Lessons Learned in Designing an Effective Early Algebra Curriculum for Grades K–5

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Abstract

We describe here lessons learned in designing an early algebra curriculum to measure early algebra's impact on children's algebra readiness for middle grades. The curriculum was developed to supplement regular mathematics instruction in Grades K–5. Lessons learned centered around the importance of several key factors, including using conceptual frameworks to design the components of our curriculum, treating early algebra content as a set of core algebraic thinking practices across several core content domains, and following particular stages in curriculum design research. We also learned the importance of recognizing and addressing "gaps" in the assessment tools available for measuring growth in students' learning, in the empirical research base for early algebra, and in support for teachers around professional learning in early algebra. We found curriculum design to be a complex and expensive process that required careful pacing and deliberation to address these gaps.

Key Words:
curriculum design, early algebra, elementary grades, learning progressions

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The Need for an Early Algebra Curriculum

In recent decades, views on teaching and learning algebra in school mathematics have changed deeply. Prompted by algebra's historical status as a gateway to academic and economic success (Schoenfeld, 1995), scholars argued that developing children's informal notions about algebraic ideas from the start of formal schooling, beginning in kindergarten, would better prepare them for success in formal algebra in later grades. Such a substantial change in school mathematics has significant costs that include "deep curriculum restructuring, changes in classroom practice and assessment, and changes in teacher education—each a major task" (Kaput, 2008, p. 6). This raises fair questions about the impact that algebraic thinking in the elementary grades (or, *early algebra*) might have on children's algebra readiness for middle grades. In essence, as we have asked elsewhere (Blanton et al., 2019), does early algebra matter?

In considering the tension around the costs and benefits of early algebra, we began a program of research over a decade ago to better understand early algebra's impact. We wanted to know what difference a comprehensive, research-based approach to instruction around early algebraic concepts across elementary grades might make in children's algebraic thinking as they entered middle grades. Our immediate challenge was clear: Elementary schools were not yet equipped to develop children's algebraic thinking in the deep, systemic way that we felt was needed to fairly measure early algebra's impact.

First, elementary teachers are critical to algebra reform, yet they have not historically been provided with sufficient professional learning opportunities to develop classroom instruction that fosters the rich and connected kinds of algebraic thinking that constitute early algebra (e.g., Greenberg & Walsh, 2008; Kaput & Blanton, 2005). Second, existing curricula have not adequately addressed early algebraic concepts and practices in a manner that focuses on important ways of thinking algebraically. Even now, widely used mathematics curricula for elementary grades too often treat algebra as a collection of "things to do" (e.g., solve an equation) rather than as a set of practices or habits of mind, such as generalizing mathematical relationships, to develop in children's thinking across elementary grades.

Further, mathematics curricula have not always addressed core algebraic concepts with sufficient depth. Consider the concept of mathematical equivalence. The equal sign symbolizes an equivalence relation that indicates two mathematical objects are equivalent (Jones et al., 2012) and should be understood *relationally* in elementary grades as meaning the two expressions in an equation have the same value. Yet many students view this symbol *operationally* as a prompt to perform the computation indicated in the expression to the left of the equal sign (Jacobs et al., 2007). Research shows that students' operational misconceptions about the equal sign are present as early as kindergarten (Blanton, Otalora et al., 2018) and persist in later elementary grades (e.g., Stephens et al., 2013). In a recent randomized study, we found that 80% of Grade 3 students exhibited operational misconceptions about the equal sign, even when their mathematics instruction used *Common Core*-aligned curricula. These operational misconceptions have been found to persist into middle grades and negatively impact students' success solving algebraic equations (e.g., Knuth et al., 2006). Taken together, these studies suggest that understanding the meaning of the equal sign requires sustained attention over many years. Yet, well-designed curricula largely address this concept relationally—if at all—in Grade 1, likely in alignment with its treatment in Grade 1 by the *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices and Council of Chief State School Officers [NGA Center & CCSSO], 2010).

As this illustrates, it is difficult to adequately prepare elementary students for algebra with curricula that do not deeply address core algebraic concepts through a sustained, coordinated approach across elementary grades. Thus, if we were to fairly understand early algebra's impact, we needed to develop an early algebra curriculum for Grades K–5 that could supplement regular mathematics curricula with a rigorous, research-based approach to developing children's algebraic thinking. In response to this need, we developed *LEAP: Learning through an Early Algebra Progression* (Blanton et al., 2021a–c; 2022a–c) as a supplemental curriculum for Grades K–5[1].

The Early Algebra "LEAP" Curriculum

Overview of the LEAP Curriculum

Our development of the LEAP curriculum has progressed in phases over the last 15 years, beginning with the development of the Grades 3–5 curriculum and followed by the development of the Grades K–2 curriculum. The LEAP curriculum consists of 18–20 lessons per grade level—60-minute lessons in Grades 3–5 and 35-minute lessons in Grades K–2—that are taught throughout the school year. Lessons begin with a *Jumpstart* to review concepts in previous lessons and, in Grades K–2, include a *Launch* to introduce the lesson focus. These are followed by small-group investigations (*Explore and Discuss*) in which students explore algebraic ideas and share their mathematical thinking. Lessons conclude with a *Review and Discuss* to summarize key ideas and formatively assesses students' thinking. To illustrate, the Appendix provides samples of these lesson components using Grade 1 lessons on the equal sign.

We designed the curriculum using a spiraled approach so that students could continually revisit algebraic concepts and practices year-to-year, refining their understanding using

[1]The LEAP Curriculum is available at https://www.didax.com/leap.

increasingly sophisticated concepts, representations, and ways of thinking. Lessons are scaffolded with teacher questioning strategies to foster rich mathematical conversations around students' algebraic thinking. They also focus on the use of meaningful problem contexts and—particularly in the early grades—concrete and visual tools linked to abstract representations to help students develop mathematical meaning for algebraic ideas.

Does the LEAP Curriculum Work?

Curriculum development should extend beyond simply designing instructional materials to examining whether the materials are effective. Given the lack of research using rigorous, experimental designs to evaluate the effect of curricular approaches on students' mathematical achievement (Clements, 2002; U.S. Dept of Education, 2008), particularly with underserved populations of students (Clements, 2007), we wanted to know whether the LEAP curriculum was effective for students across all demographics and academic abilities. Evidence that LEAP made a significant difference in students' understanding of core algebraic concepts and practices when taught as part of students' regular instruction would also provide a measure of early algebra's impact. In what follows, we share evidence to date of LEAP's effectiveness.

We recently conducted a large-scale, randomized study of the intervention's effectiveness in Grades 3–5, where the intervention was taught by classroom teachers as part of their regular mathematics instruction. (See also Stylianou et al., this volume.) The study was conducted in 46 elementary schools using a demographically diverse sample of students from urban, rural, and suburban populations. To improve fidelity of implementation (FOI), teachers were provided with professional development throughout the intervention focusing on their early algebra knowledge, their understanding of students' early algebraic thinking, and how tasks and instruction could support the development of this thinking. An analysis of teachers' FOI showed a significant positive relationship between components of teachers' implementation and students' performance (Stylianou et al., 2019).

We found that students who were taught the intervention as part of regular instruction significantly outperformed their peers who received only regular instruction (Blanton et al., 2019). At each of Grades 3–5, significant differences were found in students' knowledge of algebraic concepts and practices (i.e., item correctness) as well as their use of algebraic strategies to solve tasks. These significant differences persisted at the end of Grade 6, one year after the intervention ended (Stephens, Stroud et al., 2021). Further, in a comparison of a subset of treatment and control schools where the majority of students were from underserved communities (e.g., 100% low SES, over 90% students of color), treatment students also significantly outperformed control students at each of Grades 3–5 in both item correctness and use of algebraic strategies (Blanton et al., 2019). Thus, our findings suggest that students, regardless of demographic, are better prepared for algebra upon entering middle grades if they are taught the LEAP curriculum as part of regular instruction. With this evidence, we seemed to have a promising curricular approach from which we might understand early algebra's impact.

Curriculum design—even for curricula focused on a particular strand such as early algebra—is a complex process. We are not curriculum designers by training, so the last 15 years have taught us much about designing effective curricula, even as we recognize that there is yet much to learn. Among the lessons learned, some have been serendipitous in nature in that we came to value a particular approach or lens used that we did not fully appreciate "in the moment." We consider some of these lessons here.

Lesson Learned: Frameworks, Frameworks, Frameworks

One aspect of our curriculum development that has come to be the most valuable to us is that it has been "frameworks" driven. Frameworks provided critical scaffolding around how we designed our curriculum, how we addressed early algebra content in the curriculum, and the different stages in our curriculum design research. The use of frameworks was not always as explicit or intentional at the start of our work as it came to be as our work matured. We learned to appreciate how frameworks helped focus and stabilize our process as well as define more clearly the steps we needed to take. In what follows, we look at the critical ways in which frameworks informed our work.

A Learning Progressions Approach as a Framework for Designing LEAP

Learning progressions have become increasingly important as a research tool because of their ability to inform the design of standards, curricula, assessments, and instruction (Daro et al., 2011). From the start of our work, we adopted a *learning progressions approach* (e.g., Shin et al., 2009, Stevens et al., 2009) in which our curricular design involved the development of several core components (e.g., Clements & Sarama, 2004; see also Fonger et al., 2018): (1) empirically-derived learning goals around early algebra content; (2) grade-level instructional sequences (referred to here as the LEAP curriculum) designed to address these learning goals; (3) validated assessments to measure students' understanding of core algebraic concepts and practices as they advance through the instructional sequences; and (4) progressions that specify increasingly sophisticated levels of thinking students exhibit about algebraic concepts and practices in response to an instructional sequence.

This approach, which appealed to us in part because it already had significant traction in educational research, provided a critical over-arching framework that helped us think more systematically about the design process. It pushed us past simply pulling together interesting algebra tasks we might assimilate into some type of sequence and, instead, slowed down the design process to focus our attention on first building the scaffolding (i.e., learning goals) from which tasks and lessons could emerge in a coordinated way. That is, a learning progressions approach helped define our first objective—the development of grade-level learning goals—which, in turn, helped us build a more coherent and connected foundation for our curricular content than we might otherwise have done.

The learning progressions approach also helped us think about what we should consider in developing learning goals. Because learning progressions prioritize empirical research on children's thinking around specific content domains (Baroody et al., 2004), this approach directed us first to analyzing empirical research on children's understanding of core algebraic concepts and what we might expect regarding their algebraic thinking at particular grades. We then reviewed available national and state curricular frameworks and standards, including NCTM's *Principles and Standards for School Mathematics* (NCTM, 2000), the *Curriculum Focal Points* (NCTM, 2006) and, later, the *Common Core State Standards for Mathematics* [NGA Center & CCSSO, 2010], for their treatment of algebra at Grades K–8. We analyzed curricular materials for Grades K–8 (e.g., *Growing with Math, Everyday Mathematics*, *Singapore Math, Investigations)* according to their treatment of algebra content, and we considered mathematical perspectives on the sequencing of algebra content by examining formal algebra textbooks at both secondary and postsecondary levels. From this analysis, we looked for coherency between empirical research on algebra learning in Grades K–8 and benchmarks of algebra learning advocated in curricula and state and national frameworks, keeping in mind that research would likely be "ahead" of learning standards. We then synthesized our findings to develop grade-level learning goals that would form the backbone of our curriculum. A learning progressions approach helped us be more systematic and intentional in our analysis than we might otherwise have been. Moreover, it kept empirical research on children's algebraic thinking at the forefront of our design. With our learning goals in place, we could then develop grade-level instructional sequences (the heart of the LEAP curriculum) along with assessments to measure students' learning as they advanced through these sequences.

A Conceptual Framework for Early Algebra

As our work progressed, we came to appreciate how the design of our curriculum and assessments flowed from our learning goals, underscoring for us the importance of the learning progressions approach we used to develop these goals. Moreover, as we developed learning goals, a central question for us was "How should we characterize the algebra that we want young children to learn?" There is a *lot* of algebra content available in learning standards, curricula, and research, and we needed a way to organize this content.

Members of our research team had experience in early algebra research prior to our curriculum design work and, from this experience, brought views on the nature of early algebra that aligned with Kaput's (2008) widely acknowledged conceptual analysis of algebra content. Kaput's content analysis of algebraic thinking involves two *core aspects*: (1) making and expressing generalizations in increasingly formal and conventional symbol systems; and (2) acting on symbols within an organized symbolic system through an established syntax, where conventional symbol systems available for use in elementary grades are interpreted broadly to include "[variable] notation, graphs and number lines, tables, and natural language forms" (p. 12). From these two core aspects, we identified four essential algebraic thinking practices that defined part of our conceptual framework for early algebra content (Blanton et al., 2011; Blanton et al., 2018): *generalizing, representing, justifying,* and *reasoning with* mathematical structure and relationships.

Kaput further argued that these core aspects occur across three *key strands*:

"1. Algebra as the study of structures and systems abstracted from computations and relations, including those arising in arithmetic (algebra as generalized arithmetic) and quantitative reasoning.

2. Algebra as the study of functions, relations, and joint variation.

3. Algebra as the application of a cluster of modeling languages both inside and outside of mathematics." (Kaput, 2008, p. 11)

Early algebra research has matured around several core content areas relative to these key strands. In developing our conceptual framework, we parsed these key strands around three predominant domains of early algebra research: (1) generalized arithmetic; (2) equivalence, expressions, equations, and inequalities; and (3) functional thinking. We see domains (1) and (2) as aligned with Kaput's key strand (1), while domain (3) aligns with strands (2) and (3).

The four algebraic thinking practices (e.g., generalizing mathematical structure and relationships) and the three content areas where these practices can occur (e.g., generalized arithmetic) defined our conceptual framework for early algebra content. This framework was critical because it helped organize all our curricular content around algebraic *ways of thinking*. That is, rather than viewing curricular content as a set of things to do (e.g., solve equations), we viewed it through the lens of developing thinking practices or habits of mind.

Tasks and lessons were created with an eye towards how well they attended to these practices within the different content domains. What (and where) were opportunities for generalizing? What kinds of representations could be used and how could tasks build representational fluency across different forms, such as words, drawings, tables, and variable notation? How could tasks build students' capacity for developing strong arguments to justify general mathematical claims? What kinds of tasks promoted reasoning with generalizations? Questions such as these helped us design content systematically and comprehensively around algebraic thinking practices. Regardless of grade level, we thought about how each lesson might support students in developing the capacity to *generalize*, to *represent* their generalizations in different ways, to *justify* their claims with strong mathematical arguments, and to *reason with* the generalizations they built. To illustrate how we thought about lesson content within this conceptual framework, Table 1 highlights selected curricular content themes within the four algebraic thinking practices and their occurrence across the Grades K–5 LEAP curriculum.

Table 1. Occurrence of selected curricular content within algebraic thinking practices (ATP).

ATP	Curricular Content Themes within Algebraic Thinking Practices (ATP)	GRADE					
		K	1	2	3	4	5
Generalize	Develop generalizations about						
	Properties of operations	X	X	X	X	X	X
	Sums of evens/odds	X	X	X	X	X	X
	Rules for growing patterns	X					
	Functional relationships between two quantities	X	X	X	X	X	X
Represent	Represent generalizations about structure/relationships with words						
	Expressions & Equations	X	X	X	X	X	X
	Prop. of operations/arithmetic relationships	X	X	X	X	X	X
	Functions	X	X	X	X	X	X
	Represent generalizations about structure/relationships with variables						
	Expressions & Equations		X	X	X	X	X
	Prop. of operations/arithmetic relationships			X	X	X	X
	Functions			X	X	X	X
	Represent generalizations about functional relationships with tables	X	X	X	X	X	X
	Represent generalizations about functional relationships with graphs				X	X	X
	Represent relationships between quantities as equations in non-standard forms (i.e., a = a, a = b + c, and/or a + b = c + d)	X	X	X	X	X	X
Justify	Develop representation-based arguments for						
	Mathematical claims about specific but uncounted cases	X	X	X			
	General mathematical claims (generalizations)		X	X	X	X	X
	Identify best (general) arguments for general claims			X	X	X	X
Reason	Develop a relational view of '='						
	Find if equations in non-standard form (e.g., a = a, a = b + c, a + b = c + d) are true or false	X	X	X	X	X	X
	Find a missing value in equations in non-standard forms	X	X	X	X	X	X

	(e.g., $a = a$, $a = b + c$, $a + b = c + d$)						
	Explicitly identify properties of operations to justify computational work		X	X	X	X	X

We had not explicitly defined this conceptual framework *a priori*, although it was already an underlying lens in our thinking given our prior research. Making the explicit connection between the need for a way to organize algebra content and our existing way of thinking about early algebra content was a clarifying moment in our design work. We were fortunate that this framework already informed our thinking about algebraic content, and the design process helped solidify our understanding. The lesson learned here concerns the value of having a specific framework in mind prior to the design process for identifying curricular content.

We also found it helpful that the framework was organized around ways of thinking rather than specific content. This helped us connect content *across* Grades K–5. For example, the practice of generalizing arithmetic relationships about operations on evens and odds, how to develop arguments for these relationships, and how to use these relationships as building blocks to reason in novel situations, was a learning thread developed with increasing complexity in each grade across Grades K–5. We distinguish this from an approach that treats concepts related to even and odd numbers in isolated ways at a particular grade or grades.

A Framework for Curriculum Research

As described above, we began our curriculum development by adopting a learning progressions approach as a framework to guide our thinking about the components needed (e.g., learning goals, instructional sequences, assessments) and how to best design these components in a way that prioritized empirical research on children's algebraic thinking. Along the way, we clarified our conceptual framework for early algebra content based on our already existing views on what it meant to think algebraically. But what informed the steps in our research process? The benefit of a framework that outlines the components of a research process for developing a curriculum and testing its effectiveness had not occurred to us before our work began. We simply kept moving forward in what felt like obvious "next steps." Looking back, we are better able to articulate our process.

As we prepared to report on the effectiveness of our Grades 3–5 curriculum, we found Clements's (2007) Curriculum Research Framework (CRF) to be particularly relevant for retrospectively characterizing the stages of our work. The CRF consists of three phases: (1) *a priori foundations*, (2) *learning model*, and (3) *evaluation*. The *a priori foundations* phase involves identifying subject matter and relevant research in teaching and learning to inform the innovation's design. *Learning model* involves the design and sequencing of lesson content to align with empirical models of children's thinking. The *evaluation* component involves the use of multiple methodologies to evaluate the appeal, usability, and effectiveness of an innovation.

We see the stages of our work as aligned with the CRF's phases for developing research-based curricular innovations (Blanton et al., 2019). Our conceptual framework for early algebra derives from Kaput's (2008) subject matter analysis of the content and practices of algebra. This, in conjunction with our analysis of empirical research on children's algebraic thinking, state and national learning standards and frameworks, existing K–8 regular mathematics curricula, as well as the canonical development of algebra as a mathematical discipline (Battista, 2004), provided the *a priori foundation* (Clements & Sarama, 2008) for the design of our intervention.

As with the *learning model* phase (Clements, 2007), the grade level instructional sequences in our curriculum were designed as conjectured routes whose sequencing was based on known or hypothesized progressions in children's thinking about our targeted subject matter domain (algebraic concepts and practices), with tasks sequenced to advance students' knowledge along a progression (Blanton et al., 2019). For example, research on the development of children's understanding of the equal sign as a relational symbol suggests that the use of true/false and open equation tasks, as well as the sequencing of such tasks by layering in the use of operations in an equation (from a simple equation with no operations, to an equation with operations only to the left or right of the equal sign, to a complex equation with operations on both sides of the equal sign), can challenge children's operational thinking (Rittle-Johnson et al., 2011). Instructional sequences across Grades K–5 were designed to account for this type of empirical research in grade appropriate ways.

Finally, as with the CRF's *evaluation* phase, our evaluation involved the use of design studies to test our instructional sequences, quasi-experimental cross-sectional and longitudinal studies to examine the curriculum's potential, and, to date, a large-scale randomized study in Grades 3–5 with a follow-up retention study in Grade 6 to examine its effectiveness.

Table 2 shows the stages of our work for the Grades 3–5 design and how this aligns with the CRF. In retrospect, we see value in a framework such as the CRF that identifies steps in conducting research on curriculum development. Designing curricula is a lengthy, complex, and expensive process. A framework such as the CRF can help organize and sequence the design process and the acquisition of funding to support it.

Table 2. Alignment of our work with the CRF (Clements, 2007).

Stage of Work (Grades 3–5)	Alignment with Phases of CRF
Identification of subject matter (early algebra concepts and practices) through analysis of empirical research, national and state learning frameworks, regular K–8 mathematics curricula, and disciplinary knowledge; Analysis of the alignment of these products with our conceptual framework for algebra	*A priori foundation*
Initial design of Grades 3–5 components: learning goals, grade level instructional sequences (curriculum), assessments	*Learning model*
Design studies for preliminary testing of the curricular design	

Small-scale quasi-experimental cross sectional and longitudinal studies of the curriculum's potential and feasibility	*Evaluation*
Large-scale, randomized (CRT) study of the curriculum's effectiveness in Grades 3–5	
Retention study of the curriculum's effectiveness in middle grades (Grade 6)	

Lesson Learned: Addressing the Gaps

Addressing the Gaps in Assessments

As we noted earlier, more rigorous experimental studies that show whether curricular approaches are effective in increasing students' mathematical achievement are needed (e.g., U.S. Dept of Education, 2008). Robust measures of learning are essential for such studies. When developing a curriculum, it is important to consider whether there are adequate, validated measures that might be used to assess its effectiveness. Existing standardized assessments might be used, but one should consider if they are sensitive enough to measure the concepts the curriculum addresses. In our case, there was a significant gap in available measures around early algebra. Even existing state standardized assessments aren't calibrated closely enough to the algebraic concepts and practices LEAP addresses at a given grade level. LEAP accelerates algebraic development, so its content is sometimes beyond what students typically encounter in a given grade. This made typical standardized assessments for elementary grades an inadequate measure of LEAP's effectiveness. Moreover, standardized assessments are sometimes only available in upper elementary grades, leaving assessment gaps in the earlier grades. To measure LEAP's effectiveness (and to help us understand early algebra's impact), we needed assessments that could closely measure growth in understanding of the particular algebraic concepts and practices we hoped the LEAP curriculum would foster in students' thinking.

Here again, our use of a learning progressions approach was fortuitous, as this approach prioritizes assessments that measure growth along an instructional sequence. This primed our thinking to be intentional about assessment development from the start of the design process. Moreover, our early algebra framework helped us think conceptually about the content of the assessment items, as we had with the curriculum itself. That is, rather than just selecting different types of ubiquitous algebra tasks (e.g., solving an equation) for our assessments, we thought about assessment design in terms of our conceptual framework around algebraic thinking practices. For example, did LEAP assessments measure students' capacity to generalize or to represent generalizations in different ways? Did they measure students' capacity to justify mathematical claims with strong (non-empirical) arguments? In essence, we learned that it is as important to think about assessment design in terms of the conceptual framework as it is the curriculum itself. Moreover, designing and validating assessments is a complex process. We came to appreciate the importance of devoting sufficient energy and resources to developing good, validated assessments from the start of the design process.

Addressing Gaps in Research on Students' Thinking

Another area for which it is important to consider whether gaps might exist is the (empirical) research base on students' thinking about curricular content. We started our research on early algebra's impact and the development of the LEAP curriculum in Grades 3–5 because there was a much more robust and stable early algebra research base in this grade domain than in Grades K–2. Thus, while our work in Grades 3–5 progressed, we simultaneously began to think about learning progressions around early algebraic concepts and practices for young (Grades K–2) learners in anticipation of extending our work to these earlier grades.

Gaps in the empirical research base for Grades K–2 led to several research projects in which we built on the early algebra research base concerning students' relational understanding of the equal sign (Blanton, Otalora Sevilla et al., 2018; Stephens et al., 2022; Stephens, Veltri Torres et al., 2021), generalizing functional and arithmetic relationships (Blanton et al., 2015; Ucles et al, 2020), using and interpreting variable and variable notation (Blanton et al., 2017; Brizuela et al., 2015; Veltri Torres et al., 2019; Ventura et al., 2021), and developing arguments for general mathematical claims (Blanton et al., 2022). This foundational research was critical in later developing our Grades K–2 learning goals, instructional sequences (curriculum), and assessments. It also points to the complexity of curriculum development. More than simply pulling tasks together (even if done in a cohesive and meaningful way), curriculum design needs to incorporate a relevant and robust research base. In our case, this required us to slow down and help build the research base needed to support the curriculum.

Addressing the Gaps in Teachers' Professional Learning Opportunities

Designing a curriculum in a way that supports teachers' learning is not a new idea. In recent years, researchers have advocated for the development of *educative* curricula (e.g., Davis & Krajcik, 2005) that "incorporate elements that are meant specifically for *teacher* learning" (Stein et al., 2007, p. 334). Early algebra is in a particularly challenging position regarding teachers' access to professional learning opportunities. Elementary teachers are critical to algebra reform, yet a disproportionately large number of pre-service and in-service elementary teachers have deeply rooted anxieties about mathematics (Battista, 1986; Haycock, 2001)—particularly algebra—that can impact the confidence with which they teach children (Bursal & Paznokas, 2006). Thus, while early algebra is now part of the discourse of reform, elementary teachers still need significant support in understanding how to integrate early algebra into their daily instruction in routine ways. At the same time, district priorities for teacher professional development often elevate literacy over mathematics (e.g., Bassok & Rorem, 2014; Wrabel et al., 2015), making it challenging for teachers to get the support they need, particularly in algebra—mathematical content they likely did not imagine teaching in elementary grades. The scarcity of resources for professional learning around teaching mathematics, along with any anxieties elementary teachers might hold about algebra, can imperil teachers' ability to build early algebra-rich classrooms and impede reforms that introduce algebra in the elementary grades.

We designed early algebra lessons with these challenges in mind, considering how we might frame content in the curriculum as if this was *all* teachers saw. What if the curriculum was their sole professional learning opportunity? How could we design it in an educative way, to support their understanding of early algebra content, students' thinking about that content, and instructional practices that support children's algebraic thinking? There were several ways we did this. First, each lesson contains a section "Addressing Common Difficulties" that describes insights from empirical research about challenges students might face in thinking about lesson concepts and how teachers might address these. Each lesson also contains a "Teaching Support" section that provides insights around general practices to support students' thinking about a particular concept or practice. Figure 1 illustrates this material for the concept of mathematical equivalence.

Each lesson also includes a rationale for task designs so that teachers can better understand the purpose of a task in developing students' understanding of particular content. Figure 2 illustrates this with a "Rationale for the Tasks" taken from a Grade 1 lesson on equivalence. Additionally, each lesson includes specific questions in boldface for teachers to ask students that can help scaffold or pace mathematical conversations. Teacher questions are followed by descriptions of what teachers might expect to learn about students' thinking. (See the selected Launch in the Appendix for an illustration of teacher questions.)

Figure 1. Students' difficulties and how teachers can support learning: Example from a Grade 1 lesson about the equal sign (Blanton et al., 2022b).
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Addressing Common Difficulties

Students with an operational view of the equal sign will get "stuck" with equations that have operations on either side of the equal sign. They may reason that an equation such as $8 + 2 = 5 + 5$ is false because $8 + 2 \neq 5$, ignoring the "+ 5" on the right of the equation. Similarly, they might reason that $8 + 2 = 10 + 3$ is true, since $8 + 2 = 10$. Using tools such as cubes can help students see that either expression in an equation may have operations (or no operation at all). Cube towers provide a concrete way to reason about a relationship apart from its abstract representation (equation).

Students with an operational view of the equal sign sometimes add all numbers in an equation that doesn't make sense to them. Similarly, when building towers, they will combine towers representing each side of an equation into a single, large tower. It is important to encourage students to compare the heights of towers representing the expressions in an equation because this helps them to think about the two expressions as quantities to be compared, not combined.

Teaching Support

Using Tools

This lesson explores comparing the heights of towers of cubes as a way to compare the values of expressions in an equation. While some students will be able to reason from the equation itself, using concrete tools to compare the expressions in an equation is an important way to visualize equations to determine if they are true or false. Using concrete tools can also help students think about the more complicated equations that can promote a relational understanding of the equal sign.

Developing a Relational Understanding of the Equal Sign

Do not be surprised if students still demonstrate an operational view of the equal sign. Examining true/false equations will help them continue to develop a relational understanding. Using equations where operations occur on both sides of an equation, where operations occur on only the right side of an equation, or where there are no operations at all, is important for challenging students' operational thinking.

Asking Questions and Listening to Students' Thinking

Lessons depend on rich mathematical discussions focused around asking students questions and listening to their thinking. Ask questions that guide students' thinking rather than telling them how to solve a particular problem. For example, the question "How do your towers represent the equation?" encourages students to explain how they can represent an expression in an equation. For the equation $9 = 6 + 4$, questions such as "How could we change your towers to make their heights the same?" and "How does this change your equation?" encourage students to think about connections between the concrete and symbolic representations.

Figure 2. "Rationale for tasks" from a Grade 1 lesson about the equal sign (Blanton et al., 2022b).
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Rationale for the Tasks

- The use of true/false equations in forms other than standard form encourages students to think relationally, rather than operationally, about the equal sign.
- The use of concrete tools such as cubes can help students reason about abstract representations such as equations.

To be clear, we do not claim that these aspects of LEAP lessons are unique to the LEAP curriculum. We highlight these aspects to illustrate a lesson we learned, namely, that curriculum design is strengthened when thought about in the educational context in which the curriculum will be used and with the goal of supporting teacher—not just student—learning. The challenges of early algebra in implementation heightened our concern about teachers' professional learning opportunities and how we might use the curriculum's design features to offset these challenges.

There are other contextual issues that we did not consider but that, in retrospect, are important to bear in mind. We designed our curriculum for a print format rather than an interactive digital one. The latter requires a different way of developing content, and LEAP would have benefited had we considered both formats simultaneously. For example, a digital format would have given us remote learning options that have been particularly needed in recent years. Planning in advance for different ways teachers might access the curriculum and how the design or format used can support teachers' learning can improve the curriculum's feasibility.

Conclusion

When we first began this journey, our plan was to do “all” of the work around understanding early algebra’s impact in one 4-year funded research project. In retrospect, this was far too ambitious. Among other things, we did not appreciate how long it would take to simply develop the tools (e.g., curriculum, assessments) from which we could begin to understand early algebra’s impact. While we have made significant progress, we still have work to do. We learned that with curriculum design (as with research), it pays to pace the work so that it can be thoughtfully carried out in clear, methodical steps. We learned that developing a curriculum can involve taking detours to first do the research needed to understand what the curricular content should be. In closing, we frame what we have learned as a series of questions to be considered when engaging in curriculum development. In particular, what framework will guide how the content of the curriculum is conceptualized? What framework will guide how the curricular components are designed and the research process for its development? What are gaps that need to be addressed in areas such as curricular tools, cognition, and implementation support? For example, are there available assessments for measuring growth in children’s thinking? Are there gaps in the empirical research base on children’s thinking about curricular content? Are there gaps in the learning opportunities teachers will have around implementing the curriculum and, if so, how can the curriculum be designed in ways that support teacher learning?

We continue to be amazed at how young children learn and grow algebraically, including through their experiences with LEAP. We are optimistic that the LEAP curriculum will be implemented in schools in ways that impact students’ opportunities for success in algebra. Most importantly, it is promising that schools now have effective curricular options such as LEAP for improving students’ algebraic thinking. As Jim Kaput might say, that is a happy story.

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Appendix

Sample excerpts of lesson components (*Jumpstart, Launch, Explore and Discuss,* and *Review and Discuss*) taken from Grade 1 lessons about the equal sign (Blanton et al., 2022b). Reprinted with permission from Didax, Inc.

Jumpstart

Complete the Jumpstart.

Display the equal sign symbol. Students should be familiar with it from Kindergarten **LEAP** lessons.

1. Do students describe the equal sign as meaning "the answer" or "the total"? This indicates they see it as an operational symbol and do not yet have a relational understanding. Look for descriptions such as the equal sign means "balance" or "the amounts on either side of the equal sign are the same."
2. Select students to share their equations. Notice whether students' examples are equations in standard form, with operations only to the left of the equal sign. Discuss why an equation with no operations, such as 8 = 8, is valid.

Jumpstart

1. What does this symbol mean?

   =

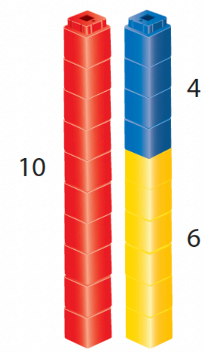
2. Give an example of how you would use it.

Launch

What does it mean for an equation to be true? Do students indicate an equation is true "if you get the right answer"? This indicates operational rather than relational thinking.

Is the equation 4 + 4 = 8 true? Is the equation 4 + 7 = 8 true? Display the equations for students. These will likely be easy for students to answer because the first equation is a doubles fact and both equations are in standard form.

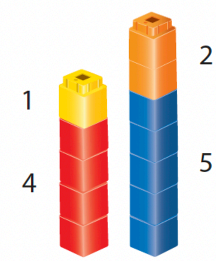
Is the equation 10 = 6 + 4 true? Let's use cubes to think about whether 10 has the same value as 6 + 4. Display the equation. Review the meaning of the equal sign as a symbol that indicates two expressions have the same value. Using language such as "has the same value as" can reinforce this understanding.



With students, use cubes to build a 10-tower and a 6 + 4-tower. Use different colors for each addend to help students visualize the addends.

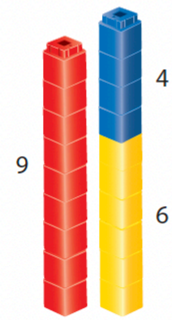
How can we use our towers to see if the equation is true? Do students compare the heights of the towers and notice they are the same? Discuss this strategy.

Is the equation 4 + 1 = 5 + 2 true? Do students think that the equation is false because "you can't have a plus sign after the equal sign"? This type of operational thinking is not uncommon at this point. Framing the question as "Is 4 + 1 the same amount as 5 + 2?" can help students think about the equation relationally.



Ask students how to build towers that represent the expressions in their equation (4 + 1 and 5 + 2). Notice whether they compare their heights. Relate their findings to the equation: 4 + 1 = 5, 5 + 2 = 7, and 5 ≠ 7, so the equation is false.

Explore and Discuss



Place students in partner pairs and give each group a set of cubes. It is helpful to represent a particular addend with the same color of cubes, so be sure students have a sufficient number of cubes of a given color.

Give students an equation and ask them to use the tower strategy discussed in the Launch to determine whether the equation is true or false.

Display each equation, one at a time:

$9 = 6 + 4$

$3 + 2 = 1 + 4$

$8 = 3 + 2$

$3 + 4 = 7 + 2$

> "I built a 9-tower and a 6 + 4-tower. When I held them up beside each other, the 9-tower was shorter. So, the equation is false."

After each equation, select students to share their strategies and the towers they built. Do students represent each expression in the equation with a tower? Do they compare their heights to determine if the equation is true or false? Keep in mind that equations with operations on either side are still new for students, and they might need additional support.

You might use additional equations to give students more practice. Be sure the addends are small enough that students can reasonably represent them with cubes. Use equations where operations are not only on the left side of the equal sign.

Thinking about Student Responses

Students who see the equal sign as an operational symbol might exhibit different misconceptions:

- They sometimes add all the numbers in an equation rather than compare the value of each side. With towers, these students might build a single tower showing all the addends (for example, a $9 + 6 + 4$ tower for the equation $9 = 6 + 4$) rather than compare the heights of the towers representing each expression in the equation.
- They may think an equation such as $4 + 1 = 5 + 2$ is true because $4 + 1 = 5$, essentially ignoring "+2" in the equation. Similarly, they may think an equation such as $3 + 2 = 1 + 4$ is false, since $3 + 2 \neq 1$.
- They may see an equation such as $5 = 3 + 2$ as "backwards" because there is no operation to the left of the equal sign.

Review and Discuss

Describe the strategy we used today to find if an equation is true or false.