RESEARCH PAPER

Diagrams in Mathematics: What Do They Represent and What Are They Used For?

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Diagrams are succinct, visual representations of real-world or mathematical objects that serve to communicate properties of the objects and facilitate problem solving. This study explored perceptions of mathematics teachers in Singapore, who are heads of mathematics departments in their respective schools, related to diagrams and their use in the teaching of mathematics. An open-ended survey was adopted to illicit responses to i) what is a diagram, and ii) when do you use diagrams in your mathematics instruction. The findings of the study show that participants generally viewed diagrams as visual representations of real life or mathematical objects and they used them mainly as visual aids when illustrating mathematical concepts and relationships.

The revised school mathematics curriculum for secondary schools in Singapore places emphasis on Big Ideas in School Mathematics (BISM) (Ministry of Education, 2018). One BISM is diagrams for representation and communication of mathematical objects. Though in the ‘eyes’ of mathematicians and curriculum developers the role of diagrams may be apparent the same may not be for mathematics teachers in general. To facilitate development of mathematics teachers enacting the revised curriculum in Singapore secondary schools a study was carried out amongst school leaders, specifically the heads of mathematics departments from twenty secondary schools to ascertain their understanding of diagrams specific to the teaching and learning of mathematics.

The Study

Diagrams in Mathematics

> Diagrams are succinct, visual representations of real-world or mathematical objects that serve to communicate properties of the objects and facilitate problem solving. For example, graphs in coordinate geometry are used to represent the relationships between two sets of values, geometrical diagrams are used to represent physical objects, and statistical diagrams are used to summarize and highlight important characteristics of a set of data. Understanding what different diagrams represent, their features and conventions, and how they are constructed helps to facilitate the study and communication of important mathematical results. (Ministry of Education, 2018, p. S2-5)

For the purpose of this study, we adopt Winn’s (1987) definition that a diagram is a 2-dimensional, visual representation, that exploits spatial layout in a meaningful way, enabling complex processes and structures to be represented comprehensively. In the teaching and learning of geometry, diagrams are simultaneously concepts and spatial representations of abstract ideas (Gagatsis et al., 2010). This facilitates the construction, argumentation, and understanding of geometrical ideas (Dimmel & Herbst, 2015). Diagrams have also been the oldest form of communication to convey formally or informally mathematical concepts and proofs (Cellucci, 2019; Shin et al., 2018). In mathematical proofs, diagrams may illuminate key inference steps, but they do not replace the rigour of axiomatic presentations (Hilbert, 2004; Shin et al., 2018). However, the heuristic conception of diagrams (Cellucci, 2019), stemming from the analytic method of mathematics involving both deductive and non-deductive rules resonates with diagrams as a heuristic tool in mathematical problem solving (Polya, 1945). Drawing diagrams is a significant problem-solving heuristic, and research shows

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that a visual representation through a model or diagram is the most effective amongst others for problem-solving (Hembree, 1992; Uesaka & Manalo, 2012; Wong, 1999).

Larkin and Simon (1987) noted a substantial connection between diagram use and problem solving, specifically, when the diagrammatic forms are representative of a cognitive process or are schematic in nature. Stylianou (2011) examined the functions of representations in mathematical problem-solving. In doing so, she summarised roles of such representations in students' problem-solving activities. Representations act as tools that support understanding where different aspects of the problem can be combined to observe how they interact. Representations also serve as a means to record information and reduce the cognitive load imposed on one's working memory. Regarding visual representations, Stylianou (2011) noted that diagrams are flexible exploration devices that allow a solver to generate new information about a problem or reveal information that is not immediately obvious. Additionally, representations may also be used to monitor and assess progress in problem-solving. The findings from Stylianou's study evidently show how representations take on different utility value as the objective of a problem-solving activity changes. They illustrate the different roles they play and highlight the benefits of employing them.

De Toffoli (2018) distinguished between different types of representations and purported that choosing the right one is essential. One benefit of visual representations such as diagrams can be credited to its capability to transform a mathematical idea from one form to another. They are useful in providing a complementary language to sentential representations of the same knowledge. Nelson (1993) cited in Small (2012), states that "diagrams, either with or without accompanying words, can be extremely powerful tools for reasoning and explaining" (p.21). Moreover, they make apparent those quantifiable relationships that define a problem (Sunzuma et al., 2020). This alternate form of representing the same knowledge facilitates the process of sense-making and knowledge construction (Mudaly, 2012) which illuminates why diagrams have received much attention from many research areas.

A cognitive advantage of diagrams is that they act as external sketches where interconnected pieces of information can be put together and therefore relieve students' working memory load (van Essen & Hamaker, 1990). A diagram facilitates conceptualisation of the problem structure. It serves as an intermediate step between a mental representation and a physical representation of a concept. Diagrams highlight important relations between quantities and operations in a given problem and assist students to extract pertinent information (Larkin & Simon, 1987). Particularly, some cognitive scientists have concentrated on the role diagrams play in various cognitive activities such as memory, perception, inference and problem-solving (Hamami & Mumma, 2013; Shin, 2015). This brings about the discussion about working memory and the cognitive load imposed on a learner when faced with a problem. Diagrams are able to reduce cognitive demands and alleviate working memory (Murata, 2008; Ngu et al., 2014). Furthermore, they facilitate connections between concrete and abstract representations which support students' problem-solving activities. They allow both teachers and students to view a problem in its entirety as all parts are displayed on the diagram at the same time (Sunzuma et al., 2020).

The Research Questions

The study reported in this paper is part of a larger study (Manoharan, 2021) that investigated teachers' perceptions of diagrams for the teaching and learning of mathematics in Singapore secondary schools. The research questions that guide the study reported in this paper are:

1. *What are teachers' perceptions about what diagrams in mathematics represent?*
2. *When do teachers use diagrams in their mathematics instruction?*

The Instrument

The instrument, a pen-and-paper survey comprising an open-ended questionnaire, was used to probe the understanding of mathematics teacher leaders about diagrams and their role in the teaching and learning of mathematics. Details about the survey are reported in Manoharan (2021). To answer the research questions, responses to prompts in the first section of the survey were coded. The questions were:

- What do diagrams in mathematics represent? [Item 1]
- In your teaching of mathematics when do you use diagrams and why? [Item 2a]
  Do you use diagrams to make connections between mathematical ideas? [Item 2b]

Subjects

A total of 20 heads of mathematics departments whom we refer to as mathematics teacher leaders (MTL) from secondary schools in Singapore participated in the study. These MTLs had taught mathematics across grades 7 to 10 for at least 3 years prior to their participation in the study. The participants were in-service teachers, attending higher degree courses or professional development sessions at the National Institute of Education when they completed the survey. The collection of data was governed by the IRB of the university.

Data and Analysis

A hybrid process of deductive and inductive thematic analysis was used to facilitate the coding process. A thematic analysis, as noted by Braun and Clarke (2006) is a useful and flexible method that can potentially provide a rich and detailed account of data for qualitative research. A hybrid approach employed for this study incorporated both the 'bottom-up' inductive approach of Boyatzis (1998) and the "top-down" deductive approach outlined by Crabtree and Miller (1999). The process of data extraction, coding and categorisation was divided into two stages:

Stage 1: A set of *priori* codes were created drawing from the literature reviewed. This allowed for the coding process to be structured and grounded in existing theories as an initial coding cycle (Linneberg & Korsgaard, 2019).

Stage 2: During the process of coding, it became apparent that not all the responses could be captured by the set of *priori* codes. Through iterative rounds of revising the codes and coding the data, a final set of *posteriori* codes was derived. Swain (2018) notes that this approach of encoding the data results in theory being a precursor to, and an outcome of, data analysis. The set of codes derived from both literature and the actual data, serves as a conceptual framework which guided the process of analysis. Table 1 shows the sets of *priori* and *posteriori* codes. Table 2 shows samples of responses and corresponding codes.

Table 1
Priori and Posteriori Codes

Priori Category Code Description		Posteriori Category Code Description	
[Item 1] Category Code: Representations (R)			
R1	Source of information (Stylianou, 2011)	R1	External representations (Winn, 1987)
R2	Means to record information (Stylianou, 2011)	R2	Source of information (Stylianou, 2011)
R3	A heuristic used in any part of problem-solving (Cellucci, 2019; Polya, 1945; Uesaka & Manalo, 2012)	R3	A heuristic used in any part of problem-solving (Cellucci, 2019; Polya, 1945; Uesaka & Manalo, 2012)

R4	Simplifying tool – known and unknown quantities can be represented. (Stylianou, 2011)		
[Item 2a] Category Code: Affordances (A)			
A1	Facilitate understanding (Cellucci, 2019; Stylianou, 2011)	A1	Facilitate understanding (Cellucci, 2019; Stylianou, 2011)
A2	Diagrams act as a means to connect between existing knowledge and skills (Mudaly, 2012)	A2	Reduce cognitive demand (Murata, 2008; Ngu et al., 2014; Stylianou, 2011)
A3	Reduce cognitive demand (Murata, 2008; Ngu et al., 2014; Stylianou, 2011)	A3	Assist with visualisation (Polya, 1945)
A4	Exploration device (Stylianou, 2011)	A4	Exploration device (Stylianou, 2011)
A5	Monitoring and assessing students' understanding (Stylianou, 2011)	A5	Monitoring and assessing students' understanding (Stylianou, 2011)
[Item 2b] Category Code: Making Connections (MC)			
MC1	Connect between existing and new knowledge (Mudaly, 2012)	MC1	Connect between existing and new knowledge (Mudaly, 2012)
MC2	Boost germane cognitive load (Ngu et al., 2014)	MC2	Boost germane cognitive load (Ngu et al., 2014)
MC3	Connect between different representations (De Toffoli, 2018)	MC3	Connect between different representations (De Toffoli, 2018)

Table 2
Sample Responses and Codes

Teacher	Response to Item 1: What do diagrams in mathematics represent?	Codes
MTL7	Diagrams in Mathematics are **visual representations** of mathematics objects, their properties and relationships with each other. Diagrams are often used as a **problem-solving tool.**	R1 R3
MTL12	Diagrams are a means to **consolidate data and information** for the ease of analysis and pattern finding. Diagrams serve to **convey** and **explicate concepts and information**. Diagrams aid in **problem solving** as information can be represented in a more **comprehensive manner to enable the problem solver to gain better perspective of the problem and form connections**	R2 R3
	Response to Item 2a: When do you use diagrams and why?	
MTL3	I will use a diagram when I can. I believe that multiple-representations and multimodal representations will help students to **make connections** between the representations and **deepen their understanding** of the concept. A diagram also **explicates the relationships** between aspects of the concepts. Different diagram can be used to **highlight certain features for pattern recognition** or for study of trends as in the case of statistical diagrams. Also, I believe a diagram helps in problem solving. When we download the information given and make sense of it in a diagram, we **reduce cognitive load** so as to **free up our mind to focus on ways to solve the problem.** The diagram may aid in **exploring various ways to solve a problem.**	A1 A3 A2 A4

Findings and Discussion

What Do Diagrams in Mathematics Represent?

Table 3 shows the main perceptions held by MTLs about what diagrams represent.

Table 3
Frequency by Category of Responses to Item 1

Survey Item	Category codes for representation	Frequency (n=20)
What do diagrams in mathematics represent?	External representation (R1)	13
	Sources of information (R2)	12
	Heuristic—problem-solving tool (R3)	6

The responses revealed three key perceptions, all of which have been identified in other theoretical or empirical work on diagrams as in the review of literature of this study. Consistent with the definition of diagrams of this study, 13 of the MTLs identified the powerful ability of diagrams to provide a visual representation of real world or mathematical objects. MTL1 stated that diagrams may "represent statistical information" and are also able to "summarise the problem" given. They make quantifiable relationships and structural features of a problem apparent (Sunzuma et al., 2020). Furthermore, 12 also recognised that diagrams provide a concise method of presenting information. Many respondents indicated that diagrams "communicate information" such as mathematical "concepts and properties" (MTL2, MTL3, MTL4, MTL20). However, only 6 perceived diagrams as a heuristic to be used in any part of problem-solving to facilitate understanding and consequently guide problem-solving activities. This finding is of concern as diagrams are a significant heuristic for problem-solving (Hembree, 1992; Polya, 1945; Uesaka & Manalo, 2012; Wong, 1999). Furthermore, mathematical problem-solving is the primary goal of mathematics instruction in Singapore schools.

When and How Diagrams are Used

Perceptions of the MTLs related to when do they use diagrams and why are shown in Table 4. The responses were coded in two categories, namely affordances of diagrams and making connections.

Table 4
Frequency by Category of Responses to Items

Item 2	Category Codes	Frequency (*n* = 20)
(2a) In the teaching of mathematics when do you use diagrams and why?	Affordances	
	Assist with visualisation (A3)	16
	Facilitate understanding (A1)	10
	Reduce cognitive demand (A2)	6
	Monitoring and assessing students' understanding (A5)	4
	Exploration device (A4)	2
(2b) In your teaching of mathematics, do you use diagrams to make connections between mathematical ideas?	Making Connections	
	Connect between existing and new knowledge (MC1)	7
	Connect between different representations (MC3)	7
	Boost germane cognitive load (MC2)	2

From Table 4, it is apparent that all the MTLs used diagrams to gain from the multitude of benefits that diagrams afford when engaging students in mathematics instruction. This finding concurs with that of earlier works established by many others (see Cellucci, 2019; Mudaly, 2012; Stylianou, 2011). Sixteen of the MTLs used diagrams to assist with visualisation - when “explaining concepts, as it helped students visualise” (MTL20) and “guided them when solving problems” (MTL17). Half of the MTLs noted that diagrams facilitate understanding by serving as external representations of interrelated concepts and properties. Transforming sentential representations to diagrammatic representations allows one to make connection between concepts. It supports schema acquisition which enables problems-solving. MTL11 mentioned that students having difficulty understanding the problem will benefit from a diagram as it “will provide a clearer picture.”

Less than a third of the MTLs, placed emphasis on the following during their mathematics instruction: (i) diagrams reduce the cognitive load of oneself when working on a mathematical task, (ii) diagrams may be used to monitor and assess one’s understanding of mathematical ideas, and (iii) diagrams as an exploration device—tool for representing, visualising, etc. of mathematical ideas.

When asked if they made connections using diagrams, all the MTLs responded positively. However as shown in Table 4, less than half of them were cognisant that diagrams facilitate connections between existing and new knowledge and does the same between different representations. MTL 6 presented Figure 1, as an example of how she engages students in reasoning with diagrams whilst connecting their knowledge about congruent figures and properties of circles.

AB is a diameter of the large circle, centre *O*.
CD is a diameter of the small circle, centre *O*.
AC and *BD* are tangents to the small circle.

Show that triangle *OAC* is congruent to triangle *OBD*.
Give a reason for each statement you make.

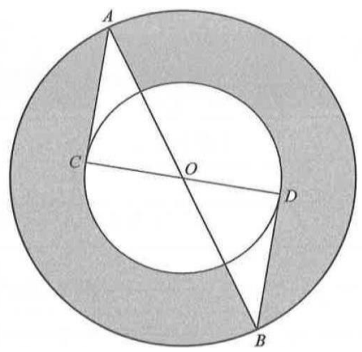

Figure 1. Reasoning with diagrams.

MTL11 was the only respondent who mentioned that diagrams can facilitate big ideas such as Proportionality. She illustrated, see Figure 2, how diagrams may be used to teach the concept of arc length of a circle. Using the diagrams and comparing some calculations, students would be guided to comprehend the proportion of arc length over the circumference of circle. In addition, students’ knowledge of fractions and circle properties are drawn on to teach the new concept of arc and sector length.

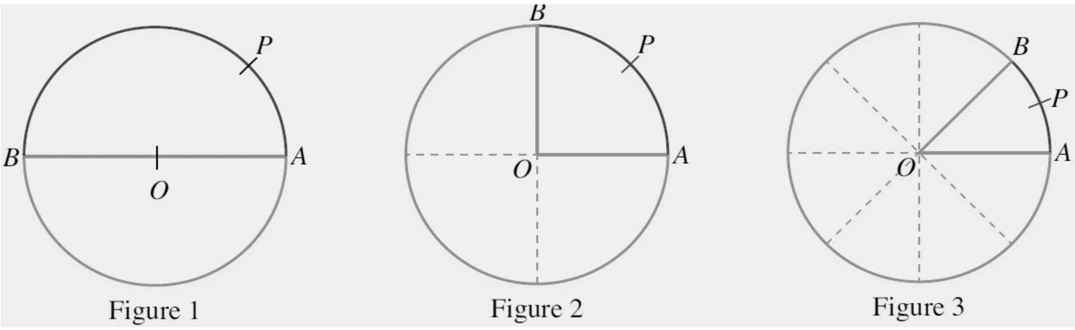

Figure 2. Arc length of a circle.

Conclusions

This study involved secondary school mathematics teacher leaders. As the revised school mathematics curriculum for implementation in schools from 2020 onwards has placed heightened emphasis on big ideas in mathematics with diagrams being one of them, the findings of this study are timely.

Diagrams as external representations and as sources of information, emerged as the dominant perceptions. MTLs acknowledged the capacity of diagrams to externalise concepts and relationships (van Essen & Hamaker, 1990) which facilitates problem-solving for learners. They also noted that critical information about a problem can be recorded in a single, coherent diagram (Stylianou, 2011). However, very few teachers were able to identify diagrams as an effective heuristic that can be used in any part of problem-solving (Polya, 1945). Indeed, most associated diagrams with primarily being external sketches that represent information but did not consider the more specific capacity of diagrams. Some distinctive functions of diagrams include, but not limited to, serving as simplifying tools which can elucidate unknown quantities (Chu et al., 2017), as exploration tools which allow for manipulation of concepts, and as monitoring tools to assess learning (Stylianou, 2011). These findings suggest that teachers may be more accustomed to using diagrams for the most recognized use – as external representations. There was also a lack of affirmation about the use of diagrams in making connections between existing and new knowledge, and also between different representations. Although aware of the connected nature of mathematics, the findings suggest that teachers lack the understanding of how to utilise diagrams as a Big Idea in making these connections thus not fully realising the potential of this vision.

Conversations about teaching towards Big Ideas and ways to enhance pedagogy will greatly impact the extent to which teaching mathematics can progress towards a larger conceptual understanding (Woodbury, 2000). Teaching towards Big Ideas affords opportunities for teachers and curriculum specialist to rethink, refine, and possibly reinvent how they communicate with the use of diagrams. To prepare teachers for the challenge ahead of illuminating diagrams as a big idea in the teaching and learning of mathematics, professional development of mathematics teacher leaders in this area is critical. Therefore, it is recommended that mathematics teacher leaders engage in professional development to deepen their knowledge on diagrams and their potential for the teaching and learning of mathematics.

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