Aligning Mathematical and Musical Linear Representations to Support Fractional Reasoning

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This paper describes the authors' journey in designing a linear representation that aligns mathematics and music to support fractional reasoning. The three authors, guided by the theoretical framing of realistic mathematics education, engaged in a task design process over a 12-month period to develop integrated resources. Data were collected in the form of Zoom meeting recordings, among other sources. This paper describes some key findings from a narrative analysis of this process. We posit that the contribution is twofold—methodological insight of a task design journey and findings around the obstacles and resolutions of aligning representations across mathematics and music, despite the obvious confluence.

The focus of this paper is the development of a key representation modelling the integration of music beats per bar and note values and fractions on a number line. Guided by principles of realistic mathematics education (RME), this linear representation was intended for use in supporting teachers' and their Grade 4–6 students' (ages 9–12 years) mathematical problem-solving across multiple constructs of fractions (a gap the authors identified in literature). This paper describes how the first author, a doctoral candidate, together with her two supervisors, embarked upon an exploratory research journey in search of a potential mathematics-music confluence. The journey involved 12 months of task design and materials development work. The tasks have been shared with ten mathematics teachers in two primary schools in the Eastern Cape province of South Africa.

Initially, identifying synergies between the mathematics and the music seemed straightforward. As the process unfolded, however, seeking alignment was not as simple as we had first anticipated. There are some obvious links, such as note values being described as fractions (for example, half notes and quarter notes). Such links do not readily translate, however, into an alignment of key representations such as the number line and the music bar. This prompted the research question: "How can mathematical and musical linear representations be aligned to support fractional reasoning?". We present a narrative analysis of the task design process as evidenced by a data set comprising journaling, e-mail communications, recordings of our Zoom meetings, trialled representations, and field notes on teacher feedback. Review of these data provide insight into the grappling that occurred during this methodological process, linking well to the MERGA44 conference theme: *Mathematical Confluences and Journeys*, and foregrounding our methodological journey. We share the rigour and structure through which we analysed the process of integrating fractional concepts in mathematics and music. The empirical contribution is, we believe, the identification of certain obstacles that may arise when designing tasks that integrate mathematics with another subject area (music, in the case of our paper).

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Literature Review

Multiple Constructs of Fractions

Fractions, an integral part of most mathematics curricula, are challenging to teach and learn. One of the reasons suggested for this challenge is the complex nature of multiple, and interrelated, constructs of fractions: fraction as measure, fraction as quotient, fraction as ratio, fraction as operator, and the part-whole fraction model (Siemon et al., 2015). In many primary-level mathematics classrooms, the part-whole construct is all too often the main, or only, focus. A single context or problem scenario is seldom considered from the perspective of multiple, interrelated constructs. Luneta (2013) argues that these multiple constructs of fractions, while derived from the ways in which they were used, should not be taught as discrete categories. Rather, students should be allowed opportunities to experience the multiple meanings of fractions so as to make sense of the same situation in various ways (Lamon, 1999). A further complexity is that, when working with fractions, the unit changes depending on the context in which the fraction is being used, creating challenges when trying to make sense of fraction representations (Lamon,1999; Siemon et al., 2015). The representations discussed in this paper draw principally on two constructs of fractions: fraction as measure (measures or distances from a given point (0) on the same scale, such as a number line) and fraction as ratio (the relationship between two quantities) (Siemon et al., 2015). We explore how the mathematics-music connection may provide rich opportunities for students to experience moving across these multiple constructs of fractions.

Task Design to Explore the Mathematics-music Connection

Literature on the connection between mathematics and music describes some of the ways in which the integration of the two can benefit education. Geist et al. (2012) note, for example, that musical elements such as rhythm, melody, pattern, and beat are known to aid mathematics learning. Benefits of such integration include increasing student motivation and engagement, as well as decreasing anxiety (Edelson & Johnson, 2003; Lovemore et al., 2021). Several studies have connected music note values with the part-whole construct of fractions. Both Courey et al. (2012) and Azaryahu et al. (2019), for example, explore ways to pedagogically use the connection between music note value names (e.g., half note) and their corresponding fraction (e.g., one half). In contrast, Cortina et al. (2015) suggest the design of tasks that instead encourage a fraction as measure construct. The current study seeks to find ways in which mathematics-music integration can support not only the part-whole construct, but also the fraction as measure and fraction as ratio constructs.

Graven and Coles (2017) describe task design as a process through which researchers and teachers develop tasks for a specific mathematics concept, learning trajectory, or group of students. Tasks are therefore designed to create particular opportunities for learning, and manipulatives are carefully selected (Jones & Pepin, 2016). Choy (2016), in studying teachers' design of fraction tasks, explains that task design is a deliberate, careful practice and should provide opportunities to work flexibly between multiple representations of a concept. Tasks for the current study have been designed to guide students in interpreting multiple representations and deepening conceptual understanding of a musical bar and fractional reasoning. Ainley et al. (2015) argue that the question, "Is this task purposeful for learners?" is important for designers to answer (purposeful here referring to an "engaging challenge for the learner within the classroom context" (p. 406) as opposed to a necessarily real-world application).

Theoretical Frame and Methodology

In grappling with the demands of integrating the mathematical and musical representations, the authors looked to RME for their theoretical framework. A primary principle of RME is the recognition of mathematics as a human activity (Freudenthal, 1991). Cobb et al. (2008) identify three key tenets of RME, the first being that the design of an instructional sequence should have an "experientially real" starting point for students in which they can actively engage in a meaningful way (p. 108). This, as van den Heuvel-Panhuizen (2003) explains, may be a mathematical activity which is realistic in daily life, or it may be a problem situation which students can "imagine", provided it is a context which is *real* in the students' experience (p. 10). This aligns with Ainley et al.'s (2015) considerations of task design. Cobb et al.'s second principle of RME is that students' informal reasoning, speaking and symbolising, established from the *real* context, should allow for progression to a more formal mathematical reasoning. And finally, Cobb et al.'s third principle is that mathematical activities should be designed in a way that supports the "process of vertical mathematisation" (2008, p. 109). This third principle was what particularly guided our end point: Students representing and working with formal mathematical modelling of fractions on a number line.

As noted, this paper reports on the initial phase of the first author's doctoral study: The task design phase of developing integrated strategies for teaching fractions and music ahead of sharing the strategies and related resources with the participating teachers. We, the three authors, are also participants in the task design process. Data were collected over a 12-month period, and included 15 hours of Zoom meeting recordings, 77 email threads, WhatsApp communications, reflective research journal entries, and field notes from meetings with the participating teachers. Ethical considerations were upheld via gatekeeper permission and informed consent.

Data analysis started as an inductive process. Data were reviewed and recurring patterns were noticed (McMillan, 2010). Segments from the transcriptions and recordings were identified as critical moments, pointing to a pattern of *obstacle—resolution—obstacle—resolution*. For each of these, the following set of three key questions came to the fore: (a) How does the task maintain the *fidelity* of the mathematics, the music, and the integration of the two? (b) How does the task *simplify* the complexity of the integration for subsequent implementation within a classroom setting? (c) What *key representation* would best support conceptual clarity? These align with the acknowledgement made by Courey et al. (2012) about the importance of designing lessons in which non-musician teachers can "deliver the intervention with fidelity" (p. 253). Categorising strategies were then used to group and compare the descriptive data. Aware of the limitation of categorising strategies (referred to by Maxwell (2013, p. 112) as "analytical blinders"), the authors also made use of connecting strategies to understand the grouped data in context and to recognise relationships between categories. This allowed for an in-depth narrative analysis and presentation of the task design grappling journey towards aligning the mathematical and musical representations.

Findings

We provide, in this section, a summary of the key obstacles and resolutions, plus describe in more detail one cycle illustrating the process of aligning the number line and music bar representations. We show how the questions of maintaining fidelity, simplification of complexity, and identification of key representations were addressed. Nine groupings of obstacle-resolution cycles were identified; the first being the selection of a starting problem scenario which, in line with RME principles, is *experientially real* for students. The chosen story-based scenario of African animals crossing a river allowed for children to jump across a constant distance, in a constant time, but with different rhythms. As is shown in Figure 1, below, this then led to a link between the river-crossing and a music bar.

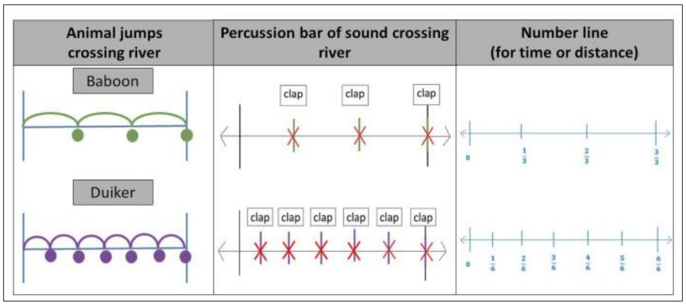

Figure 1. Integrated representations of animal river crossings, music percussion bars and number lines (adapted from Lovemore et al., 2022).

We made the decision that the unit in this scenario could be either the river-crossing (distance and time) or the music bar, so allowing for developing reasoning around both the fraction as measure (measure of time and distance) and the fraction as ratio (jumps per river-crossing or beats per bar) constructs. We saw the opportunity to align the visual representations of the animal river-crossing jumps, the music percussion line, and a number line (as is illustrated in Figure 1, above). Our grappling, informed as it was by the teachers' trialling and feedback, sought to address teachers' expressed concerns about their lack of prior musical knowledge and a lack of confidence to integrate music and fractions.

Obstacles of Aligning the Number Line and the Music Bars

We established that the whole (unit) should be clearly specified across contexts. The decision was that the river-crossing was one unit (of distance and time), and the music bar, similarly, would be one unit. The imagined distance over which the river-crossing stretched, or the time it took for an animal to jump across, could vary, just as the visual representation of distance of a musical bar or the time a bar takes to be played can vary. However, when aligning the music bar with a number line, the music notes are visually placed in the middle of the bar. This, we noted, could cause confusion as it did not align with the time and distance markers on the number line. There is a discrepancy between what one *hears* in music and how one traditionally *notates* the musical note values. In music, beats per bar are used to keep time. One would often hear musicians counting "1234, 1234," for example. In terms of a time progression (measurement of duration of time), for example, one would start at the zero seconds and hold a note for one second (0 to 1). However, this holding of the note would be counted as 1, with the 1 being said at the start of the note or the clap (the mathematical 0 point in time). A further complexity is that visually notated, the note count 1, time 0 to 1 second, is represented in the middle of the music bar, not at the starting line. So, for example, as shown in Figure 2, a whole note played for 4 beats is placed in the middle of the bar in music, yet the note is played (held) from the start of the bar to the end of the bar. Similarly, two half notes could be placed at one third and two thirds of the distance of a bar – yet these placements at thirds do not send the message that the first note needs to be played from the start of the bar to halfway (2 beats – half of 4 beats per bar) and the second is played from the middle to the end of the far (for another 2 beats).

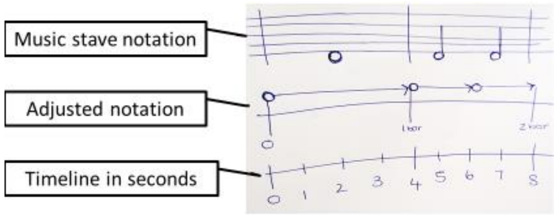

Figure 2. Representing the start and duration of a note value aligned with a timeline.

Merging of fractions on linear representations with musical representations could thus potentially create misconceptions or confusions. As is shared below, these sorts of nuanced differences resulted in much grappling for the authors.

> TL: What I'm seeing here, Mel, is where you are placing the crotchet [quarter note] or the note, it's fitting in between the lines on the timeline.
>
> MG: Sticking it between could be confusing. I think you have to actually do it on the seconds.
>
> TL: But in music we don't. In music we draw them [in the middle] …
>
> MG: Uhm, when we've got our second timeline … and when we play our music we play our crotchet [quarter note] at the start and it lasts one second …
>
> TL: So are you starting on the zero of the timeline? The first [note]?
>
> SR: You measure the second after its done. The note you measure as it starts.
>
> MG: So in relation to the timeline, we start at zero, instantly we start, 1234. And then it lasts that long.
>
> TL: So are we moving away from this idea where we place it in the middle?
>
> MG: Yes.
>
> SR: So you can't exactly align the two number lines. [Zoom meeting: 2021-04-12]

After much further deliberation, we concluded that the traditional notation of the music and the number (time) line could not exactly align. This would compromise the fidelity of both. We therefore decided to trial the adjusted notation shown in Figure 2, above, using a note value visually represented on the zero of the timeline that indicated the duration of that particular note would be four seconds. Due to concern expressed by participating teachers about their lack of confidence in working with western note value notation, we recognised that the representation in Figure 2, above, had to be reconsidered. In addition, the note values held some limitations of only working in circumstances where the music bar was written in a 4/4 time signature, meaning that each bar would consist of four beats per bar. This led us to a change of focus, looking now at the integration possibilities of fraction as ratio (beats per bar or jumps per river-crossing). After further task design grappling, we decided to use percussion beats (Xs) on a percussion line. This seemed a more obvious confluence with a number line (see Figure 1, above). This, however, led to fresh obstacles of fidelity in the musical and mathematical representations. Where, for example, would we clap together? In music, if two musicians clapped rhythms of different note values (e.g., if Musician A were to clap three beats per bar and Musician B, 6 beats per bar), their first clap together would be on count 1. However, in mathematics, the notion of equivalence would mean that Musician A's first clap (of three) would align with Musician B's second clap (of six), because $\frac{1}{3} = \frac{2}{6}$. Once again, we recognised the potential for misconceptions to occur if we were to overlay these representations. Almost a whole year after the first representations, through grappling with the confluences and contrasts, we eventually decided that we did not need to superimpose the mathematical number line representation and the music bar representation, a decision captured in the discussion below.

> TL: Yes, and we're trying to represent what we hear, to make it look the same, but it's not. We don't actually have to put the note, the X, on the number line.
>
> MG: So what we need to do here, is to draw some distinction between what we're going to do with the number line, when we're thinking distance, time, where there was a starting point of zero, whereas with the percussion we stick it in the middle of the line… We can see the similarity but we can see what's different as well… because we keep looking at this bar line and we see the similarities. But we're conflating two concepts. [Zoom Meeting: 2022-01-25]

Once this recognition of difference was established, we were able, as we discuss in our next section, to develop linked key representations, maintaining the fidelity of mathematics and music and simplifying the complexity to support moving between the constructs of fractions.

Resolution for Aligning the Number Line and the Music Bars

Having been freed from our self-imposed burden of superimposing the similar but different representations, we could focus on finding an alignment between the representations that built on the original animal river-crossing problem scenario. A key realisation was that it was not necessary to conflate the fraction as measure (measure of time and distance) and the fraction as ratio (beats per bar or jumps per river-crossing) concepts. Rather, the scenario allowed for moving flexibly *between* these interrelated concepts.

> MG: I think a big AHA is here, when we dealing with percussion claps we're dealing *4 beats per bar*— that is a *rate (emphasis).* We don't normally put rate on a number line. This is a different concept of fraction as ratio to what we're going to do on the number line. When we say how long does it take to jump? That's a length of time, that is fraction as measure. If the river-crossing is 1 unit that's fraction as measure. [Zoom meeting: 2022-01-25]

RME principles encourage using students' informal models of problem situations and then guiding them through the process of reinventing to arrive at formal representations (van den Heuvel-Panhuizen, 2003). These principles guided us in our design of the key representation and task: Students represent animal river-crossing jumps on a mathematical number line, where the unit is a river-crossing. Due to complexities of musical notation, we noted that it would need to be specified to students that they could make use of certain animals jumps aligning with musical note values i.e., in this scenario kudu jumps (1 whole jump per river-crossing is equivalent to whole notes in music), ostrich jumps (2 jumps per river-crossing are equivalent to half notes in music), zebra jumps (4 jumps per river-crossing are equivalent to quarter notes in music) or monkey jumps (8 jumps per river-crossing are equivalent to eighth notes in music). To simplify the complexity, we explained to the participating teachers that this would work for a 4/4 time signature in music, but that other, more complex, options were also possible. The first author designed a set of resources where students would be given musical note values printed on transparent cards, cut to exact size where one whole note card will fit into one whole music bar, two half note cards would fill the measure of one whole bar, and so on. This was designed to indicate the similarity between the animal jumps and the music note values, where the measure of time is considered. Students would be able to choose where they place the pitch of the note values, as long as their notes fitted into the music bar.

The resource was then further developed to take the form of a triple number line, where the fraction as measure construct could be developed. Here, the number line with the river-crossing units would remain constant. Above it would be a number line indicating distance, and below it a timeline indicating the duration of the animal jumps or the music notes. The units of the distance number line and the time number line could signify different variables creating powerful opportunities for problem solving (e.g., If the river is 10 metres wide and it takes all animals 4 seconds to cross then … etc.). Figure 3, below, provides an example of the key representation being used to compose music. The resource has a musical stave represented above the number line, not to superimpose the two, but to indicate the alignment. The figure shows how we envisaged the representation being used to solve complex problems, moving between the fraction as ratio and fraction as measure constructs.

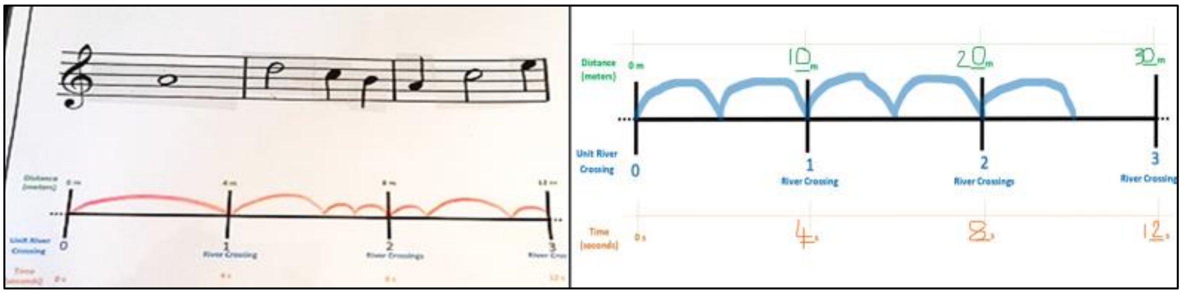

Figure 3. Example of the key representation being used to 'compose' music and the triple number line used to solve complex fractional problems.

Next, we discussed possible questions that could be posed to students: *Ostrich does two jumps per river-crossing. If one river-crossing is x metres, and it takes y seconds to cross, and ostrich jumped 5 jumps, how far would ostrich be jumping in metres? How long did it take Ostrich to get there? At what speed did Ostrich travel?* We recognised that students would be able to use the triple number line representation to support them in tackling such problems. In reflecting on the above representation, the third author commented,

> The power here is that you bring multiple aspects of fractional reasoning into play. The learning here is about shifting attention between different wholes and proportional ways of working. That's the power of this and integrating into music. [Zoom meeting: 2022-01-25]

Once we were satisfied that this key representation maintained fidelity (by aligning, but not overlaying, the linear mathematical and musical representations), we prepared sets of simplified resources to share with the participating teachers.

Conclusion

This paper has described the task design journey of searching for ways to align linear mathematical and musical representations to facilitate students' solving of problems which required moving flexibly between the constructs of fraction as measure and fraction as ratio. The process of grappling throughout this journey was meticulously documented and patterns of obstacle-resolution were identified. Three key elements guided the task design journey: Maintaining the fidelity of the mathematics and the music within the integration; simplifying the complexity of the integration for classroom implementation; selecting and designing key representations to best support conceptual understanding. In answer to our research question: "How can mathematical and musical linear representations be aligned to support fractional reasoning?", we conclude that rather than superimposing the mathematical and musical linear representations, a key presentation aligning a musical stave above a triple number line, can simplify the complexity while maintaining fidelity to support fractional reasoning. We anticipate that our findings will resonate with other researchers, task designers, or teachers integrating mathematics with other art forms or subject areas.

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