Teaching Complex Probability Problems Using Simple Simulations with Applications to the Broken Stick Problem

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Abstract: Probability is generally considered one of the most challenging areas to teach in mathematics education due to its intricate nature. However, the simulation-based teaching method can increase students' accessibility significantly to the probability problems because it enables students to resolve the problems with minimal mathematical skills. By substantially reducing the stress caused by complex mathematical calculations, simulations help students focus on the key concepts of complex probability problems. Furthermore, the programming languages, such as R and Python, can be easily implemented in classrooms to perform simulations that allow students and teachers to discuss the idea to solve the probability problems step by step. This article presents a classical probability problem known as 'the broken stick problem' to show the efficiency of the simulations in teaching probability. It contrasts the effectiveness of mathematical and computational simulation approaches to the solutions of the broken stick problems with several variations. For concrete illustrations, a series of R codes and their examples are provided at the end of the article, which can be used for the simulation study in probability class with various versions of broken stick problems. The extension of the computational approach to other historic probability problems is also discussed.

Keywords: Geometric probability, Acute, Obtuse, Heron's formula, Computational solution

Introduction

Probability and statistics literacy is essential for preparing students for life as informed citizens since it is needed in many decision-making situations such as voting, medical diagnosis, finance and insurance, and forecasting (Gal, 2005). It is important to use the opportunities technology offers for teachers and students to discuss and build models to describe real-world scenarios through simulation (Lee & Hollebrands, 2008). In this article, we discuss the classical probability problem, known as the broken stick problem. We present various versions of the broken stick problem and illustrate both the mathematical solution and the simulation-based approach.

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The broken stick problem originates from an examination of Cambridge University in the mid-19th century (Univ. of Cambridge, 1854). It says,

> *"A rod is marked at random at two points, and then divided into three parts at these points; the probability of its being possible to form a triangle with the pieces is 1/4."*

The exam presented the solution using a direct mathematical approach to solving the inequalities required to form a triangle on the $x - y$ plane. Lemoine (1875) used the combinatorial method with the discrete points evenly spaced on the stick. Poincare (1981) introduced a geometric approach, transforming the sample space of the two random points to the inside of an equilateral triangle. All methods conclude that the probability for the three pieces to form a triangle is 1/4. Since it was introduced in 1854, various versions of the broken stick problem have been proposed: for example,

- What is the probability of forming an acute (or obtuse) triangle?
- What is the probability of forming a triangle with an area less than 1?
- What is the probability of forming a triangle with the largest angle greater than 60°?

The mathematical solutions to the probability problems above can be challenging and inaccessible for high school or college students in introductory statistics courses. However, students can comprehend these problems using experimental approaches, such as the simulation study with the aid of technology. For example, we cannot conduct actual experiments for the broken stick problem, but we can have students do virtual experiments in classrooms with an elementary level of programming skills. The innovations in technology have made programming languages (such as R, Python, and JAVA) accessible to students for free. Furthermore, computers and statistical packages have been recommended as teaching tools (Carver et al., 2016). Using computational tools in probability/statistics class can be very efficient and provide great insight into understanding complex mathematical problems (Benakli et al., 2017; Koparan & Yilmaz, 2015). The simulation-based approach can be an effective alternative to help students without a solid mathematical background understand probabilistic phenomena without becoming lost in the steps of complex mathematical calculations.

Probability Questions on the Broken Sticks

The First Question: Forming a Triangle

Question 1: *A stick, dropped on the floor, breaks at random into three pieces. What is the probability that the three parts of the broken stick form a triangle?*

Let a, b, and c denote the lengths of the three pieces and, without loss of generality, let the length of the stick be one, or $a + b + c = 1$. For the three pieces to form a triangle, the sum of the two sides should be greater than that of the other side (triangle inequality theorem). Namely, for $0 < a,\ b,\ c < 1$,

$$a + b > c, \quad b + c > a, \quad c + a > b. \tag{1}$$

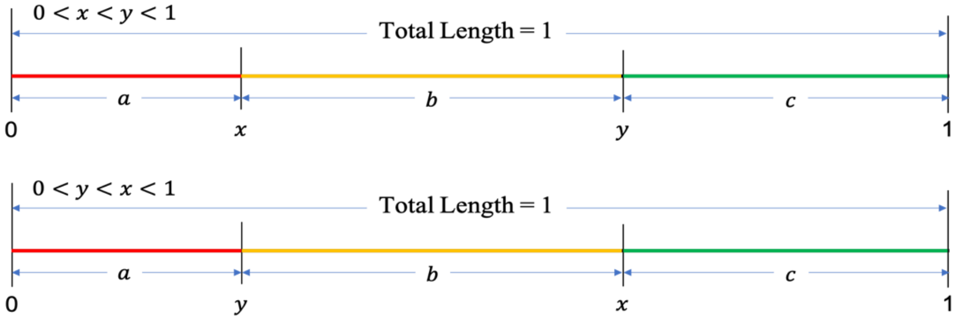

Figure 2. Three pieces of broken stick with the broken points x and y.

With $a + b + c = 1$, the condition (1) can be simplified and represented by one inequality: $\max(a, b, c) < 1/2$. But, throughout this paper, we maintain the original conditions in (1) since it is more intuitive and much easier for students to understand. In Figure 1, the points x and y represent two real numbers randomly selected in the interval (0, 1). Thus, the points x and y can be considered two random points described on the Cambridge University exam aforementioned above. Using two points x and y on the stick, we break the stick into three pieces. Then, the lengths of the three pieces a, b, and c can be written in terms of x and y:

$$a = x, \quad b = y - x, \quad c = 1 - y, \tag{2}$$

provided $0 < x < y < 1$ (see Figure 1). By substituting (2) for (1), we can see the inequalities in (1) are equivalent to

$$y > \frac{1}{2}, \quad x < \frac{1}{2}, \quad y < x + \frac{1}{2}. \tag{3}$$

We can graph these inequalities using the Desmos graphing calculator. The intersection of all three inequalities shows the region where a triangle is formed. Including the case of $0 < y < x < 1$, two regions exist where the three pieces form a triangle (Figure 2). Then, the probability of forming a triangle is the sum of the areas of the shaded regions, which is 1/4, as depicted in the graph.

To answer Question 1, we can also calculate the probability using calculus based on the probability distribution theory. Let E_1 be *an event that the three parts of the broken stick form a triangle*. Since x and y are independent uniform random variables on (0,1), the joint probability density function of x and y is given by $f(x, y) = 1$ with $0 < x < 1$ and $0 < y < 1$. Hence, the probability that the three parts of the broken stick form a triangle is obtained by solving the system of the inequalities (3). Let E_1 denote *the event that the broken sticks form a triangle*. Then

$$Pr(E_1) = 2Pr\left(y > \frac{1}{2}, \; x < \frac{1}{2}, \; y < x + \frac{1}{2}\right) = 2\int_{1/2}^{1}\int_{y-1/2}^{1/2} dxdy = \frac{1}{4}. \tag{4}$$

$Pr(E_1)$ is obtained by multiplying 2 to $Pr\ (y > 1/2, \; x < 1/2, \; y < x + 1/2\ )$ since we have two cases to consider: $0 < x < y < 1$, and $0 < y < x < 1$. Though Question 1 is the simplest case among the broken stick problems, the mathematical solution provided in (4) is not easy to understand unless students have comprehensive knowledge in many mathematic topics such as a system of the inequalities, double integral, probability distribution theory, etc.

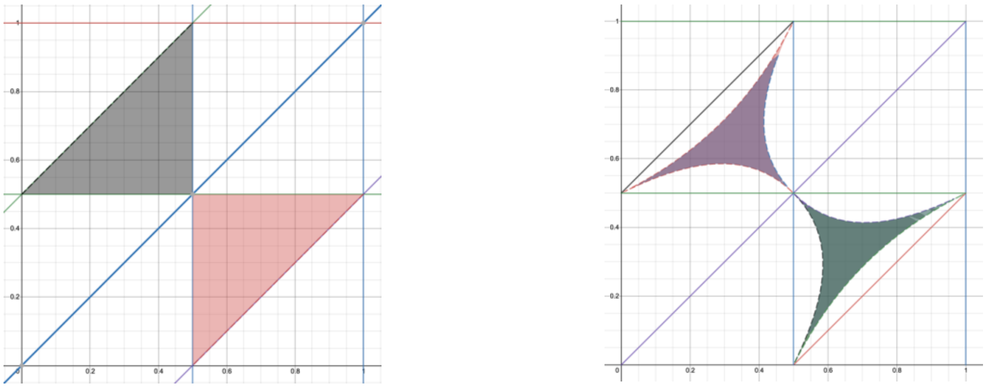

Figure 3. Geometric Representation of Question 1 (left) and Question 2 (right). The Shaded Regions İnclude All of The Pairs (x, y) Forming a Triangle (left) and An Acute Triangle (Right).

Next, we explore an alternative solution: the computational approach. Imagine having a stick with a length of 1 and breaking it at two random points to make three pieces. Since x and y are random numbers between 0 and 1, we have two cases: $0 < x < y < 1$, or $0 < y < x < 1$. Thus, the lengths of the three sides can be formulated by $a = \min(x, y)$, $b = \max(y, x) - \min(x, y)$, $c = 1 - \max(x, y)$ (see Figure 1). After we obtain the lengths of three sides (a, b, and c), we determine whether these three pieces can form a triangle and record the result by checking the inequalities in (1) are true or not (yes or no). We repeat these steps sufficiently many times, say 10,000 times. Among the 10,000 repetitions, we count the number of yeses. Then, the probability is given by the number of yeses divided by the number of repetitions. The whole procedure is summarized in Algorithm 1.

Table 1. Algorithm of The R Code for The Simulation Study: Question 1.

Algorithm 1.
[1] Set $Counter = 0$.
[2] Generate x and y from $U(0,1)$ independently.
[3] Set $a = \min(x, y)$, $b = \max(y, x) - \min(x, y)$, and $c = 1 - \max(x, y)$.
[4] If $a + b > c$ and $b + c > a$ and $c + a > b$, then $Counter = Counter + 1$.
[5] Repeat Step [2] - Step [4] N times.
[6] Result: $\Pr(E_1) = Counter / N$.

In Algorithm 1, $U(0,1)$ denotes a uniform distribution over (0,1). Thus, x and y represent the broken points in

Figure 1. Table 2 shows an R code based on Algorithm 1. Since many online R compilers are available, such as http://makemeanalyst.com/run-your-r-code, https://rextester.com/l/r_online_compiler, https://rdrr.io/snippets, and we can compile this code wherever the internet is available without installing a software for the R compiler.

Table 2. R Code for Question 1.

R code

```
# Step[1] Initialize variables
N = 10000   # The number of repetitions
Counter = 0 # Variable for counting the number of cases to form triangles

# Step[5] Repeat Step[2]-Step[4] N times
 for (i in 1:N){
# Step[2] Generate random variables x and y from uniform(0 ,1)
   x = runif(1,0,1); y = runif(1,0,1);

# Step[3]: Three sides of the triangle
   a = min(x,y)
   b = max(x,y)-min(x,y)
   c = 1-max(x,y)

# Step[4]: Condition for forming a triangle (the Core Part)
   if (a+b>c & b+c>a & c+a>b) Counter=Counter+1
}
# Step [6]: Calculate the probability to form a triangle
Prob = Counter/N
Prob   # Print out the result
```

The core part in the R code above is Step [4], which determines whether the three pieces can form a triangle or not. The condition is the same as (1). One of the advantages of the simulation-based approach is that students can compute the probabilities for different variations of the broken stick problem by modifying Step [4] only. The following sections discuss this point in detail with specific examples.

The Second Question: Forming an Acute Triangle

Question 2: *A stick, dropped on the floor, breaks at random into three pieces. What is the probability that the three parts of the broken stick form an acute triangle*?

Question 2 is one of the extensions of Question 1. For the three pieces to form an acute triangle, the sum of the squares of any two sides should be greater than that of the third side (Pythagorean inequality theorem). That is,

$$a^2+b^2>c^2,\quad b^2+c^2>a^2,\quad c^2+a^2>b^2 \tag{5}$$

and these inequalities can be represented in terms of x and y, where $0<x<y<1$, in the same manner used for Question 1 by using (2):

$$x^2+(y-x)^2>(1-y)^2,\ (y-x)^2+(1-y)^2>x^2,\ (1-y)^2+x^2>(y-x)^2 \tag{6}$$

The area of the region created by the points (x,y) satisfying the inequalities in (6) represents half of the probability of forming an acute triangle since it covers the case $0<x<y<1$ only. By including the case of $0<y<x<1$, we can find two regions where an acute triangle is formed (Figure 2). Then, the sum of the areas of the shaded regions provides the probability of forming an acute triangle. However, it is much more difficult to calculate the sum of the areas mathematically by solving a system of inequalities in (6). The equation in (7) demonstrates how it can be computed using calculus. Let E_2 denote *the event that the broken sticks form an acute triangle*. Then the probability of E_2 is given by

$$\Pr(E_2)=2\int_0^{\frac{1}{2}}\left(\frac{1}{2(1-x)}-\frac{1}{2}\right)dx-4\int_0^{\frac{1}{2}}\left(\frac{2x^2-1}{2(x-1)}-\frac{1}{2}\right)dx=\ln\left(\frac{8}{e^2}\right)=0.0794. \tag{7}$$

In the mathematical approach for $\Pr(E_2)$, it is impossible to avoid using complex integration, which makes students digress from the main idea of the problem. However, when we use the computational methods, we need to add the conditions in (6) only to Step [4] of Algorithm 1 as follows:

Step [4]: Condition for Forming an Acute Triangle (Question 2)
[4] If ($a+b>c$ & $b+c>a$ & $c+a>b$) and ($a^2+b^2>c^2$ & $b^2+c^2>a^2$ & $c^2+a^2>b^2$), then $Counter = Counter + 1$.

The Third Question: Forming an Obtuse Triangle

Question 3: *A stick, dropped on the floor, breaks at random into three pieces. What is the probability that the three parts of the broken stick form an obtuse triangle?*

Let E_3 denote *the event that the three parts of the broken stick form an obtuse triangle*. Using the Pythagorean inequality theorem, the conditions for a triangle with lengths of sides a, b, and c to be an obtuse triangle are

$$a^2+b^2<c^2,\quad b^2+c^2<a^2,\quad c^2+a^2<b^2 \tag{8}$$

or, in terms of x and y, for $0<x<y<1$,

$$x^2+(y-x)^2<(1-y)^2,\ (y-x)^2+(1-y)^2<x^2,\ (1-y)^2+x^2<(y-x)^2 \tag{9}$$

Figure 3 (left panel) presents the graphical representation of Question 3. The shaded parts indicate where an obtuse triangle forms. For example, the shaded parts of the upper triangle denote the region created by (3) and (9). The total area of shaded regions of the left panel in Figure 3 directly gives $\Pr(E_3)$, and can be calculated as follows:

$$\Pr(E_3) = 2\int_{\frac{1}{2}}^{1}\int_{0}^{1-\frac{1}{2y}} dydx + 4\int_{0}^{\frac{1}{2}}\left(\frac{2x^2-1}{2(x-1)} - \frac{1}{2}\right) dx = \frac{9}{4} - 3\ln 2 = 0.1706 \quad (10)$$

Again, we emphasize that $\Pr(E_3)$ can also be obtained without employing the complicated mathematics stated in (10). By only adding the conditions in (8) to Step [4] in Algorithm 1, we obtain the value for $\Pr(E_3)$.

Step [4] Condition for Forming an Obtuse Triangle (Question 3)
[4] If ($a+b>c$ & $b+c>a$ & $c+a>b$) and ($a^2+b^2<c^2$ & $b^2+c^2<a^2$ & $c^2+a^2<b^2$), then $Counter = Counter + 1$.

Remark: $\Pr(E_3)$ can also be calculated using the results of Questions 1 and 2. Since $\Pr(E_1) = \Pr(E_2) + \Pr(E_3)$, we can conclude $\Pr(E_3) = \Pr(E_1) - \Pr(E_2) = 1/4 - 0.07944 = 0.17056$. In the left panel of Figure 3, when a point (x, y) is exactly on the dotted curves, the three sides (a, b, and c) form a right triangle. However, since the area of the region comprising the points on the dotted curves is zero, the probability that the three sides form a right triangle is zero.

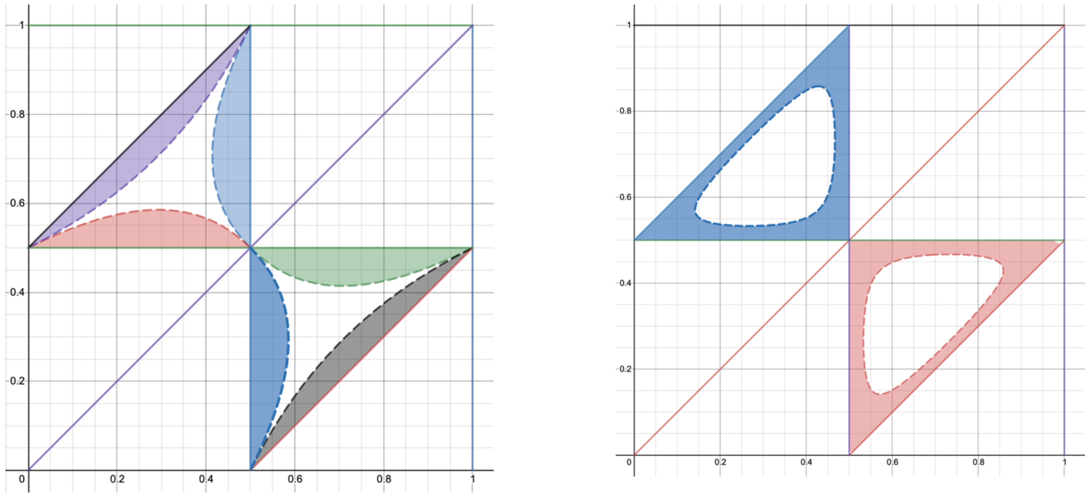

Figure 4. Geometric Representation of Question 3(left) and Question 4 (right). The Shaded Regions İnclude All of the Pairs (x, y) Forming an Obtuse Triangle (left) and a Triangle with An Area < 0.03.

Beyond Forming a Triangle

The Fourth Question: Forming a Triangle with an Area Less Than 0.03

Question 4: *A stick, dropped on the floor, breaks at random into three pieces. What is the probability that the three parts of the broken stick form a triangle with an area less than 0.03?*

Let E_4 denote *the event that the three parts of the broken stick form a triangle with an area<0.03.* Since we have the lengths of the three sides, Heron's formula can be used to solve this problem. With Heron's formula, the area of the triangle is given by

$$Area = \sqrt{\frac{1}{2}\left(\frac{1}{2}-a\right)\left(\frac{1}{2}-b\right)\left(\frac{1}{2}-c\right)} < 0.03 \qquad (11)$$

because $a+b+c=1.$ Moreover, the inequality (11) is easy to demonstrate that this is equivalent to the quadratic inequality for x and y by inserting (2) in (11), for $0<x<y<1$,

$$(2x-1)y^2 + (1-x-2x^2)y + x^2 < 0.2536. \qquad (12)$$

The right panel in Figure 3 displays the region created by the points (x, y) satisfying the obtuse triangle conditions. The upper-left part is for $0<x<y<1,$ and the lower-right part is for $0<y<x<1$. Solving the quadratic inequality (12) for y (using the quadratic formula) results in the complement set of $b_1(x)+b_2(x)<y<b_1(x)-b_2(x)$. Here, $b_1(x)$ and $b_2(x)$ are defined as follows:

$b_1(x) = \frac{2x^2+x-1}{2(2x-1)}$ and $b_2(x) = \frac{\sqrt{(2x^2+x-1)^2-4(2x-1)(x^2-0.2536)}}{2(2x-1)}$.

Hence, with the condition (3), the following algebraic derivation provides the probability that the three parts of the broken stick form a triangle with an area less than 0.03:

$$\Pr(E_4) = \frac{1}{4} - 2\int_{x_1}^{x_2}\int_{b_1(x)+b_2(x)}^{b_1(x)-b_2(x)} dydx = \frac{1}{4} - 2\int_{x_1}^{x_2}\sqrt{x^2 + \frac{0.12^2}{2x-1}} = 0.1156, \qquad (13)$$

where $x_1 = 0.1418$ and $x_2 = 0.4670$, which are the solutions for $b_2(x) = 0$, for $0<x<y<1$.

We can calculate this probability through a simulation by using the condition (11) and modifying step [4] of Algorithm 1 in the following way.

Step [4] Condition for Forming a Triangle with an Area less than 0.03
[4] If ($a+b>c$ & $b+c>a$ & $c+a>b$) and $\sqrt{\frac{1}{2}\left(\frac{1}{2}-a\right)\left(\frac{1}{2}-b\right)\left(\frac{1}{2}-c\right)} < 0.03$,   then $Counter = Counter + 1$

Figure 4 displays a part of the results of the simulation study. With the number of replications large enough, each graph shows the computational solution converging to the theoretical solution of each Question 1-4.

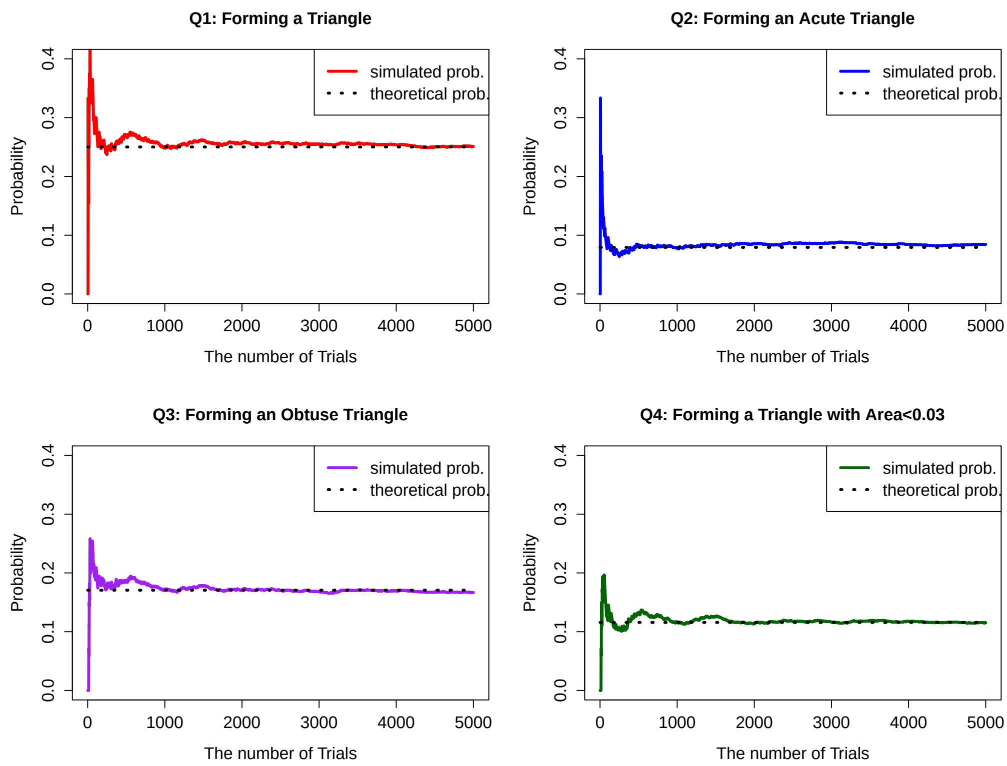

Figure 5: Simulation Results for Question 1-4. The Computational Solutions Converge to Theoretical Solutions

The Fifth Question: Mixed Conditions and Beyond the Broken Stick Problems

Question 5: *A stick, dropped on the floor, breaks at random into three pieces. What is the probability that the three parts of the broken stick form an acute triangle with an area greater than 0.03*?

Question 5 contains the conditions for Questions 2 and 4. Obviously, the mathematical approach for Question 5 is much more intractable. Since it is very challenging to deliver the mathematical procedure for Question 5within a single class period, the mathematical approach tends to make the class inefficient and demotivate the students. However, the computational method does not require substantial mathematical work and knowledge to answer Question 5. It is enough to write two conditions together in Step [4] as follows.

Step[4] Condition for Forming an Acute Triangle with an Area greater than 0.03 (Question 5)
[4] If ($a+b>c$ & $b+c>a$ & $c+a>b$) and ($a^2+b^2>c^2$ & $b^2+c^2>a^2$ & $c^2+a^2>b^2$) and $\sqrt{\frac{1}{2}\left(\frac{1}{2}-a\right)\left(\frac{1}{2}-b\right)\left(\frac{1}{2}-c\right)} < 0.03$, then $Counter = Counter + 1$

Moving from Question 1 to Question 5, we observed the mathematical solutions becoming more complicated and requiring a high level of algebra and calculus. However, the computational approach enables students to find the answers by only revising the mathematical conditions in the R code. Furthermore, the computational approach is much more intuitive since each line of code reveals the corresponding geometric expression directly. For instance, in Question 4, the mathematical solution includes complex mathematical works such as (11) to (13), which are hard to understand for most students in elementary statistics class. In contrast, the computational approach directly states Heron's formula (11) only in code, which is the core formula for this problem. Thus, for the simulation-based approach, we do not need to employ daunting mathematical works beyond Heron's formula to solve Question 4, and students can enjoy the probability problems without unnecessary stress caused by complicated mathematics.

This article illustrated the advantages of the computational approach in teaching probability problems using the broken stick problem only. However, we can apply this approach to the extended areas such as the following:

- Advanced geometric probability: forming quadrilaterals, pentagons, and n-sided polygons with numerous variations (Crowdmath, 2019).
- Historic probability problems
    - Buffon's needle (Buffon, 1777)
    - Monty Hall problem (Selvin, 1975)
    - Gambler's ruin problem (David, 1998).
- Dynamic model simulation: the wildland fire behavior and disease/virus spread.

We can create interesting and exciting classes that motivate students by implementing the computational approach to probability problems. If instructors keep developing simulation examples for various probability problems, we can expect to provide students with more friendly probability classes in the near future.

Conclusion

In modern society, it is crucial to have the ability to solve problems in a multi-dimensional approach. Often, we think of only mathematical rules and formulas to solve problems, but today's advances in technology offer further opportunities to comprehend problems from multiple points of view. In general, solving a probability problem may require advanced knowledge of mathematics, and it is still essential for students to understand and value mathematical solutions. However, using technological tools that are available online and offline, students can solve complex mathematic problems with minimal effort. Moreover, providing students with different approaches to solving problems helps them be more flexible and creative in problem-solving. As shown in 'the broken stick problem', in teaching probability, the simulation-based approach can be an attractive alternative for transforming a heavy and flat math class into an inspiring and exciting event.

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Appendix (R code for simulation study)

```
# A function obtaining the simulated probability -------------------------------

Triangle <- function(n) {
# Generate two random variables between 0 and 1
  v <- runif(n,0,1); w <- runif(n,0,1);

# The Lengths of three broken pieces
  a <- pmin(v,w); b <- pmax(v,w)-pmin(v,w); c <- 1-pmax(v,w)

# Sort a, b, c in Increasing Order
  s1 <- pmin(a,b,c)  # The shortest side
  s3 <- pmax(a,b,c)  # The longest side
  s2 <- 1-s1-s3      # The middle length side
```

```
  s  <- 1/2          # Half perimeter for Heron's formula
Trials<-seq(1,n,1)

# 1. Q1: Forming a Triangle
  count1 <- ((s1+s2)>s3)
  cumul1 <- cumsum(count1)
  Prob1  <- cumul1/Trials

# 2. Q2: Forming an Acute Triangle
  count2 <- (((s1+s2)>s3)&((s1^2+s2^2)>s3^2))
  cumul2 <- cumsum(count2)
  Prob2  <- cumul2/Trials

# 3. Q3: Forming an Obtuse Triangle
  count3 <- (((s1+s2)>s3)&((s1^2+s2^2)<s3^2))
  cumul3 <- cumsum(count3)
  Prob3  <- cumul3/Trials

# 4. Q4: Forming a Triangle with Area<0.03
  count4 <- (((s1+s2)>s3)&(sqrt(abs(s*(s-s1)*(s-s2)*(s-s3)))<0.03))
  cumul4 <- cumsum(count4)
  Prob4  <- cumul4/Trials

Prob=cbind(Prob1, Prob2, Prob3, Prob4, Trials)
Prob      # Return the results
}        # The end of Triangle Function

# Graph function ----------------------------------------------------------------
  Graph   <- function(Prob,Qnum,title){
    color <-c("red","blue","purple","darkgreen","black")
thsol <-c(0.25, 0.0794, 0.1706, 0.1156)

  plot(Prob[,5],Prob[,Qnum], type="l", lty=1, lwd=3, ylim=c(0,0.4),
  xlab="The number of Trials", ylab="Probability", col=color[Qnum],
  cex.axis=1.3, cex.lab=1.3, cex.main=1.3, main=title)
    lines(Prob[,5],rep(thsol[Qnum],n), type="l", lty=3, lwd=3, col=color[5])
  legend("topright", legend=c("simulated prob.", "theoretic prob."),
  lty=c(1,3), lwd=3, col=c(color[Qnum],color[5]),cex=1.3)
}
```

```
# The end of Graph function
#--------------------------------------------------------------------------------

# Execute the functions: 'Triangle' and 'Graph' ---------------------------------
  n     <- 5000        # The number of simulations
  Prob  <- Triangle(n)
#--------------------------------------------------------------------------------

# Print the results out: Probabilities
  paste("The probability to form a triangle is", Prob1[n])
  paste("The probability to form an Acute triangle is", Prob2[n])
  paste("The probability to form an Obtuse triangle is", Prob3[n])
  paste("The probability to form a triangle with an area<0.03 is", Prob4[n])

# Print the results out: Graphs
  par(mfrow=c(2,2))
  Graph(Prob,1,"Q1: Forming a Triangle")
  Graph(Prob,2,"Q2: Forming an Acute Triangle")
  Graph(Prob,3,"Q3: Forming an Obtuse Triangle")
  Graph(Prob,4,"Q4: Forming a Triangle with Area<0.03")
```