Equivalence Tasks in a Digital Algebraic Notation System Promotes Performance in Middle School Mathematics

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Abstract: The concept of equivalence can be elusive to students and can be confounded with unproductive understandings of the equals sign. Using the game-based digital algebraic notation system, From Here to There! (FH2T), students explore ideas of equivalence by dynamically transforming expressions or equations among mathematically equivalent states. In fall of 2019, 409 middle-school students completed a randomized control trial where they worked in either GM or an online problem set control over four half-hour sessions during their math block. We found that students in the FH2T condition showed enhanced performance over the control (p=.043, η_p^2 =.010). We will describe the FH2T technology, describe our results, and discuss implications for the usefulness of digital environments in mathematics education.

Keywords: Algebra and Algebraic Thinking, Technology, Instructional Activities, Middle School Education

Introduction

Student misunderstandings or misconceptions about equivalence and the equals sign have been noted as inhibiting success in upper-level STEM disciplines (National Mathematics Advisory Panel, 2008). A common misconception that students have is viewing the equals sign as marking or calling for a computation, such as interpreting “4+1=5” as “four and one makes five.” While this description is accurate in this case, over identifying “makes” with the equals sign can be unproductive. For example, students who rigidly associate “makes” with the equals sign may claim equations like “4+1=2+3” as not valid because there is a sum on the right-hand side of the equals sign, rather than a single number. More generally, these types of operational understandings are associated with difficulty in equation solving (Knuth, Stephens, McNeil, & Alibali, 2006). A robust perspective, in contrast, is when the equals sign is viewed as a relational symbol such that there is an equivalence relation between the expressions on each side of the symbol (Kieran, 2007).

Much of the research on students’ understanding of mathematical equivalence overlaps with students’ understanding of the equals sign (Blanton, Stephens, Knuth, Gardiner, Isler, 2015; Kieran, 2007; Knuth, et al., 2006). Notably, Rittle-Johnson and colleagues (2011) point out that by 4 years of age, most children have understandings of numerical equivalence – whether two sets have equal quantities of items – but issues of equivalence become entangled with issues of notation as students work with bigger numbers, more complex operations, and generalized forms with variables. In our work, we untangle equivalence from the equals sign with *goal state* tasks in FH2T. In FH2T, students used Graspable Math (GM), a dynamic algebraic notation tool, to transform an initial expression into a mathematically equivalent goal state using permissible gesture-actions (Figure 1). Numbers and expressions are virtual objects on the screen and can be moved, combined, or re-represented (through factoring or decomposition, for example) in GM and FH2T. Th task provides opportunity to create and notice equivalent states of expressions throughout transformations.

Figure 1. The FH2T game has 14 worlds (left), with 18 problems per world (center). A sample FH2T task with a transformable expression “0+1•1+2•2+3•3+4•4+5” and a Goal of “15+20” (right).

In fall of 2019, we conducted a randomized control trial with middle-school students. Students were randomly assigned to one of two interventions: FH2T and an online problem set control. Student performance on two arithmetic equivalence and four algebraic conceptual items in pre- and post-tests were compared within and across conditions to explore how practice with FH2T may be associated with performance.

Theoretical framework

The operational vs. relational dichotomy in students' perspectives of the equals sign is well-documented (Blanton et al., 2015; Carpenter, Franke, & Levi, 2003; McNeil, Grandau, Knuth, Alibali, Stephens, Krill, 2006). Stephens and colleagues (2013) adds nuance to that discussion by differentiating between a *relational-computational* view, where students understand that two sides of the equals sign calculate to the same value, and a *relational-structural* view, where students understand that a particular expression is just one of myriad ways to represent that quantity. This subtle difference is tied to a structural understanding of algebra (Kieran, 2007), where students may see some algebraic expressions as "single objects or as being composed of several objects" (CCSS.Math.Practice.MP7).

Landy, Allen, and Zednik (2014) proposed that sense making of symbolic notation with its structures and rules happens through perceptual and sensorimotor systems. Sensorimotor mechanisms are not simply ways to translate marks on a page or readying them to be cognitively recognized, they are constitutive aspects of symbol manipulation. Thus, the capacity for symbolic reasoning is in part the ability to "perceptually group, detect symmetry in, and otherwise perceptually organize symbolic notations," (Landy et al., 2014, p. 1) – to notice and manipulate objects within the notation. Potentially, training one's perceptual and sensorimotor systems in symbolic notation may result in more effective reasoning about the relationships represented by the symbols (Kellman, Massey, & Son, 2010). This is not to say that blindly or rotelya manipulating equations will result in a robust understanding of algebraic notation, but rather to point out that sensorimotor experiences of procedural routines can help to reinforce understandings of allowable transformations and support developing conceptual understandings.

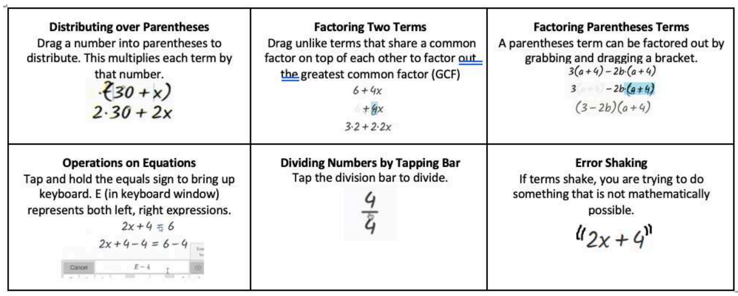

Figure 2. Sample transformations from a FH2T gesture guide.

Grounded in this theory, GM is designed as a virtual environment where algebraic objects (numbers, terms, or expressions) can be manipulated by a mouse click or a finger on a touchscreen, and the system behaves dynamically according to the mechanics of algebraic notation (Figure 2; Weitnauer, Landy, Ottmar, 2016). For instance, enacting operations such as division involves dropping one object on top of another or touching the operator in between two adjacent objects. Other mouse- or touch-actions enact additional algebraic transformations such as distribution, factoring, and properties of equality. When a mathematically impermissible move is attempted, the system gives a perceptual response by shaking those terms and does not enact the move. GM provides an environment that allows for playing with algebraic expressions and equations as virtual objects. Prior work revealed that GM may be effective for decreasing notation errors and improving mathematical understanding for elementary and middle school students (e.g., Daigle et al.,2019). For this study, the GM environment is presented to students through *From Here to There!* (FH2T, Ottmar, et al., 2015), a hierarchical, self-paced goal state tasks arranged in worlds (i.e., math topics) inside a universe.

Research methods

Our study involved a student-level between-subjects randomized trial of FH2T and problem sets in ASSISTments, an online homework platform (Heffernan and Heffernan, 2014). We utilized a pretest-intervention-posttest design with four 30-minute intervention sessions covering the four operations, order of operations, and balancing equations, and two 45-minute assessment sessions before and after the intervention. Students in the FH2T condition solved goal state tasks, while students in the problem set control worked through items compiled from three open-source mathematics curricula: Engage NY (2014), Utah Math Project (2016), and Illustrative Math (2017). The control condition included a combination of computations and word problems, with answer types including short answer, multiple choice, and open response. The control students saw one item at a time, and had the opportunity to request three hints and the final solution. The mathematical performance assessment consisted of six items: two items on mathematical equivalence (Rittle-Johnson et al., 2011), and four items on the conceptual understanding of equivalence in algebra (Star et al., 2014).

All participants came from a large, urban district in southeastern United States, and the analytic sample consisted of 409 middle school students (55% White, 23% Asian, 14% Hispanic and 7% Black; 227 males 182 females) who completed pre- and post-tests. Teachers were self-selected to have their classes participate, and the majority of students (343) were in 6th grade advanced level classrooms. The student population of the participating schools is comprised of 10% English Language Learners and about 30% identified as high needs, including but not limited to low income, limited English proficiency, or having identified learning disabilities.

Results

First, we examined whether students in the two conditions performed comparably on the pretest. An independent sample t-test on the pretest scores revealed that the baseline scores did not significantly differ between conditions, p = .982. Next, we examined the effects of condition on students' performance by conducting a 2 (pre- vs. post-tests) × 2 (FH2T vs. control) repeated measures ANOVA on the scores. The analysis revealed a main effect of time on the scores suggesting that students in both conditions showed improvement from pretest (M = 3.89, SD = 1.63) to posttest (M = 4.23, SD = 1.57), $F(1, 407) = 28.73$, $p < .001$, $\eta_p^2 = .066$.. The significant Time × Condition interaction was an interaction of magnitude, $F(1, 407) = 4.14$, p = .043, $\eta_p^2 = .010$. Students in the FH2T condition showed *greater* improvement compared to students in the problem set condition (Figure 3). The effect of condition was not significant, p = .370.

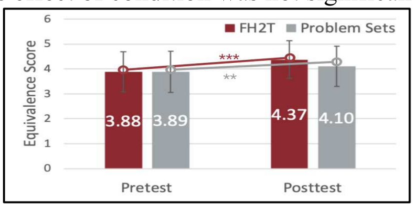

Figure 3. Time by Condition interaction on test scores. Error bars represent the standard deviation.

Last, we estimated the effects of condition by conducting a multiple linear regression predicting posttest score. The baseline model estimating the influences of pretest, instruction level, and grade on the posttest score revealed that all three variables were significant predictors of the posttest scores, and together they accounted for 50.7% of the variance in the posttest score. Adding condition to the model revealed that students in the FH2T condition scored 0.15 standard deviation (SE=.07) higher on the posttest compared to students in the problem set condition, and the condition accounted for an additional 0.6% of the variance in students' posttest scores above and beyond the covariates, p=.03. In summary, we found (a) students in both conditions improved their performance on mathematical equivalence from pretest to posttest and (b) students in the FH2T condition showed greater improvement on the posttest compared to students in the problem set condition.

Conclusion

Despite major efforts in research, curricula development, and policy, students still struggle with understanding equivalence and the equals sign. This study showed improved performance on arithmetic and algebraic equivalence items using FH2T, a gamified version of GM as compared to a control condition of online problem sets with hints and feedback. Most of the student participants were considered strong in mathematics, bringing into question the generalizability of these results, but the comparison with the control condition indicates students using FH2T experienced some benefit over traditional problem sets with extra support. Considering that FH2T (and GM) was built to augment perceptual learning of algebraic notation, these results provide some evidence supporting the notion that students' algebraic reasoning may be partially comprised of perceptual-motor routines (Goldstone et al., 2010, 2017). On first glance, FH2T (and GM) seems to have students mechanically match active expressions to goal states, but this study suggests that there is something more going on. Potentially, students' experience of moving and transforming algebraic objects reinforced by the error feedback and visual changes to the expressions helps students to attend to relevant details and generalize notation mechanics. Furthermore, the experience of transforming expressions and equations into perceptually different but mathematically equivalent states may support conceptual understanding of equivalence. Future directions include explorations of clickstream data that were produced through this intervention. Through this data, students' actions within each task and solution choices can be to reconstructed and synthesized, and we can begin to uncover potential mechanisms of learning in online platforms. Additionally, we are interested in interviewing students as they work through tasks to develop understandings of how students formulate strategies for reaching the goal states, and how they choose to work within the FH2T goal state environment.

References

Blanton, M. L., Stephens, A., Knuth, E. J., Gardiner, A., & Isler, K. (2015). The development of children's algebraic thinking: The impact of a comprehensive early algebra intervention in third grade. *Journal for Research in Mathematics Education*, 46(1), 39–87.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.

Daigle, M., Harrison, A., Braith, L., Ottmar, E., Hulse, T., & Manzo, D. (2019). From here to there! Elementary: a game-based approach to developing number sense and early algebraic understanding. *Educational Technology Research and Development*, *67*(2), 423–441.

Goldstone, R. L., Marghetis, T., Weitnauer, E., Ottmar, E. R., & Landy, D. H. (2017). Adapting Perception, Action, and Technology for Mathematical Reasoning. *Current Directions in Psychological Science, 26*(5), 434–441.

Goldstone, R. L., Landy, D. H., & Son, J. Y. (2010). The education of perception. *Topics in Cognitive Science, 2*(2), 265–284.

Heffernan, N. T., & Heffernan, C. L. (2014). The ASSISTments ecosystem. *International Journal of Artificial Intelligence in Education, 24*(4), p. 470 – 497.

Kellman, P. J., Massey, C. M., & Son, J. Y. (2010). Perceptual learning modules in mathematics: Enhancing students' pattern recognition, structure extraction, and fluency. *Topics in Cognitive Science*, 2(2), 285–305.

Kieran, C. (2007). Learning and teaching of algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 2, pp. 707–762). Charlotte, NC: Information Age.

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does Understanding the Equal Sign Matter? Evidence from Solving Equations. *Journal for Research in Mathematics Education*, 37, 297–

Landy, D., Allen, C., & Zednik, C. (2014). A perceptual account of symbolic reasoning. *Frontiers in Psychology*, 5(April), 1–10.

McNeil, N. M., Grandau, L., Knuth, E. J., Alibali, M. W., Stephens, A. C., & Krill, D. E. (2006). Middle-School Students' Understanding of the Equal Sign: The Books They Read Can't Help. *Cognition and Instruction*, 24(3), 367–385.

National Mathematics Advisory Panel. (2008*). Foundations for Success: The Final report of the National Mathematics Advisory Panel*, U. S. Department of Education: Washington, DC.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics.* Washington, DC: Council of Chief State School Officers. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf, February 2019.

Ottmar, E. R., Landy, D., Goldstone, R. L., & Weitnauer, E. (2015). Getting From Here to There ! : Testing the Effectiveness of an Interactive Mathematics Intervention Embedding Perceptual Learning. In *Proceedings of the 37th Annual COGSCI*. Pasadena, CA.

Rittle-Johnson, B., Matthews, P., Taylor, R. S., & McEldoon, K. (2011). Assessing knowledge of mathematical equivalence: A construct-modeling approach, *Journal of Educational Psychology*, *103*(1), 85–104.

Stephens, A. C., Knuth, E. J., Blanton, M. L., Isler, I., Gardiner, A. M., & Marum, T. (2013). Equation structure and the meaning of the equal sign: The impact of task selection in eliciting elementary students' understandings. *The Journal of Mathematical Behavior*, *32*(2), 173–182.

Star, J. R., Pollack, C., Durkin, K., Rittle-Johnson, B., Lynch, K., Newton, K., & Gogolen, C. (2014). Learning from comparison in algebra. *Contemporary Educational Psychology*, *40*, 41–54.

Weitnauer, E., Landy, D., & Ottmar, E. R. (2016). Graspable Math: Towards dynamic algebra notations that support learners better than paper. In *Future Technologies Conference* (pp. 406–414). San Francisco.

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