Thinking About Quantity:

The Intertwined Development of Spatial and Numerical Cognition

Nora S. Newcombe

Temple University

Susan C. Levine

University of Chicago

Kelly S. Mix

Michigan State University

AUTHOR NOTE

Abstract

There are many continuous quantitative dimensions in the physical world. Philosophical, psychological and neural work has focused mostly on space and number. However, there are other important continuous dimensions (e.g., time, mass). Moreover, space can be broken down into more specific dimensions (e.g., length, area, density) and number can be conceptualized discretely or continuously (i.e., natural versus real numbers). Variation on these quantitative dimensions is typically correlated, e.g., larger objects often weigh more than smaller ones. Number is a distinctive continuous dimension, because the natural numbers (i.e., positive integers) are used to quantify collections of discrete objects. This aspect of number is emphasized by teaching of the count word sequence and arithmetic during the early school years. We review research on spatial and numerical estimation, and argue that a generalized magnitude system is the starting point for development in both domains. Development occurs along several lines: (1) changes in capacity, durability and precision, (2) differentiation of the generalized magnitude system into separable dimensions, (3) formation of a discrete number system, i.e., the positive integers, (4) mapping the positive integers onto the continuous number line, and (5) acquiring abstract knowledge of the relations between pairs of systems. We discuss implications of this approach for teaching various topics in mathematics, including scaling, measurement, proportional reasoning, and fractions.

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      A challenge in modern air travel is getting luggage onto planes reliably and cheaply. Different airlines have different regulations: only two bags, total luggage taking up a specified volume, each bag no more than a certain weight, and so on. However, often there are few consequences of variations in these regulations, because the dimensions of number, volume and weight are correlated. For example, three suitcases are likely to take up more volume than two suitcases and they also probably weigh more. But number, volume and weight are not perfectly correlated. Consider the situation of a traveler carrying four small bags, each containing a few light items of clothing. She will be penalized by a number rule, but fare well under a volume or weight rule. On the other hand, a traveler carrying one medium-sized suitcase filled with books would fare well under a number or volume rule, but be penalized by a weight rule. By adulthood, most of us understand these trade-offs, and know that various quantitative dimensions are distinct, even if correlated, and that they can vary in occasionally surprising ways. But how do we reach this point? How and when do distinct dimensions of quantity become differentiated? Very different answers to these questions have been given over the years, and the differences among these approaches touch on many of the most challenging issues in the contemporary study of cognitive development.

      A classic approach came from Piaget (1952), who argued that true quantification is not observed at all in infancy and preschool, and emerges during the elementary school years. He observed that children cannot measure, even in a rough way, e.g., they cannot judge the relative heights of two towers of blocks if one tower is on the floor and the other tower is on a table. In the spatial domain, he reported that young children encode space only topologically. In his conservation of number task, he found that children say that the number of objects in a line of objects changes when the objects are spread out, and the length of the line increases while the density decreases; children often focus on length, disregarding both density and number. However, despite these striking (and replicable) observations, Piaget's view of the development of quantitative reasoning is no longer widely accepted, for many reasons. One vital issue is that he vastly underestimated the strength of the starting points for cognitive development (e.g., Gelman & Baillargeon, 1983; Newcombe, 2013).

      There are two contrasting contemporary approaches to quantitative development, both of which embrace strong starting points. One view builds on the idea of a generalized magnitude system extending across various dimensions of continuous quantity (e.g., Petzschner, Glasauer & Stephan, 2015; Walsh, 2003) to postulate that infants begin with this system (e.g., Lourenco & Longo, 2011; Mix, Huttenlocher & Levine, 2002). In this view, development consists of increasing precision in estimation, differentiation of the correlated dimensions, formation of the discrete number system, in part but not entirely through acquisition of culturally-transmitted symbol systems (e.g., Mix, Huttenlocher & Levine, 2002) and eventual remapping of the quantitative dimensions with formal specification of how they are related. An alternative view is the core knowledge view, which holds that infants begin life with separable modules that form the core components of mature quantitative cognition, with two of these distinct modules involving number and space, namely the approximate number system (ANS) as well as the geometric module (e.g.,

Dehaene & Brannon, 2011; Gallistel & Gelman, 1992; Spelke & Kinzler, 2007). In this view, development depends largely on the acquisition of culturally-transmitted symbol systems, notably language, and on increasing precision in the ANS. The aim of this paper is to make a case for the first view and to explore its implications for education and instruction. In unfolding this story, we also critique the core knowledge proposal regarding number, while indicating ways in which elements of that approach are potentially compatible with our own. The hypothesis of a geometric module is, however, discussed (and questioned) elsewhere (Twyman & Newcombe, 2010).

Origins and Development of Spatial Estimation

Spatial estimation is the basis for eventual coordination of various quantitative dimensions, so knowledge concerning its developmental trajectory is crucial. Research on this topic has largely concentrated on length (or distance), in response to Piaget's claim that spatial coding is topological, and hence non-metric, for the first decade of life. As researchers developed new techniques to study infants and toddlers, his conclusion was called into question, and new ways of conceptualizing spatial location coding and developmental change were proposed. More recently, convergences and points of contact among these different views of quantitative development have become apparent.

Early Spatial Estimation

Children remember spatial location metrically, at least in simple tasks. For example, children between the ages of 18 and 24 months can search accurately for an object hidden in a 5-foot long sandbox, first touching the sand only 3 to 4 inches from the correct location (Huttenlocher, Newcombe & Sandberg, 1994). Once toddlers move around the box, errors get larger, but children are still reasonably accurate, far better than Piaget would have predicted (Newcombe, Huttenlocher, Drummey & Wiley, 1998). Even 5-month-olds look longer at hiding-and-finding events in a 30-inch long box when objects emerge from locations 8 to 12 inches away from the hiding location rather than where they had disappeared (Newcombe, Huttenlocher & Learmonth, 1999; Newcombe, Sluzenski & Huttenlocher, 2005). Infants are also sensitive to vertical as well as horizontal extent, as shown by their reactions to containers that were ¼ and ¾ filled with bright red liquid (Gao, Levine & Huttenlocher, 2000).

Metric coding is not the only way to code spatial location, however. Categorical location is also important because it is easy to remember, e.g., people are more likely to know that their keys are somewhere on the coffee table than to know exactly where they are. In an influential model, Huttenlocher, Hedges and Duncan (1991) proposed that fine-grained estimations are combined with memories for the spatial category in which a location appeared, according to a Bayesian combination rule. Initial experiments involved the location of a dot in a circle, in which the spatial categories are the quadrants defined by horizontal and vertical axes. Subsequent work extended the model to maps (Newcombe, Huttenlocher, Sandberg, Lie & Johnson, 1999), photographs of real-world scenes (Holden, Curby, Newcombe & Shipley, 2010) and the three-dimensional world (Holden, Newcombe & Shipley, 2013). Applied to thinking about development, this model suggests that toddlers' bias patterns for search in the sandbox may index the early availability of Bayesian

combination of categorical and fine-grained metric information. Specifically, search is biased toward the center of the box, suggesting that toddlers use the sandbox as a category (Huttenlocher et al., 1994).
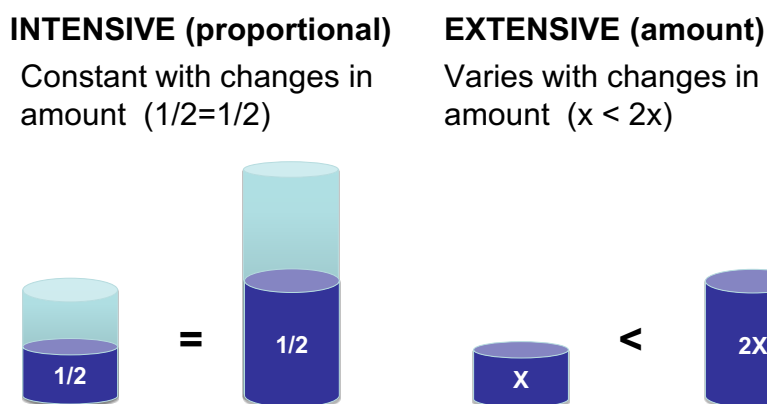
Developmental Change

Research on infants' and children's coding of location in terms of length and height has not simply shown early competence. The studies also delineate several lines along which children change from a less accurate or less flexible representational system to a more mature one. One set of changes involves improvements in the Bayesian system. While infants seem to encode both metric and categorical location and to combine them, they do none of this in an optimal way. First, the capacity and durability of the system is limited. When more than one object is hidden at a time in the sandbox paradigm, or there is a longer waiting period, they do poorly. The ability to remember two objects, or one object for two minutes, develops only gradually over the preschool years (Sluzenski, Newcombe & Satlow, 2004). Second, when there are two dimensions to consider (e.g., radial distance and angle) rather than just one, children cannot coordinate categorical and metric coding until about 9 years of age (Sandberg, Huttenlocher & Newcombe, 1996). Third, the spatial categories used become smaller with age, and hence more informative; adjustment by a smaller category draws estimates to a prototype value closer to the actual location. Sub-division of a space into more than one category appears between 4 to 8 years, depending on the size of the space, and results in a distinctive bias pattern in the sandbox best described as a quintic function (Huttenlocher et al., 1994; see Figure 1 showing spatial memory in 10-year-old children in the sandbox task).

Figure 1
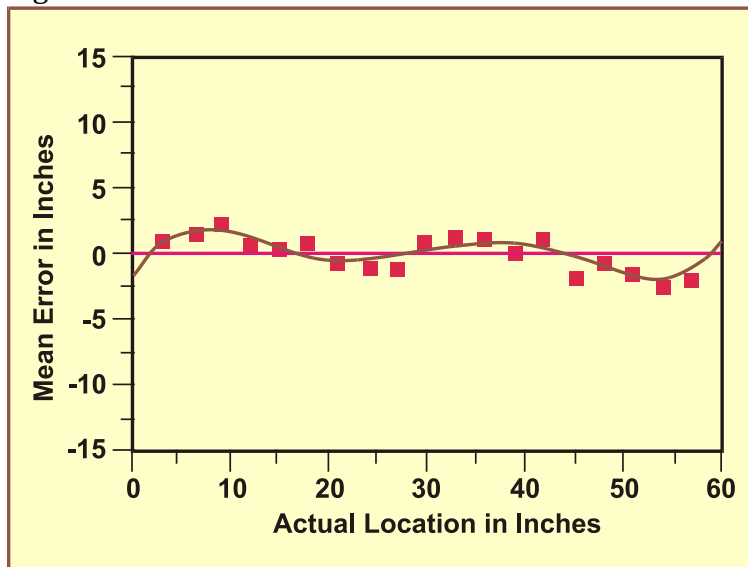
A second set of changes in early spatial representation involves the fact that providing an enclosing frame, such as a sandbox or a container, is essential for infants' success in metric estimation (Duffy, Huttenlocher, Levine & Duffy, 2005; Huttenlocher, Duffy & Levine, 2002). That is, infants rely on intensive (or proportional) coding using a

perceptually-available standard against which to estimate extent, and cannot succeed without it. This reliance on relative coding of amount extends through the preschool years and into early elementary school (Duffy, Huttenlocher & Levine, 2002; Huttenlocher et al., 2002; Huttenlocher, Levine, Ratliff, 2010; Vasilyeva, Duffy & Huttenlocher, 2007). The ability to code extent in the absence of a salient perceptual standard, or extensive coding of amount, only begins to emerge at age 4 or 5, and is more fully developed by age 8 or so (Duffy et al., 2005; Vasilyeva et al., 2007). The change from an intensive to an extensive coding system is not all-or-none, but rather entails the addition of the flexibility to use extensive coding when appropriate. Each system is relevant in some situations and for different mathematical calculations, as illustrated in Figure 2.

INTENSIVE (proportional)
Constant with changes in amount (1/2=1/2)

EXTENSIVE (amount)
Varies with changes in amount (x < 2x)

1/2 = 1/2   X < 2X

There is an important educational implication of the fact that intensive coding is available early: it is exactly what is needed for intuitive scaling and proportional reasoning. In fact, soon after children become able to appreciate symbolic representations at all (for an overview, see DeLoache, 2011), 3-year-old children can use a small representation of the sandbox to locate toys buried in the sandbox (Huttenlocher, Newcombe & Vasilyeva, 1999). Initially, finding hidden objects using a map or model is harder than placing visible objects in accord with a map or model, but this difference passes quickly (Huttenlocher, Vasilyeva, Newcombe & Duffy, 2008). Indeed, four-year-olds can use a small-scale representation to find objects in larger-scale spaces when there is only a small difference in scale (1:6); by 5 years, children succeed even when the difference is more dramatic (1:19.2; Vasilyeva & Huttenlocher, 2004). Precision in scaling tasks proceeds regularly and sequentially from 3 to 5 years, and can be assessed with paper-and-pencil tasks (Frick & Newcombe, 2012). But even later, at least through age 10, a larger scaling factor results in lower accuracy on a proportional matching task (Boyer & Levine, 2012). Performance seems to be supported by proportional perceptual estimation—roughly a quarter of the way across on a map implies roughly a quarter of the way across on a referent space (Möhring, Newcombe & Frick, 2014). These findings suggest that scaling, one of the

important aspects of mathematical development according to recent mathematical standards, can and should be supported by preschool play activities, such as using maps to solve maze puzzles (Jirout & Newcombe, 2014).

<div align="center">Origins and Development of Numerical Estimation</div>

Infants have been found to encode size, area, contour length and volume as well as distance (Brannon, Lutz & Cordes, 2006; Hespos, Dora, Rips & Christie, 2012). They also encode time (Addyman, Rocha & Mareschal, 2014; Brannon, Suanda & Libertus, 2007) and speed, a ratio of distance and time (Möhring, Libertus & Bertin, 2012). But no domain has attracted more interest than number, in which reconsideration of Piaget's claims of protracted development began four decades ago, but about which there continues to be considerable controversy. Initial experiments on number focused on preschool children (e.g., Gelman, 1972), but experimenters soon began to study infants.

The Innate Number Hypothesis

The core knowledge view is that infants spontaneously notice and process discrete number because humans are naturally wired to perceive it (e.g., Feigenson, Dehaene & Spelke, 2004; Gallistel & Gelman, 1992; Spelke & Kinzler, 2007). While the first reports of sensitivity to number involved numbers within the subitizing range from 1 to 4 (e.g., Starkey, Spelke & Gelman, 1983), subsequent work involved larger numbers that involve a separate system often called the approximate number system (ANS) (e.g., Xu & Spelke, 2000; Xu, Spelke & Goddard, 2005). Because accumulating evidence shows that small and large number representations differ at both the behavioral and the neural levels in infants as well as adults (e.g., Cordes & Brannon, 2009; Hyde & Spelke, 2008, 2011), the appearance of strong starting points in both systems is interesting and important (Carey, 2009).

There are, however, several reasons to suggest that the findings in these studies may show sensitivity to continuous magnitude rather than discrete number (Cantrell & Smith, 2013; Leibovich & Henik, 2013; Mix et al., 2002). Cantrell and Smith (2013) offer an exceptionally clear recent review of the voluminous literature. They argue that "discrete quantity in the environment is correlated with other stimulus dimensions; as the number of discrete elements in a set increases, other perceptual properties change as well, and although one might control one of these properties in any one experiment, all of them cannot be controlled simultaneously" (p. 332). While there are various ways of attempting to control for these correlated dimensions, none of them is perfect, and in fact Cantrell and Smith (2013) note that each method has its own distinctive drawbacks. Thus, the demonstrations of infant sensitivities to discrete number, both small and large, could reflect the operation of a quantification mechanism that attends to correlated quantitative dimensions in the world. In fact, work with adults provides evidence that adults may persist in using correlated visual cues in number judgment tasks (Gebuis & Reynvoet, 2012a, b).

While one possible rebuttal point is that number is easier to process than other dimensions of quantity, such as area (Cordes & Brannon, 2008; Libertus, Starr & Brannon, 2014), there are conflicting reports about judgments of area (Odic, Pietroski, Hunter, Lidz & Halberda, 2013) and it is not clear these studies ruled out other variables, such as contour

length (Clearfield & Mix, 1999; Mix, Huttenlocher, & Levine, 2002). Importantly, Cantrell and Smith (2013) do not claim that any single spatial dimension (e.g., area) trumps number, or underlies number. Instead, the idea is that there are a host of quantitative variables so correlated with one another as well as with discrete number that babies may not initially disentangle them, or do so only weakly and with difficulty. The natural prediction of this position is that adults, children and (perhaps especially) infants should all show cross-dimensional generalization.

<u>A Generalized Magnitude System</u>

Both adults and preschool children have in fact been found to link number and space (e.g., de Hevia & Spelke, 2009; Hurewitz, Gelman & Schnitzer, 2006; Soltesz, Szucs & Szucs, 2010), although there are dissenting opinions (e.g., Anobile, Cicchini & Burr, 2013). There are a variety of kinds of evidence. For example, behaviorally, children between 2 and 4 years show very similar Weber fractions for number and area with similar growth patterns for each, and they apply the word "more" accurately in both number and area contexts, beginning at the same ages (Odic et al., 2013; for slightly different results, see Odic, Libertus, Feigenson & Halberda, 2013). In adults as well, there is evidence of links between number and space, both from these studies and from neuroimaging studies, where it has been shown that topographic field maps based on non-numerical sensory information and discrete number cannot be disentangled (Cohen Kadosh & Walsh, 2009; Gebuis, Gevers & Kadosh, 2014).

Research with infants using looking time paradigms also suggests a generalized magnitude system. Infants form expectations about number based on length and temporal duration, as well as vice versa (de Hevia & Spelke, 2010; Hyde, Porter, Flom & Stone, 2013; Lourenco & Longo, 2010; Srinivasan & Carey, 2010). Moving beyond space, number and time to the auditory dimension, infants may show cross-modal transfer when using pitch (Dolscheid, Hunnius, Casasanto & Majid, 2014), although Srinivasan and Carey found no transfer between length and loudness.

Collectively, these findings make a strong case for a generalized magnitude system that is present in infancy and persists into adult life. Further work is needed to specify for what dimensions, and whether some dimensions are primary. Intriguingly, space is arguably the best candidate for "basic" status, if there is a basic dimension. Children can map length to number and (partially) length to brightness, but they do not seem to relate brightness to number (de Hevia, Vanderslice and Spelke, 2012). Along similar lines, human adults show larger effects of space on time than time on space, although symmetric effects appear in rhesus monkeys (Merritt, Casasanto & Brannon, 2010).

<u>Developing Understanding of Discrete and Continuous Number</u>

If quantitative thinking begins with a generalized magnitude system, children must overcome several obstacles to achieve a mature understanding of quantity, some of which are included in a developmental model proposed by Leibovich and Henik (2013). Specific challenges include: (1) differentiating the correlated dimensions; (2) understanding the positive integers, which allow for the determination of exact set size; (3) using and coordinating the approximate and symbolic number systems; (4) mapping the positive

integers onto notions of continuous or approximate quantity, in order to deal with mathematical topics that involve continuous quantity, such as measurement and any topic involving the rational numbers; and (5) systematically mapping the differentiated continuous dimensions onto each other, to understand abstractly the relation between, for example, surface area and volume.

Differentiation of Dimensions

Because the idea that infants begin with a generalized magnitude system has only recently been taken seriously, there is little evidence bearing directly on the issue of how a generalized magnitude system differentiates into distinct dimensions over developmental time. Cantrell and Smith (2013) suggest a Signal Clarity hypothesis, in which the correlated quantitative dimensions proceed from an integral to a separable state (cf. Garner, 1974). Such a developmental sequence has been studied already at other ages and with respect to different kinds of stimuli (e.g., Smith & Kemler, 1978). Some of the predictions of Signal Clarity have already been confirmed, e.g., Cantrell and Smith's Hypothesis 1, the malleability of the Weber fraction (Odic, Hock & Halberda, 2014), and new research is appearing (Cantrell, Boyer, Cordes & Smith, 2015). Dramatic cases of inverse correlation might be especially helpful for children learning to differentiate different dimensions of quantity, e.g., when 100 ants take up much less space than 1 elephant. The approach is broadly compatible with ideas about statistical learning, which has been extensively studied in other domains (e.g., Aslin & Newport, 2012). Much more research is needed, however, to understand how statistical learning supports differentiation of quantitative dimensions, if indeed it does.

Identifying Discrete Number and Mapping the Count Words

Counting ability precedes discrete quantification in object-based tasks (Huttenlocher, Jordan & Levine, 1994; Mix, Huttenlocher & Levine, 1996). The initial stages of counting system acquisition may involve acquiring the meanings of the words *one, two* and *three*, and then learning the cardinal principle, i.e., that each successive number in one's count list refers to a set size that is one more than the previous number (Carey, 2009; Wynn, 1990). Because the small numbers map to numbers within the subitizing range, whose conceptual representations may depend on object files and/or formation of easily-recognized shapes (e.g., Carey, 2009; Choo & Franconeri, 2014), it is tempting to conclude that mapping small numbers to corresponding count words is easy, but that is not the case. The words *one, two* and *three* are not acquired simultaneously but rather slowly and sequentially over an extended time period (e.g., Mix, 2009; Wynn, 1990).

There is a considerable literature at this point concerning whether the count words are initially mapped to the ANS (or AMS), to a separate system in which the small numbers are maintained as object files, or to both. An influential paper by Le Corre and Carey (2007) supported the importance of the initial mapping to an object-based system (enriched by linguistic quantifiers) acquisition of a cardinality principle based on that acquisition, and only subsequent mapping of the count words to the approximate system. Other papers have provided evidence for mapping to an analogue system before acquisition of the cardinality principle (e.g., Wagner & Johnson, 2011) although there are doubts (e.g., Negen & Sarnecka,

2015). It is possible that mapping of small and large number words onto quantity may occur independently of each other, with approximate mapping of larger count words to large sets being related to the child's age rather than to their mapping of smaller count words to small sets (Gunderson, Spaepen, & Levine, 2015). In this case, the mapping of large number words to large sets might occur for some children prior to acquiring the cardinal principle, and for others after acquiring the cardinal principle. Yet another view is that a step-by-step process of mapping to an object-based system eventually transfers to an analogue system, perhaps based on the fact that the count words seem to apply to both (e.g., Spelke & Tsivkin, 2001). In support of this view, a key turning point may occur for children who know *three* as well as *one* and *two*, who then acquire or can be trained to use *four* in a fashion that generalizes (Huang, Spelke & Snedeker, 2010; Mix, 2009). Thus, 1, 2 and 3 in the object-file system may be progressively mapped to their count words by associative mechanisms, then 4 may also be associatively mapped, but to distributions in the approximate system. Once the associative system is included, with the addition of 4, generalization and inference is possible. Finally, 5- to 7-year-old children may become able to relate the larger count words to the approximate system using structure mapping (Sullivan & Barner, 2014). Eventually, conceptual understanding of numerical relations appears to move away from associations between numerical symbols and their concrete referents and move to associations among numerical symbols (e.g., Lyons, Ansari & Beilock, 2012), a view that is consistent with theories of symbol grounding (e.g., Lakoff & Nunez, 2000).

Development of the Approximate and Symbolic Number Systems

The core knowledge view proposes that the ANS is innately-specified (e.g., Gallistel & Gelman, 1992), but investigators working within this tradition have also investigated individual differences and malleability in the ANS. They have reported evidence that individual differences in ANS precision are linked to mathematics achievement (e.g., Halberda, Mazzoco & Feigenson, 2008; Mazzoco, Feigenson & Halberda, 2011; Starr, Libertus & Brannon, 2013), that culture and education enhances ANS precision (Piazza, Pica, Izard, Spelke & Dehaene, 2013), that ANS precision waxes and wanes within a session depending on the individual's history of making easy or difficult judgments (Odic, Hock & Halberda, 2014) and that symbolic arithmetic can be improved by ANS training (Hyde, Khanum & Spelke, 2014; Park & Brannon, 2013). However, an alternative point of view is that facility with symbolic number — not proficiency with nonverbal processes—predicts later mathematical achievement (e.g., Bartelet, Vaessen, Blomert & Ansari, 2014; Gobel, Watson, Lervag & Hulme, 2014; Lyons & Beilock, 2011; Szucs, Devine, Soltesz, Nobes & Gabriel, 2014). This conclusion is supported by a recent review (De Smedt, Noel, Gilmore & Ansari, 2013).

One way to reconcile these suggestions is to posit that the ANS influences mathematical achievement by exerting an influence on the early development of symbolic number abilities. Data are accumulating to support this argument. For example, a mediation analysis of longitudinal data within the preschool age range showed that ANS acuity did not predict mathematics achievement with symbolic mediators in the model (vanMarle, Chu, Li & Geary, 2014). Indeed, the relations may change developmentally, with the ANS important in preschool and early elementary school in supporting learning of symbolic arithmetic, but

later overshadowed in importance by knowledge of the symbolic system. Findings that symbolic and nonsymbolic systems are correlated in younger children but not older children or adults provide support for this view (Fazio, Thompson, Bailey & Siegler, 2014; Gilmore, Attridge, De Smedt & Inglis, 2014).

Number Lines: Mapping the Positive Integers onto Continuous Quantity

Our argument so far suggests that children in elementary school have both a continuous quantity estimator and an increasingly robust and separate system of discrete number, which they are beginning to link to symbolic representations. However, in some cases, continuous quantity and discrete number may compete, as we shall explore in the next section. In addition, the two systems need to be coordinated, as when placing numbers on a number line that spans some range, such as 0 to 10 or 0 to 100. If the numbers are placed correctly, they should be evenly spaced, resulting in a perfectly linear relation between number and position. Performance on these number line tasks has now been extensively studied (e.g., Opfer & Siegler, 2007; Siegler & Opfer, 2003; Thompson & Opfer, 2008). Children initially tend to space small numbers farther apart than they should be, and bunch together the larger numbers, a pattern of responses best fit by a logarithmic function. Responses shift to a more mature, linear function, but the logarithmic-to-linear shift depends on the number range. Although 7-year-olds respond linearly on the 0 to 100 number line, they respond logarithmically on the 0 to 1,000 number line. By 9 years of age, children respond linearly on the 0 to 1,000 number line, but respond logarithmically on the 0 to 10,000 number line, and so on. Number line performance is associated with better subsequent learning and performance in mathematics more generally (e.g., Booth & Siegler, 2006; Ramani & Siegler, 2008; Schneider, Grabner & Paetsch, 2009), and children with poor number line performance are more likely to have mathematics learning disabilities (Geary, Hoard, Byrd-Craven, Nugent & Numtee, 2008).

While the descriptive facts about developing number line representations are reasonably clear, there are many differing interpretations of the effects. Siegler and colleagues conceptualize the logarithmic-to-linear shift in terms of representational change. When errors are logarithmic, the underlying representation itself is skewed, based on an approximate sense of quantity that distorts and compresses larger nonverbal quantities. However, a second possibility is that the data reflect the existence of two (or more) linear segments of knowledge about certain ranges of numbers, each with its own slope (e.g., one slope for the numbers within a child's counting range, and a different slope for larger numbers, outside the child's counting range; different slopes for different areas of the count list based on different levels of familiarity and fluency) (Ebersbach, Luwel, Frick, Onghena & Verschaffel, 2008; Landy, Charlesworth & Ottmar, 2014; Lourenco & Longo, 2009; Moeller, Pixner, Kaufmann & Nuerk, 2009). A third possibility is that number placements are based on proportional reasoning using the endpoints of the line, sometimes in connection with use of the midpoint, therefore obeying a cyclic power law. This model can also generate the logarithmic-to-linear shift, but in addition predicts that seemingly-linear patterns actually contain subtle deviations from linearity, as seen in Figure 3, Panels B and C (Barth & Paladino, 2011; Slusser, Santiago & Barth, 2012).

Figure 3
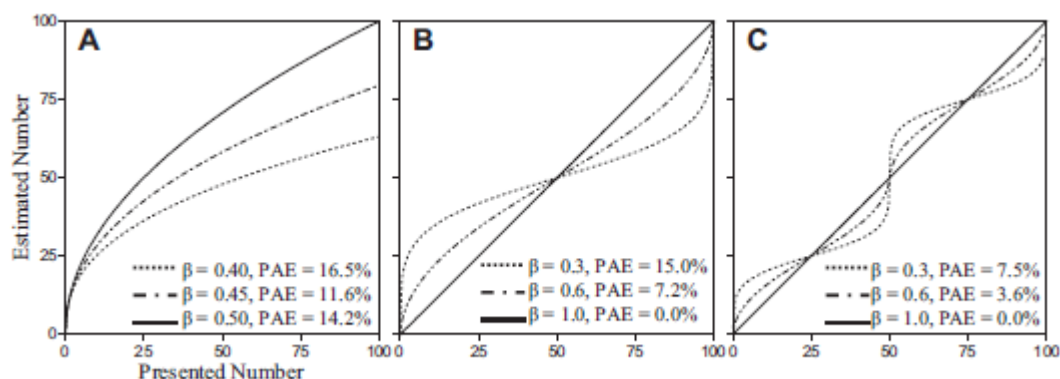
Only the third explanation explicitly emphasizes the spatial aspect of the number line task. But the large literature on the spatial-numerical association of response codes (SNARC) effect certainly suggests linkages, as do other findings regarding space and number such as the fact that preschool spatial skills longitudinally predict number line accuracy (Gunderson, Ramirez, Beilock & Levine, 2012; for an overview of space-number effects, see McCrink & Opfer, 2014). Along these lines, it is striking that the patterns in Figure 1 and Figure 3, Panel C appear identical. The quintic function reflects the division of a range of numbers into halves, which would aid scaling and proportional reasoning. Because scaling and proportional reasoning are likely to be especially challenging for unfamiliar number ranges, aspects of the second explanation (that responses differ based on fluency in a given numerical range) can even be integrated with the third (that responses are proportional).

There is evidence in support of this line of thought. Adults who respond linearly on a 2,000 to 3,000 number line respond logarithmically for the same range of quantities placed on a 1,639 to 2,897 number line (Hurst, Monahan, Heller & Cordes, 2014). Adults also respond logarithmically with unfamiliar symbolic number formats (Chesney & Matthews, 2013) and with large or even fictitious numbers defining the right end of the line (Rips, 2013). Similarly, even first grade children are sensitive to the number line endpoints on the 0-100 task, suggesting an early application of proportional reasoning, and older children (up to 5th grade) achieve greater accuracy by mentally imposing a mid-point anchor (Rouder & Geary, 2014). Further, children perform better when given a line without a rightmost boundary, and when provided with a measurement unit (Cohen & Sarnecka, 2014).

The number line is, of course, a cultural invention, although arguably one that leverages the deep associations between space and number. (For contrasting views of how unschooled indigenous people use number lines, see Dehaene, Izard, Spelke & Pica, 2008, and Nunez, Cooperrider & Wasserman, 2012). But people have great difficulty in understanding magnitudes that are outside their experience, both very big magnitudes, as in billions of dollars or sizes of planets, and very small magnitudes, as in nanoseconds or sizes of atomic particles (Landy, Silbert & Goldin, 2013). This barrier is a real challenge to understanding politics and economics (e.g., the federal budget) and also science, where many quantities are outside the range of human experience (e.g., nanoseconds, light years). One way to address problems with number lines of this kind is to nest time scales that students do understand to build other scales using techniques of analogical learning (Resnick, Shipley, Newcombe, Massey & Wills, 2012); the crucial goal is to establish salient

markers on the otherwise difficult-to-understand continuum.

<div align="center">Discrete and Continuous Number</div>

The theoretical position presented so far suggests the possibility that learning the discrete number system could overshadow the use of continuous magnitude, and thus interfere with important kinds of mathematical learning. In this section, we show that in fact there is evidence for such interference, in three different and important areas of early mathematics: learning to measure, learning to think about proportions and probabilities, and learning to calculate with fractional quantities. These difficulties occur despite the fact that discrete number and continuous quantity seem to have common developmental origins in a generalized magnitude system and that number and proportions have common neural coding schemes (Jacob, Vallentin & Nieder, 2012). The challenge, it seems, is to coordinate these systems once discrete and continuous quantities are differentiated.
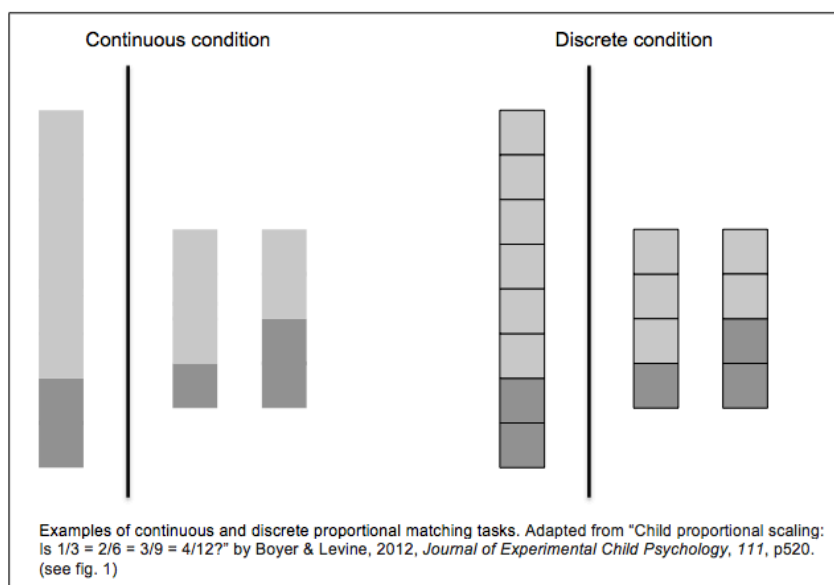
<u>Learning to Measure</u>

Piaget was correct when he observed that young children find it difficult to measure. Surveys of the mathematical abilities of American children consistently show difficulties in understanding ruler measurement, which persist at least through the fourth grade (e.g., Gonzales, Williams, Jocelyn, Roey, Kastberg & Brenwald, 2008). Consider the sample test item shown below, in which children are shown a crayon and a ruler, but the crayon's left end is not at the zero point. When children are asked to select a number that captures the length of the crayon from the 4 response choices shown, some of them simply report the number at one end (i.e., 5 in the example below), or (in a slightly more sophisticated fashion) count the hash marks on the ruler from initial to final (i.e., 4 in the example below). These problems have been shown in experimental settings as well as national assessments (Huttenlocher, Levine, & Ratliff, 2011; Levine, Kwon, Huttenlocher, Ratliff & Deitz, 2009; Solomon, Vasilyeva, Huttenlocher, & Levine, under review). Teaching measurement effectively is an important target of instruction because the persistence of difficulties into late elementary school poses challenges to instruction, for example in science classes and labs that assume that children understand measurement and units of measure.

Consistent with the theoretical framework outlined in this article, Solomon et al.'s (under review) findings show that children's problems in measurement are at least partially attributable to the difficulty of thinking about discrete units in the context of a continuous measurement instrument such as a ruler. Children's strong impulse to count something that looks like a discrete object gets in the way of learning to count the spatial intervals demarcated by these numbers. Interestingly, kindergarten and second grade children perform much better on misaligned problems when the units are discrete objects (adjacent pennies) that make clear what the countable unit should be, exactly because children are used to counting objects. Other techniques leveraging this insight have recently been devised to teach measurement more effectively in the early grades by utilizing misaligned ruler problems and emphasizing that the relevant countable units are spatial extents, even though those units do not look like discrete objects (Congdon, Gibson, & Levine, 2013; Levine, Kwon, Huttenlocher, Ratliff, & Deitz, 2009).

Correct   **Hash-mark Error**   **Random**   **Read-off Error**
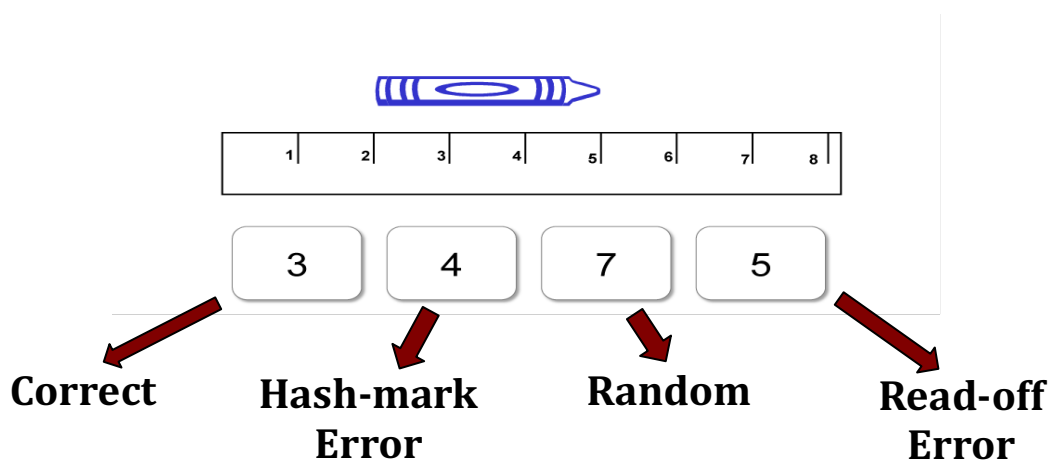
<u>Proportional and Probabilistic Reasoning</u>

In learning to measure, elementary school children have difficulty conceptualizing continuous spatial intervals as countable units of measurement because they are focused on the idea of numbers as enumerating a set of discrete objects. In proportional reasoning, there is a similar problem. When countable units are salient, children as old as 5th graders have difficulty concentrating on spatial extent when they should do so. For example, consider the two problems shown below, in which people are asked to select which of the two alternatives best matches a standard. Adults and children both do well with the problem at the left where length is quite salient and they can use an intuitive perceptual strategy based on intensive coding; these problems are very similar to scaling problems although the cover story for the task is typically quite different. Adults also do well with the problem at the right, but children often say that the correct answer is the alternative that shows 2 units because the standard has 2 units. Children are seduced by this error until they are 8 or 9 years of age (Boyer, Levine & Huttenlocher, 2008; Boyer & Levine, 2012).

Examples of continuous and discrete proportional matching tasks. Adapted from "Child proportional scaling: Is 1/3 = 2/6 = 3/9 = 4/12?" by Boyer & Levine, 2012, *Journal of Experimental Child Psychology, 111*, p520. (see fig. 1)

The same pattern is seen in probabilistic reasoning. Children were shown two donut-shaped forms that were divided into red and blue regions, each with a spinner in the center. Their task was to decide for which donut the spinner was most likely to land on one color or the other. Performance on this task was above chance by age 6 years, but only when the different colored regions were presented as continuous amounts. When the red and blue regions were divided into several equal-sized, bounded units, children performed worse and did not begin to succeed until 10 years of age (Jeong, Levine & Huttenlocher, 2007). Again, the continuous task is easier because these quantities can be mapped onto approximate magnitude representations more readily, and the task with units is hard because of children's impulse to count anything that is countable and to erroneously base their responses on these counts.

Overall, the developmental challenge is to impose discrete, countable, equal-size units onto continuous amounts and to know which system to use, when, and how. Formal instruction in proportional and probabilistic reasoning may be helped by building on children's intuitions about continuous amounts and intuitive proportional reasoning and then provide strong analogies to these same amounts with discrete units imposed (Boyer & Levine, in press).

Understanding Fractions

Just as for proportional reasoning and measurement, children struggle to see fractions in terms of countable units. One indication is that they initially ignore portion size in tests of fraction comprehension. For example, when dividing sets of different sized "candies" among recipients, children doled out an equal number of candies with no regard for size, even when it would have been straightforward to equate the total amount for each recipient by allocating the small and large candies in a 2:1 ratio (Frydman & Bryant, 1988; see Sophian, Garyantes & Chang, 1996, for a similar pattern of findings with different materials). Children also exhibit a whole number bias when interpreting written and spoken fraction names. Specifically, they tend to match fraction names to pictures that show the cardinal number of pieces for both the numerator and denominator (e.g., a picture of 3 shaded and 5 unshaded parts to represent the fraction name, "three-fifths" rather than the correct choice of 3 shaded and 2 unshaded parts) (Mix & Paik, 2008; Paik & Mix, 2003).

Despite these difficulties, studies that show children readily acquire the meanings of common fractions, such as one-half, but are limited to demonstrating this understanding on tasks that require an approximate sense of ratio, such as matching equivalent fraction pictures (Spinillo & Bryant, 1991), estimating the results of additions and subtractions (Mix, Levine, & Huttenlocher, 1999), or completing pictorial analogies based on equivalent fractions (Singer-Freeman & Goswami, 2001). This approximate ability emerges years earlier than children can complete more precise, symbolic fraction tasks. Young children also can use approximate comparisons to make equal shares of continuous amounts by comparing the sizes of the shares (Hunting & Sharpley, 1988; Miller, 1984)—procedures not too far removed from conventional measurement. Perhaps competence on these nonverbal tasks reflects children's ability to recruit generalized magnitude representations as referents at the same time they have difficulty understanding of the referents for fraction symbols (i.e., unit counts).

This pattern is also evident in research showing that children perform intuitive fraction tasks better when the quantities are continuous (i.e., spatially contiguous), rather than discrete (i.e., unitized). Hunting and Sharpley (1988) found that 35% of 4- to 7-year-olds successfully divided a clay sausage in half, but only 11% of the same children did so for a deck of 12 cards. This is noteworthy because units come for free in discrete sets—units that could support more precise divisions than one could achieve for unmeasured, continuous amount. Yet, children performed better without the built-in units, perhaps because they based their responses on magnitude representations and these continuous quantities were, thus, easier to map.

Finally, there is evidence that adults and 11- to 13-year-old children can represent the meanings of fraction symbols as magnitudes on a mental number line (DeWolf, Grounds, Bassok, & Holyoak, 2014; Schneider & Siegler, 2010; Siegler, Thompson, & Schneider, 2011).   This mapping is based on a rough estimate of the absolute quantity represented by a fraction, rather than precise, part-whole relations based on unit measures. Such a representation appears to be quite effortful but unlike erroneous strategies such as simply comparing numerators, supports accurate performance.

Taken together, these studies suggest that adults and children map fraction meanings to their quantitative referents in a holistic way. However, holistic mappings only go so far. Recognizing physical situations that can be called "half," is not the same as mapping a numerator and denominator onto their specific referents. To achieve this mapping, children need to understand measurement units and how these units represent the hierarchical relations between parts and wholes, as well as the way the numbers of units and their sizes are represented in symbolic fractions. Although rough part-whole concepts emerge by preschool in nonverbal tasks, children struggle to master fraction notation throughout the elementary grades and into adolescence (National Mathematics Advisory Panel, 2008; Post, 1981). Even when they attain some competence, it is often based on rote application of procedures and whole number confusion. For example, Kerslake (1986) observed that approximately 1 in 5 students between the ages of 12 and 15 years erroneously claimed that $1/3 + 1/4 = 2/7$. Understanding the generalized magnitude system and its differentiation during development may be a key part of gaining a deep understanding of units and fractions (Schneider & Siegler, 2010).

In particular, it may not be clear to children that the numerals in fractions stand for counts. That is, the denominator stands for the number of divisions of the whole that were made to yield equal-sized units, and the numerator stands for the number of these equal-sized pieces in the total quantity. This failure to interpret fraction symbols in terms of measurement units has been long-recognized as a major obstacle to understanding (Baroody & Hume, 1991; Behr, Harel, Post, & Lesh, 1992; Steffe, 1988) and may be rooted in the same challenges of separating and integrating number and spatial extent noted previously, for measurement and proportional reasoning. Importantly, recent research suggests that successfully navigating this integration is more predictive of subsequent success in mathematics (e.g., algebra) than knowledge of magnitudes per se (Booth & Newton, 2012).

Indeed, recent evidence demonstrating a connection between early fraction ability and later achievement (e.g., Siegler, Thompson & Schneider, 2011; Siegler, Duncan, et al., 2012) seems to be based on facility with this mapping—the mapping of numerals to units (Hecht & Vagi, 2010; Siegler, 2009; Sophian, 2007. In these studies, early fraction ability

was measured various ways, such as matching written fractions to their pictorial representations, judging which of two written fractions represents the larger quantity, and placing written fractions on a number line, but all required children to interpret written fraction symbols in terms of the number and size of units represented, and to do so for fractions beyond the range that is typically acquired intuitively (e.g., beyond ¼ and ½). This is an important sense in which the kind of fraction understanding that is being assessed differs from the tasks used to demonstrate earlier emerging competence with fractions, and may be the critical reason they are both relatively challenging and also so predictive.

<div align="center">Conclusion</div>

Magnitude estimation begins with a strong starting point, namely a generalized magnitude system that allows for many different kinds of judgment about the numerical, spatial and physical world. However, the dimensions are not sharply differentiated, estimation is imprecise, and estimation is relative. Preschool children can use these systems to perform simple scaling tasks and to reason numerically, sometimes even about fractional quantities. The challenges of the school years are to decrease reliance on perceptually available standards, to learn formal systems, such as measurement devices and calculation, sort out which dimensions of magnitude are relevant for which problems, understand fractions, and (ideally) learn how to reason about very small- and very large-scale magnitudes. Understanding how development occurs can allow for tailoring mathematical education to build on children's natural strengths.

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