior to writing the proof	64	15
Students attended to rigor in sub-arguments	100	4
Students attended to triangle congruence criteria	79	15
Students attended to the "Prove" statement in explicit ways	64	23

* Indicates the main findings with the largest percentage gap between SPs and NPs (> 75%)

All SPs accurately marked the diagrams for both proofs (i.e., 100% frequency). The smartpen technology enabled us to see noticeable differences in the ways this occurred. For Task 6, SPs tended to mark the diagram in two distinct ways. Either they worked through the details of the proof, marking off congruent parts as they made their inferences, or they marked the congruent parts, using the diagram as a checklist to show that they had proven the triangles congruent. For Task 7, three SPs seemed to immediately recognize how to solve the proof, so they explained a plan for the proof *and* marked the congruent parts prior to beginning the proof (see Figure 2). Each proof task included some type of bisector in the "Given." SPs were clear about what type of bisector they were working with 100% of the time (e.g., line segment bisector). They explicitly indicated what was being bisected 92.9% of the time.

When SPs wrote or articulated their warrants (i.e., reasons for their statements), they typically indicated the typology in explicit ways 85.7% of the time. SPs appropriately connected claims to definitions 85.7% of the time, sometimes even stating the exact definition (78.6%). For both proofs, the concept of congruent triangles was critical toward developing a valid proof. All SPs wrote CPCTC as the warrant for their triangle congruence statements. When asked what it meant or stood for, 100% of SPs were able to articulate what CPCTC stood for or explain what it meant (i.e., Corresponding Parts of Congruent Triangles are Congruent).

As they were thinking aloud, SPs' explanations contained logical connectives, such as "next," "and then," and "we can conclude," in 92.9% of the PTIs. The P18-T6 excerpt above is a good example of this. Also, P20-T6 used logical connectives "then" and "so" in various ways:

> <u>So, we have</u> this is congruent to that and we have that this is perpendicular to that <u>so I guess we could use</u> the right angles theorem to prove that these are congruent <u>and then we could prove</u> that this is congruent by the reflexive property of congruence. <u>And then we can get</u> the angles congruent by C-P-C-T-C.

Although, in this explanation, P20 seemed to skip over the step of stating that the triangles were congruent, it was included in the written proof.

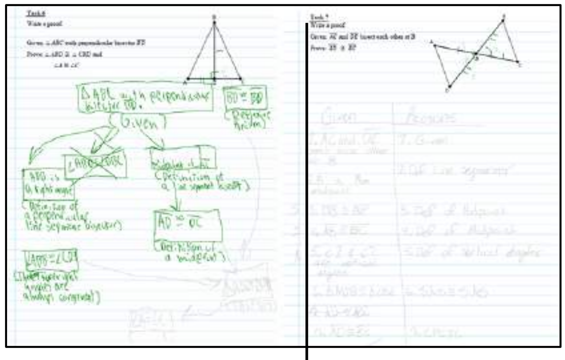

Figure 2: P15-T6 Used Diagram as a Checklist & P20-T7 Used Diagram as a Planning Tool

Arrow 1	Arrow 2	Arrow 3 and Triangle Congruence Statement
		

Figure 3: P16-T6 attends to sub-arguments and triangle congruence criteria

SPs attended to the details of their proofs in multiple ways. They articulated a plan for the proof before writing the proof 64.3% of the time. SPs attended to rigor in their sub-arguments 100% of the time. They attended to triangle congruence criteria in explicit ways 78.6% of the time. And they explicitly attended to the "Prove" statement 64.3% of the time. P16-T6's work provides evidence of attending to sub-arguments (i.e., branches of a proof claim and consequence) and triangle congruence criteria. His written work (see Figure 3) is very methodical in that he established three congruent parts prior to drawing in the arrows in his flow proof to connect the three congruent statements to the triangle congruence statement:

> Ok so then we have our three parts [draws three arrows]. So, we know that these are congruent and then we can say that triangle ABD is going to be congruent to triangle C, CBE. [Pause] I had to take a moment there to see which point was corresponding with point A. So, then we have our two triangles. And we can say this, because of S-A-S theorem. And after this, we can use my favorite theorem again to say that line segment AD is congruent to line segment EC because of C-P-C-T-C. So yeah.

The smartpen allowed us to see how the student worked out the three congruence statements prior to writing and then drawing arrows to the triangle congruence statement. We can see from the combination of transcript and smartpen images that he attended to triangle congruence criteria when he said: "we have our three parts," paused to accurately write the triangle congruence statement in a way that matched up the corresponding parts, and drew the three arrows before writing the triangle congruence statement.

Competencies and Behaviors of Students Who Were Unsuccessful with the Proofs

Large discrepancies between SPs' and NPs' behaviors were noted in the data. Table 4 includes frequencies from the PTIs (as percentages) for both groups for each finding and sub-finding. The differences in occurrences of the six main findings range from 35-92% with a gap of more than 80%

for 3 of the 6 main findings (as indicated in the table by *). For the second research question, it is not so productive to go through each finding one-by-one because there is little to say about the absence of something. Instead, we address the kinds of behaviors observed in NPs given the absence of many of the competencies exhibited by high percentages of SPs. Because the percentages of occurrences of common behaviors were much lower in this data set, we chose a lower threshold of >20% for the codes that we discuss in this section.

First, there were multiple issues noted with the ways in which students dealt with the "Given" information. In particular, 23.1% of the time, NPs incorrectly stated the "Given" when they read it aloud. The most common error for both tasks was saying "line" rather than "line segment." Second, 46.2% of the time, NPs omitted notation or information when they wrote the "Given" statement in the first line of their proof.

NPs' warrants were vague 61.5% of the time. They often did not identify warrants as postulates, definitions, or theorems, and they frequently did not seem to know definitions of relevant concepts. For example, P12-T6 wrote "Definition of bisect" as a reason for Line 3. Yet, in order for the corresponding statement, $\angle ABD \cong \angle CBD$, to be true, $\overline{BD}$ would have had to have been an *angle* bisector rather than a *perpendicular* bisector. In the next line of the proof, the student's warrant references the particular diagram, rather than a definition.

NPs displayed a lack of confidence in 34.6% of the PTIs. They sometimes would say that they could not remember definitions or reasons for their statements. For example, when asked about a reason for one of the steps in her proof, one NP said, "I don't have one." This statement also indicates a lack of using logical reasoning. Other comments made by NPs as they shared their thinking included: "I don't know how to do this one" [P1]; "I don't know how to explain it" [P4]; and "I remember having a lot of trouble on this because I didn't understand" [P11].

Discussion and Conclusions

Making use of smartpen technology, we explored the competencies and behaviors of high-attaining students with the expectation that even high-attaining students would have gaps in their abilities to prove. Clear differences were noted in the approaches taken by students who were successful with the two proof tasks, compared to students who were unsuccessful with those tasks. These findings contribute to the research on the teaching and learning of proof in geometry, specifically, the first part of Battista's (2007) question: What components of proof are difficult for students and why? There are several important take-aways from these findings.

First, when we compare the frequency percentages of competencies and behaviors observed in the two student groups in Table 4, the differences are relatively large. For example, the difference between how often SPs were observed productively attending to "Given" information compared to how often NPs did so was 77%, with SPs doing so 100% of the time and NPs only 23% of the time. This finding indicates that more work is needed to support students to productively attend to proof assumptions.

With respect to geometric diagrams, 100% of SPs marked their diagrams, made valid claims supported by assumptions about diagrams, and used the diagrams as a check list or a planning tool, particularly when solving a proof at the "global" level (Koedinger & Anderson, 1990; Cai, 1994). In contrast, NPs did not always mark their diagrams; they sometimes used the diagrams to plan; and they rarely made appropriate assumptions about the diagrams. Rather than using the "Given" information to draw valid conclusions, NPs put forth inappropriate inferences which often seemed to come from what the diagram "looked like." Doing so follows the perceptual proof scheme described by Harel & Sowder (2007).

There was also ample evidence to suggest that SPs had strong understandings related to the typology of warrants and that they understood how to use them to develop proof arguments. In contrast, this competency was rarely observed in the PTIs of NPs. Instead, NPs were sometimes observed justifying statements using reasons related to particular diagrams (e.g., B is the midpoint). Additionally, at times, NPs simply wrote, as their warrants, some mathematical object (e.g., bisector) without writing anything further, such as clearly identifying the type of bisector or explicitly stating whether they were thinking about a theorem or definition. NPs explicitly connected claims to definitions, stated definitions, and were able to articulate a definition of congruent triangles less than 15% of the time. These issues connect to Reiss and colleagues’ (2001) claims about the importance of coordinating declarative knowledge with higher order skills. Regarding higher order skills, evidence that students were reasoning through their proofs in a logical manner was observed 93% of the time for SPs, compared to only 8% of NPs.

The likelihood of SPs attending to important details in their proofs was also much greater than the likelihood of NPs doing so. The area of greatest difference was with respect to sub-arguments. SPs attended to sub-arguments in rigorous ways 100% of the time; NPs did so only 4% of the time. This finding relates to logical reasoning in that a sub-argument is a chain of reasoning that begins with an assumption and involves more than one deduction (Cirillo, Murtha, McCall, & Walters, 2017). SPs were also observed attending to other kinds of details in ways that were not observed in the NP data, such as: developing a plan for their proof (i.e., thinking at the global rather than local level); attending to triangle congruence criteria; and attending to the “Prove” statement. SPs were observed doing these things most of the time; while NPs were not. These findings are consistent with Healy and Hoyle’s (1998) descriptions of what makes proving so complex. In particular, they argued that proving involves: sorting out what is “Given” from what can be deduced and then organizing the conclusions that can be drawn from the “Given” into a coherent and complete argument that meets the proof goal.

The use of smartpen technology allowed us to “see” and analyze the data in ways that we would not have been able to see or do without it. For example, the active ink feature in the pencast videos allowed us to track the ways in which students marked their diagrams and how they shifted back and forth between diagram and proof. We conjecture that this kind of analysis is only the tip of the methodological iceberg in terms of what is possible to do with this and other tracking technologies. What kinds of insights into student thinking might be gained from more studies like this one, is an open but exciting question.

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