Key Shifts in Thinking in the Development of Mathematical Reasoning

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This symposium will draw on the evidenced-based learning progressions for multiplicative thinking, algebraic reasoning, geometrical reasoning, and statistical reasoning presented at previous MERGA conferences (see references by symposium authors in the papers that follow). The four papers will consider key shifts in thinking identified within each progression, without which students' progress may be seriously constrained.

Paper 1: *A Disposition to Attend to Relationships: A Key Shift in the Development of Multiplicative Thinking*
[Dianne Siemon]

This paper draws on multiple data sources to better understand the shift from additive to multiplicative thinking, which is crucial to all further participation in school mathematics.

Paper 2: *Key Shifts in Students' Capacity to Generalise: A Fundamental Aspect of Algebraic Reasoning*
[Max Stephens, Lorraine Day, & Marj Horne]

This paper will elaborate five levels of algebraic generalisation and two key understandings based on an analysis of students' responses to RMFII algebraic reasoning tasks.

Paper 3: *Cognitive Flexibility and the Coordination of Multiple Information in Geometry and Measurement*
[Rebecca Seah & Marj Horne]

This paper analyses students' solutions to problems in geometry and measurement situations in order to identify key components needed to nurture reasoning.

Paper 4: *Facilitating the Shift to Higher-order Thinking in Statistics and Probability*
[Rosemary Callingham, Jane Watson, & Greg Oates]

Students have difficulty moving from concrete representations and procedural mathematical statistics to context-based appreciation of data. This paper examines the barriers to this shift to higher-order thinking based on the Statistical Reasoning Learning Progression.

2022. N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys* (Proceedings of the 44th Annual Conference of the Mathematics Education Research Group of Australasia, July 3–7), p. 41–57. Launceston: MERGA.

Cognitive Flexibility and the Coordination of Multiple Information in Geometry and Measurement

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Building from the evidence-based learning progression in geometric reasoning from the RMFII project, this paper presents data from students' solutions to three problems in geometry and measurement situations to identify key components needed to nurture reasoning. To show emerging analytical reasoning students must coordinate multiple pieces of information and demonstrate cognitive flexibility in their use of visualisation, diagrams, language, and symbols.

Understanding, fluency, problem-solving and reasoning are an integral part of becoming numerate. Good problem solvers exhibit cognitive flexibility, the ability to coordinate number skills, visual-spatial and other cognitive processes such as organising multiple pieces of information (Ionescu, 2012). Given the considerable difficulty Australian students face with solving problems and justifying their mathematical thinking (Thomson et al., 2017), we seek to identify key components needed to nurture reasoning. Geometric reasoning is the ability to critically analyse axiomatic properties, formulate logical arguments, identify new relationships, prove propositions, and used geometric knowledge in solving measurement problem situations (Seah & Horne, 2021b). A draft learning progression was developed based on Battista's (2007) exposition of Van Hiele levels of geometric thinking. Analysis of student data produced an evidenced based learning progression comprise of eight thinking zones: Zone 1: Pre-cognition; Zone 2: Recognition; Zone 3: Emerging informal reasoning; Zone 4: Informal and insufficient reasoning; Zone 5: Emerging analytical reasoning; Zone 6: Property-based analytical reasoning; Zone 7: Emerging deductive reasoning; Zone 8: Logical inference-based reasoning. We analyse student work in depth to determine how to nurture increasingly sophisticated reasoning from informal (Zone 3) through to emerging deductive reasoning (Zone 7).

Method

The data source used for this analysis is taken from the Reframing Mathematical Future II project. The results of these findings have been published elsewhere. Our aim here is to identify significant changes in student thinking by finding factors that cause a shift from Zone 3 to Zone 7. We do this by analysing students' responses to three tasks: 1) reasoning about nets (Seah & Horne, 2020), 2) making deductions of angle magnitudes (Seah & Horne, 2021a), and 3) enlarging a logo and determining its area (Seah & Horne, 2021b) (Figure 1). The geometric contexts of these tasks allow students to demonstrate their knowledge and understanding. The reasoning required for the net task is Zones 2, 3 and 5. The angle magnitudes task is Zones 2, 5, 6, and 8. The logo drawing task is Zone 3 and 5. The logo area task is Zones 4 and 7.

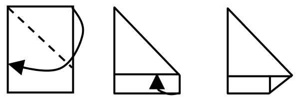
In designing the rubric, we determined that a zero score is given for no response or irrelevant responses. A '1' score denoted some recognition of the concepts but not full application. A maximum score would be given for a correct response with sound reasoning. Scores in between, the number of which depended on the complexity and the context of the task, would be given for partially correct answer and reasoning. For example, GCRD1 requires either a correct or incorrect enlargement logo drawn so the ceiling score is 2. Conversely, it was possible to get some of the angle magnitudes (GANG4) correct and give partial reasoning, thus requiring more gradation with a score of 4 being the ceiling. The data analysed came from students in 12 trial schools and 32 project schools.

2022. N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys* (Proceedings of the 44th Annual Conference of the Mathematics Education Research Group of Australasia, July 3–7), pp. 50–53. Launceston: MERGA.

GNET 4. Sam thinks he has drawn a net of a cube using six squares but it does not fold up to make a cube. What might Sam's drawing look like? Explain how you know.

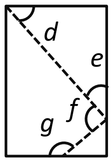
Geometric Angles 2

A four-sided shape is folded from a sheet of A4 paper using the following instructions.



a [GANG3]
What is the name of this shape?
____
Explain your reasoning.

b [GANG4]
Unfold the paper and find the size of each marked angle.
Angle *d* = ____________ Angle *e* = ____________
Angle *f* = ____________ Angle *g* = ____________



Explain your reasoning.

LOGO

A designer draws a triangular logo on grid paper. He wants to enlarge the logo so the sides are twice as long.

a. [GCRD1] Draw his enlarged logo on the graph.

b. [GCRD2]. Write the coordinates of the corners A', B', and C' of the new large triangle**:**

c. [GCRD3] If the area of the original logo is 2.25m^2, what will the area of the new logo be? Explain how you know?

Figure 1. Sample of assessment tasks on geometric reasoning.

Findings

Overall Results

Students' responses to the tasks reflected not only their ability to reason, but the extent of the task requirement and the exposure they had with the concepts. As shown in Table 1, by the number of no responses and correct responses received, the GNET task was the easiest whereas GCRD3, which required students to find the area of the enlarged shape, was the hardest.

Table 1
Breakdown of Student Responses for Each of the Questions (percentage)

	GNET4		GANG4		GCRD1		GCRD3	
	Trial	Project	Trial	Project	Trial	Project	Trial	Project
Score	*n = 233*	*n = 566*	*n = 157*	*n = 270*	*n = 118*	*n = 328*	*n = 118*	*n = 328*
0	13.2	9.1	38.5	16.3	17.8	30.8	37.3	47
1	11.4	10.7	28.9	27.4	53.4	38.4	52.5	47
2	36.5	30.4	18.6	19.6	28.8	30.8	1.7	2.1
3	38.9	49.8	10.9	22.6			8.5	4
4			3.9	14.1				

In GNET4, 48% of the students used just the information in the question by either drawing six squares that would fold into a cube or drew a correct shape but did not provide a reason. In the trial data, 39% of students gave a correct response. This improved in the project data. Students who gave a correct reason went beyond the information given in the question and called on other knowledge, such as visualising the nets from different perspectives. Compared to GNET task, the number of no response or irrelevant responses was higher in the GANG4. Around 29% of the students showed partial angle knowledge by providing a label (e.g., acute, or obtuse) or recognising one angle magnitude; 19% showed emerging analytical reasoning giving two angles correctly, with some explanation; and 11% trial and 23% project students correctly calculated the angles giving some reasons though often sparce. Logical inference-based reasoning, albeit about a simple situation, was shown by 4% and 14% of trial and project students respectively who reasoned correctly and deduced all angle magnitudes.

In the logo task, 18% of the trial students did not draw an enlarged logo and 37% did not attempt to calculate the area. More than half of the students (53%) operated within the information given in the question by drawing a larger logo in some form although incorrectly either by enlarging one dimension only or a larger logo with no attention to the magnitude of the enlargement. A similar number (52%) gave a response to the area question that was incorrect, often just using the numbers given in the question by doubling 2.25 and did not provide units or gave little reasoning. Around 29% correctly enlarged the logo and just over 2% were able to give a correct area measurement, often using a procedural explanation. Just over 8% were able to reason correctly, giving an explanation that recognised that doubling the length of all the sides quadrupled the area, thus showing emerging deductive reasoning.

Types of Reasoning

Table 2 shows the responses to the three questions. The questions are shown with the score given following the dot so that GANG4.1 means a sore of 1 on the question GANG4.

Table 2
RMFII Zones of Geometric Thinking

Zone 2. Recognition	GNET4.1	GANG4.1		
Zone 3. Emerging informal reasoning	GNET4.2		GRD1.1	
Zone 4. Informal and insufficient reasoning			GRD3.1	
Zone 5. Emerging analytical reasoning	GNET4.3	GANG4.2	GRD1.2	
Zone 6. Property-based analytical reasoning		GANG4.3		
Zone 7. Emerging deductive reasoning			GRD3.2	GRD3.3
Zone 8. Logical inference-based reasoning		GANG4.4		

We can see that student responses to these three questions spread across the zones of reasoning. For GNET, the move to analytical reasoning appeared to occur with a response scored of 3. The two student responses in Figure 2 demonstrate this. Student A used recognition of a taught prototype. Student B used visualisation and then used a combination of diagram and language to explain the image in their mind and hence their reasoning.

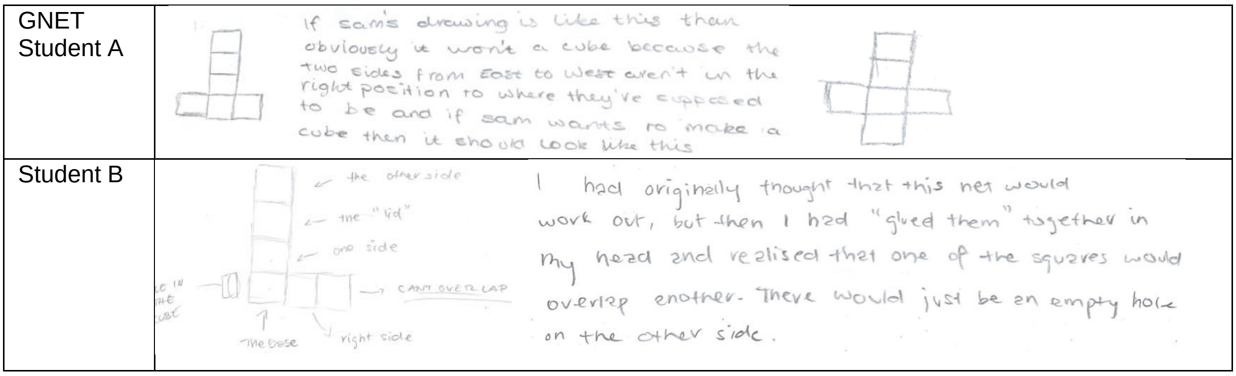


Figure 2. Students' responses on the GNET4 task.

In GANG4, analytical reasoning emerged with a response score of 2 where students gave partially correct answers (usually 45° with no explanation). Some students were starting to make connections but tended to explain using benchmarks such as 90°, as demonstrated here by student C who used no diagrams.

Student C: *d* and *e* has the same size angle as you can see, *f* as everyone knows that it is 90° because it's a right angle and *g* is an obtuse, which is 180° (wrote 45°, 45°, 90°, 180°).

Limited ability to explain, use diagram effectively and present a sequential argument show clearly in the attempts of the students. The few students who were able to reason deductively justified 45° as half of the corner right angle and calculated the 135° either by using the interior angles or the straight angle with 45°. For GCRD3, student 10JW27701 shows an attempt to calculate area but is just using the numbers given in the question rather than demonstrating analytical reasoning in the solution (Figure 3). Meanwhile, student 10YL4700 demonstrates sound deductive reasoning showing explanations both algebraically and in words.

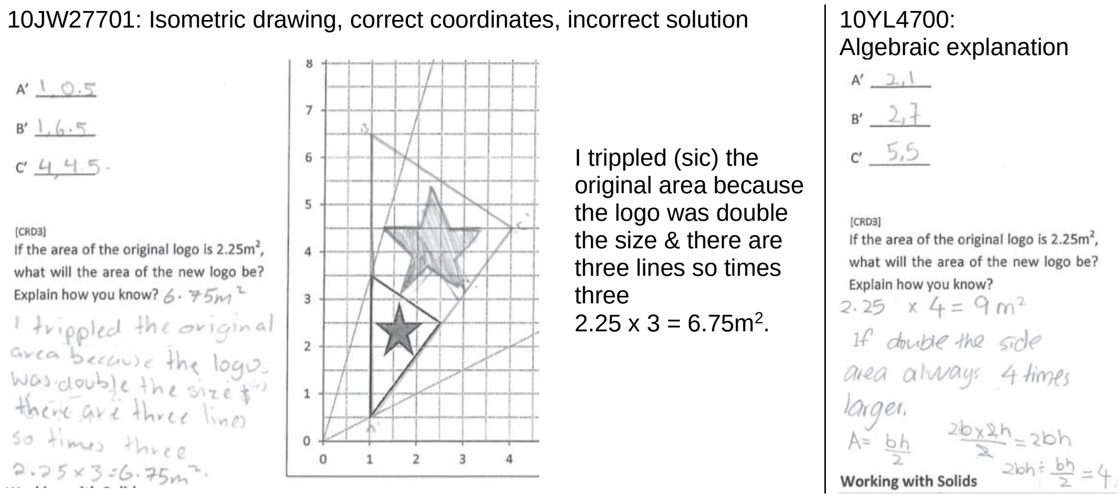


Figure 3. Students' responses on the GCRD task.

To reason analytically or deductively, coordination between the information presented in the question with the network of one's own conceptual understanding is needed. While knowing the mathematical concepts is important, the results here demonstrate that students needed to visualise the problem in situ, coordinate the information in the question with their prior knowledge to obtain a solution and present their argument using diagrams, language, and symbols flexibly. Finally, they need to be able to check that their reasoning is sound. In short, they need cognitive flexibility. These things need to be explicitly in the curriculum. At the moment, visualisation and the flexible use of communication tools is absent.

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