JUNIOR HIGH SCHOOL MATHEMATICS
BOOK III

WENTWORTH-SMITH MATHEMATICAL SERIES

JUNIOR HIGH SCHOOL MATHEMATICS

BOOK III

BY

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PREFACE

While this book presents a thoroughly practical kind of mathematics, as do also Books I and II, it is the purpose of the book to make the treatment sufficiently formal to enable the student to appreciate more fully the nature of pure mathematics. It is only by so doing that the door of the science can be opened sufficiently to enable him to determine whether he should pursue the subject further. In Book I the work in arithmetic was extended, the subject of intuitive geometry was introduced, and the algebraic formula was used when needed; in Book II the work in arithmetic was continued, particularly as it refers to the problems of everyday life, and such algebra as is essential in various practical lines was set forth; and now Book III offers a fitting close to an introductory course in mathematics by extending the work in practical algebra, by showing the nature and some of the practical uses of trigonometry, and by introducing the student to the first steps of demonstrative geometry.

The student who expects to enter college will find that the algebra given in this series satisfies the requirements in many cases and that even the highest requirements in both algebra and geometry can be met in a year or a year and a half more. The authors have had in mind the needs not only of this class of students but also of those students who do not expect to enter college and yet who wish for and are entitled to have a general survey of elementary algebra, an introduction to the meaning and the practical uses of trigonometry, and an idea of scientific demonstration as it appears in its most available form, the elements of geometry.

As to sequence, algebra has the first place in this book for the reason that the student is already familiar with the subject and needs to use it in the trigonometry. Trigonometry is next studied for the reason that it requires algebra but in its first stages, which are here presented, makes no use of demonstrative geometry, depending rather upon the intuitive geometry already studied. Demonstrative geometry comes last for the reason that it requires more maturity of judgment than the kind of algebra and trigonometry given in this book. It is feasible, however, to carry the algebra and geometry parallel if desired, or to reverse the order.

Special attention is called to the sequence of the work in the first steps in demonstrative geometry, independent deduction preceding the formal proof and a large number of practical exercises following each proposition. The exercises are simple in their nature, as they should be at this early stage, but they are sufficient to encourage that independence of mind which is far more valuable than a knowledge of a conventional number of formal propositions. In the treatment of the basal propositions themselves an attempt has been made to depart from that extreme formality that is often discouraging to students at this stage of their development.

The authors wish to express their indebtedness to Mr. T. M. Cleland for his artistic treatment of the full-page illustrations. They feel that these illustrations have not merely an esthetic value but an important educational value as well, showing as they do the geometric figures in immediate relation to their practical applications.

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PART I. ALGEBRA

PART II. TRIGONOMETRY

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SYMBOLS AND ABBREVIATIONS

The following are the most important symbols used:

$+$	plus	$\parallel$	parallel
$-$	minus	$\perp$	perpendicular
$\times$ or $\cdot$	times	$\angle$	angle
$\div$, /, or :	divided by	$\triangle$	triangle
$\sqrt{\ }$	square root of	$\square$	rectangle
$\sqrt[n]{\ }$	nth root of	$\square$	parallelogram
$=$	is equal to, equals, is equivalent to	st.	straight
		rt.	right
a^2	square of a	Ax.	axiom
a^n	nth power of a	Post.	postulate
$\ldots$	and so on	Ex.	exercise, example
$>$	is greater than	Const.	construction
$<$	is less than	Def.	definition
$\therefore$	therefore	Cor.	corollary

Symbols of aggregation are used as explained in the text.

There is no generally accepted symbol for "is congruent to." The sign $=$ is commonly employed, the context telling whether equality, equivalence, identity, or congruence is to be understood; but some teachers use $\leqq$, $\cong$, or $\equiv$ for congruence.

SUGGESTIONS AS TO OMISSIONS

Students who have completed Books I and II may briefly review pages 1–34 and begin their advanced work on page 35 In geometry they may briefly review pages 135–164 and begin with demonstrative geometry on page 165. The more difficult propositions in the exercises in geometry are placed towards the end of each exercise and may be assigned to specially qualified students. Pages 95–106 and 257–270 may be omitted if desired.

JUNIOR HIGH SCHOOL MATHEMATICS

BOOK III

PART I. ALGEBRA

I. INTRODUCTION

1. Nature of the Work. Students who have completed Books I and II have learned the nature of algebra. They have learned its importance in the understanding of formulas, graphs, and equations; the use of the negative number; and the applications of the science to measurements of various kinds and to the solution of the problems of arithmetic. They are therefore prepared to undertake the study of a more advanced kind of algebra.

Just as those who know short methods in arithmetic can frequently obtain a result in less time than those who do not know them, so those who know even the first part of algebra can often save much time in the solution of problems in arithmetic. Those who study more algebra will be still better prepared. Such students will better understand the formulas which they will meet in books and periodicals relating to mechanical work, domestic art, construction, aviation, and similar subjects. They will also be the better able to develop for themselves such formulas when they need them.

2. Shorthand of Algebra. Algebra employs a universal shorthand of great labor-saving value. We may not be able to read or to speak a foreign language; but anyone who has studied a little algebra and who sees the expression

$$2x + 7 = 19$$

in a book in such a language knows exactly what it means and what he is expected to do with it. It is a shorthand way of writing this problem: Find the number which becomes 19 when 7 is added to its double. We may state it in other words and in other languages, but everyone in the world who knows algebra knows its meaning.

3. Rules in Shorthand. Practically all of our computations are made without stopping to think of the reasons involved. If we wish to find the area of a rectangle, we simply multiply the length by the width; we would waste time if we stopped to think of the reason, after having once understood it and learned the rule. We may not remember the exact words of the rule in the book that we studied, but we remember the idea.

We learned this rule for finding the area of a triangle:

To find the area of a triangle take half the product of the base and height.

Algebra enables us to express this in shorthand, thus:

$$A = \frac{1}{2}bh.$$

In expressing this rule in shorthand we let A stand for *area*, but we may use a small letter if we prefer. We let b stand for *base* and h stand for *height*. When a number and a letter or when two letters have no sign between them, multiplication is always to be understood.

Thus ab means $a \times b$; $2ab$ means $2 \times a \times b$; $\frac{1}{2}bh$ means $\frac{1}{2}$ of $b \times h$.

Exercise 1. Shorthand of Algebra

Examples 1 to 18, oral

1. As in § 2, state the meaning of the expression $3x = 6$.

2. State the meaning of the expression $x + 5 = 17$.

State the meaning of each of the following expressions:

3. $x - 2 = 9$. **5.** $\frac{1}{4}x = 5$. **7.** $2x + 5 = 11$.

4. $x \div 2 = 9$. **6.** $\frac{3}{4}x = 6$. **8.** $3x - 5 = 13$.

9. If a stands for 5 and b stands for 7, what is the value of ab?

We usually say, "If $a = 5$ and $b = 7$, then $ab = 5 \times 7 = 35$."

10. If $a = 7$ and $b = 9$, what is the value of ab?

If $a = 5$ and $b = 10$, state the value of each of the following:

11. ab. **13.** $a + b$. **15.** $b \div a$. **17.** $\frac{1}{5}a$.

12. $\frac{1}{2}ab$. **14.** $b - a$. **16.** $a \div b$. **18.** $\frac{1}{5}b$.

19. Write the following rule in algebraic shorthand: To find the area of a rectangle take the product of the base and height.

20. If $a = 37$ and $b = 42$, find the value of $\frac{1}{2}ab$.

21. If $a = 22\frac{1}{2}$ and $b = 39$, find the value of ab.

22. If r represents the radius of a circle and c represents the circumference, express in algebraic shorthand this rule:

To find the circumference of a circle take $6\frac{2}{7}$ times the radius.

23. Two of the sides of a rectangle being a and b, write in algebraic shorthand the rule for finding the perimeter p.

4. Symbols of Algebra. Most of the symbols used in algebra are the same as those used in arithmetic. The most important of these symbols are given below.

Addition. Addition is indicated by the plus sign, +.

This is not always the case. In arithmetic we write $2\frac{1}{2}$ when we mean $2 + \frac{1}{2}$. In algebra we write $a + \frac{1}{2}$ if we wish to indicate the sum of $\frac{1}{2}$ and some number which we represent by a.

Subtraction. Subtraction is indicated by the minus sign, −.

Thus, $a - b$ means that a number indicated by b is to be subtracted from a number indicated by a. If a is 10 and b is 3, then $a - b$ means $10 - 3$, and its value is 7.

Multiplication. Multiplication is indicated in algebra by the symbol ×, by a dot, or by the absence of a sign.

Thus, the product of two numbers which we may represent by b and h may be expressed by $b \times h$, $b \cdot h$, or bh. The last of these forms is the one generally used in algebra, but in arithmetic we cannot write 35 for 3×5, because 35 means $30 + 5$.

Division. Division is rarely indicated in algebra by the symbol ÷. The division of a by b is more commonly indicated by the fraction form, $\frac{a}{b}$; for convenience in printing, however, it is often indicated by a/b or by $a : b$.

Equality. Equality is indicated, as in arithmetic, by the symbol =, read "is equal to" or "equals."

For example, in arithmetic we have $2 + 3 = 5$. In algebra, if $a = 7$ and $b = 4$ in some problem, then

$$a + b = 7 + 4 = 11,$$

$$a - b = 7 - 4 = 3,$$

$$ab = 7 \times 4 = 28,$$

and

$$\frac{a}{b} = \frac{7}{4} = 1\tfrac{3}{4}.$$

5. Symbols of Powers and Roots. In arithmetic we rarely use any symbols except those given on page 4, but in algebra many practical problems involve powers and roots.

Power. If we multiply a number by itself, we may indicate the operation in various ways. For example, the product of 4 and 4 may be indicated by 4×4 or by 4^2, the latter being read "4 square" or "4 to the second *power*." In algebra we usually write a^2 for aa, while in arithmetic we usually write 4×4 instead of 4^2.

There are other powers besides the second power. Thus,

$aaa = a^3$, read "a cube" or "a to the third power";
$xxxx = x^4$, read "x to the fourth power,"

and so on to whatever power we wish to use.

The product of two or more equal factors is called a *power* of any one of the equal factors.

In a case like m^5 the 5 is called an *exponent*.

If there is no exponent, the exponent 1 is understood; thus, $a = a^1$.

Root. One of the equal factors whose product is a given number is called a *root* of the number; one of the two equal factors of a number is called the *square root* of the number; one of three equal factors, the *cube root*; one of four equal factors, the *fourth root*; and so on.

We indicate the square root of 2 by the symbol $\sqrt{2}$, the cube root of a by the symbol $\sqrt[3]{a}$, and so on.

Because $3 \times 3 = 9$, we see that 3 is the square root of 9.

Because $4^3 = 64$, we see that 4 is the cube root of 64.

Similarly, $\sqrt[4]{16} = \sqrt[4]{2 \times 2 \times 2 \times 2} = 2$, and $\sqrt[5]{m^5} = m$.

In a case like $\sqrt[n]{a}$, the n is called the *index of the root.*

The index 2 is not written in the case of a square root.

A number may not have an exact root. Thus $\sqrt{2} = 1.41 \cdots$, and we say that 1.4 is the square root of 2 to one decimal place; 1.41, to two decimal places; and so on.

Exercise 2. Symbols

All work oral

1. What is meant by the expression b^2+2? What is its value when $b=5$? when $b=7$? when $b=10$?

2. What is meant by the expression n^2-7? What is its value when $n=10$? when $n=20$? when $n=30$?

3. What is meant by the expression $3x$? What is its value when $x=5$? when $x=10$? when $x=1000$?

4. What is meant by the expression x^3? What is its value when $n=2$? when $n=3$? when $n=10$?

State the meaning of each of the following expressions and state its value when $a=10$ and $b=2$:

5. $a+b^2$. **7.** $a-b^2$. **9.** ab^2. **11.** $a/10$. **13.** $30:a$.

6. $\frac{a}{5}$. **8.** $\frac{6}{b}$. **10.** $\frac{a}{b}$. **12.** $\frac{b}{a}$. **14.** $\frac{ab}{10}$.

15. What is the exponent in the expression 2^3? What does it indicate? State the value of the expression.

16. Read the expression a^2 and state its value when $a=5$; when $a=6$; when $a=8$; when $a=10$.

17. State the value of m^4 when $m=1$; when $m=3$.

Notice that $m^4=mmmm=m^2m^2$. Hence $3^4=3^2\times 3^2$.

18. What is the exponent in the expression x^4? What does it indicate? State the value of x^4 when $x=10$.

19. Read the expression $\sqrt{49}$, and state its value.

A table of square roots and cube roots is given on page 280, and may be used at any time.

20. Read the expression $\sqrt[4]{16}$; state the index of the root and also the value of the expression.

6. Formula. When we state a rule in the shorthand of algebra, we call the expression a *formula.*

The student is already familiar with many rules used in measuring, either from the preceding books of this series or from arithmetic. A few of these rules will now be stated, and the formula will be given under each.

The student should be certain that he understands how the formula represents the rule.

While it is better to use initial letters in a formula, other letters may be used if desired.

There is no general custom as to using capital letters or small letters. Each represents a number.

The area of a rectangle is equal to the product of the base and height.

$$A = bh.$$

This means that the number of square units of area is equal to the product of the number of linear units of the base by the number of linear units in the height, and so for other similar cases.

The area of a triangle is equal to half the product of the base and height.

$$A = \tfrac{1}{2}bh.$$

The volume of a rectangular solid is equal to the product of the length, width, and height.

$$V = lwh.$$

The circumference of a circle is equal to π times the diameter.

$$c = \pi d.$$

The Greek letter π (pī) stands for a number that is approximately 3.1416. For common measurements we use $3\frac{1}{7}$, or $\frac{22}{7}$, for π, and the student may do this unless required to use 3.1416.

Since d is twice the radius, we may write the formula $c = 2\pi r$.

The area of a circle is equal to π times the square of the radius.

$$A = \pi r^2.$$

Exercise 3. Formulas

Write the following rules as formulas, using initial letters:

1. The area of a parallelogram is equal to the product of the base and height.

2. The area of a square is equal to the second power of a side.

This is why the second power of a number is called its square, and similarly for the cube in Ex. 3.

3. The volume of a cube is equal to the third power of an edge.

4. The volume of a cylinder is equal to the product of the height and π times the square of the radius.

The result is the product of the height and πr^2. It may be indicated by $h\pi r^2$, but it is customary to write π first, thus: πhr^2, or $\pi r^2 h$, either being proper. Now write the equation for V.

The teacher should observe that only right circular cones and right circular cylinders are considered in this book.

5. The volume of a pyramid is equal to one third the product of the base and height.

6. The volume of a cone is equal to one third the product of the height and π times the square of the radius.

7. The curve surface of a cylinder is equal to the product of the height and π times the diameter.

Use S or C for curve surface.

8. The curve surface of a cone is equal to half the product of the slant height s and π times the diameter.

The slant height is the length of the slanting line from the point of the cone to the circumference of the base.

9. The volume of a sphere is equal to $\frac{4}{3}$ of π times the cube of the radius.

7. Formulas and Rules. We have seen how rules that are already made can be expressed much more conveniently as formulas. We shall now proceed to make formulas, and then to write rules from these formulas.

1. Write a formula for the cost of a certain number of articles when the cost of one is known. From the formula thus obtained write a rule.

In all such cases the articles are supposed to be alike in kind and in price unless the contrary is stated.

Suppose that each article costs d dollars and that there are n articles. Then n articles will cost nd dollars.

That is, the total cost C is given by the formula

$$C = nd.$$

To find the cost of a given number of articles of the same kind, multiply the cost of each article by the number of articles.

2. Write a formula for reducing yards to inches, and then express the formula as a rule.

Since in 1 yd. there are 36 in., evidently i, the number of inches, is 36 times y, the number of yards; that is,

$$i = 36\,y.$$

To find the number of inches in a given number of yards, multiply the number of yards by 36.

Of course in arithmetic we multiply 36 in. by the number of yards, but in algebra we do not ordinarily label the numbers.

3. In cooking a certain kind of beef there should be allowed a quarter of an hour for every pound, and then twenty minutes in addition to this. Express this rule as a formula.

If p is the number of pounds and t is the time in hours, then, since 20 min. = $\frac{1}{3}$ hr.,

$$t = \tfrac{1}{4}p + \tfrac{1}{3}.$$

The number of hours required to cook this kind of beef is found by dividing the number of pounds by 4 and then adding $\frac{1}{3}$.

Exercise 4. Formulas and Rules

Write formulas and rules for finding the following:

1. The number of nickels equal in value to a given number of dollars.

2. The number of dollars equal in value to a given number of nickels.

3. The cost of each article, given the total cost and the number of articles.

4. The number of inches in a given number of feet and inches.

Let n be the number of inches required, f the given number of feet, and i the given number of inches.

5. The number of pounds in a given number of ounces.

6. John's age, it being known that he is 28 yr. younger than his father, and his father's age being known.

Let J represent the number of years in John's age, and F the number of years in his father's age.

7. The profit that a grocer makes on a certain number of eggs, the profit per dozen being known.

8. The cost of an excavation l feet long, w feet wide, and d feet deep, at c cents per cubic yard.

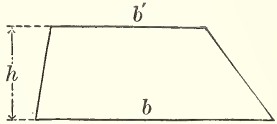
9. The cost C of keeping house one week for n persons at $2.75 per person and $7.50 for general expenses.

10. The cost of running the schools of a city, given the average cost per pupil and the number of pupils.

11. The cost of coal for the schools of a city, given the cost per ton and the number of tons.

12. The number of cubic feet to be allowed for a room in a new school building, the number of pupils and the number of cubic feet per pupil being known.

8. Parentheses. The area of a trapezoid is equal to half the product of the height multiplied by the sum of the bases. If we try to express this rule as a formula, we find that we need another symbol; for if we use h to represent the height, b and b' (read "b prime") the bases, and A the area, we have



$$A = \tfrac{1}{2}\,h \times \text{the sum of } b \text{ and } b'.$$

This is an awkward expression, and therefore we use parentheses to assist us, thus:

$$A = \tfrac{1}{2}\,h\,(b + b').$$

This means that the operation indicated within the parentheses is to be performed first.

For example, $a(b + c)$ means a times the sum of b and c, while $ab + c$ means that c is added to the product of a and b. If $a = 2$, $b = 5$, and $c = 3$, then

$$a(b + c) = 2 \times (5 + 3) = 2 \times 8 = 16,$$

and $$ab + c = 2 \times 5 + 3 = 10 + 3 = 13.$$

Sometimes we need to inclose parentheses within parentheses, thus: $a(b + (c - a))$. In such a case we avoid confusion by using brackets for the outside parentheses, thus: $a[b + (c - a)]$.

Occasionally we use other symbols for this purpose. Thus, the following all have the same meaning: $(a + b)\,x$, $[a + b]\,x$, $\{a + b\}\,x$, and $\overline{a + b}\,x$.

Usually, however, the parentheses answer all purposes, because we do not often need very complicated expressions.

Such symbols as these, used to collect certain terms, are called *symbols of aggregation*. The word *parentheses* is often used to mean all kinds of symbols of aggregation.

9. Order of Operations. In our study of algebra we are beginning to meet with expressions that are sufficiently complicated to require that we should know the order in which the indicated operations are to be performed. Unless we do this we shall not know whether in such an expression as $4 + 8 \div 2 \times 3^2$ we are first to add 4 and 8, or first to divide 8 by 2, or first to perform some other operation.

The order in which the operations are to be performed is a matter of agreement among mathematicians. This agreement is substantially that this order is to be followed:

1. *Powers and roots.*
2. *Multiplications and divisions in the order they occur.*
3. *Additions and subtractions in the order they occur.*

The above order may be modified by the use of parentheses, the operations indicated within the parentheses taking the precedence.

Thus,
$$\begin{aligned} 4 + 8 \div 2 \times 3^2 &= 4 + 8 \div 2 \times 9 \\ &= 4 + 4 \times 9 \\ &= 4 + 36 \\ &= 40, \end{aligned}$$
while
$$\begin{aligned} (4 + 8) \div (2 \times 3^2) &= 12 \div (2 \times 9) \\ &= 12 \div 18 \\ &= \tfrac{2}{3}. \end{aligned}$$

The fraction bar not only indicates division but it also has the force of parentheses.

Thus $a + \dfrac{b+c}{d}$ means that b is first to be added to c, this sum is then to be divided by d, and the quotient is to be added to a. If $a = 2$, $b = 7$, $c = 3$, and $d = 5$, we have

$$2 + \frac{7+3}{5} = 2 + \frac{10}{5} = 2 + 2 = 4.$$

Exercise 5. Order of Operations

Examples 1 to 30, oral

State the value of each of the following expressions:

1. $2^2 + 3$.	**6.** $3 + 2 \times 5$.	**11.** $5^2 - 10 \div 2$.
2. $2 + 3^2$.	**7.** $(3 + 2) \times 5$.	**12.** $(5^2 - 10) \div 2$.
3. $(2 + 3)^2$.	**8.** $8 + 4 \div 2$.	**13.** $2 + \sqrt{25} - 3$.
4. 2×3^2.	**9.** $(8 + 4) \div 2$.	**14.** $2 + \sqrt{25 - 9}$.
5. $(2 \times 3)^2$.	**10.** $8 \div 2 + 4$.	**15.** $3^2 + \sqrt[3]{8} + 1$.

If $a = 4$, $b = 1$, $c = 2$, *state the values of the following:*

16. $a^2 + b$.	**21.** $a + b \div c$.	**26.** $a(b + c)$.
17. $a - c^2$.	**22.** $(a + b) \div c$.	**27.** $ab + c$.
18. $(a - c)^2$.	**23.** $\sqrt{a} + 5b$.	**28.** $a - c - b$.
19. $\sqrt{a} + 12$.	**24.** $\sqrt{a + 5b}$.	**29.** $a - (c - b)$.
20. $\sqrt{a + 12}$.	**25.** $(b + c)a$.	**30.** $a - (c + b)$.

If $a = 9$, $b = 8$, $c = 32$, *find the values of the following:*

31. $\sqrt{a} + \sqrt[3]{b}$.	**35.** $\sqrt[5]{c} + \sqrt[3]{b}$.	**39.** $c - a - b$.
32. $\sqrt{a} + \sqrt[5]{c}$.	**36.** $b^2 \div c + 1$.	**40.** $c - (a - b)$.
33. $\dfrac{a + b - 1}{c}$.	**37.** $\dfrac{b^2 \div (a - 1)}{b}$.	**41.** $\dfrac{a + b + c}{7}$.
34. $\dfrac{8b}{c} + a$.	**38.** $\dfrac{c}{4b} + \dfrac{a + 7}{b}$.	**42.** $\dfrac{b + c}{a + 1}$.

Taking the value of n as given, and using the table of roots on page 280 when necessary, find the value of each of the following expressions, carrying the result to three decimal places:

43. $\sqrt{n}$; $n = 7, 15, 30$.	**45.** $\sqrt[3]{n}$; $n = 2, 5, 17$.
44. $4\sqrt{n}$; $n = 5, 20, 32$.	**46.** $n^2\sqrt[3]{n}$; $n = 4, 6, 15$.

10. Equation. When we state the formula for the circumference,

$$c = \pi d,$$

we assert that c is equal to πd, and we may therefore think of the formula as an *equation.*

In this case c is the *first member* of the equation and πd is the *second member.*

We also speak of the members of an equation as the *sides* of the equation, using either "members" or "sides" as we prefer.

If we know the diameter of a circle, we can find the circumference from this formula. If we know the circumference, it is often necessary to find the diameter. This can easily be done; for, evidently, if we divide equal numbers by the same number, the results will be equal. We therefore divide these two equals by π and we have

$$\frac{c}{\pi} = d.$$

We must remember that $\pi = \frac{22}{7}$, approximately, so that we have $c \div \frac{22}{7} = d$, or $\frac{7}{22} c = d$.

For those who have studied Book II of this series the above solution will be easily understood. Others will see from it the necessity for studying the equation, and for them the subject must be treated in the simple manner shown on pages 15–20.

11. Unknown Quantity. In the equation $x = \pi d$ we can find the value of x if we know the value of d. In this case, d is a *known quantity* because it is stated that "we know the value of d," and x is an *unknown quantity.*

The unknown quantity is often called the *unknown.*

The finding of the value of the unknown quantity is called the *solution* of the equation. When this value is found, the equation is *solved*, and the value itself is often called the solution of the equation.

12. Letters used in Equations. In the equation $c = \pi d$ we have used initial letters, and this is a common custom. When it is not convenient to use initials the letter x is generally used for the unknown quantity.

In the case of easy equations we can often state at sight the value of the unknown quantity. Thus, if

$$x + 5 = 9,$$

we see at once that $x = 4$, because $4 + 5 = 9$.

Similarly, if $\frac{20}{x} = 5$, then $\frac{4}{x} = 1$, and so x must be 4.

In the following exercise the students are simply expected to use their judgment, unhampered by any idea of formal axioms.

Exercise 6. Solving Equations at Sight

State at sight the value of the unknown quantity in each of the following equations:

1. $x + 5 = 8.$	**13.** $7x = 14.$	**25.** $20 = 10x.$
2. $x + 3 = 5.$	**14.** $7x = 35.$	**26.** $30 = 3x.$
3. $x + 9 = 12.$	**15.** $8x = 24.$	**27.** $40 = 10x.$
4. $x + 7 = 15.$	**16.** $9x = 90.$	**28.** $30x = 30.$
5. $x - 1 = 1.$	**17.** $x + 8 = 18.$	**29.** $12x = 24.$
6. $x - 5 = 2.$	**18.** $8 + x = 10.$	**30.** $100 = 10x.$
7. $x - 7 = 3..$	**19.** $x - 4 = 11.$	**31.** $500 = 5x.$
8. $x - 10 = 8.$	**20.** $9 - x = 1.$	**32.** $25x = 50.$
9. $2x = 6.$	**21.** $3x = 300.$	**33.** $25x = 75.$
10. $3x = 15.$	**22.** $10x = 300.$	**34.** $100x = 50.$
11. $\frac{1}{4}x = 7.$	**23.** $\frac{1}{3}x = 10.$	**35.** $\frac{1}{5}x = 2.$
12. $\frac{8}{x} = 4.$	**24.** $\frac{15}{x} = 5.$	**36.** $\frac{100}{x} = 10.$

13. Axioms. In solving the equations on page 15 no explanation was necessary, the solutions being quite evident. What is really done, however, is to consider the two equal members as just balancing, thus:

$x + 5$ just balances 8

It is evident that if we take 5 from each of these two equals, the results will just balance; that is,

$$x \text{ just balances } 8 - 5,$$

or

$$x = 3.$$

In this case we have assumed that if equals are subtracted from equals, the remainders are equal.

Such a common-sense statement is called an *axiom.*

The axioms needed at present are as follows:

1. *If equals are added to equals, the sums are equal.*

That is,	$4 = 4$
Adding	$3 = 3$
we have	$7 = 7$

2. *If equals are subtracted from equals, the remainders are equal.*

That is,	$7 = 7$
Subtracting	$4 = 4$
we have	$3 = 3$

3. *If equals are multiplied by equals, the products are equal.*

The student should illustrate axioms 3 and 4 as above, taking $9 = 9$ and multiplying each member of the equation by 3 and also dividing each member by 3.

4. *If equals are divided by equals, the quotients are equal.*

Exercise 7. Uses of Axioms

Examples 1 to 6, oral

1. If $3x = 15$, what is the value of x? What axiom is used in finding the value?

2. If $\frac{1}{3}x = 7$, what is the value of x? What axiom is used in finding the value?

3. If $x - 7 = 10$, what is the value of x? What is added to $x - 7$ to make x? What must then be added to 10? What axiom is used in finding the value?

4. If $x + 8 = 18$, what is the value of x? What axiom is used in finding the value?

5. If $2x + 3 = 13$, what is the value of $2x$? What axiom is used? If $2x = 10$, what is the value of x? What axiom is used in finding the value?

6. If $\frac{3}{4}x - 4 = 5$, what is the value of $\frac{3}{4}x - 4 + 4$, or $\frac{3}{4}x$? What must then be done to find the value of $3x$? What must then be done to find the value of x? What axiom is used in each case?

Find the value of x in each of the following equations, stating the axiom or axioms used:

7. $5x = 75$.

8. $9x = 333$.

9. $17x = 102$.

10. $\frac{3}{2}x = 3$.

11. $\frac{21}{10}x = 21$.

12. $2\frac{1}{10}x = 21$.

13. $2.8x = 19.6$.

14. $7.4x = 22.2$.

15. $8\frac{2}{3}x = 52$.

16. $\frac{3}{4}x = 9$.

17. $\frac{2}{3}x = 16$.

18. $\frac{5}{8}x = 15$.

19. $\frac{4}{5}x = 24$.

20. $\frac{3}{5}x = 90$.

21. $\frac{3}{10}x = 90$.

22. $0.4x = 1.6$.

23. $0.9x = 8.1$.

24. $0.6x = 21$.

25. $x + 17 = 17$.

26. $2x + 9 = 27$.

27. $x - 9 = 10$.

28. $2x - 7 = 15$.

29. $\frac{1}{2}x + 4 = 9$.

30. $\frac{3}{4}x - 2 = 7$.

31. $\frac{5}{8}x + 3 = 28$.

32. $3\frac{2}{3}x - 7 = 5$.

33. $\frac{5}{16}x - 1 = 9$.

14. Simple Equation. An equation which contains the first power and no higher power of the unknown quantity is called a *simple equation.*

A simple equation is also called a *linear equation* or an *equation of the first degree.*

15. Identity. An equation which is true for any value whatsoever of any letter or letters is called an *identity.*

For example, $(a+b)^2 = a^2 + 2ab + b^2$ is an identity, for it is true whatever values we give to a and b.

An equation involving only known numbers, like $3 + 7 = 10$, is also called an identity.

An identity is sometimes expressed by the symbol $\equiv$, as in $x^2 + y^2 \equiv y^2 + x^2$, read "$x^2 + y^2$ is identical to $y^2 + x^2$."

16. Root of a Simple Equation. The quantity that substituted for the unknown quantity reduces a simple equation to an identity is called the *root* of the equation and is said to *satisfy* the equation.

Thus, if $x + 3 = 7$, we see that $x = 4$; that is, 4 is the root of the equation and it satisfies the equation.

If $nx = n^2$ and we consider x as the unknown quantity, we divide both members by n and we then see that $x = n$. Hence n is the root of this equation.

17. Transposition. Suppose that we have to solve the equation

$$x - 7 = 18.$$

If we add 7 to each member, we have

$$x - 7 + 7 = 18 + 7,$$

or

$$x = 18 + 7.$$

We then say that we have *transposed* 7.

The word "transpose" is evidently not necessary, since we can use "subtract" or "add" in its stead, but it is a word that is commonly employed in algebra.

18. Illustrative Problems. **1.** Six times a certain number is 54. Find the number.

Let $x =$ the number.

Then $6x =$ six times the number.

But $54 =$ six times the number.

Therefore $6x = 54$.

Dividing by 6, $x = 9$. Ax. 4

Therefore the required number is 9.

Check. Substituting in the statement of the problem, $6 \times 9 = 54$.

2. Nine times a certain number is equal to the number increased by 96. Find the number.

Let $x =$ the number.

Then $9x =$ nine times the number,

and $x + 96 =$ the number increased by 96.

Therefore $9x = x + 96$.

Subtracting x, $8x = 96$. Ax. 2

Dividing by 8, $x = 12$. Ax. 4

Therefore the required number is 12.

Check. Substituting in the statement of the problem,

$$9 \times 12 = 12 + 96.$$

3. The sum of two numbers is 37, and one of them is nine more than three times the other. Find the numbers.

Let $x =$ the smaller number.

Then $3x + 9 =$ the larger number.

Therefore $3x + 9 + x = 37$.

Combining, $4x + 9 = 37$.

Subtracting 9, $4x = 28$. Ax. 2

Dividing by 4, $x = 7$. Ax. 4

Then $3x + 9 = 3 \times 7 + 9 = 30$.

Therefore the required numbers are 7 and 30.

Check. $7 + 30 = 37$, and $30 = 3 \times 7 + 9$.

Exercise 8. Problems

1. Five times a certain number is equal to 15,145. Find the number.

2. Five times a certain number is equal to the same number increased by 16. Find the number.

3. If from seven times a certain number we subtract 2, the result is 54. Find the number.

4. If to nine times a number we add 7, the result is 106. Find the number.

5. The sum of two numbers is 18, and one of them is four more than the other. Find the numbers.

6. The sum of two numbers is 57, and one of them is two more than ten times the other. Find the numbers.

7. The difference between two numbers is 6, and one of them is four times the other. Find the numbers.

8. One of two numbers is $\frac{1}{10}$ of the other, and the sum of the numbers is 88. Find the numbers.

9. One of two numbers is 10% of the other, and the difference between the numbers is 153. Find the numbers.

10. If we divide 180 by a certain number, the quotient is 36. Find the number.

If $180/x = 36$, what two operations are necessary in order to find the value of x? What axioms are used?

11. If we divide 182 by a certain number, the quotient is 45.5. Find the number.

12. One student asked another to think of a number, multiply it by 7, add 4, add twice the original number, subtract 2, subtract nine times the original number, and gave the result as 2. How did he know the result?

13. Make a puzzle problem like the one given in Ex. 12.

19. Further Uses of Axioms. In Exercise 7 the axioms chiefly used were those of subtraction and division, although some use was made of the axiom of multiplication. In actual work with practical formulas all four axioms are needed, as will be seen in the following solutions:

1. From the formula $W - 25\,c = b$, find a formula for W.

Adding $25\,c$ to each member, we have

$$W = b + 25\,c, \qquad \text{Ax. 1}$$

for it is easily seen that $W - 25\,c + 25\,c$ is the same as W.

This is a formula which may be used in finding the weight of a box containing 25 cans, the weight of the empty box being b pounds and the weight of each can being c pounds. Similarly, the other formulas given on this page have practical uses, although we are not at present concerned with the nature of these uses.

2. From the formula $W = b + nc$, find a formula for b.

Subtracting nc from each member, we have

$$W - nc = b, \qquad \text{Ax. 2}$$

or

$$b = W - nc,$$

since, in an equation, we may evidently interchange the two members.

3. From the formula $t = V/s$, find a formula for V.

Multiplying each member by s, we have

$$ts = V, \qquad \text{Ax. 3}$$

or

$$V = ts.$$

4. From the formula $t = (V - 40)/s$, find a formula for s.

Multiplying each member by s, we have

$$ts = V - 40. \qquad \text{Ax. 3}$$

Dividing each member by t, we have

$$s = \frac{V - 40}{t}. \qquad \text{Ax. 4}$$

20. Formulas used in Industries. The formulas given in the following exercise are all used in various industries. It is not necessary for our present purposes to explain the meaning of each formula, but it suffices to say that the formulas represent real cases.

Exercise 9. Formulas used in Industries

Given the following formulas, derive the formulas required, and state the axiom used in each step:

1. $T = WV^2/32.2$, a formula used in connection with the pulley. Derive a formula for W.

2. From the formula in Ex. 1 derive a formula for V.

3. $L = 3\frac{3}{8}(R + r) + 2\,d$, a formula used in connection with a crossed belt on two pulleys. Derive a formula for d.

4. From the formula in Ex. 3 derive a formula for R.

5. $\text{H.P.} = D^2N/2.534$, a formula used in connection with power boats. Derive a formula for N.

In this case H.P. stands for horse power and should be considered as a single letter.

6. From the formula in Ex. 5 derive a formula for D.

7. $T = 33{,}000\ \text{H.P.}/2\,\pi rS$, a formula used in connection with revolving shafts. Derive a formula for S.

See the note under Ex. 5.

8. $H = 0.02\,LV^2/64.4\,D$, a formula used in connection with pumps. Derive a formula for V.

9. $P(2^n - 1) = W$, a formula used in connection with pulleys. Derive a formula for P; for 2^n.

10. $Sr^4 = nW \times 33 \times 10^8$, a formula used in connection with steel springs. Derive a formula for W; for r.

21. Graph of a Formula. On page 9 the formula derived for the cost of n articles at d dollars each was

$$C = nd.$$

Taking some particular value for d, say $1\frac{1}{4}$, the formula becomes

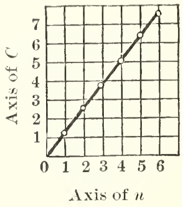
$$C = 1\tfrac{1}{4}n,$$

it being the custom to write the number before the letter.

If we give different values to n, we can find the corresponding values of C, as shown in the following table:

$n =$	0	1	2	3	4	5	6
$C =$	0	$1\frac{1}{4}$	$2\frac{1}{2}$	$3\frac{3}{4}$	5	$6\frac{1}{4}$	$7\frac{1}{2}$

We may now represent the formula by a figure. Taking paper ruled in squares as here shown, we mark the values of n on the line marked *axis of n*, and the corresponding values of C on the lines drawn perpendicular to this axis. That is, when $n = 1$, $C = 1\frac{1}{4}$, so we make a dot or a small circle $1\frac{1}{4}$ units above the axis of n and 1 unit to the right of the axis of C; when $n = 2$, $C = 2\frac{1}{2}$, so we make a dot or a small circle $2\frac{1}{2}$ units above the axis of n and 2 units to the right of the axis of C; and similarly for other values of n.



Connecting these points, we have, in this case, a straight line which is called the *graph* of the equation $C = 1\frac{1}{4}n$.

If we wish to find the value of C when $n = 7$, we may simply extend the graph and see where it cuts the perpendicular from the point 7 on the axis of n.

If necessary the teacher should explain the axis of C. The importance of graphs should be explained as in Book II of this series.

22. Curve Graph. Graphs are not always straight lines; indeed, they are generally not straight lines except when both letters are of the first power (§ 5).

Consider, for example, the graph of the important formula for the area of a circle,

$$A = \pi r^2.$$

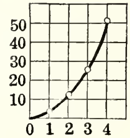
Since we may take $3\frac{1}{7}$ for π, this formula becomes

$$A = 3\tfrac{1}{7}\, r^2.$$

Giving to r various values, we have this table:

r =	0	1	2	3	4
A =	0	$3\frac{1}{7}$	$12\frac{4}{7}$	$28\frac{2}{7}$	$50\frac{2}{7}$

Since the values of A increase very rapidly, we may take a different unit in measuring on the axis of A, or the A-axis. Letting one space represent 10 units on the A-axis, we have the graph here shown. This line is, then, the graph of the equation



$$A = 3\tfrac{1}{7}\, r^2.$$

If the same unit were taken on the axis of A as on the axis of r, the figure would be ten times as high as this one, but the general nature of the graph would be the same. Students should use their judgment as to taking different units; if the numbers are all small, the same unit should be used. In most of the examples given in this elementary presentation of the subject the same unit can be used for both axes. In Exs. 9 and 10 on page 25, however, it is legitimate to use different units.

For example, if the equation were $P = \frac{1}{2} n^2$, we should have

n =	0	1	2	3	4
P =	0	$\frac{1}{2}$	2	$4\frac{1}{2}$	8

In this case we could easily use the same unit for both axes.

Exercise 10. Graphs

Represent by graphs the following formulas:

1. $A = 3h$. **2.** $A = \frac{1}{2}h$. **3.** $C = 3d$. **4.** $n = 5c$.

5. Represent by a graph the equation $A = 2c$ and from the graph determine the value of A when $c = 2\frac{1}{2}$; when $c = 3\frac{1}{4}$; when $c = 4\frac{1}{2}$.

This is done by measuring the distance to the graph from the points $2\frac{1}{2}$, $3\frac{1}{4}$, and $4\frac{1}{2}$ on the c-axis.

6. Draw the graph of $c = \frac{3}{4}t$. From the graph determine the value of c when $t = 3\frac{1}{2}$; when $t = 4\frac{1}{4}$.

7. Represent graphically the equation $x + 3 = y$. From the graph determine the value of y when $x = 1\frac{1}{2}$.

Three different expressions relating to the drawing of graphs have been used in Exs. 5–7, and the student should become familiar with each of them.

8. In a certain shop the workmen receive 45¢ per hour; that is, $W = 0.45h$, where W is the total amount of a man's wages and h is the number of hours he works. Draw a graph of the formula for $h = 0, 1, \cdots, 8$. From the graph determine his wages for 3 hr.; 5 hr.; 7 hr.; $2\frac{1}{2}$ hr.; $3\frac{1}{2}$ hr.; $5\frac{1}{2}$ hr.

The student should now see that he can save time and labor by first letting $h = 0$, then letting $h = 8$, and joining the corresponding points for W by a straight line. This could not be done if W and h were not both of the first power.

9. Draw a graph of the formula $S = 4\pi r^2$. From the graph find the approximate value of S when $r = \frac{1}{2}$; when $r = \frac{3}{4}$; when $r = 1\frac{1}{4}$; when $r = 1\frac{1}{2}$.

10. Draw a graph of the formula $V = \frac{4}{3}\pi r^3$. From the graph find the approximate value of V when $r = \frac{1}{2}$.

23. Function. In the formula for the circumference of a circle, $c = \pi d$, we see that c has different values as d takes different values. That is, if $d = 1$, $c = 3\frac{1}{7}$; if $d = 2$, $c = 6\frac{2}{7}$; if $d = 3$, $c = 9\frac{3}{7}$; and so on. In other words, c depends on d for its value.

When one quantity depends on another for its value, it is said to be a *function* of that other quantity.

In the case of $A = \pi r^2$, A is a function of r; in the case of $C = nd$, C is a function of both n and d.

In the case of $A = \pi r^2$, r is also a function of A, for

$$r = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{A}{3\frac{1}{7}}},$$

so that as A changes in value, r also changes. Similarly, in the case of $c = \pi d$, d is a function of c.

In every example in arithmetic we found that the result depended on some other quantity given in the example; in other words, the examples in arithmetic illustrate the great importance of functions in mathematics.

We may say, "We will buy it if we have the money," the act of buying depending upon the amount of money we have. In the same way we say in algebra, "The circumference of the circle is $3\frac{1}{7}$ if the diameter is 1," the circumference depending on the diameter. Likewise, we say, "The man's wage will be $4 to-day if he receives 50¢ per hour and works 8 hr."; that is, the wage for the day is a function of the wage per hour and also of the number of hours.

To express the fact that W is a function of x, we write $W = f(x)$ and read this, "W is a function of x."

If x is a function of y and z, we write $x = f(y, z)$ and read this: "x is a function of y and z."

Mathematics is largely a science of functions. It is not necessary to make frequent use of the word "function" or of the symbols $f(x)$ and $f(y, z)$, but it is advantageous to have the idea well fixed in mind, particularly if the student is to take up the study of trigonometry as presented in Part II of this book.

Exercise 11. Functions

Examples 1 to 4, oral

In each of the following formulas state one letter that is a function of another:

1. $c = 2\pi r$. **2.** $V = e^3$. **3.** $V = \frac{4}{3}\pi r^3$. **4.** $S = 4\pi r^2$.

Draw graphs of the following formulas:

5. $d = 10c$. **6.** $A = 6s$. **7.** $A = s^2$. **8.** $A = 6s^2$.

From each of the following formulas derive a formula for x, and then write a statement that x is a function of some special letter:

9. $5x = 7y$. **10.** $x + 3 = y + 8$. **11.** $2x + 7 = a + 9$.

12. From the equation $x + 2 = y + 6$ write a statement showing that one letter is a function of the other, draw a graph of the equation, and from the graph find the value of x when $y = 3\frac{1}{2}$.

In each of the following formulas write a statement that one letter is a function of two other letters, using for this purpose the form $x = f(y, z)$:

13. $3x = 7y + z$. **14.** $A = \frac{1}{2}bh$. **15.** $V = \pi r^2 h$.

16. Draw a graph of the formula $V = e^3$ and from the graph find an approximate value of V when $e = 2\frac{1}{2}$.

17. Draw a graph of the formula $V = \frac{1}{3}\pi r^2 h$ when $h = 6$.

This evidently reduces to the problem of drawing a graph of the formula $V = \frac{1}{3}\pi r^2 6 = 2\pi r^2 = 2 \times 3\frac{1}{7}r^2$; that is, of $V = 6\frac{2}{7}r^2$.

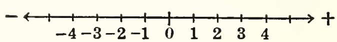
18. Express in algebraic shorthand the statement that m is a function of n, and then write three equations in which m is such a function.

24. Negative Number. When the temperature is 70° above zero we say that the temperature is 70°. If the temperature falls 70°, we then say that it is zero, or 0°. If it falls 10° more, we say that it is 10° below zero, and we usually indicate this by the symbol $-10°$. We read this symbol "minus 10°" or sometimes "negative 10°." This does not mean that 10° has been subtracted from anything; it simply means that 10° is to be considered below zero. We call -10 a *negative number*.

Ordinary numbers, like 5, $2\frac{1}{2}$, and 47.25 are called *positive numbers*. It is not the custom to write the + sign before a positive number, but it is always necessary to write the − sign before a negative number.

Sometimes a negative number has a meaning that is easily understood, and sometimes it has no such meaning. For example, it means nothing to say that there are -10 students in a recitation room, but it is very convenient to speak of $-10°$ of temperature.

It is convenient to think of a negative number as simply a number on the other side of zero from a positive number. Thus, if we speak of west longitude as positive, then east longitude is negative; whereas if we speak of east longitude as positive, then west longitude is negative.



The following are other simple illustrations of positive numbers and negative numbers, or of *directed numbers*, as the two together are called:

Direction. If north is positive, south is negative; if up is positive, down is negative.

Money. If a man's resources or assets are positive, his debts are negative.

Weight. If weight (downward force of gravity) is positive, an inflated balloon may be thought of as having negative weight.

25. Addition involving Negative Numbers. Some operations with negative numbers are very simple, while others require some explanation. For example, if a man is \$10 in debt, we may say that he is worth – \$10. If he gets \$20 more in debt, we may think of his capital as – \$10 and – \$20; that is, he is \$30 in debt, and we say that – \$10 + (– \$20) = – \$30.

If the man now earns \$50, he can evidently pay up his debt and have \$20 over; that is, – \$30 + \$50 = \$20. If, however, a man has – \$30 (is \$30 in debt) and earns \$25, he can evidently pay up \$25 of his debt and then have – \$5 (be \$5 in debt); that is, – \$30 + \$25 = – \$5.

To add a negative number to a negative number, add as if the numbers were positive and then prefix the negative sign.

Thus, $-7 + (-9) = -16.$

The – 9 is inclosed in parentheses to avoid the confusion of signs.

To add a positive number to a negative number, consider the numbers as positive, find their difference, and then prefix the sign of the numerically larger number.

Thus, $-7 + 9 = 2,$

and $-7 + 4 = -3.$

The numerically larger number is the number that is larger without reference to the sign, – 3 being numerically larger than 2.

Addition involving negative numbers will be more clearly understood by answering the following questions:

If your score in a game is – 10 and your next mark is – 5, what is then your total score?

If your score is – 10 and you make 4, what is then your score?

If your score is – 15 and you make 25, what is then your score?

26. Subtraction involving Negative Numbers. If my score in a game is -4 and yours is 12, your score is how much more than mine? Evidently I must make 4 to get up to zero, and 12 more to get up to your score, so I must add 16 to my score to get up to yours. That is,

$$12-(-4)=16.$$

The subtraction of a negative number is the same as the addition of the positive number having the same numerical value.

As will be inferred from page 29, by the numerical value of a number is meant the value of the number without reference to the sign.

Now consider another case: What must be added to $5 to make $-$15; that is, if a man has $5, how much must he lose so as to be $15 in debt? Evidently the man must lose $5 to be worth exactly nothing, and he must go in debt $15, making a total loss of $20, to be $15 in debt. That is, if we take $5 from $-$15, we have $-$20.

Hence $$-\$15-\$5=-\$20.$$

The subtraction of a positive number is the same as the addition of the negative number having the same numerical value.

It is not necessary to remember these principles, for all we need do in any case is to see what number must be added to the subtrahend to make the minuend.

Before considering Exercise 11 answer these questions and illustrate each by drawing a diagram as on page 28:

What must be added to $-9°$ to make $-3°$? How much is $-3-(-9)$?

What must be added to $-12°$ to make $10°$? How much is $10-(-12)$?

How much is $10-(-20)$? $10-(-30)$?

Exercise 12. Addition and Subtraction

1. In playing a game the scores made were -2, 6, -4, 8, 7, -5, 3. How much was the total score?

2. A ship sails from a place in $-8° 30'$ latitude to a place in $+11° 30'$. Through how many degrees of latitude has it sailed? Draw a rough diagram.

3. On a winter morning in Maine the temperature was $-7°$, but at noon it was $19°$. How many degrees had the temperature risen? Draw a rough diagram.

4. In Ex. 3, if the temperature fell $20°$ during the afternoon, what was it then? Draw a diagram.

5. A man having a balance of \$1250 on his account book incurs a debt of \$1500. If he enters this debt on his book, what is then his balance?

Perform the following additions and subtractions:

6. $82 + (-70)$.

7. $82 - (-70)$.

8. $150 - 320$.

9. $-150 - 320$.

10. $-150 + 320$.

11. $-150 - (-320)$.

12. $-320 - (-150)$.

13. $275 - 300$.

14. $275 - (-300)$.

15. $-275 + 300$.

16. $-275 - 300$.

17. $42.8 - 68.2$.

18. $-42.8 + 68.2$.

19. $-42.8 - (-68.2)$.

Performing the operations in the order indicated, find the value of each of the following:

20. $370 + 280 - 460 + (-20) - (-40)$.

21. $127.8 - 26.3 + 41.9 - 482.4 - (-48.3)$.

22. $67 - 82.9 + 320 - (-40.4) + 72.8 - 482.6$.

23. $289.42 + 3.4 - 82.08 + (-43.8) - (-275.4)$.

27. Multiplication involving Negative Numbers. By considering the following problems we can easily see what meaning we should give multiplication when negative numbers are involved:

1. If a man saves \$15 a month, how much better off will he be 6 mo. hence; that is, + 6 mo. from now?

Evidently he will be better off by 6 × \$15, or \$90.

2. If a man saves \$15 a month, how much better off was he 6 mo. ago than he is now; that is, – 6 mo. from now?

Evidently he was worse off by 6 × \$15, or \$90; that is,

$$(-6) \times \$15 = -\$90.$$

3. If a man wastes \$15 a month, that is, if he saves – \$15 a month, how much better off will he be 6 mo. hence?

Evidently he will be worse off by 6 × \$15, or \$90; that is,

$$6 \times (-\$15) = -\$90.$$

4. If a man wastes \$15 a month, that is, if he saves – \$15 a month, how much better off because of the waste was he 6 mo. ago, that is, – 6 mo. from now?

Evidently he has wasted 6 × \$15, or \$90; that is,

$$(-6) \times (-\$15) = \$90.$$

28. Laws of Signs. From the above problems we infer the following laws:

Plus × *plus* = *plus*,

Minus × *plus* = *minus*,

Plus × *minus* = *minus*,

Minus × *minus* = *plus*.

These laws may be stated more concisely, thus:

In multiplication two like signs produce plus; two unlike signs produce minus.

Exercise 13. Multiplication

1. The water in a reservoir is d feet deep and increases i inches per day. Find the depth when $d=18$, $i=4$, and the number of days is -8. Explain what is meant.

2. In Ex. 1 find the depth when $d=10$, $i=-2$, and the number of days is 6. Explain what is meant.

3. In Ex. 1 find the depth when $d=20$, $i=-6$, and the number of days is -3. Explain what is meant.

4. In measuring the depth of the water in a harbor a sailor finds that the depth at a point P is d fathoms and that it increases regularly f fathoms for every $\frac{1}{8}$ mi. to the south and decreases at the same rate to the north. Write a formula for the depth m miles south of P. Find the depth when $d=20$, $f=2$, and $m=3$; when $d=10$, $f=2$, and $m=-2$; when $d=14$, $f=-3$, and $m=1\frac{1}{2}$; when $d=12$, $f=-2\frac{1}{2}$, and $m=-1\frac{1}{4}$.

5. If a man saves d dollars per day, how much will he save in n days? Find the amount the man will save when $d=2.50$, $n=7$; when $d=3.25$, $n=-4$. Explain the meaning of the second of these numerical cases.

Perform the following multiplications:

6. $75 \times (-37)$.

7. $-75 \times (-37)$.

8. $4.8 \times (-2.9)$.

9. $-4.8 \times (-2.9)$.

10. $6.82 \times (-3.75)$.

11. -6.72×4.8.

12. $-29.8 \times (-2.76)$.

13. $-3\frac{4}{5} \times (-2\frac{3}{5})$.

14. If a dirigible balloon makes d miles a minute against the wind, how far will it go in m minutes? Find the distance when $d=0.9$ and $m=7\frac{1}{2}$. Find the distance when $d=-0.1$ and $m=3\frac{1}{4}$, explaining the meaning of this case.

29. Division involving Negative Numbers. Since division is the opposite of multiplication, we can find the laws of signs very easily. That is,

Because $2 \times 4 = 8$, we see that $8 \div 4 = 2$.
Because $(-2) \times (-4) = 8$, we see that $8 \div (-4) = -2$.
Because $(-2) \times 4 = -8$, we see that $(-8) \div 4 = -2$.
Because $2 \times (-4) = -8$, we see that $(-8) \div (-4) = 2$.

We may state these laws as follows:

Plus ÷ *plus* = *plus*,
Plus ÷ *minus* = *minus*,
Minus ÷ *plus* = *minus*,
Minus ÷ *minus* = *plus*.

These laws may be stated more concisely, thus:

In division two like signs produce plus; two unlike signs produce minus.

Exercise 14. Division

1. A ship sails from longitude 0° to longitude $-42° 30'$. After sailing one fourth the distance, what is its longitude?

We may consider $-42° 30'$ to mean $42° 30'$ E., and by distance we mean the distance measured in degrees of longitude.

2. The longitude of A is 42° and that of B is $-68°$. Find the longitude of a place halfway from A to B.

Perform the following divisions:

3. $-3456 \div 144$.

4. $-6912 \div (-288)$.

5. $-17.28 \div 1.2$.

6. $-3.456 \div (-1.44)$.

7. $13{,}824 \div (-14.4)$.

8. $2560 \div (-0.16)$.

9. $-98.04 \div (-7.6)$.

10. $44.622 \div (-13.4)$.

11. $15.96 \div (-0.038)$.

12. $-0.6384 \div 0.42$.

13. $-11.374 \div 4.7$.

14. $-22.748 \div (-2.2)$.

II. ADDITION AND SUBTRACTION

30. Terms used. In the Introduction we have used chiefly the terms that are familiar from the work in arithmetic. There are, however, certain other terms that we must know if we are to proceed further in algebra.

If the student is familiar with Book II of this series, he knows these terms, but it is well that he should study the formal definitions here given. For this reason certain terms already familiar to the student of arithmetic are now defined. It is not so important to memorize a definition as to be able to use the word intelligently.

31. Monomial. An algebraic expression in which the parts are not separated by the signs + or − is called a *monomial.*

Thus a, $-x^2$, and $\sqrt{m}$ are monomials.

32. Polynomial. An algebraic expression consisting of two or more monomials separated by the signs + or − is called a *polynomial.*

Thus $a-b$, $4x^2+7$, and $3a+2b^2-\sqrt{c}$ are polynomials.

33. Terms of a Polynomial. The monomials that make up a polynomial are called the *terms* of the polynomial.

Thus a and $3x^2$ are the terms of the polynomials $a+3x^2$ and $a-3x^2$. Since we may write $a-3x^2$ in the form $a+(-3x^2)$, we may also speak of a and $-3x^2$ as the terms of $a-3x^2$.

34. Binomial and Trinomial. A polynomial of two terms is called a *binomial*; of three terms, a *trinomial.*

Thus $5a-7$ is a binomial, and $2x^2-3x+1$ is a trinomial.

35. Absolute Term. If a polynomial contains a numerical term, that term is called the *absolute term.*

Thus, in the polynomial a^2+3a-4 the absolute term is -4, and in the polynomial $3x^4+\sqrt{2}$ the absolute term is $\sqrt{2}$.

36. Similar Terms. Monomials that have a common factor are called *similar terms* or *similar monomials* with respect to the common factor.

Thus $2x$, $4x$, and $-9x$ are similar with respect to x; $4\sqrt{3}$ and $-\frac{4}{5}\sqrt{3}$ are similar with respect to $\sqrt{3}$; $5\cdot(-8)$ and $6\cdot(-8)$ are similar with respect to -8; $\frac{1}{2}ax$ and $\frac{1}{2}bx$ are similar with respect to x; and $3x^2$ and bx^2 are similar with respect to x^2.

Terms that are not similar are said to be *dissimilar*.

37. Factor. Any one of two or more algebraic expressions is called a *factor* of the product of those expressions.

Thus, just as 3 and 5 are factors of 15, so x and y are factors of xy; $(b+b')$ and h are factors of $(b+b')h$; and x and x are factors of x^2.

The name "factor," familiar from arithmetic, has already been introduced informally in the preceding exercises.

38. Coefficient. If an expression is the product of two factors, either factor is called the *coefficient* of the other.

Thus in the expression ab, a is the coefficient of b, and b is the coefficient of a.

In any algebraic expression the factor that is considered the coefficient is usually written first. Thus, in $2\pi r$, 2 is the coefficient of πr, and 2π is the coefficient of r.

The coefficient 1 is omitted, x being the same as $1x$.

39. Exponent. The number or letter placed to the right and slightly above another expression of number to indicate a power is called an *exponent*.

This was less formally defined on page 5.

Thus in x^3, 3 is the exponent of x; in $(a+b)^n$, n is the exponent of $a+b$; and in $3ab^x$, x is the exponent of b.

The exponent 1 is omitted, x^1 being the same as x.

The terms "coefficient" and "exponent" should be carefully distinguished. *The coefficient shows the number of equal addends; the exponent shows the number of equal factors.*

In $3a^2$, 3 is the coefficient of a^2, and 2 is the exponent of a.

40. Algebraic Sum. The result obtained by adding two or more numbers, considered with respect to their signs as well as to their values, is called their *algebraic sum.*

Thus the algebraic sum of 2 and -3 is -1, although $2+3=5$. Just as 2 ft. + 3 ft. = 5 ft., so $2 \cdot 5 + 3 \cdot 5 = 5 \cdot 5$, and $2x + 3x = 5x$. Similarly, just as 3 ft. + 4 ft. − 2 ft. = 5 ft., so $3m + 4m - 2m = 5m$.

41. Addition of Monomials. *To add similar monomials, find the algebraic sum of the coefficients of the common factor and prefix this sum to the common factor.*

For this purpose we may consider an expression like $3(a+b)$ as a monomial, as in the case of $3(a+b) + 7(a+b) = 10(a+b)$.

To add dissimilar monomials, write the terms one after another, each with its proper sign.

Thus the sum of x, $3y$, and $-z$ is $x + 3y - z$.

Exercise 15. Addition of Monomials

Examples 1 to 8, oral

1. Add 7 lb. and 3 lb.; $7l$ and $3l$; $7 \cdot 20$ and $3 \cdot 20$.

2. Add 3 yr., 2 mo., and 8 da., stating the result as a compound number. Add $3y$, $2m$, and $8d$.

Notice that the plus signs are not necessary in one result.

3. Add $3x$, $7x$, $-2x$, y, and $9y$.

Add the following:

4. $4x$, $3x$, $-7x$.

5. $9a$, $7a$, $4a$.

6. $4mn$, $7mn$, $9mn$.

7. $-2x^2$, $-3x^2$, x^2.

8. $17a^2b$, $16a^2b$, $19a^2b$.

9. $16ax^2$, $-4ax^2$, ax^2.

10. $8(a+b)$, $-2(a+b)$.

11. $3\frac{3}{4}x^2$, $-4\frac{1}{2}x^2$, $6x^2$.

12. The lengths of two lines are $2x$ and $3y$. How long a line can be formed by these lines together?

42. Addition of Polynomials. If we have to add 3 ft. 4 in. and 8 ft. 2 in., we simply add the feet and the inches separately, the result being 11 ft. 6 in. In the same way, we add algebraic expressions by adding the similar terms. For example:

$$\begin{array}{r} 2x+3y \\ 5x+\ \ y \\ \hline 7x+4y \end{array} \qquad\qquad \begin{array}{r} 9x-7y+3z \\ x+9y-2z \\ \hline 10x+2y+\ \ z \end{array}$$

To add polynomials write similar terms in the same column and add these terms, writing their sums as a polynomial.

In a case involving different powers of some letter, like x, it is customary to write the highest power first, then the next highest, and so on, as in $x^2 + 7x + 3$; or to proceed in the opposite direction as in $2 + 4x - 3x^2$.

43. Check. An operation that tends to prove the correctness of a result is called a *check* upon that result.

Thus in addition we may check the result by adding in the other direction from that first taken.

One of the best checks on the operations in algebra is the substitution of any values we please for the letters, as in the following example, where we let $x = 1$, $y = 1$, and $z = 1$.

OPERATION	CHECK
$4x+2y-4z+5$	$4+2-4+5=7$
$2x-7y+9z-3$	$2-7+9-3=1$
$6x-5y+5z+2$	$6-5+5+2=8$

Here we have simply put 1 in place of x, y, and z, in the addends and in the sum, and we have $7 + 1 = 8$. Therefore the work *checks*.

We may have an error in spite of this check, as would be the case if we should write $6y - 5x + 5z + 2$ instead of $6x - 5y + 5z + 2$, or if we should make an error in computation in the check. In case of doubt, especially in case of exponents, use other values than 1.

Exercise 16. Addition of Polynomials

1. Add the following:

$$\begin{array}{lr} 7h + 4t + 3u & 743 \\ \underline{9h + 2t + 4u} & \underline{924} \end{array}$$

Compare the two problems and the two results, and state what values must be given to h, t, and u in order to make the first the same as the second.

2. Add the following and discuss as in Ex. 1:

$$\begin{array}{lr} 9T + 3h + 5t + 2u & 9352 \\ \underline{18T + h \qquad + 7u} & \underline{18107} \end{array}$$

3. Add the following and explain the difference in the forms of the results:

$$\begin{array}{lr} 8T + 2h + 7t + 5u & 8275 \\ \underline{9T + 3h + 6t + 9u} & \underline{9369} \end{array}$$

4. Add the following and explain the difference in the forms of the results:

$$\begin{array}{ll} 3y + 2f + 7i & 3 \text{ yd. } 2 \text{ ft. } 7 \text{ in.} \\ \underline{2y + 2f + 6i} & \underline{2 \text{ yd. } 2 \text{ ft. } 6 \text{ in.}} \end{array}$$

Add the following expressions and check the results:

5. $3x^2 + 2xy + y^2$, $x^2 - 4xy + y^2$, $x^2 + 7xy + y^2$.

6. $4x^2 + y^2$, $x^2 - 2y^2$, $3x^2 + y^2$, $x^2 + 4y^2$, x^2, $5y^2$.

7. $8x^3 + x^2y + 7xy^2 + y^3$, $3x^3 - 8x^2y + 9xy^2 - 5y^3$.

8. $9x^2 - 8x + 7$, $8x^2 - 9x + 4$, $x^2 + 2x - 8$.

9. $x^4 + 3x^3 - 3x^2 + 4x - 1$, $x^3 + 3x + 1$.

10. $x^5 + x^3 - 1$, $x^4 + x^2$, $x + 2$.

11. $3x^4 + 2x^2 - 7$, $x^4 + 4x^3 + 2x^2 + 4x + 11$.

12. $x^5 - x^4 + x^3 - x^2 + x - 1$, $x^4 - x^3 + x^2 - x + 2$.

Rearrange the terms as necessary, add, and check:

13. $3x^2 + 4x - 8,\ 5x^2 - 7x + 9,\ x^2 + 4 - 5x.$

14. $4x^2 + 6x + 7,\ 9x^2 - 8x + 6,\ x - 4 + x^2.$

15. $5x - 4x^2 - 6,\ 3x^2 - x + 7,\ x^2 + 2 - 4x.$

16. $-3x^2 + 7x + 5,\ -x^2 - 2x + 3,\ 4x + 2x^2 + 8.$

17. $-8x - 7x^2 + 4,\ -20x - 15x^2 + 23,\ 3x^2 - 9.$

18. $a^2 + 2ab + b^2,\ a^2 - 2ab + b^2,\ -a^2 + 2ab - b^2,\ a^2 + b^2.$

19. $3a^3 + a^2 + 2a + 4,\ a - a^3 + a^2 - 3,\ a + a^2 - a^3 + 3.$

20. $x^2 + y^2 + z,\ x^2 - y^2 + z,\ -x^2 + y^2 - 3z,\ -x^2 - y^2 - 8z.$

21. $3a^2b + ab^2 + 8,\ a^2b + ab^2 - 9,\ a^2b - 4ab^2,\ -8a^2b.$

Simplify the following by combining similar terms:

22. $x^2 + 4y^2 - 4z^2 + 2x^2 - 7y^2 + 3x^2 - 4y^2 + z^2 - 3x^2 + 9.$

23. $4a^3 + 2a^2b + 3ab^2 + b^3 - 4a^3 + 2a^2b - 3ab^2 - b^3.$

24. $3x^3 + 7x^2 + 2x - 5 + 3x + 8x^2 + 6 - x^3 - 9x^2 - 6x.$

25. $4m^2n + mn^2 + 3mn - 2mn^2 - mn + 5m^2n.$

26. Add $5t^3 + 4t^2 + 6t + 3$ and $3t^3 + 2t^2 + 2t + 4$; also 5463 and 3224. What do the polynomials equal if $t = 10$?

27. How much is 5 ft. + 8 ft.? $5f + 8f$? a feet + b feet? $af + bf$? $ax + bx$? $am + bm$? $5 \cdot 5 + 8 \cdot 5$? $a \cdot 5 + b \cdot 5$?

We may think of the sum of a feet and b feet as $(a + b)$ feet.

Similarly, $$af + bf = (a + b)f.$$

That is, we do not know the numerical value of the coefficients a and b, so we *indicate* their sum.

Add as directed, indicating the sums of the coefficients:

28. p miles + q miles; $pm + qm$; $px + qx$; $p \cdot 8 + q \cdot 8.$

29. $ax + bx$; $ay + by$; $ax^2y + bx^2y$; $a \cdot 3 \cdot 4 + b \cdot 3 \cdot 4.$

30. $x\sqrt{y} + 3\sqrt{y}$; $mp^2 + np^2$; $x\sqrt{a^2 + b} + y\sqrt{a^2 + b}.$

Exercise 17. Equations involving Addition

Examples 1 to 9, oral

1. What must be added to each member of the equation $9x - 7 = 4 - 3x$ in order to have the x's in one member and the absolute terms in the other?

Reduce to a form convenient for solving, as in Ex. 1:

2. $4x - 2 = 4 - 3x$.

3. $7x - 20 = 40 - 2x$.

4. $9x - 30 = 60 - 4x$.

5. $3x - 20 = 80 - 8x$.

6. $21x + 4x = 16$.

7. $30x - 8 + 2x = 42$.

8. $10x - 4 + 3x = 70$.

9. $40x - 8 + 4x = 80$.

Solve the following, and check each result by substituting in the original equation:

10. $3x + 7 = 2x + 9$.

11. $7x + 2 = 2x + 77$.

12. $9x - 8 = 3x + 40$.

13. $4x - 32 = 48$.

14. $4.7x - 9.2 = 3.7x + 7$.

15. $26x - 122 = -18$.

16. If from six times a certain number we subtract 16.4, the result is 13.6. What is the number?

17. If from nine times a certain number we subtract 4.3, the result is 20. What is the number?

18. If from 42 times a certain sum of money we take \$10, there is left \$200. What is the sum of money?

19. If from 16 times a certain distance we take 29.9 ft., there is left 50.1 ft. What is the distance?

20. If to twice a certain number we add 28, the result is 308. Find the number.

21. Write a problem similar to one of the above, and then write the equation and solve it.

44. Subtraction of Monomials. We find the difference between two monomials very much as we find the difference between 5 ft. and 3 ft.; that is, just as 5 ft. – 3 ft. = 2 ft., so $5f - 3f = 2f$, $5x - 3x = 2x$, and $5\sqrt{a} - 3\sqrt{a} = 2\sqrt{a}$.

The simplest way to obtain the result is to find the quantity which added to the subtrahend will produce the minuend. In the case of $5x - 3x$ we see that $2x$ must be added to $3x$ to make $5x$.

To subtract a monomial from a similar monomial, find the difference between the coefficients of the common factor and multiply this difference by the common factor.

Thus, $12x - x = 11x$ and $mx^3 - 8nx^3 = (m - 8n)x^3$.

To subtract a monomial from a dissimilar monomial, merely indicate the subtraction.

Thus, as we may write 9 ft. – 4 in., so we may write $9f - 4i$, or $9x - 4y$, or $9\sqrt{x^2 + y} - 4\sqrt{x + y^2}$.

Exercise 18. Subtraction of Monomials

Examples 1 to 3, oral

1. What must be added to 9 to make 16? to $9a^2b^3$ to make $16a^2b^3$? How much is $16a^2b^3 - 9a^2b^3$?

2. What must be added to $-4°$ to make $9°$? to $-4x$ to make $9x$? How much is $9p - (-4p)$?

3. What must be added to $-8°$ to make $5°$? to $-8x$ to make $5x$? How much is $5x - (-8x)$?

Find the value of each of the following:

4. $\frac{3}{8}a - \frac{1}{3}a$.

5. $4.1\text{ M} - 0.7\text{ M}$.

6. $\sqrt{a} - 3\sqrt{a}$.

7. $4a - (-5a)$.

8. $8\sqrt{ab} - 9\sqrt{ab}$.

9. $3(a^2 - b^2) - 7(a^2 - b^2)$.

10. The line x being longer than the line y, express the difference in their lengths.

45. Subtraction of Polynomials. If we have to subtract 3 ft. 7 in. from 12 ft. 11 in., we simply subtract the feet and inches separately, the result being 9 ft. 4 in. In the same way we subtract polynomials. For example:

21 hr.	16 min.	38 sec.		$21x+16y+38z$
12 hr.	9 min.	29 sec.		$12x+9y+29z$
9 hr.	7 min.	9 sec.		$9x+7y+9z$

To subtract one polynomial from another, arrange similar terms under one another and subtract these terms separately.

For example, subtract $5x^2-8xy+2$ from $9x^2-7-4xy$. Rearranging, we have the following:

Operation	Check
$9x^2-4xy-7$	$9-4-7=-2$
$5x^2-8xy+2$	$5-8+2=-1$
$4x^2+4xy-9$	$4+4-9=-1$

Here the work is checked by letting $x=1$ and $y=1$. If a check for the exponents is desired, use other values than 1 for x and y.

Exercise 19. Subtraction of Polynomials

1. Perform the following subtractions:

$$\begin{array}{r} 9h+3t+7u \\ 6h+2t+u \\ \hline \end{array} \qquad \begin{array}{r} 937 \\ 621 \\ \hline \end{array}$$

Compare the two problems and the two results, and state what values must be given to h, t, and u in order to make the first the same as the second.

2. Subtract and discuss as in Ex. 1, explaining the difference in the forms of the results:

$$\begin{array}{r} 15T+9h+7t+3u \\ 6T+9h+4t+7u \\ \hline \end{array} \qquad \begin{array}{r} 15{,}973 \\ 6{,}947 \\ \hline \end{array}$$

Rearrange the terms as necessary, subtract, and check:

3. $x^2 - 1 + x$ from $2 - x^2 + 3x$.

4. $m^2 + n + 1$ from $3 + 7m^2 + 4n$.

5. $8a^2 - 17a + 3$ from $52a - 9a^2 + 46$.

6. $15M^2 - 14M + 7$ from $27M^2 - 14M + 2$.

7. $a^3 - 3a^2b + b^2$ from $a^3 + 3a^2b - b^2$.

8. $17p^2 + 15pq - q^2$ from $7p^2 - 15pq + 15q^2$.

9. $3(a + b) + 5$ from $9(a + b) + 27$.

10. $9(a + b) + 27$ from $3(a + b) + 5$.

11. $ax^2 + bx + c$ from $px^2 + qx + r$.

12. $17am + 19pq + 3$ from $20am - 8pq + 7$.

13. $42a^3 - 17a^2b + 15ab^2 - b^3$ from $a^3 - b^3$.

14. $a^3 - 3a^2b + 3ab^2 - b^3$ from $3a^2b - 3ab^2$.

15. $P^2 - 3Px + 0.7x^2$ from $P^2 + 2Px - 7$.

16. $K^2 + 0.32K$ from $K^3 + K^2 - 0.32K + 7$.

17. $4a^2 - 8c^2 + 7b^2$ from $9b^2 - 6c^2 - 8a^2$.

18. $3ax + cz - 2by$ from $ax + 4by - 6cz$.

19. $a^2b^2 + 3m^2n^2 - 4x^2y^2$ from $6m^2n^2 + 8x^2y^2 - 2a^2b^2$.

20. $a^2 + 4b^2 - 3c^2 + 9d^2$ from $8a^2 - 2d^2 - 6c^2$.

If $A = 4a^2 + 2ab - b^2$, $C = -3a^2 + 8ab$,
$B = 6a^2 - 5ab + 8b^2$, $D = -2a^2 - 5ab - 3b^2$,

find the result in each of the following cases, and check:

21. $D - C$. **26.** $A - B + C$. **31.** $A + B + C + D$.

22. $C - D$. **27.** $A + C - B$. **32.** $A + B - C + D$.

23. $A - C$. **28.** $B + C - A$. **33.** $A - B + C - D$.

24. $C - A$. **29.** $B - A + C$. **34.** $A - B - C - D$.

25. $B - C$. **30.** $C - D + A$. **35.** $A + B + C - D$.

46. Removal of Parentheses. We have found that we do not need to change the signs of any terms when we add. For example, if we add $7a + b$ to $a + 3b$, the result is $8a + 4b$. If we should write this in the form

$$(7a + b) + (a + 3b),$$

we could just as well remove the parentheses and write

$$7a + b + a + 3b.$$

It is therefore easy to see that

$$\boldsymbol{a + (b + c) = a + b + c,}$$

and $$\boldsymbol{a + (b - c) = a + b - c.}$$

If the parentheses inclosing an expression are preceded by the positive sign, the parentheses may be removed without any change in the signs of the terms.

We may treat in this manner any other symbol of aggregation. Thus $8 + \overline{2 + 6} = 8 + 2 + 6$, and $32 + [8 - 5] = 32 + 8 - 5$.

If we subtract $b + c$ from a, and $b - c$ from a, we have the following results:

$$\begin{array}{l} a \\ \underline{b + c} \\ a - b - c \end{array} \qquad\qquad \begin{array}{l} a \\ \underline{b - c} \\ a - b + c \end{array}$$

We therefore see that

$$\boldsymbol{a - (b + c) = a - b - c,}$$

and $$\boldsymbol{a - (b - c) = a - b + c.}$$

If the parentheses inclosing an expression are preceded by the negative sign, the parentheses may be removed, provided the sign before each term is changed.

Thus $15a^2 - 17ab + b^2 - (3a^2 - b^2) = 15a^2 - 17ab + b^2 - 3a^2 + b^2$
$= 12a^2 - 17ab + 2b^2.$

Exercise 20. Removal of Parentheses

Examples 1 to 7, oral

Remove the parentheses and simplify the results:

1. $9+(8+2)$.
2. $6+(5+3)$.
3. $8+(7-4)$.
4. $8x+(4x+x)$.
5. $3p+(9p+4p)$.
6. $8a^2+(3a^2-2a^2)$.

7. How much does $10+4$ become when increased by $8+3$? How much is the sum of 10, 4, 8, and 3?

Remove the parentheses and simplify the results:

8. $(3a+a)+(8a+a)$.
9. $(a+6a)+(4a-a)$.
10. $(8x+y)+(x-3y)$.
11. $3a-b+(a+4b)$.
12. $32a-(8a-b)+(4a-3b)$.
13. $35xy-(25xy+8)-(6xy-9)$.
14. $75a^2b^2-(a^2b^2+c)+(8a^2b^2-3c)$.
15. $(a^2-b)-(b-c)-(c-d)-(d-e)-(e-a^2)$.
16. $a^2-(b-c)+b-(c-d)+c-(d-a^2)$.
17. $a^2-2ab+b^2-(a^2+2ab+b^2)$.
18. $a^3-3a^2b+3ab^2-b^3-(a^3+3a^2b+3ab^2+b^3)$.
19. $P^4+P^3-3P^2-4P+7-(4P^3+2P^2-4P-7)$.
20. $4a^2-17a+16-(3a^2+16a-4)+(17a-3)$.
21. $6x+8x^2-9-(x+3)+(x^2-7)-(7-x)$.
22. $4+3x-(x^2-1)+(x+7)-(x^2-3x+4)+(x+3)$.
23. $x^3-3x^2y+3xy^2-y^3-(x^3+3x^2y+3xy^2-y^3)$.
24. $a^2b^2c^2-(a^2+b^2+c^2)-(1-a^2b^2c^2)-2a^2b^2c^2$.

25. Indicate by using parentheses the subtraction of x^3+x^2-7 from x^3-2x^2+7. Then remove the parentheses and simplify the result.

47. Removal of Several Symbols of Aggregation. The symbols of aggregation used most frequently in algebra, and mentioned informally on page 11, are the following:

Parentheses, as in $a-(b+c)$ *Brackets*, as in $p-[q-r]$
Bar or *vinculum*, as in $x-\overline{y-z}$ *Braces*, as in $m-\{m+n\}$

When one symbol of aggregation incloses another, it is usually simpler to remove the inner symbol first.

1. Simplify the expression $10-(4-\overline{3-2})$.

$$\begin{aligned}10-(4-\overline{3-2}) &= 10-(4-1)\\ &= 10-3\\ &= 7.\end{aligned}$$

2. Simplify the expression $20a-[10a-(a-b)]$.

$$\begin{aligned}20a-[10a-(a-b)] &= 20a-[10a-a+b]\\ &= 20a-[9a+b]\\ &= 20a-9a-b\\ &= 11a-b.\end{aligned}$$

Cases in which more than one set of symbols of aggregation is inclosed within another are so rare that they need have no serious consideration in the school. In the important work with formulas the above examples suffice as types.

When symbols of aggregation are removed, the signs of the terms affected should be changed if necessary.

48. Insertion of Parentheses. From what we have learned of the removal of parentheses we see that

Two or more terms may be inclosed in parentheses preceded by a plus sign without changing the signs of the terms.

Two or more terms may be inclosed in parentheses preceded by a minus sign, provided the sign of each term is changed.

We rarely have to insert parentheses, and the case may be dismissed with brief mention.

Exercise 21. Symbols of Aggregation

Remove the symbols of aggregation and simplify:

1. $7-(3-2)+12-(5+3)$.

2. $19-3(2+4)+5(8-3)$.

3. $6x^2-[x^2-(x^2-4xy+y^2)+7xy-6y^2]$.

4. $9a+8-(3a-6)-[a-8-(4a-5)]$.

5. $7x^3-(8x^2y-4xy^2)+[y^3-3x^3+(6x^2y-xy^2+2y^3)]$.

6. $5a-[a+4b-(c-3d+e-2a)+c]-5b-3c+d$.

7. $45a^2-8a^2+(b-c-8c)+(6a-b-\overline{3c-2b})$.

8. $9a-(3a-b)+2c+4c-(4a+\overline{a-b}+6c)$.

9. $5ab+(7ab-ab+1)+[ab-(ab+ab-3)]$.

10. $(a^2+2ab+b^2)-(a^2-\overline{2ab+b^2})-(-a^2+\overline{2ab-b^2})$.

11. $35a^2b-(25ab^2+32)-[5a^2b-2+(5ab^2-2)]$.

Remove the parentheses, leaving the brackets:

12. $[x^2-(3x^2+y^2)]\times[x^2-(3x^2-y^2)]$.

13. $[a^2b^3-(c^3d^4+8)]\times[a^2b^3-(c^3d^4-8)]$.

14. Inclose the last two terms of $x^3-3x^2y+3xy^2-y^3$ in parentheses, without changing the value of the expression. Do the same for $x^3+3x^2y-3xy^2+y^3$.

In Exs. 14 and 15 change the signs when necessary.

15. In the expression $a^4-4a^3b+6a^2b^2-4ab^3+b^4$ inclose the last three terms in parentheses without changing the value. Do the same for the last two terms.

Remove the symbols of aggregation and simplify:

16. $8.2x^2-3.4x+5.8-[3.4x^2-(7.8x-4.3)]$.

17. $5.43m^3-[3.82m^3-4.3m^2-(2.9m^3+0.4m^2-9)]$.

18. $4.8x^2y^2+1.7-[3.8x^2y^2+(8.2x^2y^2+3)]$.

Exercise 22. Equations involving Subtraction

Examples 1 to 10, oral

1. In the equation $8x = 15 + 3x$, what do we first subtract from both members? By what do we then divide?

Solve the following equations:

2. $4x = 18 + x$.

3. $6x = 16 + 2x$.

4. $15x = 32 + 7x$.

5. $35x = 36 + 17x$.

6. $42x = 81 + 33x$.

7. $33x = 3 + 3x$.

8. $22x = 2.1 + x$.

9. $35x = 3.5 + 28x$.

10. In the equation $9x + 35 = 5x + 39$, what literal term should we first subtract from both members? Then what numerical term? By what do we then divide?

Solve the following equations:

11. $23x + 4 = 3x + 84$.

12. $42x + 8 = 3x + 47$.

13. $19x + 5 = 15x + 25$.

14. $7.2x + 4 = 6.9x + 16$.

15. $1.4x + 9 = 0.8x + 45$.

16. $4.5x + 6.8 = 3.9x + 12.8$.

17. If 10 is added to 16 times a certain number, the result is 58. What is the number?

18. If 14 is added to 12 times a certain number, the result is 86. What is the number?

19. If from 12 times a certain number twice the number is taken, the result is 700. What is the number?

20. If from 30 times a certain number 14 times the number is taken, the result is 48. What is the number?

21. If five times a boy's age plus seven times his age is 180 yr., how old is the boy?

22. Make and solve a problem similar to Ex. 21.

Exercise 23. Review

Add the following expressions and check the results:

1. $4x^2 - 7x + 2$, $3x^2 - 5x - 6$, and $x^2 + 4x - 5$.

2. $7 + 3x - x^2$, $3x - 4x^2 + 2$, and $8 - 2x + 5x^2$.

3. $\frac{1}{2}x - \frac{1}{3}y$, $\frac{1}{4}x + \frac{1}{5}y$, $3x + y$, and $\frac{13}{15}y + \frac{1}{4}x$.

Subtract in the following cases and check the results:

4.	**5.**	**6.**
$7b + 3p + 2q$	$2h + 7t + 5u$	$5h + 9t + 6u$
$5b - 2p - 6q$	$h + t + u$	$3h + 2t + 4u$

In Ex. 5 notice the similarity to the case of 275 − 111.

7. A coal dealer bought a carloads of coal averaging 42 tons each and b carloads averaging 48 tons each. He sold $3a$ tons to one customer, 8 tons to another, and $7b$ tons to another. How many tons had he left?

Remove the parentheses and simplify the following:

8. $7t + 8u + (5t + u) - (4t + 5u)$.

9. $5a^2 - 7ab - 3ac - (2a^2 + 4ab - 3ac)$.

10. $2p - (3q + 4p) + (4q - p) - (q - 3p)$.

11. $5a^2 - (3ab + b^2) - (4a^2 - \overline{2ab - 3b^2})$.

12. Express 321, 473, 502, and 365 algebraically, using h, t, and u to represent hundreds, tens, and units respectively. Then add and check the result.

13. What must be added to $4a^2 - 3b + c$ to make zero?

14. What must be subtracted from $4a^2 - 3b + c$ to leave zero?

15. What must be added to $4x^2 - 3xy + y^2$ to make $-3x^2$?

III. MULTIPLICATION

49. Multiplication. We often need to multiply one polynomial by another, but before we can state these cases so that they will be clearly understood we should know how to multiply one monomial by another.

In the simplest cases this is quite like the multiplication of denominate numbers.

For example, just as 3×5 ft. $= 15$ ft., so $3 \times 5x = 15x$.

If negative signs are involved, we proceed as in § 28.

For example, just as $-3 \times -7 = 21$, so $-3 \times (-7x) = 21x$.

If exponents are involved, we need only consider what they mean, and the multiplication is usually very simple.

For example, $a^3a^4 = aaa \times aaaa$; that is, a is used seven times as a factor, and so the result is a^7.

50. Laws of Multiplication. From the above examples the following laws of multiplication are easily understood.

1. *In multiplying monomials the exponent of any letter in the product is equal to the sum of the exponents of that letter in the factors.*

That is, $$a^m \cdot a^n = a^{m+n}.$$

This is apparent from § 49, because if a is used m times as a factor, and also n times as a factor, it is used in all $m + n$ times as a factor.

2. *A power of a power of a number is equal to the power of the number indicated by the product of the exponents.*

That is, $$(a^m)^n = a^{mn}.$$

For example, $(a^3)^2 = (aaa)^2 = aaa \times aaa = aaaaaa = a^6$.

3. *In multiplication two like signs produce plus; two unlike signs produce minus.*

That is, $$+a \cdot (+b) = +ab \qquad +a \cdot (-b) = -ab$$
$$-a \cdot (-b) = +ab \qquad -a \cdot (+b) = -ab$$

51. Multiplication of a Polynomial by a Monomial. If we multiply 4 ft. 3 in. by 2, we have 8 ft. 6 in. In the same way, if we multiply 4 times one number plus 3 times another number by 2, we evidently have 8 times the first plus 6 times the second. That is,

4 ft. 3 in.	$4f+3i$	$4x+3y$
2	2	2
8 ft. 6 in.	$8f+6i$	$8x+6y$

In the same way we have the following:

Operation	Check
$a^2 - 2ab + 3b^2$	$1-2+3=2$
ab	$1=1$
$a^3b - 2a^2b^2 + 3ab^3$	$1-2+3=2$

In algebra it is just as convenient to write the multiplier at the left and work from left to right if one cares to do so.

If a check upon the exponents is desired, let $a=2$ and $b=2$, or take other values. In case either factor becomes zero in the check, use some other values for the letters.

To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial and add the partial products.

Exercise 24. Multiplication by a Monomial

Examples 1 to 4, oral

1. $10(10+7)$.

2. $-8x^2(x^2-x)$.

3. $p(p+q-r)$.

4. $-3a(a^2-2)$.

5. $-8a^2b(a^2-2ab+b^2)$.

6. $25x^2(x^2-2xy+y^2)$.

7. $75a^3x^3(a^3-3a^2x+3ax^2-x^3)$.

8. $-22x^ny^n(x^ny^n-3xy-20)$.

9. What is the area of a rectangle whose base is $a+3b$ and height $2a$?

52. Multiplication of a Polynomial by a Polynomial. If we multiply 43 by 21, we multiply first by 1 unit and then by 2 tens, and then add the partial products, thus:

Multiplicand	43	$40 + 3$
Multiplier	21	$20 + 1$
Multiplying by 1 unit	43	$40 + 3$
Multiplying by 2 tens	86	$800 + 60$
Sum of partial products	903	$800 + 100 + 3$

In a similar way we multiply $4x^2 + 3x$ by $2x^2 + x$, thus:

Operation	Check
$4x^2 + 3x$	$4 + 3 = 7$
$2x^2 + x$	$2 + 1 = 3$
$8x^4 + 6x^3$	21
$4x^3 + 3x^2$	
$8x^4 + 10x^3 + 3x^2$	$8 + 10 + 3 = 21$

To multiply a polynomial by a polynomial, multiply the multiplicand by each term of the multiplier and add the partial products.

Before multiplying arrange both polynomials *according to the descending powers* of some letter. If we prefer, we may arrange both polynomials *according to the ascending powers* of some letter.

For example, in multiplying $7 - x^2 + 8x$ by $x + 2 - x^2$ we arrange the two polynomials in either of the following ways, preferably the first:

$-x^2 + 8x + 7$	$7 + 8x - x^2$
$-x^2 + x + 2$	$2 + x - x^2$

While it would be possible to perform the multiplication without this systematic arrangement, the work would be more confusing and the chance for errors would be greater.

It often happens that a polynomial can be arranged according to the descending powers of one letter and the ascending powers of another letter, as in the case of $x^3 - 3x^2y + 3xy^2 - y^3$.

Exercise 25. Multiplication of Polynomials

Perform the following multiplications, and check:

1. $(x+y)(x+y)$.
2. $(x+y)(x-y)$.
3. $(x-y)(x+y)$.
4. $(x-y)(x-y)$.
5. $(2x+y)(2x-y)$.
6. $(a+b)(a^2+b^2)$.
7. $(a^2+b^2)(a^2+b^2)$.
8. $(x^2+y^2)(x^2-y^2)$.
9. $(2x^2+y)(2x^2-y)$.
10. $(-3x^2+5)(3x^2-5)$.

11. Multiply a^2b^2+1 by a^2b^2-1, and then multiply the product by a^4b^4+1.

Perform the following multiplications, and check:

12. $(a+b)(a^2+2ab+b^2)$.

13. $(a-b)(a^2-2ab+b^2)$.

14. $(x+y)(x+y)(x+y)$; $(x+y)(x^2+2xy+y^2)$.

15. $(x-y)(x-y)(x-y)$; $(x-y)(x^2-2xy+y^2)$.

16. $(2P^2+3P+4)(4P^2-7P+6)$.

17. $(125m^2n+32mn^2)(m^3-2m^2n+3mn^2-n^3)$.

18. $(a^3-3a^2b+3ab^2-b^3)(a^2-2ab+b^2)$.

19. $(a^2b^2c^2-3abc+4)(a^2b^2c^2+4abc-7)$.

Arrange the following in convenient form and then perform the multiplications:

20. $(a^3-4b^3+3ab^2-4a^2b)(a^2b-a^3+b^3+ab^2)$.

21. $(x^4+y^4-3x^3y-3xy^3)(4xy^3+2x^3y+x^4-y^4)$.

22. $(x^4+3xy^2+x^2y^2-3x^2y+y^4)(xy+y^2+x^2)$.

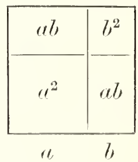
23. Multiply $3h+4t+2u$ by $6t+7u$, and also multiply 342 by 67. State why the coefficients in the first product are not the same as the figures in the second product.

53. Square of the Sum or Difference of Two Numbers. The product of $a+b$ by $a+b$ is $a^2+2ab+b^2$, and the product of $a-b$ by $a-b$ is $a^2-2ab+b^2$, as shown below.

$$\begin{array}{lll} a & + & b \\ a & + & b \\ \hline a^2 & + & ab \\ & & ab+b^2 \\ \hline a^2 & + & 2ab+b^2 \end{array} \qquad\qquad \begin{array}{lll} a & - & b \\ a & - & b \\ \hline a^2 & - & ab \\ & - & ab+b^2 \\ \hline a^2 & - & 2ab+b^2 \end{array}$$

The square of the sum of two numbers is the square of the first, plus twice their product, plus the square of the second.

That is, $\mathbf{(a+b)^2 = a^2 + 2ab + b^2}$.



This is easily seen from the annexed figure.

For example, $16^2 = (10+6)^2$

$= 10^2 + 2 \times 10 \times 6 + 6^2$

$= 100 + 120 + 36 = 256.$

The square of the difference of two numbers is the square of the first, minus twice their product, plus the square of the second.

That is, $\mathbf{(a-b)^2 = a^2 - 2ab + b^2}$.

For example, $47^2 = (50-3)^2 = 50^2 - 2 \times 50 \times 3 + 3^2 = 2209.$

Exercise 26. Square of the Sum or Difference

All work oral

State the results of the following:

1. 11^2.	**5.** $(x+a)^2$.	**9.** $(p-q)^2$.	**13.** $(2m+5)^2$.
2. 21^2.	**6.** $(x+1)^2$.	**10.** $(2p-q)^2$.	**14.** $(5m-2)^2$.
3. 31^2.	**7.** $(a+5)^2$.	**11.** $(5m-1)^2$.	**15.** $(2-5m)^2$.
4. 22^2.	**8.** $(p+q)^2$.	**12.** $(5m+1)^2$.	**16.** $(x+\frac{1}{2})^2$.

17. What is the area of a square whose side is $b+d$?

54. Square Root of a Trinomial. Since we have shown

that $(a+b)^2 = a^2 + 2ab + b^2$

and $(a-b)^2 = a^2 - 2ab + b^2$,

we see that $a + b = \sqrt{a^2 + 2ab + b^2}$

and $a - b = \sqrt{a^2 - 2ab + b^2}$.

Therefore, if a trinomial is a perfect square, that is, if it is the product of two equal binomials, we can easily factor it and thus find its square root. Thus we see that

$$\sqrt{25x^2 + 30x + 9} = \sqrt{(5x+3)(5x+3)} = 5x + 3$$

because $\sqrt{25x^2} = 5x$, $\sqrt{9} = 3$, and $30x = 2 \times 5x \times 3$.

The student should compare $25x^2 + 30x + 9$ with $a^2 + 2ab + b^2$ and understand clearly how the square root is found.

Although $4 = (-2) \times (-2)$ as well as 2×2, so that the square root of 4 is either 2 or -2, often written ± 2 and read "plus or minus 2," we do not use the sign $\pm$ before the 2 unless we place it before the $\sqrt{4}$; that is, $\sqrt{4} = 2$ and $-\sqrt{4} = -2$.

Exercise 27. Square Roots

By the aid of factoring find the square root of each of the following and check the result by squaring the root found:

1. $x^2 + 6x + 9$.
2. $p^2 + 14pq + 49q^2$.
3. $49p^2 - 14pq + q^2$.
4. $36x^2 + 12x + 1$.
5. $4t^2 + 4t + 1$.
6. $4 \cdot 10^2 + 4 \cdot 10 + 1$.
7. $121m^2 - 44m + 4$.
8. $225x^2 + 30x + 1$.

9. Find the side of a square of area $16a^2 - 40ab + 25b^2$.

10. Find the square roots of $9x^2 + 6x + 1$ and 961.

11. Find the square roots of 441, 121, and 2704.

12. What term added to $x^2 + 4x$, $x^2 - 6x$, and $a^2 + 8a$ will make each a perfect square?

55. Product of the Sum and Difference of Two Numbers. If we multiply $a - b$ by $a + b$, or $a + b$ by $a - b$, we find that the product is $a^2 - b^2$, as is here shown.

$$\begin{array}{ll} a - b & \\ a + b & \\ \hline a^2 - ab & \\ \quad\; ab - b^2 & \\ \hline a^2 \quad\;\; - b^2 & \end{array} \qquad\qquad \begin{array}{ll} a + b & \\ a - b & \\ \hline a^2 + ab & \\ \quad - ab - b^2 & \\ \hline a^2 \quad\;\; - b^2 & \end{array}$$

Therefore, *the product of the sum and difference of two numbers is the difference of their squares.*

That is, $$(a + b)(a - b) = a^2 - b^2.$$

For example, $(7x + 3b)(7x - 3b) = 49x^2 - 9b^2$,

and $43 \times 37 = (40 + 3)(40 - 3) = 1600 - 9 = 1591$.

Exercise 28. Product of the Sum and Difference

All work oral

State the following products:

1. $(a + b)(a - b)$.
2. $(m + n)(m - n)$.
3. $(a + 5)(a - 5)$.
4. $(a + 9)(a - 9)$.
5. $(3a + 4)(3a - 4)$.
6. $(a^4 + 2)(a^4 - 2)$.
7. $(3a^2 + 1)(3a^2 - 1)$.
8. $(2p^2 + 1)(2p^2 - 1)$.
9. $(5x^3 + 1)(5x^3 - 1)$.
10. $(x + 1)(1 - x)$.
11. $(a + b)(b - a)$.
12. $(a^m b^m + 1)(a^m b^m - 1)$.
13. Multiply $30 + 2$ by $30 - 2$; 32 by 28.

State the following products:

14. 31×29.
15. 33×27.
16. 42×38.
17. 51×49.
18. 61×59.
19. 62×58.
20. 71×69.
21. 82×78.
22. 97×103.

56. Factoring the Difference of Two Squares. The difference of two squares is always factorable (§ 55).

The difference of the squares of two numbers is the product of the sum and difference of the numbers.

That is, $a^2 - b^2 = (a + b)(a - b).$

Factor the binomial $49x^2 - 36y^2$.

The sum of the square roots of $49x^2$ and $36y^2$ is $7x + 6y$, and their difference is $7x - 6y$.

Therefore $49x^2 - 36y^2 = (7x + 6y)(7x - 6y)$.

One great object in factoring is to arrange an expression in a form more suitable for computation.

Thus, to find the value of $75^2 - 15^2$, we can more easily use the form $(75 + 15)(75 - 15)$, for this reduces to 90×60, or 5400.

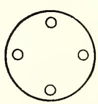
Exercise 29. Factoring the Difference of Two Squares

Examples 1 to 10, oral

Factor the following expressions:

1. $p^2 - q^2$.	6. $19^2 - 6^2$.	11. $144x^2 - 25$.
2. $a^2 - 9$.	7. $25 - x^2y^2$.	12. $169a^2 - 36b^2$.
3. $17^2 - 5^2$.	8. $36p^2 - 4q^2$.	13. $196p^2 - 9q^2$.
4. $81^2 - 9^2$.	9. $49 - 4a^2b^2$.	14. $\frac{4}{9}x^2y^2z^2 - \frac{9}{4}$.
5. $73^2 - 7^2$.	10. $a^2b^2c^2 - d^2$.	15. $2\frac{1}{4}a^2 - 6\frac{1}{4}b^2$.

16. Arrange the expression $127.8^2 - 29.4^2$ in a form more convenient for computation.



17. A metal plate is 16 in. in diameter and has four holes, each of which is 2 in. in diameter. Arrange the statement of the area of the surface in a convenient form for computation, and then find this area.

57. Special Case of the Difference of Two Squares. The terms mentioned in § 56 may be the squares of polynomials.

1. Factor $(x + 3y)^2 - 16z^2$.

Taking the square roots of $(x + 3y)^2$ and $16z^2$, we have

$$(x + 3y)^2 - 16z^2 = (x + 3y + 4z)(x + 3y - 4z).$$

2. Factor $a^4b^4c^4 - (x - y)^2$.

$$\begin{aligned} a^4b^4c^4 - (x - y)^2 &= [a^2b^2c^2 + (x - y)][a^2b^2c^2 - (x - y)] \\ &= (a^2b^2c^2 + x - y)(a^2b^2c^2 - x + y). \end{aligned}$$

3. Factor $(a + b)^2 - (x - y)^2$.

$$\begin{aligned} (a + b)^2 - (x - y)^2 &= [(a + b) + (x - y)][(a + b) - (x - y)] \\ &= (a + b + x - y)(a + b - x + y). \end{aligned}$$

Exercise 30. Factoring the Difference of Two Squares

Examples 1 to 3, oral

1. Factor $(x + y)^2 - z^2$, $(x - y)^2 - z^2$, and $(a + b)^2 - 4$.
2. Factor $(p + q + r)^2 - a^4$ and $(p - q + r)^2 - m^2$.
3. Arrange $(3x + y)^2 - z^2$ for convenient computation.

Factor the following expressions:

4. $(a + 3)^2 - b^2$.
5. $(a + b)^2 - 9$.
6. $(a + b)^2 - 4c^2$.
7. $(a - b)^2 - 4c^2$.
8. $(x - y)^2 - (a + b)^2$.
9. $(x - y)^2 - (a - b)^2$.

Arrange in form convenient for computation:

10. $8^2 - (5 - 2)^2$.
11. $9^2 - (6 + \frac{3}{4})^2$.
12. $75.6^2 - (3.4 + 6.3)^2$.
13. $(25.8 + 4.2)^2 - (9.7 + 6.3)^2$.

Reduce to lowest terms by canceling common factors:

14. $\dfrac{p^2 - (q + r)^2}{(p + q + r)^2}$.
15. $\dfrac{(w + x)^2 - (y + z)^2}{w + x + y + z}$.

58. Product of Two Binomials. Consider the product of two binomials having a common term, thus:

$$\begin{array}{ll} x+5 & \\ x+3 & \\ \hline x^2+5x & \\ \quad\; 3x+15 & \\ \hline x^2+8x+15 & \end{array} \qquad \begin{array}{l} x+a \\ x+b \\ \hline x^2+ \quad ax \\ \qquad\quad bx+ab \\ \hline x^2+(a+b)x+ab \end{array}$$

The product of two binomials having a common term is the square of the common term, plus the product of the common term by the sum of the other terms, plus the product of the other terms.

That is, $\mathbf{(x+a)(x+b)=x^2+(a+b)x+ab}$.

Thus $(x+5)(x-3)=x^2+(5-3)x+5(-3)=x^2+2x-15$.

Similarly, $(10+a)(10+b)=100+10(a+b)+ab$

$=10(10+a+b)+ab$.

Hence $17\times 19=10(10+7+9)+63$

$=260+63=323$.

Exercise 31. Product of Two Binomials

Examples 1 to 15, oral

State the product in each of the following cases:

1. $(x+1)(x+3)$.
2. $(x+7)(x-3)$.
3. $(x+4)(x+6)$.
4. $(a+7)(a-2)$.
5. $(10+2)(10-1)$.
6. 12×17.
7. 13×15.
8. 16×16.
9. 17×18.
10. 15×19.
11. 12×14.
12. 13×14.
13. 13×17.
14. 16×18.
15. 16×19.

Find the product in each of the following cases:

16. $(x+24)(x+37)$.
17. $(x-58)(x+75)$.
18. $(p+120)(p+140)$.
19. $(a-135)(a-250)$.

59. Factoring a Quadratic Trinomial. A trinomial of the form $x^2 + bx + c$ is called a *quadratic trinomial.*

We consider first the product of two binomials.

$$\begin{array}{ll} x + 4 & x + a \\ x + 2 & x + b \\ \hline x^2 + 4x & x^2 + ax \\ \quad\; 2x + 8 & \qquad\; bx + ab \\ \hline x^2 + 6x + 8 & x^2 + (a + b)x + ab \end{array}$$

Hence the factors of $x^2 + 6x + 8$ are $x + 4$ and $x + 2$, and the factors of $x^2 + (a + b)x + ab$ are $x + a$ and $x + b$.

Hence, if a trinomial of the form $x^2 + px + q$ is factorable, the first term of each factor is x; and the second terms of the factors are the two numbers whose product is q and whose sum is p, the coefficient of x.

Factor $x^2 - 2x - 15$.

Since the product of the second terms is -15, one of these terms must be positive and the other negative. The sum of the two numbers is -2, hence the negative number must have the greater numerical value.

The two numbers whose product is -15 and whose sum is -2 are evidently 3 and -5.

Therefore $x^2 - 2x - 15 = (x + 3)(x - 5)$.

We check the result by multiplying $x - 5$ by $x + 3$.

60. Directions for Factoring $x^2 + bx + c$. In factoring a trinomial of the form $x^2 + bx + c$, we proceed as follows:

Find two monomials whose product is the absolute term with its proper sign, and whose sum is the coefficient of x with its proper sign.

Write for the factors two binomials, the first term of each being x, and the second terms being, respectively, the monomials thus found.

Exercise 32. Factoring Quadratic Trinomials

Examples 1 to 8, oral

1. Name two numbers such that their sum is 11 and product 28; such that their sum is 3 and product -28.

2. Name two numbers such that their sum is -1 and product -42; such that their sum is 13 and product 42.

Name two numbers whose sum s and product p are:

3. $s=7,\ p=12$.

4. $s=8,\ p=16$.

5. $s=5,\ p=-36$.

6. $s=-5,\ p=-36$.

7. $s=-3,\ p=-70$.

8. $s=-1,\ p=-72$.

Factor the following and check the results:

9. a^2+a-72.

10. a^2-a-72.

11. x^2+5x+6.

12. x^2-x-6.

13. p^2+p-6.

14. $p^2-13p+42$.

15. $a^2+11ab+28b^2$.

16. $x^2+16xy+63y^2$.

17. $x^2+16x+15$.

18. $x^2+5x-50$.

19. $x^2-4x-77$.

20. $x^2-9xy-22y^2$.

21. $x^2+9xy-22y^2$.

22. $x^2-10x-39$.

Factor both terms of each of the following fractions and then reduce the fraction to lowest terms:

23. $\dfrac{x^2+9x+14}{x^2+5x+6}$.

24. $\dfrac{p^2+15p+56}{p^2+17p+72}$.

25. By trying the factors of the first and last terms find the factors of $10x^2+17x+3$. Verify the result.

26. Find the dimensions of a rectangle whose area is $x^2+9x+20$. What are the dimensions if $x=3$?

61. Cube of a Binomial. If we multiply $a^2 + 2ab + b^2$, or $(a+b)^2$, by $a+b$, we have $(a+b)^3$, thus:

$$\begin{array}{l} a^2 + 2ab + b^2 \\ a + b \\ \hline a^3 + 2a^2b + \quad ab^2 \\ \qquad\quad a^2b + 2ab^2 + b^3 \\ \hline a^3 + 3a^2b + 3ab^2 + b^3 \end{array}$$

The cube of the sum of two numbers is the cube of the first, plus three times the square of the first multiplied by the second, plus three times the first multiplied by the square of the second, plus the cube of the second.

That is, $\mathbf{(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3}$.

For example, $(a+5)^3 = a^3 + 3a^2 \cdot 5 + 3a \cdot 5^2 + 5^3$

$= a^3 + 15a^2 + 75a + 125.$

Similarly, $\mathbf{(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3}$.

Exercise 33. Cubing Binomials

Examples 1 to 8, oral

State the cube of each of the following:

1. $a+m$.	**3.** $x+1$.	**5.** $p+q$.	**7.** $10+1$.
2. $a-x$.	**4.** $x-1$.	**6.** $p-q$.	**8.** $10-1$.

Find the cube of each of the following and check the result:

9. $2p-q$.	**11.** $1-2x$.	**13.** $1+p^2$.	**15.** p^2+1.
10. m^2+n.	**12.** $1-3y$.	**14.** $1-p^2$.	**16.** p^2-1.

17. Cube $t+2$ and $10+2$; $t+3$ and 13.

18. Cube $h+7$ and 107; $h+5$ and 105.

19. Cube $6t+7$ and from the result find the cube of 67.

62. Factoring a Perfect Cube. We have found (§ 61) that

$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3$$

and
$$(a-b)^3 = a^3 - 3\,a^2b + 3\,ab^2 - b^3.$$

These statements may be combined, thus:

$$(a \pm b)^3 = a^3 \pm 3\,a^2b + 3\,ab^2 \pm b^3.$$

The upper signs of $\pm$ go together and the lower signs go together.

Hence we see that *the cube root of a polynomial in the form* $a^3 \pm 3\,a^2b + 3\,ab^2 \pm b^3$ *is of the form* $a \pm b$.

Factor $8\,x^3 + 12\,x^2 + 6\,x + 1$ and thus find its cube root.

Since this polynomial is equal to $(2\,x)^3 + 3\,(2\,x)^2 + 3\,(2\,x) + 1$, it is equal to $(2\,x+1)^3$. The factors are therefore $2\,x+1$, $2\,x+1$, and $2\,x+1$. The factored form is $(2\,x+1)^3$ and the cube root is $2\,x+1$.

Exercise 34. Factoring Perfect Cubes

Examples 1 to 6, oral

1. Factor $p^3 + 3\,p^2q + 3\,pq^2 + q^3$; $p^3 - 3\,p^2q + 3\,pq^2 - q^3$.
2. Factor $x^3 + 3\,x^2 + 3\,x + 1$; $x^3 - 3\,x^2 + 3\,x - 1$.
3. Factor $1 - 3\,x + 3\,x^2 - x^3$; $1 + 3\,pq + 3\,p^2q^2 + p^3q^3$.

Factor each of the following:

4. $5^3 + 3 \cdot 5^2 + 3 \cdot 5 + 1$.
5. $6^3 + 3 \cdot 6^2 + 3 \cdot 6 + 1$.
6. $8^3 + 3 \cdot 8^2 + 3 \cdot 8 + 1$.
7. $a^3 + 6\,a^2 + 12\,a + 8$.
8. $a^3 + b^3 + 3\,ab(a+b)$.
9. $x^3 + 6\,x^2 + 12\,x + 8$.
10. $x^3 - 6\,x^2 + 12\,x - 8$.
11. $27\,p^3 + 27\,p^2 + 9\,p + 1$.
12. $27\,p^3 - 27\,p^2 + 9\,p - 1$.
13. $8\,x^3 + 12\,x^2y + 6\,xy^2 + y^3$.

Find the cube root of each of the following:

14. $1 - 6\,xy + 12\,x^2y^2 - 8\,x^3y^3$; $8\,x^3y^3 + 12\,x^2y^2z + 6\,xyz^2 + z^3$.
15. $8\,x^3y^3 - 12\,x^2y^2z + 6\,xyz^2 - z^3$; $27\,m^3 + 27\,m^2 + 9\,m + 1$.

IV. DIVISION

63. Division of a Monomial. If we have a formula like $V = \frac{4}{3}\pi r^3$, it often becomes necessary to divide both members of the equation by some monomial like 4π, and hence we need to know how to proceed. That is, the division of one monomial by another is often necessary in practical work with formulas.

The first thing that we have need to review is the law of signs, already studied in § 29, page 34.

Since $+a \cdot +b = +ab$, we see that $\dfrac{+ab}{+a} = +b$;

since $-a \cdot -b = +ab$, we see that $\dfrac{+ab}{-a} = -b$;

since $+a \cdot -b = -ab$, we see that $\dfrac{-ab}{+a} = -b$;

since $-a \cdot +b = -ab$, we see that $\dfrac{-ab}{-a} = +b$.

That is, as already stated on page 34,

Plus ÷ *plus* = *plus*,

Plus ÷ *minus* = *minus*,

Minus ÷ *plus* = *minus*,

Minus ÷ *minus* = *plus*.

These laws may be stated more concisely, thus:

In division two like signs produce plus; two unlike signs produce minus.

This law may be illustrated numerically thus:

$$\frac{+27}{+9} = +3, \qquad \frac{-27}{+9} = -3,$$

$$\frac{+27}{-9} = -3, \qquad \frac{-27}{-9} = +3.$$

64. Law of Exponents. Division being the inverse of multiplication, we have the following:

Since $a^2 \cdot a^3 = a^{2+3} = a^5$, we have $a^5 \div a^3 = a^{5-3} = a^2$;
since $a^x \cdot a^y = a^{x+y}$, we have $a^{x+y} \div a^y = a^x$.

That is, $\boldsymbol{a^m \div a^n = a^{m-n}}$.

The exponent of any letter in the quotient is equal to the exponent of that letter in the dividend minus the exponent of that letter in the divisor.

For example, $20\,a^7 \div 4\,a^3 = 5\,a^{7-3} = 5\,a^4$
and $-35\,x^7y^4 \div 7\,x^5y = -5\,x^2y^3$.

Exercise 35. Division of Monomials

Examples 1 to 5, oral

Perform the divisions indicated below:

1. $\dfrac{a^2b^2c^2}{abc}$.
2. $\dfrac{a^2m^3n^4}{amn}$.
3. $\dfrac{x^3y^3z^3}{xyz}$.
4. $\dfrac{x^9y^8z^7}{x^2y^2z^2}$.
5. $\dfrac{22.2\,x^5y}{3\,x^2y}$.
6. $\dfrac{225\,a^4b^3c^2}{25\,ab^2c}$.
7. $\dfrac{72\,x^3y^4z^5}{-6\,xyz}$.
8. $\dfrac{-238\,m^6n^5}{14\,m^3n^2}$.
9. $\dfrac{288\,a^7b^7c^7}{-12\,abc}$.
10. $\dfrac{-325\,x^2y^4z^8}{-25\,x^2yz^6}$.
11. $\dfrac{-825\,p^8q^8r^8}{25\,p^6q^6r^6}$.
12. $\dfrac{333\,p^2qr^2}{-9\,p^2r^2}$.
13. $\dfrac{-702\,x^{10}y^9z^6}{9\,x^9y^8z^6}$.
14. $\dfrac{125\,a^mb^mc^m}{-25\,abc}$.
15. $\dfrac{-225\,m^2n^3p^4}{-25\,mn^2p^4}$.

Arrange each of the following expressions in form more convenient for computation:

16. $\dfrac{-V^3r^8}{\frac{1}{2}Vr^2}$.
17. $\dfrac{-4\,\pi\,8.3^3}{6\,\pi\,8.3}$.
18. $\dfrac{a^3(b+c)^4}{a^2(b+c)^3}$.

65. Division of a Polynomial by a Monomial. If we divide 10 ft. 5 in. by 5, we have 2 ft. 1 in. Similarly,

3)12 ft. 6 in.	8)40 yd. 16 in.	5)25 tens + 5
4 ft. 2 in.	5 yd. 2 in.	5 tens + 1
$3)\underline{12f+6i}$	$8)\underline{40\cdot 7+16\cdot 9}$	$5)\underline{25t+5}$
$4f+2i$	$5\cdot 7+\ 2\cdot 9$	$5t+1$

That is, to divide a polynomial by a monomial,

Divide each term of the dividend by the divisor and add the partial quotients.

If we divide zero by any number, the quotient is zero. Division by zero has no meaning.

Exercise 36. Division by a Monomial

Examples 1 to 9, oral

1. Divide by 2: 6 ft. 4 in.; $6f+4i$; 6 tens + 4; 64.

2. Divide by 3: 6 mi. 75 rd.; $6m+75r$; $6xy+75xy$.

3. Divide by p: $px+py$; $pbc+pxy$; $p\sqrt{x}+p\sqrt{y}$.

4. Divide ax^2+ay^2 by a; $mxy+m^2n$ by m; $apq-axy$ by a; $4x^3y-12x^3z$ by $4x^3$.

5. Divide $-px+py$ by p; $-px+py$ by $-p$.

Perform the divisions indicated:

6. $\dfrac{72a^2b^2+36ab}{36ab}$.

7. $\dfrac{125p^2q^2r^2+50pqr}{25}$.

8. $\dfrac{-8pq^2r-12p^2qr}{-4pq}$.

9. $\dfrac{-18abc^2+6a^2b^2c}{-6c}$.

10. Since $125=100+20+5$, show that the division of 125 by 5 may be considered a special case of the division of a polynomial by a monomial, and divide accordingly.

66. Division of a Polynomial by a Binomial. The cases given below illustrate the case of the division of a polynomial by a binomial and the check for the work.

1. Divide $x^2 - 5x - 84$ by $x + 7$ and check the result.

Operation	Check
$$\begin{array}{r\|l} x^2 - 5x - 84 & x+7 \\ x^2 + 7x & \overline{x - 12} \\ \hline -12x - 84 & \\ -12x - 84 & \\ \hline \end{array}$$	$$\frac{-88}{8} = -11$$

We may check the result by carefully reviewing the work; or by multiplying the quotient by the divisor, the product being the dividend; or by substituting values for the letters.

Applying the second of these checks, we can easily show that $(x + 7)(x - 12) = x^2 - 5x - 84$.

Applying the third check, we let $x = 1$, and have $1 - 5 - 84 = -88$, $1 + 7 = 8$, $-88 \div 8 = -11$; and the quotient, $x - 12$, becomes -11.

2. Divide $a^3 - b^3$ by $a - b$ and check the result.

Operation	Check
$$\begin{array}{l\|l} a^3 - b^3 & a - b \\ a^3 - a^2b & \overline{a^2 + ab + b^2} \\ \hline \quad a^2b - b^3 & \\ \quad a^2b - ab^2 & \\ \hline \qquad ab^2 - b^3 & \\ \qquad ab^2 - b^3 & \\ \hline \end{array}$$	$a = 2$, $b = 1$ $$\frac{7}{1} = 7$$

In the check we cannot let $a = 1$ and $b = 1$ because this would make the divisor zero. We therefore let $a = 2$ and $b = 1$.

Since we cannot divide by zero, in checking we use some values for the letters that do not make the divisor zero.

Since algebraic division rarely requires a divisor of more than two terms, we shall consider only monomial and binomial divisors.

Exercise 37. Division of Polynomials

Examples 1 to 5, oral

1. In dividing $p^3 + 3p^2q + 2pq^2$ by $p^2 + 2pq$, what is the first term of the quotient?

2. How do you check the work in division? Illustrate.

State the first term of the following quotients:

3. $x^2 - 7x + 10$ divided by $x - 5$; by $x - 2$.

4. $6a^2 + 11ab + 3b^2$ divided by $3a + b$; by $2a + 3b$.

5. $8p^2 - 26p + 15$ divided by $-2p + 5$; by $-4p + 3$.

Divide the following, checking the results:

6. $x^2 + 5x + 4$ by $x + 1$.
7. $x^2 + 3x + 2$ by $x + 1$.
8. $a^2 + 8a + 12$ by $a + 6$.
9. $a^2 - 2a - 15$ by $a - 5$.
10. $c^2 + 9c + 14$ by $c + 7$.
11. $p^2 - 3p - 40$ by $p - 8$.
12. $q^2 - 11q + 18$ by $q - 9$.
13. $b^2 - 7b - 30$ by $b + 3$.
14. $x^2 - 4xy - 5y^2$ by $x + y$.
15. $p^2 - 6pq + 5q^2$ by $p - q$.
16. $x^2 - 2x - 24$ by $x - 6$.
17. $x^2 - 9x + 20$ by $x - 4$.
18. $x^2 + xy - 20y^2$ by $x + 5y$.
19. $p^2 + pq - 20q^2$ by $p - 4q$.
20. $x^2 + 11x + 28$ by $x + 4$.
21. $l^2 + 8l - 33$ by $l + 11$.

Arrange according to descending powers and divide:

22. $x^3 + x^4 - 16x - 4 - 9x^2$ by $x + 2$.

23. $-8x^3 - 25x + 12 + 31x^2$ by $x - 3$.

24. $7y^3 + 3y - 7y^2 + 4 - 7y$ by $y - 1$.

25. $m^5 + 4m^4 - 7m^3 - 15m^2 + 31m - 42$ by $m + 3$.

26. Divide $t^2 + 4t + 4$ by $t + 2$, and divide 144 by 12.

This illustrates once more the relation of algebra to arithmetic.

27. Divide $t^3 + 6t^2 + 9t + 2$ by $t + 2$, and 1692 by 12.

67. Fraction in the Quotient. In algebra, as in arithmetic, the quotient may contain a fraction.

For example, divide $a^3 + b^3$ by $a - b$.

OPERATION

$$
\begin{array}{l|l}
a^3 + b^3 & a - b \\
\cline{2-2}
a^3 - a^2b & a^2 + ab + b^2 + \dfrac{2b^3}{a-b} \\
\cline{1-1}
\quad a^2b + b^3 & \\
\quad a^2b - ab^2 & \\
\cline{1-1}
\qquad ab^2 + b^3 & \\
\qquad ab^2 - b^3 & \\
\cline{1-1}
\qquad\quad 2b^3, \text{ remainder} &
\end{array}
$$

CHECK

$a = 2,\ b = 1$

$\frac{9}{1} = 7 + \frac{2}{1} = 9$

Here the remainder is $2b^3$. If we continue the division, the next term of the quotient is $\dfrac{2b^3}{a}$, and all the other terms are fractional. We therefore simply express the division by writing the remainder over the divisor, thus: $\dfrac{2b^3}{a-b}$. The quotient is therefore usually written in the form $a^2 + ab + b^2 + \dfrac{2b^3}{a-b}$.

The subject of fractions is treated later. For the present we may write a fraction as in arithmetic.

Exercise 38. Fraction in the Quotient

1. Divide $a^3 + 1$ by a; by a^2; by a^3; by $a - 1$.
2. Divide $p^4 + 1$ by p^2; by p^4; by $p + 1$; $p^3 - 1$ by $p + 1$.
3. Divide $m^2 + 2m + 2$ by $m + 1$; by $m - 1$; by $m + 2$.

Divide the following:

4. $p^2 + 3p + 1$ by $p - 1$.
5. $a^2 + 3a - 10$ by $a - 3$.
6. $x^2 - 7x + 12$ by $x - 5$.
7. $x^2 + 4x + 4$ by $x - 3$.
8. $p^2 + 3p - 4$ by $p + 2$.
9. $m^2 + 4m - 9$ by $m + 3$.
10. $m^2 + 5m - 7$ by $m + 3$.
11. $x^2 + 3xy + y^2$ by $x + y$.

V. FRACTIONS

68. Algebraic Fraction. An expression in the form $\frac{a}{b}$, in which either a or b is an algebraic expression, or both are algebraic expressions, is called an *algebraic fraction.*

For example, $\frac{3}{x}$, $\frac{x^2+y}{x-y^2}$, and $\frac{3x+y}{2a}$ are algebraic fractions.

Since we cannot divide by zero, the denominator cannot be zero.

Because of its relation to factoring we have already, on page 59, introduced some simple work in reduction of fractions.

The general principles of algebraic fractions and the terms used are the same as those with which the student is familiar from the study of arithmetic.

69. Reduction of a Fraction to Lowest Terms. A fraction is said to be reduced to *lowest terms* when the numerator and denominator have no common factor.

To reduce a fraction to lowest terms, divide both numerator and denominator by their common factors.

When a line is drawn through the factors by which both terms of the fraction are divided, the factors are said to be *canceled.*

1. Reduce $\frac{35a^2b^3c^4}{45a^4b^3c^2}$ to lowest terms.

Dividing both terms by 5, a^2, b^3, and c^2 we have $\frac{7c^2}{9a^2}$.
In practice we actually divide by $5a^2b^3c^2$.

2. Reduce $\frac{a^2+ab-2b^2}{a^2-b^2}$ to lowest terms.

$$\frac{a^2+ab-2b^2}{a^2-b^2}=\frac{\cancel{(a-b)}(a+2b)}{\cancel{(a-b)}(a+b)}=\frac{a+2b}{a+b}.$$

A fraction like $\frac{a^2b^3}{x^3y^2}$ is often printed a^2b^3/x^3y^2. In such an expression it is understood that a^2b^3 is divided by x^3y^2 and not merely by x^3. That is a^2b^3/x^3y^2 means $a^2b^3/(x^3y^2)$.

Exercise 39. Reduction to Lowest Terms

Examples 1 to 3, oral

1. Reduce to lowest terms: $\frac{2}{4}$; $\frac{6}{8}$; $\frac{a}{ab}$; $\frac{a^2b}{ab^2}$.

2. Reduce to lowest terms: $\frac{2x}{3x}$; $\frac{3x^2}{5x}$; $\frac{4a^2}{8a^3}$; $\frac{a+b}{(a+b)^2}$.

3. Reduce to lowest terms: $\frac{pqr}{p^2q^2r^2}$; $\frac{m^2n}{n^3}$; $\frac{(a-b)^2}{2(a-b)}$.

Reduce the following fractions to lowest terms:

4. $\frac{16p^2}{24p^3}$.
6. $\frac{36m^5n^5}{48m^6n^6}$.
8. $\frac{21a^2b^3c^4}{35a^4b^3c^2}$.
10. $\frac{2\pi r}{4\pi r^2}$.
5. $\frac{25a^2b}{75ab^2}$.
7. $\frac{72m^2n^{10}}{16m^{10}n^2}$.
9. $\frac{72a^mb^m}{81ab}$.
11. $\frac{\frac{1}{2}ab}{2ab}$.

12. Reduce $\frac{a+b}{a^4-b^4}$ to lowest terms.

Notice that $a^4 - b^4 = (a^2)^2 - (b^2)^2 = (a^2+b^2)(a^2-b^2)$
$= (a^2+b^2)(a+b)(a-b)$.

Reduce the following fractions to lowest terms:

13. $\frac{x^2-y^2}{x+y}$.
18. $\frac{a^2-b^2}{a^4-b^4}$.
23. $\frac{x-4}{x^2-6x+8}$.
14. $\frac{x^2-y^2}{x-y}$.
19. $\frac{a^2+b^2}{a^3+ab^2}$.
24. $\frac{a+b}{a^2+2ab+b^2}$.
15. $\frac{x-y}{x^2-y^2}$.
20. $\frac{a^2-b^2}{a^2b-b^3}$.
25. $\frac{x^3+x^2y}{x^2+2xy+y^2}$.
16. $\frac{a^4-b^4}{a^2-b^2}$.
21. $\frac{x+2}{x^2+3x+2}$.
26. $\frac{(a+b)^2}{(a+b)^3}$.
17. $\frac{a^2+b^2}{a^4-b^4}$.
22. $\frac{x+3}{x^2-x-12}$.
27. $\frac{\frac{1}{2}h(B+b)}{4(B+b)}$.

70. Sign of a Fraction. The plus sign or the minus sign before a fraction is called the *sign of the fraction.*

If there is no sign expressed, the plus sign is understood as usual.

71. Changing Signs in the Terms. Since, from the law of signs in division,

$$\frac{a}{b}=\frac{-a}{-b}=-\frac{-a}{b}=-\frac{a}{-b},$$

The value of a fraction is not altered by changing the signs of both the numerator and the denominator, by changing the signs of both the fraction and the numerator, or by changing the signs of both the fraction and the denominator.

To change the sign of the numerator means that we must change the sign of *every term* of the numerator, and similarly for the denominator. Failure to do this is the cause of many errors.

For example, $$-\frac{a-b}{a+b}=\frac{-a+b}{a+b}.$$

Sometimes, when the method of factoring is not apparent, it is well to try dividing both terms by the numerator or the denominator.

Exercise 40. Reduction to Lowest Terms

Examples 1 and 2, oral

1. Reduce to lowest terms: $\frac{x-y}{3x-3y}$; $\frac{x-y}{3y-3x}$.

2. Reduce to lowest terms: $\frac{a+b}{a^2-b^2}$; $\frac{a+b}{b^2-a^2}$; $\frac{b-a}{a^2-b^2}$.

Reduce the following fractions to lowest terms:

3. $\frac{x^2-y^2}{(y-x)^2}$.

4. $\frac{p^2-q^2}{q^2-p^2}$.

5. $\frac{x^2-4}{2-x}$.

6. $\frac{m^2-n^2}{n^4-m^4}$.

7. $\frac{n^2+1}{1-n^4}$.

8. $\frac{x^2+y^2}{y^4-x^4}$.

9. $\frac{a^5-1}{1-a^{10}}$.

10. $\frac{p^2-9}{3-p}$.

72. Addition and Subtraction of Fractions. In algebra, as in arithmetic, in adding and subtracting fractions we first express them with the same denominator.

If we add $\frac{3}{4}$ and $\frac{2}{3}$, we must give these fractions some other name than fourths or thirds, and we do this by reducing them to the same denominator. This denominator might evidently be 12, 24, 36, 48, and so on, but it is also evident that we shall save work by using 12.

No extended treatment of the lowest common denominator is necessary at this time.

1. Add the fractions $\dfrac{x}{y^2}$ and $\dfrac{x^2}{2y^3}$.

If we multiply both terms of $\dfrac{x}{y^2}$ by $2y$, we shall have $\dfrac{2xy}{2y^3}$. Since this has the same denominator as $\dfrac{x^2}{2y^3}$, we may now add, thus:

$$\frac{x}{y^2}+\frac{x^2}{2y^3}=\frac{2xy}{2y^3}+\frac{x^2}{2y^3}=\frac{2xy+x^2}{2y^3}=\frac{x^2+2xy}{2y^3}.$$

2. From $\dfrac{a+b}{a-b}$ subtract $\dfrac{a-b}{a+b}$.

We can reduce the fractions to the lowest common denominator a^2-b^2, the product of the given denominators. We then have

$$\frac{a+b}{a-b}=\frac{(a+b)^2}{a^2-b^2}, \text{ by multiplying both terms by } a+b;$$

$$\frac{a-b}{a+b}=\frac{(a-b)^2}{a^2-b^2}, \text{ by multiplying both terms by } a-b.$$

Hence the difference is found as follows:

$$\begin{aligned}\frac{(a+b)^2}{a^2-b^2}-\frac{(a-b)^2}{a^2-b^2}&=\frac{(a+b)^2-(a-b)^2}{a^2-b^2}\\&=\frac{a^2+2ab+b^2-a^2+2ab-b^2}{a^2-b^2}\\&=\frac{4ab}{a^2-b^2}.\end{aligned}$$

For purposes of computation it is better to leave the denominator in factored form, thus: $\dfrac{4ab}{(a+b)(a-b)}$.

Exercise 41. Addition and Subtraction

Examples 1 to 3, oral

1. Add $\frac{1}{4}$ and $\frac{1}{4}$; $\frac{1}{a}$ and $\frac{1}{a}$; $\frac{3}{a+b}$ and $\frac{5}{a+b}$.

2. From $\frac{7}{8}$ take $\frac{5}{8}$; from $\frac{a}{a^2+b^2}$ take $\frac{b}{a^2+b^2}$.

3. State the sum of $\frac{x}{x^2-1}$ and $\frac{-1}{x^2-1}$, and then reduce the result to lowest terms.

Add the following as indicated, reducing all results to lowest terms:

4. $\frac{x+y}{8}+\frac{3x-3y}{4}$.

5. $\frac{x-y}{2}+\frac{y-x}{3}$.

6. $\frac{p+q}{4}+\frac{p-q}{6}$.

7. $\frac{a^2+b^2}{a+b}+\frac{a^2-b^2}{b+a}$.

8. $\frac{x}{y}+\frac{y}{x}$.

9. $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$.

10. $\frac{a}{xy}+\frac{b}{yz}+\frac{c}{zx}$.

11. $\frac{p+q}{p-q}+\frac{p-q}{p+q}$.

Perform the operations indicated:

12. $\frac{2}{x^2}-\frac{3}{xy}+\frac{4}{y^2}$.

13. $\frac{a}{bc}-\frac{b}{ca}-\frac{c}{ab}$.

14. $\frac{7}{p-q}-\frac{8}{q-p}$.

15. $\frac{x+2}{(x-2)^2}+\frac{x-2}{(x+2)^2}$.

16. $\frac{x}{x-y}-\frac{y}{y-x}$.

17. $\frac{x+y}{x-y}-\frac{y-x}{y+x}$.

18. $\frac{x+3}{x+5}-\frac{1}{x^2+2x-15}$.

19. $\frac{x+7}{3-x}-\frac{3}{x^2+4x-21}$.

Reduce the following expressions to fractional form:

20. $p+q+\dfrac{p^2+q^2}{p-q}$.

We may consider this as a case in the addition of fractions, $p+q$ being a fraction with denominator 1. We then have

$$\frac{p+q}{1}+\frac{p^2+q^2}{p-q}=\frac{(p+q)(p-q)}{p-q}+\frac{p^2+q^2}{p-q}$$
$$=\frac{p^2-q^2+p^2+q^2}{p-q}.$$

Hence $\quad p+q+\dfrac{p^2+q^2}{p-q}=\dfrac{2p^2}{p-q}$.

21. $3a+\dfrac{5a}{8}$.

22. $3xy+\dfrac{5}{8xy}$.

23. $5(a+b)-\dfrac{a^2+b^2}{a-b}$.

24. $p-q+\dfrac{p^2+q^2}{p+q}$.

25. $m+2n+\dfrac{4n^2}{m-2n}$.

26. $a+b+\dfrac{a^2+2ab-b^2}{a+b}$.

Perform the operations indicated:

27. $\dfrac{1}{a}+\dfrac{1}{-a}-\dfrac{1}{a}$.

28. $\dfrac{1}{a+b}+\dfrac{1}{a-b}$.

29. $\dfrac{1}{a+b}-\dfrac{1}{a-b}$.

30. $\dfrac{a^2+b^2}{a+b}-\dfrac{a^2+b^2}{b-a}$.

31. $\dfrac{x}{3x+7}-\dfrac{3x-7}{3x}$.

32. $\dfrac{3}{x-2}+\dfrac{2}{3-x}$.

33. $\dfrac{2}{x-3}-\dfrac{3}{2-x}$.

34. $\dfrac{3}{x-5}+\dfrac{1}{5-x}$.

35. $\dfrac{7p}{p^2-4}+\dfrac{8}{p+2}$.

36. $\dfrac{8T^2}{T^4-16}-\dfrac{5}{4-T^2}$.

37. $\dfrac{5}{p^2-14p+45}+\dfrac{2}{9-p}-\dfrac{3}{p-5}$.

73. Multiplication of Fractions. We multiply fractions in algebra in the same manner as in arithmetic.

Just as $5 \times \frac{2}{3} = \frac{10}{3}$, so $a \cdot \frac{b}{c} = \frac{ab}{c}$. Therefore

To multiply a fraction by an integral expression, multiply the numerator by the integral expression, writing the product over the denominator.

Just as $\frac{2}{3}$ of $\frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15}$, so $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. Therefore

To multiply a fraction by a fraction, multiply the numerators together for a new numerator and the denominators together for a new denominator.

Cancel factors common to any numerator and any denominator before multiplying.

1. Simplify $\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{d} \cdot \frac{d}{a^2}$.

Indicating the multiplication, we have $\frac{abcd}{bcda^2}$.

Canceling common factors, we have $\frac{1}{a}$.

2. Simplify $\frac{x^2y}{2\,m^2n} \times 4 \times \frac{5\,mn^3}{6\,xy^4} \times \frac{3\,y^2}{5\,m}$.

Indicating the multiplication, and canceling, we have

$$\frac{3 \cdot 4 \cdot 5 \cdot mn^3x^2yy^2}{2 \cdot 5 \cdot 6 \cdot m^2mnxy^4} = \frac{n^2x}{m^2y}.$$

3. Simplify $\frac{x^2 - x - 6}{x^2 + x - 6} \cdot \frac{x^2 + 2\,x - 8}{x^2 - 2\,x - 8} \cdot \frac{x^2 - x - 12}{x^2 + x - 12}$.

Factoring, for ease in canceling, and then canceling common factors, we have

$$\frac{(x+2)(x-3)}{(x-2)(x+3)} \cdot \frac{(x-2)(x+4)}{(x+2)(x-4)} \cdot \frac{(x+3)(x-4)}{(x-3)(x+4)} = 1.$$

Exercise 42. Multiplication of Fractions

Examples 1 to 12, oral

Perform the multiplications indicated:

1. $2 \times \frac{1}{2}$.

2. $y \times \frac{x}{y}$.

3. $8a \times \frac{1}{4a}$.

4. $8ab \times \frac{a}{4b}$.

5. $mn \times \frac{1}{m}$.

6. $mn \times \frac{m}{n}$.

7. $p^2q^2 \times \frac{1}{pq}$.

8. $p^2q^2 \times \frac{p}{q}$.

9. $\frac{a}{x} \cdot \frac{b}{y}$.

10. $\frac{a}{x} \cdot \frac{x}{y}$.

11. $\frac{a}{x} \cdot \frac{x}{a}$.

12. $\frac{ab}{xy} \cdot \frac{ax}{by}$.

13. Find the product of $\frac{ax^2}{by^2}$, $\frac{by}{ax}$, and $\frac{y}{x}$, and check.

Perform the multiplications indicated and check the work:

14. $\frac{p^2q}{m^2n} \cdot \frac{q^2m}{p^2n} \cdot \frac{mn^2}{q^2}$.

15. $\frac{-p^2m}{qn^2} \cdot \frac{-q^2n}{pm^2} \cdot \frac{-mn}{pq}$.

16. $\frac{ab^2}{c^2d} \cdot \frac{bc^2}{d^2a} \cdot \frac{cd^2}{a^2b} \cdot \frac{da^2}{b^2c}$.

17. $\frac{(2ab)^2}{(3cd)^2} \cdot \frac{(2cd)^2}{(3ab)^2}$.

18. $\frac{-p}{q^2} \cdot \frac{-q}{r^2} \cdot \frac{-r}{s^2} \cdot \frac{-s}{p^2}$.

19. $\frac{x^2-y^2}{x^2+y^2} \cdot \frac{x+y}{x-y}$.

20. $\frac{p^2+q^2}{p^2-q^2} \cdot \frac{p+q}{p-q}$.

21. $\frac{x+3}{x-3} \cdot \frac{x^2-9}{4}$.

22. $\frac{1-m^2}{m^2+1} \cdot \frac{m+1}{m-1}$.

23. $\frac{a^2-b^2}{a^2+b^2} \cdot \frac{a^4-b^4}{ab}$.

24. $\frac{x+3}{x-3} \cdot (x^2-6x+9)$.

25. $\frac{4p}{3p-5} \cdot \frac{25-9p^2}{8p^2}$.

26. $\frac{m^2+7m+12}{4-m^2} \cdot \frac{m-2}{m+4}$.

27. $\frac{a^4-16}{a^2} \cdot \frac{4a^4}{4-a^2} \cdot \frac{a+2}{a^2+4}$.

74. Division of Fractions. The method of dividing algebraic fractions is similar to that of dividing numerical fractions. For example:

$\$2 \div \$3 = \frac{2}{3}$, the denomination "dollars" disappearing in the quotient;

$\frac{2}{5} \div \frac{3}{5} = \frac{2}{3}$, the denomination "fifths" disappearing.

Since $\frac{a}{b} = \frac{ad}{bd}$, as in arithmetic, to divide $\frac{a}{b}$ by $\frac{c}{d}$ is the same as to divide $\frac{ad}{bd}$ by $\frac{bc}{bd}$.

$\frac{ad}{bd} \div \frac{bc}{bd} = \frac{ad}{bc}$, the denomination "$bd$ths" disappearing.

But $\frac{ad}{bc} = \frac{a}{b} \cdot \frac{d}{c}$. Therefore we obtain the same result in dividing $\frac{a}{b}$ by $\frac{c}{d}$ that we get by multiplying $\frac{a}{b}$ by $\frac{d}{c}$.

The fraction $\frac{d}{c}$ is called the *reciprocal* of the fraction $\frac{c}{d}$.

To divide by a fraction, multiply by its reciprocal.

1. Divide $\frac{p^2}{q}$ by $\frac{q^2}{p}$.

Multiplying $\frac{p^2}{q}$ by $\frac{p}{q^2}$ we have $\frac{p^3}{q^3}$.

2. Divide $\frac{x^2 + 4x - 21}{x^2 + 11x + 30}$ by $\frac{x^2 + 3x - 18}{x^2 - x - 30}$.

Factoring and then multiplying by the reciprocal of the divisor, we have

$$\frac{(x+7)(x-3)}{(x+5)(x+6)} \cdot \frac{(x-6)(x+5)}{(x+6)(x-3)}.$$

Canceling, we have

$$\frac{(x+7)\cancel{(x-3)}(x-6)\cancel{(x+5)}}{\cancel{(x+5)}(x+6)(x+6)\cancel{(x-3)}} = \frac{(x+7)(x-6)}{(x+6)^2}.$$

Exercise 43. Division of Fractions

Examples 1 to 8, oral

1. Divide $\frac{2}{3}$ by $\frac{1}{3}$; $\frac{2}{3}$ by $\frac{4}{3}$; $\frac{2}{a}$ by $\frac{4}{a}$.

2. Divide $\frac{a+b}{a-b}$ by $\frac{a^2+b^2}{a-b}$; $\frac{a-b}{a+b}$ by $\frac{3}{a+b}$.

Perform the following divisions:

3. $\frac{pq}{xy} \div \frac{p^2q^2}{xy}$.

4. $\frac{abc}{m^2} \div \frac{bcd}{m^2}$.

5. $\frac{3x}{y} \div \frac{9x^2}{y}$.

6. $\frac{25x^2}{y^2} \div \frac{5x}{y^2}$.

7. $\frac{m^2}{n} \div \frac{n^2}{m}$.

8. $\frac{4ab}{5cd} \div \frac{8a}{15c}$.

9. Divide $\frac{25a^2b^2}{16c^2d^2}$ by $\frac{5ab^2}{8c^2d^2}$ and check the result by letting $a=2$, $b=3$, $c=4$, $d=5$.

Perform the following divisions, and check the results:

10. $\frac{35abc}{36xyz} \div \frac{5a^2b^2c^2}{18x^2y^2z^2}$.

11. $\frac{19x^2y^2z^2}{25m^3n^2} \div \frac{57xy^2z^3}{75m^2n^3}$.

12. $\frac{35m^3n^4z^4}{23a^3b^4c^5} \div \frac{70mnz^4}{69a^3b^3c^3}$.

13. $\frac{a^2+b^2}{a^2-b^2} \div \frac{a-b}{a+b}$.

14. $\frac{a^2+4a-21}{a^2-4a-21} \div \frac{a^2+10a+21}{a^2-10a+21}$.

Consider each of the following as expressing the division of one algebraic expression by another and simplify:

15. $\dfrac{\frac{p+q}{p}}{\frac{p-q}{q}}$.

16. $\dfrac{m+\frac{m}{n}}{m-\frac{m}{n}}$.

17. $\dfrac{\frac{a^2}{b^2}-\frac{c^2}{d^2}}{\frac{c}{d}-\frac{a}{b}}$.

VI. SIMPLE EQUATIONS

75. Nature of the Work. We have already studied simple equations in which there is one unknown quantity the value of which we are expected to find. We shall now review this work, extending the applications to affairs of business and to matters of science, and we shall introduce the study of equations with two unknown quantities.

76. Illustrative Problems. **1.** In one week a merchant made a profit of 15% on the sales. This profit, together with \$250 already in the bank, amounted to \$2500. Find the amount of the sales.

Let $x =$ the number of dollars of sales.

Then $0.15\,x =$ the number of dollars of profit,

and $0.15\,x + 250 =$ the number of dollars stated, or 2500.

Hence $0.15\,x + 250 = 2500.$

Subtracting 250, $0.15\,x = 2250.$

Dividing by 0.15, $x = 15{,}000.$

Therefore the sales amounted to \$15,000.

It should be observed that we do not let x equal the *money*, but we let $x =$ the *number* of dollars. Then when we find that $x = 15{,}000$, we know that this is the *number* of dollars, and that \$15,000 is the amount of the sales.

2. A man bought a desk for \$70.50, which was 6% less than the marked price. What was the marked price?

Let $x =$ the number of dollars in the marked price.

Then $0.06\,x =$ the number of dollars in the discount,

and $x - 0.06\,x =$ the number of dollars paid, or 70.50.

Hence, $x - 0.06\,x = 70.50,$

or $0.94\,x = 70.50.$

Dividing by 0.94, $x = 75.$

Therefore the marked price was \$75.

Exercise 44. Problems in Simple Equations

1. If 18% of the weight is lost in grinding wheat into flour, how many pounds of wheat are ground if the loss in grinding is 720 lb.?

2. From Ex. 1, how many pounds of wheat, to the nearest pound, are needed to produce 1080 lb. of flour?

3. A set of furniture was sold for $78.20, after a discount of 8% was allowed on the marked price. What was the marked price?

4. Water in freezing expands 9% of its volume. How many cubic inches of water will make 1007.16 cu. in. of ice? Allowing 231 cu. in. to a gallon, how many gallons of water are needed?

5. The sum of the angles of any triangle is 180°. In a certain triangle ABC the angle A is 10° greater than the angle B, and the angle B is 25° greater than the angle C. Find the number of degrees in each angle.

6. One of the acute angles of a right triangle is five times the other. Find the size of each angle.

7. The width of a rectangle is 8 ft. less than its length, the perimeter being 104 ft. Find the dimensions.

Problems like Exs. 7–10 are types of interesting puzzles easily solved by algebra, but it is evident that they are not practical applications of algebraic principles.

8. A double tennis court is 42 ft. longer than wide and the perimeter is 228 ft. Find the dimensions of the court.

9. What number is doubled when it is increased by 21? when it is decreased by 21?

10. The perimeter of a rectangle is 96 in. and the base is five times the height. Find the area.

11. A train leaves Chicago for the West at the same time that one leaves for the East. The former travels at the average rate of 42 mi. an hour and the latter at the average rate of 38 mi. an hour. In how many hours will they be 240 mi. apart?

Let x = the number of hours.

In 1 hr. the trains are (42 + 38) mi. apart, or 80 mi. apart.
In x hours the trains are $x \cdot 80$ miles apart, or $80x$ miles apart.
Since they are 240 mi. apart,

$$80x = 240.$$

Dividing by 80, $x = 3$.

Therefore in 3 hr. the trains will be 240 mi. apart.

Check. $3 \times (48 + 32)$ mi. = 240 mi.

Like many other problems in algebra, this may easily be solved by simple arithmetic if desired, although the solution by algebra is usually clearer.

12. Two men start from Washington at the same time, one traveling south 41 mi. an hour, and one north 48 mi. an hour. How many miles apart will they be in 3 hr.? In how many hours will they be 445 mi. apart?

13. Two men start at the same time from the same place, one going east and the other going west, the former traveling twice as fast as the latter. In 3 hr. they are 225 mi. apart. Find the rate of each.

14. Two bicyclists start at the same time from the same place, one going north and the other going south, the former traveling 7 mi. an hour faster than the latter. In 2 hr. they are 62 mi. apart. Find the rate of each.

15. Two automobilists start at the same time from places 300 mi. apart and travel toward each other, the first traveling 5 mi. an hour faster than the second. They meet in 6 hr. Find the rate of each.

16. A Boy Scout starts out walking at the rate of 3 mi. an hour. Two hours later a second Boy Scout starts from the same place to overtake him, running and walking at the rate of $4\frac{1}{2}$ mi. an hour. In how many hours after the first starts will the second overtake him?

The second gains $(4\frac{1}{2} - 3)$ miles per hour on the first. He has 2×3 mi. to gain. How long will it take him to do this?

17. A man starts on foot and walks at the rate of 4 mi. an hour. An hour later a man sets out on horseback from the same place to overtake him and travels 6 mi. an hour. How soon will the second man overtake the first?

18. A train leaves Albany for Detroit at 8 A.M. and travels at the rate of 35 mi. an hour. After an hour and a half a second train follows it on a parallel track at the rate of 47 mi. an hour. At what distance from Albany will the second train pass the first?

19. A man invested a certain sum in bonds which paid $3\frac{1}{2}\%$, and invested twice as much in a farm which yielded 7% net. His total annual income from these investments was $700. Find the amount of each investment.

20. By the use of a humidifier, an instrument for making the air more humid, the humidity of a schoolroom was 7% more than doubled, and then was 62%. What was the per cent of humidity at first?

21. A certain grade of chocolate contains 12.9% protein, while cocoa contains 21.6% protein. How much chocolate, to the nearest hundredth of a pound, will it take to furnish as much protein as 1 lb. of cocoa?

22. The sum of $5000 is divided among three men so that the share of the first is double that of the second and the share of the third is equal to the sum of the shares of the first and second. How much is the share of the third?

77. Equations involving Fractions. In case an equation involves fractions we should use our common sense as to the best method of solving. Sometimes it is better to multiply both members at once by such an expression as shall leave no fractions, and sometimes it is better to simplify the equations before doing this.

Multiplying both members of an equation by such an expression as shall leave no fractions in the equation is called *clearing the equation of fractions.*

For example, if $\frac{3}{8}x = 7 + x,$

then $3x = 56 + 8x.$

That is, we have cleared the given equation of fractions by multiplying both members by 8.

78. Illustrative Problems. **1.** Solve the equation

$$\frac{x}{3} + 7 = 37 - 3x.$$

We see that we can clear of fractions at once, or we can first simplify the equation by subtracting 7 and $-3x$ from each member, thus saving some work in multiplication.

Then $3x + \frac{x}{3} = 30.$

Multiplying by 3, $9x + x = 90.$

Solving, $x = 9.$

Check. $\frac{9}{3} + 7 = 37 - 27 = 10.$

2. Solve the equation $\frac{x+1}{x-2} = \frac{x-7}{x-8}.$

Multiplying by $x - 2$ and $x - 8$, we have

$$x^2 - 7x - 8 = x^2 - 9x + 14.$$

Therefore $2x = 22.$

Dividing by 2, $x = 11.$

For such cases as are needed in this elementary treatment it is not necessary to consider the lowest common denominator.

Exercise 45. Equations involving Fractions

1. In the study of physics you may meet with the equation $C = en/(R + nr)$ and may have to find the value of R from this equation. Show how this may be done.

2. There is an important formula which you may find later in your work in algebra, $s = \frac{1}{2}(a + l)n$. Find the value of n in terms of s, a, and l.

This formula is given in more advanced books on algebra. It states how to find the sum of a certain series of numbers.

Solve the following equations:

3. $\frac{x+1}{x-1} = \frac{5}{4}$.

4. $\frac{x-1}{x+1} = \frac{11}{13}$.

5. $\frac{x+2}{x-2} = 1\frac{1}{3}$.

6. $\frac{x-2}{x+2} = \frac{15}{19}$.

7. $\frac{x+3}{x-5} = 1\frac{4}{7}$.

8. $\frac{x+1}{x-1} = \frac{x-4}{x-5}$.

9. $\frac{x+2}{x-3} = \frac{3x-12}{3x-17}$.

10. $\frac{x-5}{x+6} = \frac{3x-15}{3x+7}$.

11. $\frac{2y-1}{1+2y} = \frac{3y-2}{3y}$.

12. $\frac{2x+3}{2x-5} = \frac{4x-3}{4x-11}$.

13. $\frac{8x-1}{3} - \frac{3}{4} = \frac{x-2}{3} + \frac{6x+5}{4}$.

14. $\frac{3x}{5} + \frac{9}{10} - \frac{x+5}{5} = \frac{12x}{25} - \frac{x+3}{20} - \frac{1}{25}$.

15. There is an important formula found in more advanced works on algebra, $s = \frac{ar^n - a}{r - 1}$. Find the value of a in terms of r, n, and s.

As in Ex. 2, this gives the sum of a certain series of numbers.

79. Simultaneous Equations. We have learned that the circumference of a circle is equal to π times the diameter, or that

$$c = \pi d.$$

We also know that $2r = d$,

and hence $c = 2\pi r.$

These equations are all related and we may think of them as all fitting together, that is, as all being true for the same values of the letters.

In the same way we have ordinary algebraic equations that fit together. For example, in the equations

$$x + 2y = 7$$

and

$$x + y = 4$$

the unknown quantities x and y may have the same values; that is, if $x = 1$ and $y = 3$, both equations are satisfied, for in that case

$$x + 2y = 1 + 6 = 7$$

and

$$x + y = 1 + 3 = 4.$$

Two or more equations in which the unknown quantities have the same values are called *simultaneous equations.*

Not all pairs of equations are simultaneous equations.

For example, it is impossible to have both

$$2x - 3y = 8$$

and

$$4x - 6y = 10$$

at the same time. There are no values of x and y that satisfy both of these equations, and so the equations are not simultaneous.

We can solve a pair of simultaneous equations involving two unknowns if we can get rid of one unknown, for we shall then have an equation in only one unknown.

The process of causing an unknown quantity to disappear from any simultaneous equations is called *elimination.*

80. Solving Simultaneous Simple Equations. Suppose that we know that the sum of two numbers is 14 and that their difference is 4, and suppose that we wish to find the numbers. Letting x represent the larger number and y the smaller number, we evidently have

$$x + y = 14 \qquad (1)$$

and

$$x - y = 4. \qquad (2)$$

It is evident that by adding the two equations, member for member, we shall have

$$2x = 18.$$

Hence $$x = 9.$$

It is now easy to find the value of y by substituting in either of the given equations. Thus, from equation (1),

$$9 + y = 14,$$

and hence

$$\begin{aligned} y &= 14 - 9 \\ &= 5. \end{aligned}$$

Therefore the two numbers are 9 and 5. It is easily seen that these values check in both equations, for the first equation becomes $9 + 5 = 14$ and the second $9 - 5 = 4$.

Often, however, it is simpler to find the value of one of the letters in one equation and substitute it in the other. Thus, from equation (2),

$$x = 4 + y. \qquad (3)$$

Substituting in (1), so as to eliminate x, we have

$$4 + y + y = 14.$$

Hence $$2y = 10.$$

Dividing by 2, $$y = 5.$$

Substituting in (3),

$$\begin{aligned} x &= 4 + 5 \\ &= 9. \end{aligned}$$

These two general methods will now be considered.

81. Elimination by Addition or Subtraction. The method of solving a pair of simultaneous equations by means of addition or subtraction is most easily understood from the study of problems. The following are typical:

1. Solve the system of equations

$$2x + 3y = 27 \qquad (1)$$

$$5x - 2y = 1 \qquad (2)$$

Multiplying (1) by 2,	$4x + 6y = 54.$
Multiplying (2) by 3,	$15x - 6y = 3.$
Adding,	$19x = 57.$
Dividing by 19,	$x = 3.$
Substituting 3 for x in (1),	$6 + 3y = 27.$
Subtracting 6,	$3y = 21.$
Dividing by 3,	$y = 7.$

Check. Substituting 3 for x, and 7 for y, in (1) and (2), we have

$$6 + 21 = 27, \quad 15 - 14 = 1.$$

Because y was eliminated by adding two equations, member for member, we say that we have eliminated y by *addition*.

2. Solve the system of equations

$$3x + 2y = 23 \qquad (1)$$

$$2x + 3y = 27 \qquad (2)$$

Multiplying (1) by 2,	$6x + 4y = 46.$	(3)
Multiplying (2) by 3,	$6x + 9y = 81.$	(4)
Subtracting (3) from (4),	$5y = 35.$	
Dividing by 5,	$y = 7.$	
Substituting in (1) or (2),	$x = 3.$	

These results will easily be seen to check.

In this solution we have eliminated x by *subtraction*.

In case there is a minus sign before a term containing the unknown, it is often better to eliminate by addition; but otherwise it is usually better to use subtraction.

Exercise 46. Elimination by Addition or Subtraction

1. The sum of two numbers is 32 and their difference is 14. Find the numbers.

2. There are two numbers whose sum is 21, and one of the numbers is $2\frac{1}{2}$ times the other. Find the numbers.

Solve the following systems of equations:

3. $4x + y = 12$
$7x + 3y = 26$

4. $2x + 3y = 21$
$3x + 7y = 56$

5. $x + y = 0$
$5x + 7y = 2$

6. $3x + 2y = 11$
$-4x + 5y = 16$

7. $2x + 1\frac{1}{2}y = 9$
$3x - 2y = 5$

8. $x + 3\frac{1}{2}y = 17$
$7x + 2y = 74$

9. $P + 2R = 8$
$3P - 4R = 19$

10. $11m - 12n = 31$
$10m + 7n = 64$

11. $2m - 3n = 19$
$2m + 3n = 25$

12. $\frac{3}{4}x + y = 8$
$\frac{3}{8}x - 2y = -1$

13. $0.2x + 7y = 37$
$25x - 48y = 10$

14. $0.3P + 0.7Q = 17$
$0.2P - 0.1Q = 0$

15. $\frac{2}{3}K + 3M = 23$
$K + 10M = 73$

16. $7x + 2y = 63$
$8x - 49 = y$

17. The rainfall in a certain city was $\frac{4}{5}$ as much last year as it was the year before. The total rainfall for the two years was 88 in. What was the amount each year?

18. It is found that, when weighed in water, a mass of tin weighing 37 lb. loses 5 lb. in weight, a mass of lead weighing 23 lb. loses 2 lb., and a certain alloy of tin and lead weighing 120 lb. loses 14 lb. Find the number of pounds of tin and of lead in the 120 lb. of the alloy.

82. Elimination by Substitution. We may find from one equation the value of one unknown quantity in terms of the other, and substitute this value in the other equation.

1. Solve the system of equations

$$3x + 7y = 22.4 \qquad (1)$$

$$x - 5y = 6 \qquad (2)$$

From (2), $x = 6 + 5y.$ (3)

Substituting in (1), $3(6 + 5y) + 7y = 22.4,$

or $18 + 15y + 7y = 22.4.$

Subtracting 18, $22y = 4.4.$

Dividing by 22, $y = 0.2.$

Substituting in (3), $x = 7.$

Check. $3 \times 7 + 7 \times 0.2 = 22.4$, and $7 - 5 \times 0.2 = 6$.

This method is particularly advantageous when the coefficient of one of the unknown quantities is 1.

2. Solve the system of equations

$$5x + 2y = 34 \qquad (1)$$

$$7x - 3y = 7 \qquad (2)$$

From (1), $y = \frac{34 - 5x}{2}.$ (3)

Substituting in (2), $7x - 3 \cdot \frac{34 - 5x}{2} = 7.$

Multiplying by 2, $14x - 102 + 15x = 14.$

Solving, $x = 4.$

Substituting in (3), $y = 7.$

3. Solve the svstem of equations

$$x + 3y = 7$$

$$x - 2y = 2$$

We have $x = 7 - 3y$ and $x = 2 + 2y$; hence $7 - 3y = 2 + 2y$, and $y = 1$. Substituting, we have $x = 4$. This special form of elimination by substitution is sometimes called *elimination by comparison.*

Exercise 47. Elimination by Substitution

Solve the following equations by substitution:

1. $x + 7y = 26$
$2x - 3y = 1$

2. $3x + y = 48$
$2x + 3y = 39$

3. $3x + y = 18$
$5x + 2y = 32$

4. $x + 7y = -7$
$7x - y = 51$

5. $P + 4Q = 31$
$4P + Q = 19$

6. $2x + 3y = 19$
$x + 7y = 26$

7. $2x - y = -1$
$3x + 2y = 23$

8. $x + 5y = 35$
$3x - y = 9$

9. $7x + y = 58$
$3x - 2y = 3$

10. $M - 4N = 1$
$5M + 2N = 49$

11. $9x - 2y = 33$
$x - 15y = 48$

12. $7x + 3y = 37$
$8x - y = 29$

Solve the following equations by the easiest methods:

13. $x + 2\frac{1}{4}y = \frac{3}{4}$
$3x + 7y = 2$

14. $x - 9y = 23$
$7x - y = 37$

15. $x + 17y = 44$
$3x - 5y = 20$

16. $2x - y = 18$
$3x + 2y = 41$

17. $7x + y = 47$
$9x - y = 33$

18. $2x + 3y = 47$
$3x - y = 32$

19. $5x + y = 59$
$5x - y = 31$

20. $2x + \frac{1}{2}y = 35$
$3x - 2y = 25$

21. $5P - 2Q = 23$
$13P - 3Q = 51$

22. $3E + 4R = 63$
$9E - 3R = 54$

23. $1\frac{1}{2}x + 1\frac{1}{4}y = 19$
$1\frac{1}{5}x + \frac{3}{4}y = 13\frac{1}{5}$

24. $x + 1\frac{1}{2}y = 60$
$x + \frac{4}{5}y = 46$

Exercise 48. Problems

1. One angle of a triangle is twice another and the sum of the two angles is equal to twice the third angle. Find the number of degrees in each angle of the triangle.

2. A man bought 96 yd. of cloth for \$208. Part of it cost him \$2 a yard and the rest \$3 a yard. How many yards of each did he buy?

3. A dealer paid \$14 for 30 lb. of tea. He paid 50¢ a pound for part of it and 40¢ a pound for the remainder. How many pounds at each price did he buy?

4. A man has 36 coins consisting only of half dollars and dimes and amounting in value to \$12.40. How many coins of each kind has he?

5. If to twice one number we add four times a second number, the result is 15.2. If from twice the first we subtract the second, the remainder is 3.3. Find the numbers.

6. A gardener paid 5 men and 8 boys \$20.80 for a day's labor, and afterwards paid 7 men and 5 boys \$22.30 for a day's labor. Find the wages paid each man and each boy.

7. If 1 is added to the numerator of a certain fraction, the resulting fraction is equal to $\frac{3}{4}$; if 2 is added to the denominator, the resulting fraction is equal to $\frac{1}{2}$. What is the fraction?

8. The charge for admission to an entertainment was 50¢ for adults and 30¢ for children. If the proceeds from the sale of 142 tickets amounted to \$52.20, how many tickets of each kind were sold?

9. The number of boys enrolled in a certain Junior High School is $\frac{5}{6}$ as large as the number of girls. The total number of students enrolled is 528. How many boys and how many girls are enrolled?

10. At a school cafeteria $\frac{5}{8}$ as many lunches were served on Monday as on Tuesday. The total number of lunches served during the two days was 520. How many lunches were served each day?

11. The length of a rectangular playground in a certain city is 50% greater than the width and the perimeter is 175 rd. Find the dimensions.

12. A labor report states that in a certain factory 1200 men and women are employed. The average daily wage is \$3.40 for a man and \$1.80 for a woman. If the labor cost is \$3376 per day, how many men and how many women are employed?

13. The receipts from a football game were \$368.50, the general admission being 50¢ and admission to the grandstand 15¢ extra. If half as many again had purchased tickets for the grandstand, the receipts would have been \$390.25. How many grandstand tickets were sold?

14. If the greater of two numbers is divided by the less, the quotient is 2 and the remainder is 6. If 9 times the less number is divided by the greater, the quotient is 4 and the remainder is 4. What are the numbers?

15. The sums of \$900 and \$750 are invested at different rates and the combined yearly interest is \$90. If the rates were interchanged, the combined yearly interest would be \$91.50. Find the rates of interest.

16. A man invested a certain amount in First Liberty Bonds at $3\frac{1}{2}\%$ and a certain other amount in Second Liberty Bonds at 4%, his total annual income from the two being \$260. If he had invested as much in $3\frac{1}{2}$ per cents as he did in 4 per cents and as much in 4 per cents as he did in $3\frac{1}{2}$ per cents, his income would have been \$5 more. How much did he invest in each kind of bond?

VII. QUADRATIC EQUATIONS

83. Quadratic Equation. An equation which, when in simplest form, contains the second power, but no higher power, of an unknown is called a *quadratic equation.*

For example, $x^2 = 4$, $x^2 - 7x = 0$, and $x^2 - 9x + 8 = 0$ are quadratic equations in x.

A quadratic equation is also called an *equation of the second degree.*

A quadratic equation may always be reduced to the form

$$ax^2 + bx + c = 0,$$

in which a is not zero but b or c, or both b and c, may be zero.

Thus $5x^2 + 3x - 9 = 0$ is a quadratic equation in which $a = 5$, $b = 3$, and $c = -9$; $7x^2 - 5 = 0$ is a quadratic equation in which $a = 7$, $b = 0$, and $c = -5$. In $9x^2 - 4x = 0$, $a = 9$, $b = -4$, and $c = 0$; and in $9x^2 = 0$, $a = 9$, $b = 0$, and $c = 0$.

84. Coefficients of a Quadratic Equation. In the equation

$$ax^2 + bx + c = 0,$$

a, b, and c are called the *coefficients of the equation.*

If a, b, and c are numbers expressed by figures, as in $x^2 + 6x - 5 = 0$, the equation is called a *numerical quadratic*; if some or all of these coefficients are represented by letters, as in $x^2 + ax = 4$, the equation is called a *literal quadratic.*

The term represented by c is called the *absolute term,* or the *constant term,* of the equation.

85. Affected Quadratic. A quadratic equation that contains both the second and first powers of the unknown quantity is called an *affected quadratic.*

Thus $x^2 - 7x + 12 = 0$ and $5x^2 - 3x = 0$ are affected quadratics.

An affected quadratic is also called a *complete quadratic.*

86. Pure Quadratic. If the first power of the unknown quantity is missing, the quadratic is called a *pure quadratic.*

Thus $x^2 - 9 = 0$ is a pure quadratic.

A pure quadratic is also called an *incomplete quadratic.*

87. Solving a Pure Quadratic. We solve a pure quadratic as shown in the following problems:

1. Solve the quadratic equation $6x^2 + 25 = 4x^2 + 75$.

Given $6x^2 + 25 = 4x^2 + 75.$

Subtracting 25 and $4x^2$, $2x^2 = 50.$

Dividing by 2, $x^2 = 25.$

Extracting the square root, $x = \pm 5,$

read "plus or minus 5," for either $(+5)^2$ or $(-5)^2$ is equal to 25.

Check. Substitute either $+5$ or -5 for x in the *given* equation.

Then $6 \cdot 25 + 25 = 4 \cdot 25 + 75 = 175.$

2. Solve the quadratic equation $x^2 + 9 = 16$.

Given $x^2 + 9 = 16.$

Subtracting 9, $x^2 = 7.$

Extracting the square root, $x = \pm\sqrt{7}.$

Check. $(\pm\sqrt{7})^2 + 9 = 7 + 9 = 16.$

3. Solve the quadratic equation $x^2 + 5 = 0$.

Given $x^2 + 5 = 0.$

Subtracting 5, $x^2 = -5.$

Extracting the square root, $x = \pm\sqrt{-5}.$

Check. $(\pm\sqrt{-5})^2 + 5 = -5 + 5 = 0.$

In the solution of Ex. 3 we find a new kind of number, $\sqrt{-5}$. Evidently the square root of a negative number cannot be a positive number, for the square of a positive number is positive. For a similar reason the square root of a negative number cannot be a negative number. We therefore say that this equation has no *real root.*

For convenience we give a name to the indicated square root of a negative number, calling it an *imaginary number.*

We shall not discuss imaginary numbers further. In any problems that we meet at this time we shall not need these numbers. They are, however, of value in certain applications of higher algebra.

Exercise 49. Pure Quadratics

1. Find a positive number that is equal to its reciprocal.

The reciprocal of x is $1/x$, and hence the equation is $x = 1/x$.

2. Find a negative number that is equal to four times its reciprocal.

3. Find the two numbers, one positive and the other negative, that are equal to 25 times their reciprocals.

Solve the following equations:

4. $x^2 - 16 = 0.$

5. $x^2 - 64 = 0.$

6. $4x^2 - 196 = 0.$

7. $7x^2 - 175 = 0.$

8. $x^2 - b^2 = 0.$

9. $x^2 - k = 0.$

10. $x^2 + k = 0.$

11. $4T^2 + 5 = T^2 + 32.$

12. How many rods of fence will exactly inclose a square field whose area is 10 acres?

13. A class in physics found it necessary to solve for v the equation $2E/M = v^2$. Find a formula for v.

14. Given the formula $RF = mv^2$, used in physics, find a formula for v.

15. Given the formula $d = \frac{1}{2}gt^2$, find a formula for t.

Solve the following equations and check the roots:

16. $\dfrac{x}{36+x} = \dfrac{1}{x+1}.$

17. $\dfrac{x-3}{3} = \dfrac{7}{9+3x}.$

18. $\dfrac{x+m}{x} = \dfrac{3x}{m-x}.$

19. $\dfrac{a+x}{b+x} = \dfrac{x-a}{b-x}.$

20. $\dfrac{53-3x}{55-x} = \dfrac{1+x}{3x+35}.$

21. $x+3 = \dfrac{49+3x}{x}.$

22. $\dfrac{x-1}{x+1} = \dfrac{1+x+x^2}{7+5x+x^2}.$

23. $\dfrac{x-28}{14+3x} = \dfrac{3x-24}{x-2}.$

88. Solving by Factoring. In case we have an affected quadratic, it often happens that we can solve it by a simple method involving factoring. The solution of a single problem will suffice to illustrate this method.

Solve the equation

$$x^2 - 8x + 15 = 0.$$

Factoring, $(x - 3)(x - 5) = 0.$

Since the product of $x - 3$ and $x - 5$ is equal to 0, one or the other of these factors must be equal to 0, because no two numbers can have 0 for a product unless one of them is 0. Also, it makes no difference which factor is 0, since 0 multiplied by any number is 0.

If $x - 3 = 0,$

then $x = 3;$

and if $x - 5 = 0,$

then $x = 5.$

Therefore x may be either 3 or 5. Since there can be only two factors of the polynomial, there can be only two roots.

Check. Substituting 3 in the given equation,

$$3^2 - 8 \cdot 3 + 15 = 9 - 24 + 15 = 0.$$

Substituting 5, $5^2 - 8 \cdot 5 + 15 = 25 - 40 + 15 = 0.$

The same method applies to equations like

$$x^2 + 7x = 0.$$

Factoring, $x(x + 7) = 0.$

Then, as in the above solution,

either $x = 0$

or $x + 7 = 0.$

In the latter case we have $x = -7.$

Hence the two roots are 0 and -7.

Check. Substituting 0 in the given equation,

$$0^2 + 7 \cdot 0 = 0.$$

Substituting -7, $(-7)^2 + 7 \cdot (-7) = 49 - 49 = 0.$

Exercise 50. Solving by Factoring

Solve the following equations:

1. $(x-1)(x-7)=0.$
2. $(x-2)(x+5)=0.$
3. $(x+a)(x-b)=0.$
4. $(3x-4)(4x-3)=0.$
5. $x(x+17)=0.$
6. $7x(3x-33)=0.$
7. $9x(4x+64)=0.$
8. $ax(mx-m^2)=0.$

9. Find a number whose square increased by 6 is equal to five times the number.

10. Find a number whose square diminished by 1 is equal to three times the sum of the number and 1.

11. The product of a certain number and the number increased by 5 is equal to 14. Find the number.

12. The product of two consecutive numbers is 306. Find the numbers.

13. The product of two consecutive odd numbers is 143. Find the numbers.

Solve the following equations and check the results:

14. $x^2-9x+20=0.$
15. $x^2-7x-18=0.$
16. $x^2-11x+30=0.$
17. $x^2-11x+18=0.$
18. $P^2+7P-60=0.$
19. $x^2+35=12x.$
20. $m^2-45=-4m.$
21. $t^2+4t=77.$
22. $P^2-19P=-78.$
23. $M^2+24=10M.$
24. $11s=28+s^2.$
25. $x^2+5x-14=0.$
26. $M^2+16M+63=0.$
27. $P^2-13P+22=0.$
28. $Q^2-99=2Q.$
29. $2V+V^2=143.$
30. $16W+W^2+39=0.$
31. $K^2=2K+15.$
32. $45+K^2=14K.$
33. $W^2-63=2W.$

89. Completing the Square. A second method of solving the affected quadratic equation, and one commonly used, is called the solution by *completing the square.*

Since $(x+a)^2 = x^2 + 2\,ax + a^2$, we can easily see that $x^2 + 2\,ax$ lacks only a^2 of being a perfect square. Hence in an equation of the form $x^2 + 2\,ax = b$, we may make the first member a perfect square or *complete the square* by adding to each member the square of half the coefficient of x.

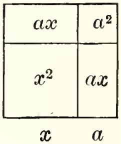
For example, given $\qquad x^2 + 14\,x = -45.$

Adding $(\frac{14}{2})^2$, or 7^2, $\qquad x^2 + 14\,x + 49 = 4.$

Extracting the square root, $\qquad x + 7 = \pm 2.$

Subtracting 7, $\qquad x = -7 \pm 2.$

That is, $\qquad x = -5$ or $-9.$



If the equation is in the form $x^2 - 2\,ax = b$, we may evidently complete the square in the same way, by adding to each member a^2, the square of $-a$. We then have

$$x^2 - 2\,ax + a^2 = b + a^2.$$

Hence $\qquad x - a = \pm\sqrt{b + a^2},$

and $\qquad x = a \pm \sqrt{b + a^2}.$

If the equation has a coefficient for x^2 other than 1, we may first divide each member by this coefficient.

For example, given $\qquad 3\,x^2 - 2\,x = 15.$

Dividing by 3, $\qquad x^2 - 2 \cdot \frac{1}{3}x = 5.$

In the last equation $-\frac{1}{3}$ equals the a of the equation $x^2 + 2\,ax = b$.

Therefore to solve an affected quadratic:

1. *Reduce the equation to the form* $x^2 + 2\,ax = b$.
2. *Add to each member the square of half the coefficient of* x.
3. *Extract the square root of each member of the result.*
4. *Solve the two resulting simple equations.*

The two equations are of the form

$$x = a + \sqrt{b + a^2} \quad \text{and} \quad x = a - \sqrt{b + a^2}.$$

Exercise 51. Affected Quadratics

Examples 1 to 5, oral

1. In order to complete the square, what must be added to x^2+2x? to x^2+4x? to x^2+10x? to x^2-10x?

Complete the square in the first member of each of the following equations by adding the same number to both sides:

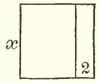
2. $x^2-2x=3$.

3. $x^2-4x=5$.

4. $x^2+8x=9$.

5. $x^2-8x=9$.

6. If from a square piece of paper I cut a strip 2 in. wide, as is shown in the figure, the area of the rest of the paper is 48 sq. in. How long is the side of the square?



Let x = the number of inches in the side.

Then x^2 = the number of square inches in the area.

Therefore $x(x-2)=48$.

Solving, $x=8$ or -6.

Of these roots -6 is inadmissible by the conditions of the problem, although it satisfies the algebraic equation. Usually only one of the algebraic roots meets the conditions of the problem.

7. The area of a rectangle is 28 sq. ft. and the width is 3 ft. less than the length. Find the dimensions.

Which one of the two roots is it reasonable to take?

8. The formula $h=a+vt-16t^2$ gives the approximate height (h feet) of a body at the end of t seconds if it is thrown vertically upward with an initial velocity of v feet per second from a position a feet high. How long will it take a rocket to reach a height of 1500 ft. if it is fired vertically upward with an initial velocity of 140 ft./sec. from a dirigible balloon 1300 ft. above the earth?

Remember that the expression 140 ft./sec. is the modern way of writing 140 ft. per second.

Solve the following equations:

9. $x^2 + 12x + 27 = 0.$

10. $x^2 - 4x = 117.$

11. $x^2 + 5x = 104.$

12. $x^2 + x = 90.$

13. $x^2 + 3x = -2.$

14. $x^2 - 7x + 6 = 0.$

15. $y^2 + 5y - 4 = 2.$

16. $m^2 - 2m + 17 = 32.$

17. $k^2 - 9k + 20 = 6.$

18. $P^2 - 11P + 50 = 22.$

19. $x^2 + \frac{1}{4}x = 17.$

20. $x^2 + 4x + 3\frac{3}{4} = 0.$

21. Two numbers differ by 3, and the sum of their squares is 225. What are the numbers?

22. Separate 42 into two such parts that the product of the parts is 392.

23. The hypotenuse of a right triangle is 5 ft. longer than the base and 10 ft. longer than the height. What is the area of the triangle?

24. Find two consecutive integers whose product is 8742.

25. The area of a certain rectangular flag is 84 sq. ft. and the perimeter is 38 ft. What are the dimensions?

26. The square of a certain number, when increased by 143, is equal to 24 times the number. Find the number. If there are two such numbers, show that each fulfills the conditions stated.

27. The square of a certain number is equal to 315 less than 36 times the number. Find the number.

28. On one side of a square piece of tin a strip 2 in. wide is soldered. The rectangular piece thus formed has an area of 8 sq. in. Find the side of the square.

29. From a square piece of tin a rectangular piece is cut by a line parallel to one edge and 4 in. from it. The rectangle remaining has an area of 12 sq. in. Find the side of the square.

90. Solving by Formula. Although the two methods of §§ 88 and 89 are sufficient for solving any quadratic equation, it is convenient to be able to write the roots at once without the trouble of factoring or of completing the square.

We could dispense with the solution of the quadratic by the formula and continue to solve by the methods already studied, but this is not advisable because it usually takes too much time.

Every quadratic equation may evidently be reduced to the form $ax^2 + bx + c = 0$, in which a, b, and c are integers.

Given $$ax^2 + bx + c = 0.$$

Subtracting c, $$ax^2 + bx = -c.$$

Dividing by a, $$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Completing the square, $$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a},$$

or $$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}.$$

Extracting the square root, $$x + \frac{b}{2a} = \pm\frac{1}{2a}\sqrt{b^2 - 4ac}.$$

Therefore $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In an equation of the form $ax^2 + bx + c = 0$,

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}.$$

If $a = 1$, then $$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

Solve the equation $2x^2 - 3x - 9 = 0$.

Here $a = 2$, $b = -3$, $c = -9$.

By the formula, $x = \frac{3 \pm \sqrt{9 + 72}}{4} = \frac{3 \pm 9}{4} = 3$ or $-\frac{3}{2}$.

The student may also solve by factoring or by completing the square.

Exercise 52. Solving by Formula

Solve the following equations by using the formula:

1. $x^2 - 4x - 45 = 0$.
2. $x^2 - 16x + 63 = 0$.
3. $x^2 + 4x - 77 = 0$.
4. $x^2 - x - 132 = 0$.
5. $x^2 + 3x - 180 = 0$.
6. $x^2 + bx + c = 0$.
7. $\frac{5}{x} + \frac{8}{x+1} = 9$.
8. $\frac{8}{x} - 1 = \frac{12}{x^2}$.
9. $6x^2 + 5x - 6 = 0$.
10. $7x^2 - 37x = 6$.
11. $2x^2 - 13x = 45$.
12. $6x^2 - 3x = 45$.
13. $3x^2 + 63 = 30x$.
14. $1.5x^2 + 5x + 1.2 = 0$.
15. $\frac{x^2}{3} - \frac{4x}{5} - 1 = 0$.
16. $\frac{x+2}{x-7} - \frac{x+5}{x-5} = 1$.

17. Find two factors of 420 whose sum is 41.

18. Separate 30 into two parts whose product is 161.

19. Solve the equation $x^2 - 4ax + 2a^2 = 0$ for x in terms of a; for a in terms of x.

20. Solve the equation $x^2 - 2(a-b)x - 4ab = 0$ for x.

21. Solve the equation $x^2 = (a+b)x$ for x.

22. Solve the equation $8a^2 + 14ab - 15b^2 = 0$ for a in terms of b; for b in terms of a.

23. Find the positive value of t in the equation $6t^2 + 11t = 121$.

24. Find the sum of the two roots of the equation $2x^2 = 1 + 7x$.

25. In Ex. 24 find the product of the two roots.

26. If a railway train had traveled 10 mi./hr. faster, it would have taken 30 min. less to travel 100 mi. Find the rate of the train.

91. Formulas used in Physics. The formulas given in the following exercise are all used in physics. It is not necessary for our present purposes to explain the meaning of each formula, since we are now concerned only with the methods of deriving new formulas from those given. If the student proceeds to the study of advanced physics, he will find it necessary to use formulas of this kind.

Exercise 53. Formulas used in Physics

Given the following formulas, derive the formulas required:

1. $F=\frac{mv^2}{R}$. Derive a formula for v; for R; for m.

2. $E=\frac{Mv^2}{2}$. Derive a formula for v; for M.

3. $G=\frac{mm'}{d^2}$. Derive a formula for d; for m; for m'.

4. The formula $S=\frac{1}{2}gt^2$ gives the number of feet S, through which a body falls in t seconds, starting from rest. Taking g as 16 ft., how long will it take a bomb to fall to the ground from an airplane 6000 ft. above the earth?

5. Given the formula $W=2.45\,(D^2-d^2)$, derive a formula for d; for D.

6. Given the formula $C=0.004\,d^2+0.14$, derive a formula for d and then find the value of d to the nearest hundredth when $C=10.24$.

7. Given the formula $2fs=v^2-u^2$, derive a formula for v; for u; for s; for f.

8. Given the formula $s=ut+\frac{1}{2}t^2f$, derive a formula for f; for u; for t.

9. Given the formula $E=kv+v^2$, derive a formula for v, and find the value of v when $E=3000$ and $k=10$.

92. Formulas used in Industries. The formulas given in the following exercise are all used in various industries, as is the case with those given on page 22.

Exercise 54. Formulas used in Industries

Given the following formulas, derive the formulas required:

1. $P = 2C - 0.01C^2$, a formula used in connection with gas engines. Derive a formula for C.

2. $V = \frac{1}{3}\pi h(R^2 + Rr + r^2)$, a formula used in connection with the volume of a portion of a cone. Derive a formula for R; for r.

3. $s = d^2w/8p$, a formula used in connection with stretching a wire. Derive a formula for d.

4. $T = a + bt + ct^2$, a formula used in connection with steel shafts. Derive a formula for t.

5. $am^2 + bm + 1/c = 0$, a formula used in connection with electricity. Derive a formula for m.

6. $S = 145.5w^2 + 120w$, a formula used in connection with electricity. Derive a formula for w.

7. $S = Vt - \frac{1}{2}gt^2$, a formula used in connection with projectiles. Derive a formula for t.

8. $S = \frac{1}{2}gt^2 + Vt$, a formula used in connection with falling bodies. Derive a formula for t.

9. $S = 0.1gt^2 + 2t + C$, a formula used in connection with a pulley. Derive a formula for t.

10. $\frac{1}{5}a = K^2 + 3K$, a formula used in finding the range of a gun. Derive a formula for K.

11. $4y = x(R - x)$, a formula used in gunnery. Derive a formula for x.

VIII. GENERAL REVIEW

Exercise 55. Review Problems

1. A cookbook states, in a certain recipe for cooking beef, that there should be allowed $\frac{1}{4}$ hr. for every pound and then 20 min. over. Express this as a formula for the number of hours h required to cook a piece of beef weighing w pounds. Find the value of h for $w = 5\frac{3}{4}$.

2. A furniture dealer advertises that he will sell furniture on monthly payments as follows:

Price of furniture	\$25	\$50	\$75	\$100	\$150	\$200
Monthly payments	\$4	\$6	\$8	\$10	\$14	\$16

Draw a graph to represent the different rates of payments and from this graph determine to the nearest quarter of a dollar the monthly payments in purchasing furniture worth \$40, \$60, \$80, \$125, and \$175.

3. Write a formula for the average amount A of air available for each of n persons in a schoolroom l feet long, w feet wide, and h feet high.

4. How many planks 10 in. wide and 12 ft. long will be required to cover a space w feet wide and l feet long, w and l being such that it is not necessary to cut any plank?

5. The safe load for an iron chain to support is given by the formula $L = 6.35\,d^2$, where L represents the load in tons and d the number of inches in the diameter of the iron rod from which the links are made. Find the safe load when the diameter is $\frac{1}{4}$ in.

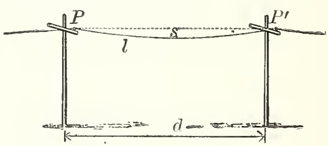
6. In Ex. 5 find the least diameter of link necessary for the support of a load of 4 tons.

7. The distance d in miles which you can see from a height of h feet, as you look out on the ocean, is given by the formula $d = 1.22\sqrt{h}$. Find the height from which you can see just 10 mi.

8. In Ex. 7 how far can you see from the top of a lighthouse, your eye being 144 ft. above sea level?

9. Plot the formula of Ex. 7 for $h = 1, 4, 9, 16, 25, 36, 49, 64, 81, 100$, and from the graph determine the approximate distances that you can see from heights of 12 ft., 20 ft., 40 ft., 75 ft.

10. A telegraph wire stretched between two points P and P' which are d feet apart sags s feet in the middle, so that the length of the wire, l feet, is greater than the distance from P to P'. If the pull with which the wire is stretched is p pounds and the weight is w pounds per foot of wire, the following formulas show various relations:



$$s = d^2w/8p \qquad l = d + 8s^2/3d$$
$$p = d^2w/8s \qquad s = \sqrt{3d(l-d)/8}$$

Remember that the expression $d^2w/8p$ means $d^2w \div (8p)$.

Suppose that the wire weighs 0.08 lb. per foot and is stretched with a pull of 260 lb. between two points 120 ft. apart; find the sag of the wire.

11. In Ex. 10 find the stretching force when the sag is 6 in., the other values being as stated.

12. In Ex. 10 find the length of the wire between P and P' when the sag is 6 in.

13. In Ex. 10 find s when $d = 150$ and $l = 150.04$.

Omit Ex. 13 except for advanced students of the class.

14. The pull of a locomotive, P pounds, is given by the formula $P = sd^2l/w$, where s is the steam pressure in pounds per square inch, d the diameter of the cylinder in inches, l the length of the stroke of the piston in inches, and w the diameter of the driving wheel in inches. Find the pull of a locomotive in which $d = 24$, $l = 46$, $s = 120$, and $w = 72$.

Notice that the solution of this problem, like those of others of the kind, requires no technical knowledge. The solution simply requires substitution in a formula.

15. An airplane is a feet above a village A. It then begins to rise and at the same time to fly in a straight line directly east until it is b feet above a village B which is d miles from A. How many miles has the airplane gone?

Notice that one set of measurements is given in feet.

16. Think of some number; multiply it by 3; add 9; multiply by 2; divide by 6; subtract the number thought of; the result must be 3. What is the explanation?

17. I am thinking of a date in American history. If I subtract 2 from the number of hundreds, 2 from the number of tens, and 1 from the number of units, and then add 445, the result is 2000. What is the date?

Use the equation $100(h - 2) + 10(t - 2) + (u - 1) + 445 = 2000$.

18. I am thinking of an important date in the history of the world. If I add 6 to the number of units and 8 to the number of tens, the result is 2000. What event is connected with the number I thought of?

19. A lighthouse 100 ft. high stands on a cliff 200 ft. above the sea. How much farther could a man see on the surface of the ocean from the top of the lighthouse than from the bottom, allowing 5 ft. for the height of the eye above the surface on which the man stands?

Use the formula $d = 1.22\sqrt{h}$ of Ex. 7.

20. Show that a number of four digits, a, b, c, d, is divisible by 9 if $a + b + c + d$ is divisible by 9.

The number of four digits is evidently $1000\,a + 100\,b + 10\,c + d$, or $999\,a + a + 99\,b + b + 9\,c + c + d$.

The rule holds for any number of digits.

21. In Ex. 20 give the proof for a number of five digits, a, b, c, d, and e.

22. One of the factors of $16\,x^2 + 9\,y^2 - 24\,xy$ is $3\,y - 4\,x$. Find the other factor. Find the value of $16\,x^2 + 9\,y^2 - 24\,xy$ if $x = 17$ and $y = 19$.

The student should remember to put the expression in the best form for easy calculation before he makes the substitutions.

23. By division show that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \frac{x^5}{1-x}$$

and verify the result by letting $x = \frac{1}{2}$.

24. In the quadratic equation $x^2 + px + q = 0$, show that the sum of the two roots is $-p$ and that their product is q. Use these facts to check the roots of the equation $x^2 + 7\,x + 6 = 0$.

25. Show that the negative roots of the two quadratic equations $x^2 + 8\,x = 33$ and $x^2 + 6\,x - 55 = 0$ are the same.

26. Show that the positive roots of the two quadratic equations $x^2 = 11\,x + 26$ and $x^2 - 169 = 0$ are the same.

27. Solve the simultaneous equations $3\,x + 7\,y - 9 = 0$, $5\,x - 9\,y - 17 = 0$.

28. Find the surface of a cube of volume 512 cu. in.

29. In Ex. 28 find the volume of the largest sphere that can be cut from the cube.

The formula for the volume of a sphere is $\frac{4}{3}\pi r^3$.

30. A boat is rowed directly across a river and soundings are taken at various distances from the bank. From the following table of the results draw a plan of the cross section of the river, showing the curve of the bed.

Distance in feet	0	5	10	15	20	25	30	35	40	45	50	60	70
Depth in feet	0	2	3	5	9	14	16	18	15	15	12	7	0

31. From the top of a cliff 1500 ft. high a bullet was fired horizontally with a velocity of 2200 ft. per second. Draw the graph of its path, the number of feet which it has fallen at the end of each second being as follows:

Seconds	0	1	2	3	4	5	6	7	8	9
Feet fallen	0	16	64	144	256	400	576	784	1024	1296

32. The space passed over by a body falling t seconds is expressed by the formula $S = 16\,t^2$, where S is the number of feet the body falls. Construct a graph of this formula, using for t the values $0, \frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}$, and 3.

33. The product of two consecutive integers is 132. Find the numbers.

Let the integers be x and $x + 1$.

34. The area of a square would be doubled if it were changed into a rectangle 6 ft. longer and 4 ft. wider than the square. Find the side of the square.

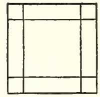
35. Find three roots of the equation $x^3 - x = 0$.

Separate the first member into three factors and place each factor equal to zero.

36. Find three roots of the equation $9\,x^3 - x = 0$.

37. Find the three roots of the equation $25\,x^3 - 16\,x = 0$, and check the results.

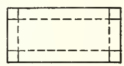
38. A square piece of tin is made into a box by cutting from the corners small squares 2 in. on a side. The box then contains 50 cu. in. Find the dimensions of the piece of tin.



39. A square piece of tin x inches on a side is made into a box by cutting from the corners small squares a inches on a side. The box then contains b cubic inches. Find the dimensions of the piece of tin. Check the work by substituting the values given in Ex. 38 and obtaining the result there found.

Notice that Ex. 39 gives a general formula for all such cases.

40. A box containing 324 cu. in. is made by cutting out the corners of a sheet of cardboard which is twice as long as it is wide, and then turning up each side 6 in. Find the dimensions of the cardboard and also of the box.



41. Expand $(3x+7)^2$. By letting $x=10$ find from the result the value of 37^2.

42. Multiply $4x+3$ by $2x+7$ and from the result find the value of 27×43.

43. The straight line AB, 1 in. long, has been divided at the point P so that $\overline{AP}^2 = AB \cdot PB$. Find the length of AP to the nearest hundredth of an inch.

44. The sum of two numbers is 10 and their product is also 10. Find the numbers.

For the square roots of small integers use the table on page 280.

45. The measures of the lengths of the three sides of a right triangle are consecutive integers. What are the lengths of the sides?

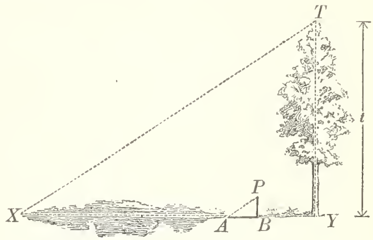
Remember that if a and b are the sides and c is the hypotenuse of a right triangle, then $a^2 + b^2 = c^2$.

PART II. TRIGONOMETRY

I. FUNCTIONS OF ANGLES

1. Shadow Reckoning. The ancients often computed the heights of trees, buildings, monuments, and the like by the aid of shadows and called the method *shadow reckoning.*

For example, if a post 4 ft. high casts a shadow 6 ft. long at the same time that a tree casts a shadow 60 ft. long, we have two similar triangles, *ABP* and *XYT*, as shown in the adjoining figure.



As stated in Book I, this means that the triangles are of the same shape but not necessarily of the same size. Such triangles have their corresponding sides proportional; that is, the quotient of any two sides of one is equal to the quotient of the corresponding sides of the other. The symbols $\angle$ and $\triangle$, and the manner of reading angles and triangles, should be explained by the teacher if necessary.

Hence $$\frac{\text{Height of tree}}{\text{Shadow of tree}} = \frac{\text{Height of post}}{\text{Shadow of post}} = \frac{t}{60} = \frac{4}{6}.$$

Therefore $$t = \frac{60 \times 4}{6} = 40.$$

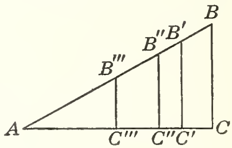
That is, the tree is 40 ft. high.

In such cases the object is assumed to stand on a horizontal plane.

In later times this method developed into one phase of trigonometry, a word derived from the Greek words *tri* (three) and *gonia* (angle), the method being based chiefly on the measurement of triangles.

2. Tangent of an Angle. In finding the height of the tree in § 1 we multiplied the length of the shadow by the ratio of the height of any post to its shadow; that is, by the quotient of height divided by shadow. The height of the post is immaterial, for we can see in this figure that

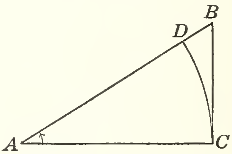
$$\frac{BC}{AC} = \frac{B'C'}{AC'} = \frac{B''C''}{AC''} = \frac{B'''C'''}{AC'''},$$



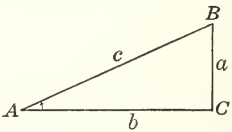
the ratio of height to shadow being the same whatever post we take.

B'' and B''' are read "B second" and "B third" respectively.

That is, in the right triangle ABC, if angle A is fixed, the ratio BC/AC is the same whatever the size of the triangle.



Since BC just touches the circle of which AC is a radius, and since "tangent" means "touching," the ratio BC/AC is called the *tangent of* A, written $\tan A$. Or, using the form of triangle generally found in trigonometries, we have

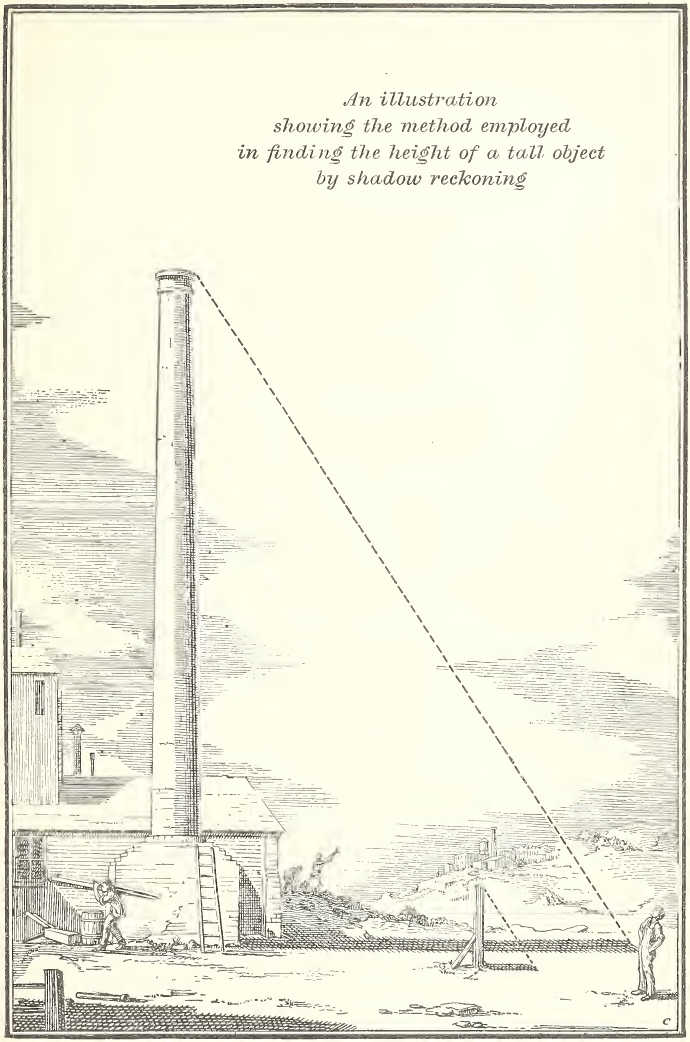


$$\tan A = \frac{a}{b},$$

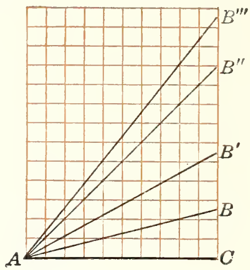
from which $$b \tan A = a.$$

We now see that if we know $\tan A$ and can measure b, we can easily compute the value of a. If, therefore, we can find the tangents of the various angles that we are likely to use, we can find the height of an object by this simple method. Our first problem, therefore, is to consider the finding of tangents of angles.

Because of the variety and nature of its applications teachers will see that the tangent is the simplest and most natural trigonometric function with which to introduce the subject.


*An illustration
showing the method employed
in finding the height of a tall object
by shadow reckoning*

3. Finding Tangents. The method of finding the tangents of various angles depends upon higher algebra, but we can determine approximately the tangent of any angle by the aid of squared paper.



In this figure angle $CAB = 14°$, and since the tangent is BC/AC we find, by counting the spaces, that

$$\tan 14° = \frac{2\frac{1}{2}}{10} = 0.25.$$

The closer approximation found by higher algebra is 0.2493.

$$\text{Similarly, } \tan CAB' = \tan 30° = \frac{5.8}{10} = 0.58,$$

$$\tan CAB'' = \tan 45° = \frac{10}{10} = 1,$$

and $$\tan CAB''' = \tan 51°\ 20' = \frac{12.5}{10} = 1.25.$$

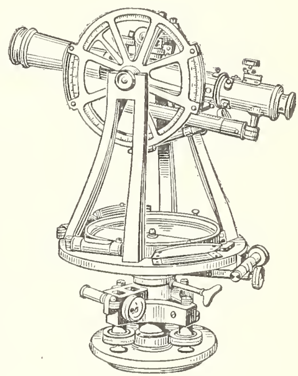
4. Table of Tangents. The following brief table of tangents of angles may be used in solving the problems in the next exercise:

Angle	tan	Angle	tan	Angle	tan
10°	0.1763	40°	0.8391	70°	2.7475
20°	.3640	50°	1.1918	80°	5.6713
30°	.5774	60°	1.7321	90°	∞

The symbol ∞ means "infinity"; that is, the tangent of 90° is infinitely long. This can easily be seen by drawing an angle of 90°.

Students should be reminded that the sum of the angles in any triangle is 180°, that the square on the hypotenuse of a right triangle is equal to the sum of the squares on the two sides, that each angle of an equilateral triangle is 60°, and that the angles at the base of an isosceles triangle are equal.

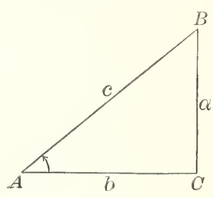
5. Measuring Angles. In outdoor work angles are commonly measured by means of a *transit* or some similar instrument. By turning the upper part of the transit here shown, horizontal angles can be measured on the horizontal plate to minutes. By turning the telescope up and down, vertical angles can be measured on the vertical circle here shown.



For school purposes a transit can be made without difficulty, using a small pipe in place of a telescope and using paper protractors for the horizontal and vertical circles.

6. Practical Use of the Tangent. Since by definition we have

$$\frac{a}{b} = \tan A,$$



we see, as in § 2, that

$$a = b \tan A.$$

Given that $b = 15$ ft. and $A = 40°$, we can find a as follows:

$$a = 15 \tan 40° = 15 \times 0.8391 = 12.5865.$$

That is, to the nearest hundredth, $a = 12.59$ ft.

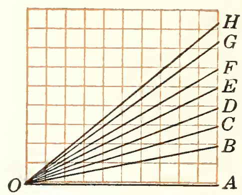
If, however, we had no instruments for measuring b more closely than to the nearest foot, then we should give 13 ft. as the value of a, since *no result can be more nearly accurate than the measurements on which it is based.*

If $b = 15$ ft. as the result of measuring with an instrument that was accurate to 0.01 ft., as we may reasonably assume, then the result for a should be given as 12.59 ft.

Exercise 1. Tangents

In this figure, given the measures of the angles as stated below, find the approximate values of the tangents of the angles:

1. Angle $AOB = 11° 19'$.
2. Angle $AOC = 16° 42'$.
3. Angle $AOD = 21° 48'$.
4. Angle $AOE = 26° 34'$.
5. Angle $AOF = 30° 58'$.
6. Angle $AOG = 36° 53'$.
7. Angle $AOH = 40° 22'$.



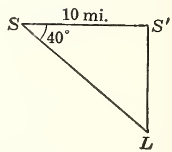
8. In the $\triangle ABC$ of § 6, suppose that $A = 40° 22'$ and $b = 35$ ft.; find the value of a.

For the tangents needed in solving Exs. 8 and 9 use the results found in Exs. 6 and 7.

9. A man standing 150 ft. from the foot of a building finds that the angle of elevation of the top is 36° 53′. If his eye is 5 ft. from the ground, how high is the building?

By the *angle of elevation* is meant the angle between a horizontal line and the line from the eye to the top of the building.

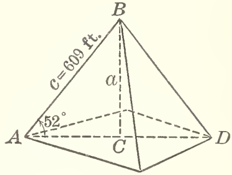
10. The captain of a ship at S observes a lighthouse L to lie 40° south of an east-and-west line. After the ship has sailed to S', 10 mi. east, the lighthouse is seen to be directly south of it. Find the distance from S' to L.



In § 6 find the value of a to four figures, given the following:

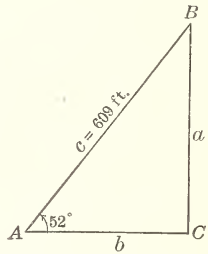
11. $b = 46$, $A = 10°$.
12. $b = 58$, $A = 20°$.
13. $b = 94$, $A = 30°$.
14. $b = 128.4$, $A = 50°$.
15. $b = 108.3$, $A = 70°$.
16. $b = 23.58$, $A = 80°$.

7. Sine of an Angle. Suppose that we know that the edge AB of the Great Pyramid is 609 ft. to the nearest foot, and that the angle DAB is 52°, measures that are easily taken. If we now knew the ratio a/c, as seen more clearly in the larger figure here shown, we could easily find the value of a. That is, because



$$a = c \times \frac{a}{c},$$

we have $$a = 609 \times \frac{a}{c}.$$



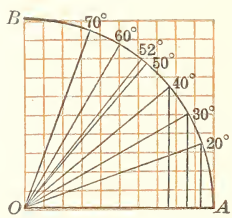
The ratio $\frac{a}{c}$ is called the *sine of A*, which is written sin A.

That is, $$\sin A = \frac{a}{c},$$

whence $$c \sin A = a.$$

In any right triangle the sine of either acute angle is the ratio of the opposite side to the hypotenuse.

8. Finding Sines. We can determine approximately the sine of any angle by the aid of squared paper. In this figure the arc AB is drawn with radius 10. Then sin 20° is equal to the perpendicular let fall on OA from the point marked 20° in this figure, divided by the radius 10. This quotient is approximately 3.4 ÷ 10, or 0.34.



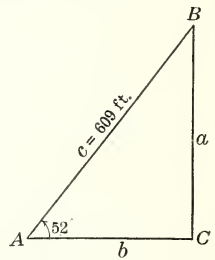
Similarly, we find the following table of sines:

Angle	sin	Angle	sin	Angle	sin
30°	0.50	50°	0.77	60°	0.87
40°	.64	52°	.79	70°	.94

9. Table of Sines. The following brief table of sines may be used in solving the problems in the next exercise:

Angle	sin	Angle	sin	Angle	sin
10°	0.1736	40°	0.6428	70°	0.9397
20°	.3420	50°	.7660	75°	.9659
30°	.5000	52°	.7880	80°	.9848
38°	.6157	60°	.8660	90°	1.0000

10. Practical Use of the Sine. We can now determine the height of the Great Pyramid referred to in § 7.



Since $\frac{a}{c} = \sin A,$

we have $a = c \sin A.$

Hence, in this case,

$$\begin{aligned} a &= 609 \sin 52^\circ \\ &= 609 \times 0.7880 \\ &= 479.892. \end{aligned}$$

Since c was given as 609 ft. to the nearest foot, we can simply say that the height is 480 ft. to the nearest foot.

In the above problem we can also, if we wish, find b. Since $B = 90° - A = 90° - 52° = 38°$, we see that

$$\begin{aligned} b &= 609 \sin 38^\circ \\ &= 609 \times 0.6157 \\ &= 374.9613. \end{aligned}$$

Hence the length of b is 375 ft. to the nearest foot.

11. Check. We may now check the three figures in each of our results by noticing that

$$a^2 = c^2 - b^2 = (c + b)(c - b),$$

whence $480^2 = (609 + 375)(609 - 375)$, approximately.

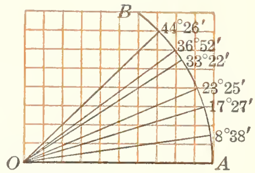
Exercise 2. Sines

In this figure, given the measures of the angles as stated below, find the approximate values of the sines of the angles:

1. 8° 38′. **4.** 33° 22′.

2. 17° 27′. **5.** 36° 52′.

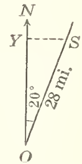
3. 23° 35′. **6.** 44° 26′.



From the above figure find the angle whose sine is:

7. 0.3. **8.** 0.6. **9.** 0.55. **10.** 0.4. **11.** 0.7.

12. After a ship has sailed N. 20° E. a distance of 28 mi., it is how many miles east of its starting point?



The expression N. 20° E. means north 20° east; that is, 20° east of a meridian *ON*. The ship sails 28 mi. from *O* to *S*. The distance *YS* is required.

In such a case the student should draw a neat diagram, but it is not necessary to draw to scale or to have the angle measured by the protractor.

Distances should be computed to the nearest unit, as shown by the given measurement; in this case, to 1 mi.

13. A submarine sails submerged N. 30° W. a distance of 10 mi. Without taking any further observations, how can the captain tell how far he is to the west of his point of departure? Find the distance.

14. In Ex. 13 how far is the submarine to the north of the point of departure?

First find the other acute angle of the triangle.

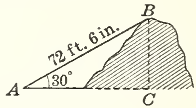
15. A cruiser starts at a buoy and sails N. 37° E., 38 mi. Find its northing and its easting.

The northing is the distance sailed to the north and the easting is the distance sailed to the east.

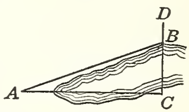
16. If a kite string 240 ft. long makes an angle of 30° with the ground, how high is the kite?

The string cannot be stretched perfectly straight, but we can form a very good approximation by this method.

17. In order to find the height of a mound which they use as an observation post some Boy Scouts stretch a string from *A* to *B*, as shown in the figure, finding *AB* to be 72 ft. 6 in. Using a protractor they find ∠*A* to be 30°. What is the height of the mound?

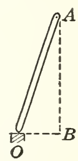


18. In planning for a pontoon bridge across the head of a lake to save marching through some swamp land a squad of Boy Scouts sighted from a point *C* due west to *A*, as shown in the figure. They then sighted from *C* directly north, thus laying out the line *CD*. On this line they took a point *B*, measured *AB*, and found it to be 1075 ft., and found ∠*A* to be 20°. Find the distance *BC*.



19. When the arm of a steel crane *OA*, 30 ft. long, makes an angle of 70° with the horizontal line *OB*, what is the vertical distance *AB*?

20. In Ex. 19 find the horizontal distance *OB*.

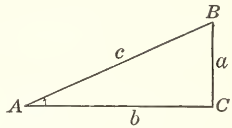


Given one acute angle and the hypotenuse of a right triangle as follows, find the other acute angle and the two sides:

21. 40°, 38 ft.	**23.** 60°, 9 mi.	**25.** 70°, 18 yd.
22. 50°, 48 in.	**24.** 30°, 18 mi.	**26.** 80°, 450 yd.

27. Make up a problem similar to Exs. 21–26, draw the figure, and solve the problem. If possible, have it relate to some measurement about the schoolhouse or in the vicinity of the school building.

12. Functions of an Angle. We have seen that the tangent of an angle changes as the angle changes. As the angle increases from 0° to 90° the tangent increases from 0 to ∞.



Similarly, as the angle increases from 0° to 90° the sine increases from 0 to 1.

In each case we have a ratio, $\frac{a}{b}$ or $\frac{a}{c}$, that depends upon the angle A for its value. Since a function of a quantity is a second quantity that depends upon the first one for its value, $\tan A$ and $\sin A$ are called *functions* of the angle A.

The angle A may evidently have other ratios which depend upon it for their values. For example, in the above figure, instead of having merely the ratios a/b and a/c we may also have such ratios as b/a, b/c, c/a, and c/b.

For our present purposes we shall find it convenient to consider two other functions of A.

13. Sine of the Complement of an Angle. In the above figure, if we have c and A given and wish to find b, we see that we must first find B, after which we shall have

$$b = c \sin B.$$

Since the sum of the two acute angles of a right triangle is 90°, we can find B from the equation $B = 90° - A$.

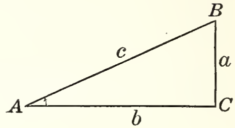
If the sum of two angles is a right angle, each angle is called the *complement* of the other.

It would be convenient if we had a table giving us the sines of the complements of angles as well as the sines themselves. We could then find b by the equation

$$b = c \times \text{sine of the complement of } A,$$

finding this sine of the complement of A from the table. On page 124 we shall find such a table.

14. Cosine of an Angle. The sine of the complement of an angle is called the *cosine* of the angle. The cosine of A, written $\cos A$, is the same as the sine of B. Expressed as an equation,



$$\cos A = \frac{b}{c},$$

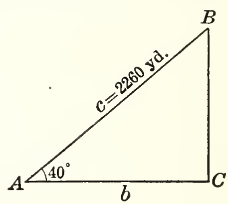
whence $$b = c \cos A.$$

The prefix *co* in "cosine" is an abbreviation of "complement."

15. Table of Cosines. In actual practice we find the cosines of angles from tables. The following is a brief table of cosines needed for the problems in the next exercise:

Angle	cos	Angle	cos	Angle	cos
10°	0.9848	40°	0.7660	70°	0.3420
20°	.9397	50°	.6428	80°	.1736
30°	.8660	60°	.5000	90°	.0000

16. Practical Use of the Cosine. An artillery officer at A receives a report from an airplane that there is a concealed battery at C, exactly east of him and exactly south of a battery B. He locates B as 40° north of a line running east from A, and his range finder shows that $AB = 2260$ yd. Find the distance AC.



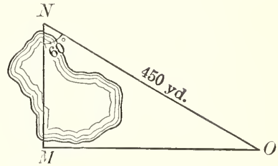
Here $c = 2260$ and $\angle A = 40°$.

Therefore $$b = c \cos A = 2260 \cos 40°$$
$$= 2260 \times 0.7660$$
$$= 1731.16.$$

Hence $AC = 1731$ yd., to the nearest yard.

Exercise 3. Cosines

1. Wishing to find the distance from M to N across a pond as here shown, some Boy Scouts sighted from M to N and then ran a line MO at right angles to MN. They measured ON and found it to be 450 yd., and they found $\angle N$ to be 60°. Find the distance MN.



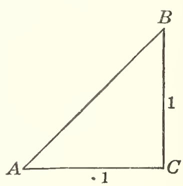
As in § 16, we have $MN = 450 \cos 60°$.

2. A boy walking along a straight road leaves it at a point P and goes along a straight oblique path 1000 ft. to a spring at S. He then takes a path that is perpendicular to the road and reaches the road at a point Q which is 866 ft. from P. Find the angle QPS which the oblique path makes with the road, and the angle at S which it makes with the other path.

First draw the figure freehand while reading the problem.

What function of $\angle P$ can be found? What is its value? What angle has this value for the function? How is $\angle S$ found?

3. In this right triangle suppose that $AC = CB = 1$ and then find the length of AB. From this find sin 45°, cos 45°, and tan 45°.



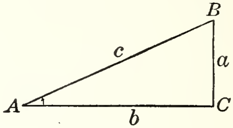
4. A ship sails N.W. 32 mi. Find its northing and also its westing.

The symbol N.W. means "northwest"; that is, N. 45° W. For cos 45°, see Ex. 3.

5. A ship sails N.E. 48 mi. Find its northing and also its easting.

6. Draw an equilateral triangle 1 in. on a side, draw a perpendicular from any vertex to the opposite side, and calculate sin 60°, cos 60°, tan 60°, sin 30°, cos 30°, and tan 30°.

17. Cotangent of an Angle. The tangent of the complement of an angle is called the *cotangent* of the angle. The cotangent of A, written $\cot A$, is the same as the tangent of B. Expressed as an equation,



$$\cot A = \frac{b}{a};$$

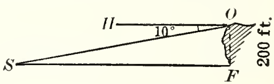
whence $$b = a \cot A.$$

18. Table of Cotangents. The following brief table of cotangents may be used in the next exercise:

Angle	cot	Angle	cot	Angle	cot
10°	5.6713	40°	1.1918	70°	0.3640
20°	2.7475	50°	0.8391	80°	.1763
30°	1.7321	60°	.5774	90°	.0000

The student should compare this table with the table of tangents given on page 116.

19. Practical Use of the Cotangent. An observer at O, the top of a cliff 200 ft. high, sees the periscope of a submarine S at an angle of depression of 10°. How far is the submarine from F, the foot of the cliff?



The *angle of depression* is the angle which the line of sight, OS, makes with the horizontal line OH, and in this figure it is the angle HOS which is equal to the angle S.

We have $$\frac{SF}{OF} = \cot S = \cot 10° = 5.6713.$$

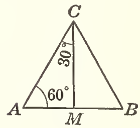
Hence $$SF = OF \times 5.6713 = 200 \times 5.6713.$$

That is, $$SF = 1134.26.$$

Therefore the submarine is approximately 1134 ft. from the foot of the cliff.

Exercise 4. Cotangents

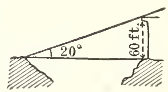
1. In planning a steel truss an equilateral triangle ABC is used, each side being 40 ft. It is required to know the height CM of the truss.



Each angle of an equilateral triangle is 60°. Hence $\angle A = 60°$ and $\angle ACM = 30°$. We also have $AM = 20$ ft. We may now find CM in several ways, as by using tan 60° or by using cot 30°.

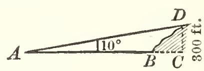
Since $CM/AM = \cot 30°$, to what is CM equal?

2. A building known to be 60 ft. high is observed from across a ravine, the angle of elevation of the top being 20°. How wide is the ravine?

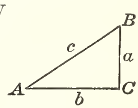


If the angle of elevation is taken from a point above the base of the building, allowance must be made for this height.

3. The top D of a hill is known to be 300 ft. above the level of a lake AB. An observer at A finds the angle of elevation of D to be 10°. Find the distance AC as shown in the diagram.



4. How far from the foot of a tree 60 ft. high must an observer lie down in order that he may see the top of the tree at an angle of 50°?



Find b, given the following:

5. $a = 47$ ft., $A = 30°$.

6. $a = 52$ in., $A = 40°$.

7. $a = 75$ yd., $A = 50°$.

8. $a = 125$ ft., $A = 60°$.

9. $a = 250$ ft., $A = 70°$.

10. $a = 12.7$ in., $A = 80°$.

11. $a = 52.8$ ft., $A = 60°$.

12. $a = 29$ ft. 6 in., $A = 40°$.

13. $a = 357$ m., $A = 40°$.

14. $a = 1275$ m., $A = 50°$.

15. $a = 3500$ m., $A = 30°$.

16. $a = 7525$ m., $A = 80°$.

Exercise 5. Review

1. If a post 4 ft. high casts a shadow 5 ft. long, find the tangent of the angle of elevation of the sun at the time.

This tangent should be expressed as a decimal fraction as usual.

2. In Ex. 1 what is the angle of elevation, stated to the nearest ten degrees?

The student should use the table given in § 4, page 116.

3. If a tree casts a shadow 75 ft. long at a time when a telegraph pole 17 ft. high casts a shadow 10 ft. long, find the tangent of the angle of elevation of the top of the tree taken at the extremity of the shadow. Find the angle of elevation and the height of the tree.

4. From the formula $\tan A = a/b$, derive a formula for b.

5. From the formulas $\sin A = a/c$ and $\cos A = b/c$ show that $\tan A = \sin A/\cos A$.

6. Show that $\cot A = 1/\tan A$ and $\cot A = \cos A/\sin A$.

7. Explain why tan 10° is the same as cot 80°, as given in the tables of § 4 and § 18. Of what angle is 0.3640 the tangent and of what angle is it the cotangent?

8. In § 9 we give 0.6428 and 0.7660 as the sines of two angles, and in § 15 we give 0.7660 and 0.6428 as the cosines of two angles, the same numbers in reverse order. Explain this coincidence.

9. One of the base angles of an isosceles triangle is 40° and the opposite side is 8 in. Find the length of the perpendicular from the vertex of the triangle to the base.

10. The angle opposite the base of an isosceles triangle is 80° and the base is 12 in. Find the length of each of the equal sides.

The perpendicular bisector of the base bisects the opposite angle.

II. TRIGONOMETRIC TABLES

20. Complementary Angles. We have now shown how some of the functions could be found to one or two decimal places by measuring certain lines, although this is not the method used in computing the tables. We have also seen that the table of tangents resembles the table of cotangents in some respects, and that the table of sines resembles the table of cosines. For example, we found that 0.1763 is given in the table of tangents as $\tan 10°$, and that the same number is given in the table of cotangents as $\cot 80°$. Indeed, all the numbers given as tangents are also given as cotangents, a circumstance that we should naturally expect for the reason that the cotangent of any angle is the tangent of 90° minus that angle. Hence we should expect to find that $\tan 10° = \cot 80°$, $\tan 20° = \cot 70°$, $\tan 30° = \cot 60°$, $\tan 45° = \cot 45°$, and so on, a function of any angle being the *cofunction* of its complement.

Exercise 6. Functions of Complementary Angles

All work oral

1. From § 9 we know that $0.9397 = \sin 70°$. Therefore 0.9397 is what other function of what other angle?

2. Since $0.7880 = \sin 52°$, it is what function of what other angle?

State the following as other functions of other angles:

3. $\sin 13° = 0.2250$.

4. $\cos 17° = 0.9563$.

5. $\tan 28° = 0.5317$.

6. $\cot 36° = 1.3764$.

7. $\sin 45° = 0.7071$.

8. $\cos 53° = 0.6018$.

9. $\tan 61° = 1.8040$.

10. $\cot 74° = 0.2867$.

11. $\sin 0° = 0.0000$.

12. $\cos 0° = 1.0000$.

Angle	sin	cos	tan	cot	
0°	0.0000	1.0000	0.0000	∞	**90°**
1°	.0175	.9998	.0175	57.2900	89°
2°	.0349	.9994	.0349	28.6363	88°
3°	.0523	.9986	.0524	19.0811	87°
4°	.0698	.9976	.0699	14.3007	86°
5°	.0872	.9962	.0875	11.4301	**85°**
6°	.1045	.9945	.1051	9.5144	84°
7°	.1219	.9925	.1228	8.1443	83°
8°	.1392	.9903	.1405	7.1154	82°
9°	.1564	.9877	.1584	6.3138	81°
10°	.1736	.9848	.1763	5.6713	**80°**
11°	.1908	.9816	.1944	5.1446	79°
12°	.2079	.9781	.2126	4.7046	78°
13°	.2250	.9744	.2309	4.3315	77°
14°	.2419	.9703	.2493	4.0108	76°
15°	.2588	.9659	.2679	3.7321	**75°**
16°	.2756	.9613	.2867	3.4874	74°
17°	.2924	.9563	.3057	3.2709	73°
18°	.3090	.9511	.3249	3.0777	72°
19°	.3256	.9455	.3443	2.9042	71°
20°	.3420	.9397	.3640	2.7475	**70°**
21°	.3584	.9336	.3839	2.6051	69°
22°	.3746	.9272	.4040	2.4751	68°
23°	.3907	.9205	.4245	2.3559	67°
24°	.4067	.9135	.4452	2.2460	66°
25°	.4226	.9063	.4663	2.1445	**65°**
26°	.4384	.8988	.4877	2.0503	64°
27°	.4540	.8910	.5095	1.9626	63°
28°	.4695	.8829	.5317	1.8807	62°
29°	.4848	.8746	.5543	1.8040	61°
30°	.5000	.8660	.5774	1.7321	**60°**
31°	.5150	.8572	.6009	1.6643	59°
32°	.5299	.8480	.6249	1.6003	58°
33°	.5446	.8387	.6494	1.5399	57°
34°	.5592	.8290	.6745	1.4826	56°
35°	.5736	.8192	.7002	1.4281	**55°**
36°	.5878	.8090	.7265	1.3764	54°
37°	.6018	.7986	.7536	1.3270	53°
38°	.6157	.7880	.7813	1.2799	52°
39°	.6293	.7771	.8098	1.2349	51°
40°	.6428	.7660	.8391	1.1918	**50°**
41°	.6561	.7547	.8693	1.1504	49°
42°	.6691	.7431	.9004	1.1106	48°
43°	.6820	.7314	.9325	1.0724	47°
44°	.6947	.7193	.9657	1.0355	46°
45°	.7071	.7071	1.0000	1.0000	**45°**
	cos	sin	cot	tan	Angle

21. Table of Functions. For our present purposes we do not need a table which gives the functions to minutes and seconds, but we shall need one which gives these functions to degrees. Such a table is given on the opposite page.

Since $\sin A = \cos(90° - A)$, we see that $\sin 50° = \cos 40°$. It is therefore apparent that we need give the functions only to 45°, the functions of angles from 45° to 90° being the same as the cofunctions of angles from 0° to 45°.

Hence in the table we may read the values of the functions from 45° to 90° by reading from the foot of the table to the top, using the degrees at the right.

For example, for cos 61°, look above cos and to the left of 61° and find 0.4848. Also, $\sin 72° = 0.9511$, $\tan 81° = 6.3138$, $\cot 66° = 0.4452$.

Exercise 7. Table of Functions

Using the table, find the following:

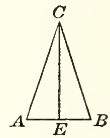
1. sin 27°.	**11.** sin 10°.	**21.** sin 78°.	**31.** sin 16°.
2. cos 37°.	**12.** cos 80°.	**22.** cos 78°.	**32.** cos 16°.
3. tan 15°.	**13.** cos 30°.	**23.** tan 78°.	**33.** sin 74°.
4. cot 28°.	**14.** sin 60°.	**24.** cot 78°.	**34.** cos 74°.
5. sin 35°.	**15.** tan 19°.	**25.** sin 45°.	**35.** tan 16°.
6. cos 55°.	**16.** cot 71°.	**26.** cos 45°.	**36.** cot 16°.
7. sin 70°.	**17.** sin 66°.	**27.** tan 45°.	**37.** tan 74°.
8. cos 20°.	**18.** cos 24°.	**28.** cot 45°.	**38.** cot 74°.
9. tan 60°.	**19.** tan 51°.	**29.** sin 0°.	**39.** tan 0°.
10. cot 30°.	**20.** cot 39°.	**30.** cos 0°.	**40.** cot 0°.

Find the angle x, given that:

41. $\sin x = 0.8387$.	**43.** $\cos x = 0.8988$.
42. $\tan x = 2.3559$.	**44.** $\cot x = 14.3007$.

Exercise 8. Review

1. The principle of a range finder is that of an isosceles triangle. The eye is at E, and an object C is reflected at both A and B to a prism at E. The instrument is arranged so that it can be adjusted to focus the lines AC and BC on the object C. If $AE = EB = 10$ in., find EC when $\angle A = \angle B = 89°$.



Practically, the distances are computed in this way when the instrument is made, and are read by the observer on a scale which accompanies the range finder.

2. In general, which is the greater, $\sin A$ or $\tan A$?

3. Show that $\sin 45° = \cos 45° = 1/\sqrt{2}$.

4. Show independently of the table that $\tan 45° = 1$.

5. From the given facts, $\sin A = a/c$, $\cos A = b/c$, and $a^2 + b^2 = c^2$, show that $\sin^2 A + \cos^2 A = 1$.

6. From the facts stated in Ex. 5 show that

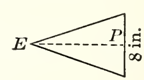
$$\sin A = \sqrt{1 - \cos^2 A} = \sqrt{(1 + \cos A)(1 - \cos A)}.$$

We can now find the sine of any angle if we know its cosine.

7. Show that $\cos A = \sqrt{(1 + \sin A)(1 - \sin A)}$, and use the formula to find $\cos 30°$, given that $\sin 30° = 0.5000$.

8. A ladder leaning against a wall makes an angle of 57° with the ground, and its foot is 8 ft. from the wall. How far up the wall does the ladder extend?

9. A boy holds a pencil 8 in. long in the position here shown, and his eye sees it under an angle of 38°, the figure forming an isosceles triangle. Find the distance from his eye to the pencil.



10. From the top of a lighthouse rising 125 ft. out of the sea the angle of depression of a boat is 18°. Find the distance of the boat from the foot of the lighthouse.

11. A ship sailed N. 47° E. and changed its latitude 30 mi. By how many miles did it change its longitude?

12. At a distance of 480 ft. the angle of elevation of the top of one of the big trees of California is 34°. Find the height of the tree.

13. When the sun's altitude is 40°, find the length of the shadow cast on level ground by a tree 60 ft. high.

14. The top of a flagstaff is partly broken off and touches the ground at a point 10 ft. from the foot of the staff. If the length of the broken part is 20 ft., what was the original height of the flagstaff?

Draw the figure; determine some function of one of the angles; find the height of the lower part of the flagstaff; and then add the length of the broken part. The problem can also be solved without trigonometry, but somewhat less readily.

15. A tree is broken by the wind, the upper part remaining joined to the lower part but tipping over so as to form the hypotenuse of a right triangle and to form an angle of 30° with the ground at a point 60 ft. from the foot of the tree. What was the original height of the tree?

16. A 40-foot ladder resting against the side of a house reaches a point 20 ft. from the ground. What angle does it make with the ground?

Find some function of the angle, and find by the table the angle which has the function with this value.

17. How far from a tree 25 ft. high must a person lie down in order that he may just see the top of the tree at an angle of elevation of 25°?

18. From the top of a lighthouse rising 100 ft. above the surface of the sea a buoy is observed to have an angle of depression of 18°. How far is the buoy from the foot of the lighthouse?

19. A ship sailing S. 48° E. changes its latitude 30 mi. in 3 hr. What is its rate of sailing per hour?

20. Seen from a point on the ground the angle of elevation of an airplane is 68°. If the airplane is 2000 ft. above the ground, how far is it in a straight line from the observer?

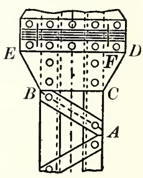
21. A barn is 36 ft. by 80 ft. and the pitch of the roof is 30°. Find the length of each rafter and the combined area of the two sloping parts of the roof.

22. Two sides of a triangle are 16 in. and 18 in., and the included angle is 40°. Find the area of the triangle.

First draw the figure as usual. Then find the height of the triangle by means of sin 40° and one side. Then recall the fact learned in arithmetic or in Book I about the area of a triangle.

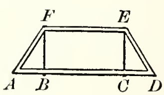
23. A man whose eye is 5 ft. above the ground stands midway between two telegraph poles which are 200 ft. apart. The angle of elevation of each pole is observed to be 50°. Find the height of each pole.

24. In this piece of construction work $BC = 1$ ft. and makes an angle of 30° with AB. Find the length of AB and also that of AC.



25. In the figure of Ex. 24 it is known that $BE = CD = 9\frac{1}{2}$ in. and that each makes an angle of 60° with BC. Find the length of the line CF.

26. A steel bridge has a truss $ADEF$ as here shown. It is known that $AD = 40$ ft., $FE = 24$ ft., and $BF = 13$ ft. 4 in. Find to the nearest degree the angle of the slope which AF makes with AD.



Find some function of the angle A and then find from the table the angle which comes nearest to having this function.

PART III. DEMONSTRATIVE GEOMETRY

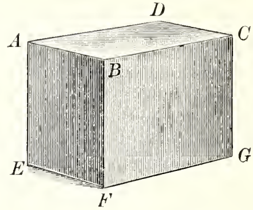
I. INTRODUCTION

1. Nature of the Work. Of the various questions that can be asked about an object, the student who has completed Book I of this series has learned that there are three which geometry can answer: What is its shape? How large is it? Where is it? To these three questions we may now add a fourth question which geometry may properly consider and answer: How do you know that the answers to these questions are correct? In Book I we considered the first three of these questions, and in this book we shall consider the fourth.

Students who have not completed Book I will be able to undertake the consideration of this fourth question, although they will first need to become familiar with the geometric forms which the other students have already learned. This can be accomplished by a careful study of this Introduction (pages 135–164).

Whether or not a student has completed Book I, some preliminary work is necessary before any actual proofs can be introduced. For some students a part of this work will be a review, requiring but little time; for others the work will be new; for all it will be an introduction to the work of demonstrative geometry. Certain terms must first be explained, however, before we can actually prove any geometric statements, since otherwise we shall not know what the statements mean or what it is in each case that we are to try to prove.

2. Solid. For our present purposes it is not necessary to define a *solid.* Cubes, spheres, blocks, pyramids, and cylinders are all examples of the solids studied in geometry. One of the most common solids is the *rectangular solid* here shown.



A *cube* is a special form of a rectangular solid.

Students should use the terms of geometry correctly, but it is not necessary that formal definitions should be learned except in cases where they are to be used as parts of a logical proof. A list of the definitions which should be memorized is given on page 162.

3. Surface. The above rectangular solid has a *surface* composed of the six *faces* of the solid. In the above figure one of the faces of the solid is the rectangle $ABCD$.

A surface has no thickness.

For example, the surface of the water in a glass has no thickness. We may *represent* a surface by a piece of paper, but a piece of paper is really a thin solid.

4. Line. In the above figure the faces $ABCD$ and $BFGC$ meet in the *line* BC.

A line has position, shape, and size, its size consisting only of its length.

We may represent a line by a mark, but a mark is really a very thin solid.

5. Point. In the above figure the lines CD and BC meet in the *point* C.

A point has position but not size.

In the above figure the point C is a *vertex* of the solid.

We may represent a point by a dot, but a dot is really a very thin solid.

6. Plane. In the figure on page 136, the face $ABCD$ is a *plane*, and the same may be said of the other five faces. Evidently a plane is a flat surface, not bending in any way.

Curve surfaces are familiar, as in the case of a sphere.

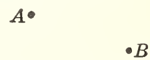
7. Plane Geometry. We are about to study figures that can be drawn in a plane, and for this reason the geometry which we shall consider is called *plane geometry.*

8. Straight Line. In the figure on page 136 the line CD is a *straight line.* A straight line does not bend in any way. Any definite part of a straight line is called a *segment* of the line.

The word "line" used alone always means a straight line, but for emphasis the word "straight" is often used with it.

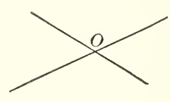
9. Properties of a Straight Line. A straight line has certain peculiarities which distinguish it from other lines and which are called its *properties.* They are as follows:

1. *One straight line and only one can be drawn through two given points.*



Thus through the two points A and B we can evidently draw any number of lines that are not straight, but we can draw only one straight line through both. This is also expressed by saying that *two points determine a straight line.*

2. *Two straight lines cannot intersect in more than one point.*



This means that one fixed line can cross another at only one point. In the figure this point is O.

3. *A straight line is the shortest path between two points.*

If one has to go from A to B, we cannot think of a shorter path than that of the line AB.

4. *If two points of a straight line lie in a plane, the whole line lies in the plane.*

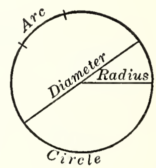
10. Other Kinds of Lines. As with a straight line, a *curve line* or *curve* is familiar to all. An illustration of a curve is here given.

CURVE

BROKEN LINE

A line made up of segments of straight lines is called a *broken line*.

11. Circle. A closed curve lying in a plane, and such that all of its points are equally distant from a fixed point in the plane, is called a *circle*.



When we draw a circle we say that we *describe a circle*. Either word, "draw" or "describe," may be used in this sense.

The terms which follow on this page are familiar to most students. They are repeated at this time merely for reference.

The point in the plane from which all points on the circle are equally distant is called the *center* of the circle.

A straight line extending from the center of a circle to the circle itself is called a *radius* (plural "radii").

A straight line through the center and terminated at each end by the circle is called a *diameter*.

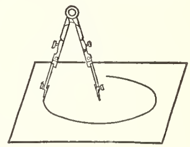
It is evident that a diameter is equal in length to two radii.

Any portion of a circle is called an *arc*.

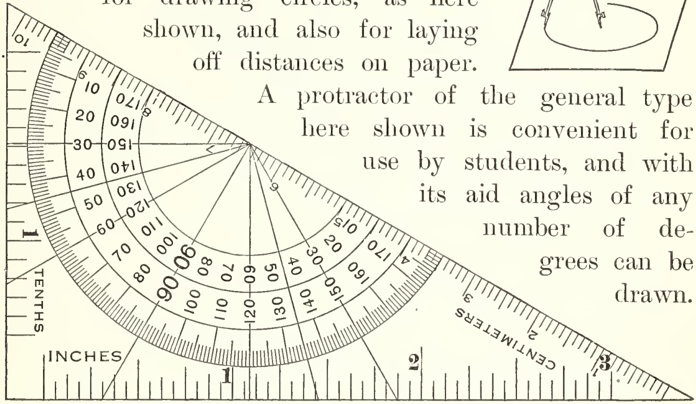
The length of the circle, that is, the distance around the space inclosed, is called the *circumference*. An arc that is half of a circle is called a *semicircle*. The length of a semicircle is called a *semicircumference*.

When a semicircle is drawn, it is the custom to draw a diameter also, the diameter of the circle being also called the diameter of the semicircle, and the radius of the circle being called the radius of the semicircle. A semicircle is a special kind of arc. Formerly the word "circle" was used to mean the part of the plane inclosed by the curve, the bounding line being then called the circumference, and this usage is still quite common.

12. Drawing Instruments. The instruments commonly used in drawing the figures in geometry are the compasses, the ruler, the protractor, and the right triangle. The compasses are used for drawing circles, as here shown, and also for laying off distances on paper.



A protractor of the general type here shown is convenient for use by students, and with its aid angles of any number of degrees can be drawn.



Each student should have a ruler, a pair of compasses, and a protractor, since the constructions studied in this book can be made only by their use.

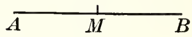
For the work in proving propositions, the compasses and ruler are the instruments used. For making ordinary geometric drawings, however, the protractor will be found very useful.

Students will find it both convenient and helpful to accustom themselves to use the metric system in measuring lengths and areas. Our recent change in international relations renders a knowledge of the common metric measures necessary, and the decimal division of the units makes the use of the system simple.

13. The Transit. For his work out of doors a surveyor measures angles and finds levels by means of a transit such as is shown on page 117.

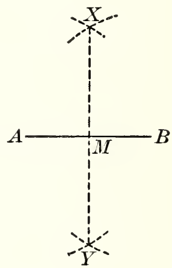
Most of the work on this page is repeated from Book I, page 115, for purposes of review for those who have studied that book and for the information of any who may not have done so.

14. Bisection of a Line. If a line is divided into two equal segments, it is said to be *bisected*, and the point of division is called the *mid-point* of the line or the *point of bisection.*



Thus, the point M bisects the line AB and is the mid-point, the bisector, or the point of bisection of AB.

15. How to Bisect a Line. One way to bisect a line is to measure its length with a ruler, divide this length by 2, and then measure off this half length. This method is, however, quite inaccurate, as may be seen by trying it and comparing the result with the better one which follows.



Let AB be the given line.

It is now required to bisect AB.

With A and B as centers and with radius greater than $\frac{1}{2}AB$ draw arcs. The most convenient radius is usually AB itself.

Call the points of intersection of the arcs X and Y.

Draw the straight line XY, and call the point where it cuts the given line M.

Then XY bisects AB at M; that is,

$$AM = BM.$$

This is much more nearly accurate than it is to measure the line with a ruler and then take half the length.

This construction is already familiar to those who have studied page 124 of Book I.

We shall now assume for the time being that

A line can be bisected.

We have not yet proved this fact, but it is easily proved a little later. For the present we shall speak of the mid-point of a line and of bisecting the line just as if we had proved that the construction is correct.

Exercise 1. Elementary Terms

1. By the aid of a ruler draw a straight line on paper and on this line mark off a segment AB, a longer segment PQ, and a shorter segment MN.

2. Draw a figure to show the number of points in which two straight lines can intersect; also one showing that a straight line may intersect a broken line or a curve in more than one point.

3. Crease a piece of paper by folding one part upon another, and show that the fourth property of § 9 applies to the crease.

4. Mark two points on a piece of paper, crease the paper through these two points, and state two properties of § 9 that apply to the crease.

5. By making two creases in a piece of paper show the application of the second property of § 9.

6. Draw a figure to show the number of points necessary to determine a straight line.

7. Draw a line 2 in. long and on it measure off 1 in. from either end. Then bisect the line by the method of § 15, and see if the point of bisection coincides exactly with the point found by measuring.

8. Draw a line 3 in. long, bisect it by § 15, and then bisect each half. Check the work by measuring lengths.

9. How many faces, vertices, and edges has a cube?

It is interesting to notice the relation of these numbers as shown in the equation $\frac{1}{6} - \frac{1}{8} = \frac{1}{8} - \frac{1}{12}$. This relation of 6, 8, and 12 was observed by the Greeks more than 2000 years ago.

10. Draw a semicircle, an arc less than a semicircle, and an arc greater than a semicircle.

11. Draw two circles with the same center, one with radius $\frac{1}{2}$ in. and the other with radius 1 in.

12. Draw a line, bisect it, and with the point of bisection as center and half the line as radius draw a circle.

If the figure is carefully drawn, the circle will pass through the ends of the line, thus giving a check on the bisection of the line.

13. In the definition of circle (§ 11) a closed curve is mentioned. Write a statement of your understanding of the meaning of the expression.

14. Write a statement of the meaning of the expression "points equally distant from a fixed point," found in § 11.

15. If a radius $1\frac{7}{8}$ in. long is used in drawing a circle, what is the length of the diameter?

16. If the diameter of a circle is $1\frac{7}{8}$ in. long, what is the length of the radius?

17. If the circumference of a circle is $\frac{22}{7}$ times the diameter, find to two decimal places the circumference of a circle whose radius is $2\frac{3}{4}$ in.

To find a number to two decimal places means to find it to the nearest hundredth. This requires the finding of the next figure beyond hundredths, rejecting it if less than 5 and increasing the hundredths by 1 if the third decimal place is 5 or more.

18. From the statement made in Ex. 17 find the semicircumference of a circle whose radius is $3\frac{1}{8}$ in., carrying the result to three decimal places.

19. If the circumference of a circle is 154 in., what is the length of the diameter? of the radius?

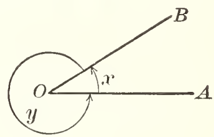
20. A chord of a circle being a straight line joining the ends of an arc, draw a circle and then draw two chords.

21. A quadrant of a circle being half of a semicircle, draw a quadrant.

16. Angle. If two straight lines meet in a point, they form an *angle.*

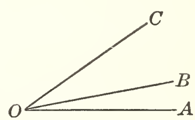
There are certain terms so simple that they cannot be defined by the use of simpler terms. Among these are point, line, surface, and angle. We can, however, make the meaning of such a term as "angle" clear by stating certain properties, as in the case of the straight line, or by discussing and explaining the term as is done on this page.

The two lines which form an angle are called the *arms*, and the point in which they meet is called the *vertex.*



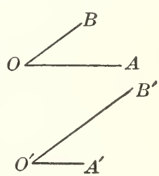
In this figure the angle x is also written $\angle O$, read "angle O," or $\angle AOB$, the letter at the vertex being read as the middle one.

17. Greater and Less Angles. From the above figure it is evident that two angles are formed; namely, the angle x and the angle y. If the arm OA revolves about the vertex O until OA falls on OB, we say that OA has *turned through* the angle x, or has *generated* it. The smaller of the two angles, x, is called *the angle* formed by the lines, or the *angle between the lines.*



The amount of turning necessary to generate the angle AOB in this figure is evidently less than the amount of turning necessary to generate the angle AOC, and so we say that angle AOC is *greater than* angle AOB, and that angle AOB is *less than* angle AOC.

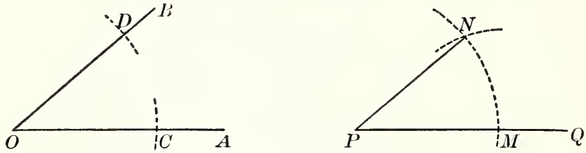
18. Equal Angles. If either of two angles can be placed on the other so that they coincide, the two are called *equal angles.*



For example, these two angles are equal, all lines being supposed to be indefinitely long. The amount of turning necessary to generate one angle is evidently the same as that necessary to generate the other.

19. Constructing an Angle equal to a Given Angle. In copying figures we often have to construct an angle equal to a given angle. This leads to the following construction:

From a given point on a given line construct a line which shall make with the given line an angle equal to a given angle.



Let P be the given point on the given line PQ and let angle AOB be the given angle.

It is required to construct a line from P which shall make with PQ an angle equal to $\angle AOB$.

With O as center and with any convenient radius draw an arc cutting OA at C and OB at D.

With P as center and with OC as radius draw an arc cutting PQ at M.

With M as center and with the straight line joining C and D as radius draw an arc cutting the arc just drawn at N, and draw PN.

Then the angle MPN is the required angle.

This construction is repeated for purposes of review from Book I.

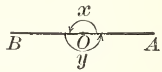
To *draw* a figure may henceforth be taken to mean the making of the figure freehand.

To *construct* a figure will be taken to mean the making of the figure accurately by the aid of compasses and ruler.

Lines used merely as aids in the construction are dotted.

In many cases it is immaterial whether the word "draw" or the word "construct" is used, as when we speak of drawing a line. When a circle is to be made, the word "describe" is often used, as in the expression "describe a circle."

20. Straight Angle. If the arms of an angle extend in opposite directions so as to be in one straight line, the angle is called a *straight angle.*

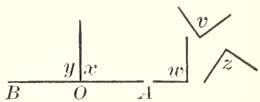


For example, each of the angles x and y in this figure is a straight angle.

Since any two straight angles may evidently be made to coincide, if placed one upon the other, we see that

All straight angles are equal.

21. Right Angle. Half of a straight angle is called a *right angle.*



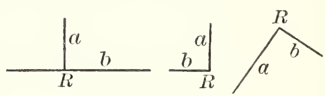
For example, x and y are evidently halves of the straight angle AOB and hence they are right angles. The angles z, v, and w are also right angles.

As you learned if you studied Book I, there are 90° in a right angle, and hence there are 180° in a straight angle. If a line OA turns all the way around the point O, it turns through 360°.

Since halves of straight angles are equal, we see that

All right angles are equal.

22. Perpendicular. If one line meets another so as to make a right angle with it, either of the two lines is said to be *perpendicular* to the other.



In each of these figures, R is the vertex of a right angle; hence in each figure a is perpendicular to b, and b is also perpendicular to a.

The line a is also called a *perpendicular* to b.

The symbol $\perp$ is used for the word "perpendicular," and therefore the expression "a is $\perp$ to b" is read "a is perpendicular to b," and the expression "a $\perp$" is read "a perpendicular."

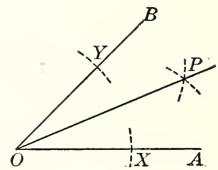
The point R in each figure is called the *foot* of the perpendicular to b, or the foot of the perpendicular to a.

The terms *horizontal, vertical,* and *slanting* are used in geometry with the usual meaning with which the student is familiar.

23. Bisecting an Angle. To draw from the vertex of an angle a line dividing the angle into two equal parts is to *bisect* the angle.

Hence we speak of the *bisection* of an angle, of the *bisector* of the angle, and of the line which *bisects* the angle.

Bisect a given angle.



Let AOB be the given angle.

What is now required?

With O as center and with any convenient radius draw an arc cutting OA at X and OB at Y.

With X and Y respectively as centers and with a radius greater than half the distance from X to Y draw arcs and call their point of intersection P.

Draw OP.

Then OP is the required bisector.

This is much more nearly accurate than it is to measure the angle with a protractor and then take half the number of degrees, as the student should show by trying both methods.

In finding the point P, a convenient radius to take is the line drawn from X to Y; that is, set one point of the compasses at X and the other at Y, construct one arc, and then construct the other.

This construction is repeated for purposes of review from Book I.

In particular we may bisect a straight angle, which is one way of accurately constructing a right angle or of constructing a perpendicular at a given point on a line.

This case is considered on the next page under the construction of perpendiculars.

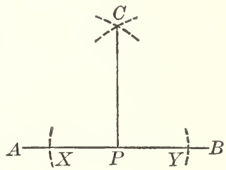
We shall now assume for the time being that

An angle can be bisected.

The proof that the above method is correct will be easily seen a little later. For the present we shall simply assume that the bisection of an angle is possible.

24. Constructing Perpendiculars. There are two convenient methods of constructing perpendiculars.

At a given point on a given straight line construct a perpendicular to the line.



Let AB be the given line and P be the given point.

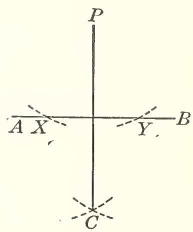
What is now required?

With P as center and with any convenient radius draw arcs intersecting AB at X and Y.

With X as center and XY as radius draw an arc, and with Y as center and the same radius draw another arc, and call one intersection of the arcs C.

Draw PC.

From a given point outside a given straight line construct a perpendicular to the line.



Let AB be the given line and P be the given point. How are X and Y fixed? Then how is C fixed? Draw PC.

These methods of construction differ from the methods of drawing which make use of a T-square or a draftsman's triangle. The methods of drawing will be reviewed later in the exercises.

25. Assumptions. We shall now assume that

1. *At a given point on a given line one perpendicular and only one perpendicular to the line can be constructed.*
2. *From a given point outside a line one perpendicular and only one perpendicular to the line can be constructed.*
3. *The shortest path from a given point to a given line is the perpendicular from the point to the line.*

The length of this path is called *the distance* to the line.

Exercise 2. Angles

1. Draw three different angles, letter each, and write them in order of size, beginning with the smallest.

2. Draw three angles, each being greater than a right angle, and proceed as in Ex. 1.

3. Draw an angle equal to three right angles.

4. Draw an angle and then construct an equal angle.

5. Construct a right angle.

6. Bisect the right angle constructed in Ex. 5, thus constructing an angle of 45°.

7. Divide an angle into four equal parts.

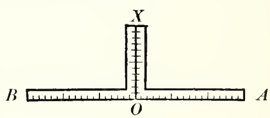
8. Construct an angle of 22° 30′.

9. Draw a line 3 in. long, bisect it, and construct a perpendicular to it at the point of bisection.

10. Draw a horizontal line, take a point below it, and from this point construct a line perpendicular to the horizontal line.

11. Draw a slanting line, take a point on the line, and at this point construct a line perpendicular to the slanting line.

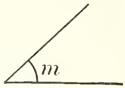
12. Draw a vertical line, take a point to the right of it, and from this point construct a line perpendicular to the vertical line.



13. Show how to test a carpenter's square by using it to draw a perpendicular OX to AB at O and then turning it over and again drawing a perpendicular to AB. If the square is true, how is this shown by § 25?

14. If the north and south line is given on a city map, show by a drawing how to determine accurately the east and west line through any point on the map.

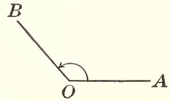
26. Acute Angle. An angle that is less than a right angle is called an *acute angle.*



For example, in this figure $\angle m$ is an acute angle.

Evidently an acute angle is an angle less than 90°.

27. Obtuse Angle. An angle that is greater than a right angle and less than a straight angle is called an *obtuse angle.*

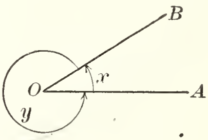


For example, in this figure $\angle O$ is an obtuse angle.

Evidently an obtuse angle is an angle greater than 90° and less than 180°.

Acute and obtuse angles are sometimes called *oblique angles*, either arm being then said to be *oblique* to the other.

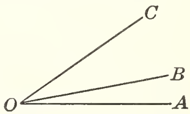
28. Reflex Angle. An angle that is greater than a straight angle is called a *reflex angle.*



For example, in this figure $\angle y$ is a reflex angle.

Evidently a reflex angle is an angle of more than 180°. Such angles are generated when a wheel turns more than halfway around, and therefore we see that we may have angles as large as we please. If a wheel turns once around, it turns through 360°; if twice around, through an angle of $2 \times 360°$, or 720°.

29. Adjacent Angles. Two angles having the same vertex and a common arm between them are called *adjacent angles.*



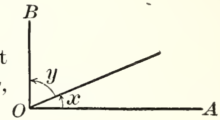
For example, in this figure $\angle AOB$ and $\angle BOC$ are adjacent angles.

If each of two adjacent angles is 90°, what kind of a figure is formed?

30. Sum and Difference of Angles. In the preceding figure $\angle AOC$ is said to be the *sum* of $\angle AOB$ and $\angle BOC$, and $\angle AOB$ the *difference* between $\angle AOC$ and $\angle BOC$.

A more formal definition is not necessary. It is simply necessary to see that angles can be added and subtracted like other magnitudes and to visualize the sum and the difference of two angles.

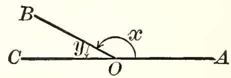
31. Complementary Angles. If the sum of two angles is a right angle, each angle is called the *complement* of the other, and the two angles are called *complementary angles.*



For example, in this figure $\angle AOB$ is a right angle. Hence $\angle x$ is the complement of $\angle y$, and $\angle x$ and $\angle y$ are complementary angles.

It is evident that the sum of two complementary angles is 90°. Hence, if one of two complementary angles is 72° 30′, the other is 90° − 72° 30′, or 17° 30′.

32. Supplementary Angles. If the sum of two angles is a straight angle, each angle is called the *supplement* of the other, and the two angles are called *supplementary angles.*



For example, in this figure $\angle AOC$ is a straight angle. Hence $\angle x$ is the supplement of $\angle y$, and $\angle x$ and $\angle y$ are supplementary angles.

It is evident that the sum of two supplementary angles is 180°. Hence, if one of two supplementary angles is 123° 5′, the other is 180° − 123° 5′, or 56° 55′.

From the definition of supplementary angles the following properties are evident:

1. *The two adjacent angles which one unlimited straight line makes with another are together equal to a straight angle.*

This means that the lines form a figure like the second one on this page, where $\angle x + \angle y = 180°$.

2. *If the sum of two adjacent angles is a straight angle, their outer arms are in the same straight line.*

That is, in the preceding figure, where $\angle x + \angle y = 180°$, the outer arms, OA and OC, are in the same straight line. If angles of 104° 20′ and 75° 40′ are placed as $\angle x$ and $\angle y$ are placed, their outer arms are in the same straight line.

3. *The supplements of equal angles are equal.*

Exercise 3. Angles

1. How many degrees are there in one fourth of a right angle?

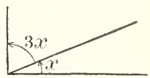
When the answer to such a question cannot be expressed exactly in degrees, minutes, and seconds, it should be carried to tenths of a second, but may be expressed in degrees and thousandths of a degree.

2. Are the angles $47° 10'$ and $43° 40'$ complementary? If not, what is the complement of $47° 10'$? of $43° 40'$?

3. Draw any acute angle and construct both its complement and its supplement.

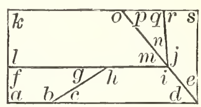
4. Are the angles $103° 42' 3.8''$ and $76° 17' 57.2''$ supplementary? If not, what is the supplement of the angle $103° 42' 3.8''$? of the angle $76° 17' 57.2''$?

5. The complement of a certain angle x is $3x$. How many degrees are there in each of the angles?



6. What kind of angle is the supplement of an acute angle? of a right angle? of an obtuse angle? Draw a figure in each case to illustrate.

7. Describe each angle in this figure with reference to the terms acute, right, obtuse, adjacent, complementary, and supplementary.

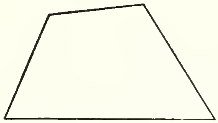


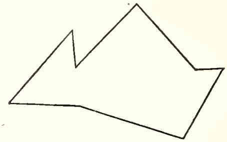
8. The supplement of a certain angle x is $4x$. How many degrees are there in each angle? Draw the figure.

9. What kind of angle is always greater than its supplement? Draw a figure to illustrate.

10. What kind of angle is formed by the sum of a right angle and an acute angle? by their difference? Draw a figure in each case to illustrate.

33. Rectilinear Figure. A closed plane figure formed by segments of straight lines is called a *rectilinear figure.*





The segments are called the *sides* of the figure.

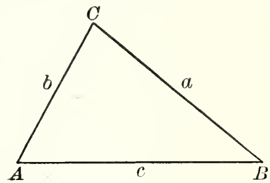
Teachers may observe that the definition might be extended to include figures that are not closed, but this is neither necessary nor desirable at this time.

34. Square. A rectilinear figure of four equal sides and four right angles is called a *square.*

This figure is too well known to require illustrating.

We shall not at present explain the word "rectangle," it being so familiar that it can be freely used in the exercises.

35. Triangle. A rectilinear figure of three sides is called a *triangle.*



In this figure the lines a, b, c are the sides of the triangle. Small letters are usually taken to represent the sides, and they correspond to the capital letters representing the opposite angles.

The sides taken together form the *perimeter* of the figure, this word being also used to represent the sum of their lengths.

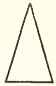
The points A, B, and C are called the *vertices of the triangle.* They are also the *vertices of the angles* A, B, and C, which are called the *angles of the triangle.*

The side AB upon which the triangle is supposed to stand is called the *base* of the triangle.

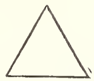
The vertex C, opposite the base of a triangle, is called *the vertex* of the triangle; that is, although a triangle has three vertices, *the* vertex is the one which is opposite the base.

These terms are familiar to students who have studied Book I.

36. Triangles classified as to Sides. A triangle is an *isosceles* triangle when two of its sides are equal; an *equilateral* triangle when all of its sides are equal.



ISOSCELES



EQUILATERAL

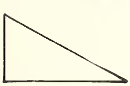
The word *equilateral* means equal-sided. It applies to any figure having equal sides.

An equilateral triangle is a special kind of isosceles triangle.

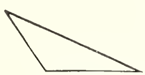
An isosceles triangle is usually represented as resting on the side which is not equal to either of the other sides. This side is called the *base* of the isosceles triangle.

If no two sides of a triangle are equal, the triangle is called a *scalene triangle*, but the term is not commonly used.

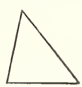
37. Triangles classified as to Angles. A triangle is a *right triangle* when one of its angles is a right angle; an *obtuse triangle* when one of its angles is an obtuse angle; an *acute triangle* when all of its angles are acute angles; an *equiangular triangle* when all of its angles are equal.



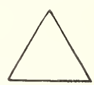
RIGHT



OBTUSE

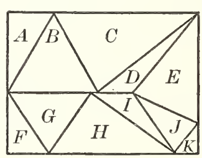


ACUTE



EQUIANGULAR

In a right triangle the side opposite the right angle is called the *hypotenuse*.

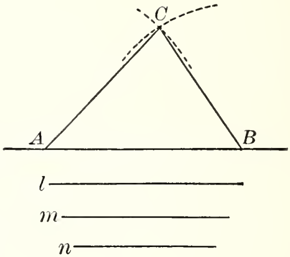


The other two sides are often called simply the *sides* of the right triangle when no confusion is likely to arise.

After he has studied the above definitions the student should be asked to describe each of the triangles which are shown in this figure, using compasses or dividers (compasses with two needle points) if necessary for comparing lines.

38. Construction of Triangles. The simple constructions given below should be studied:

1. *Construct a triangle having its sides equal respectively to three given lines.*



Let l, m, n be the given lines.

It is required to construct a triangle with l, m, n as sides.

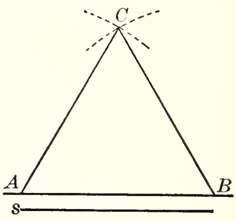
Draw a line with the ruler and on it mark off with the compasses a line segment AB equal to l.

It is more nearly accurate to do this with the compasses than with a ruler.

With A as center and m as radius draw an arc; with B as center and n as radius draw another arc cutting the first arc at C. Draw AC and BC.

Then because $AB = l$, $AC = m$, and $BC = n$ it follows that ABC is the required triangle.

2. *Construct an equilateral triangle, given one of the sides.*



Let s be the given side.

It is required to construct a triangle with each side equal to s.

Proceeding exactly as in the case above considered, except that each side is equal to s, we have the equilateral triangle required.

The student should carry out the construction in full, using the ruler and compasses.

3. *Construct an isosceles triangle, given the base and one of the two equal sides.*

This is the same as the first of the above constructions if m is made equal to n. The student should carry out the construction in full.

Exercise 4. Construction of Triangles

1. Construct a triangle having its sides respectively $\frac{1}{2}$ in., $\frac{3}{4}$ in., and 1 in.

2. Construct a triangle having its sides respectively $\frac{3}{4}$ in., 1 in., and $1\frac{1}{4}$ in.

3. What kind of triangle does the triangle in Ex. 2 seem to be? Show that your inference is correct by applying the law learned in Book I, that the square on the hypotenuse is equal to the sum of the squares on the other two sides.

4. Construct an equilateral triangle having each side 1 in. long.

5. Construct an equilateral triangle having its perimeter $3\frac{3}{4}$ in. long.

6. Construct an isosceles triangle having its base 1 in. and each of its two equal sides $1\frac{1}{2}$ in.

7. Construct a right triangle having one side 2 in. and the other side $1\frac{1}{2}$ in.

As stated in § 37, when we speak of only two sides of a right triangle we mean the two sides other than the hypotenuse.

8. Construct an equilateral triangle having each side $\frac{3}{4}$ in. long, and on each of the three sides construct another equilateral triangle outside the figure. What seems to be the nature of the entire figure thus constructed?

9. Construct an isosceles triangle having its base equal to half of one of the two equal sides.

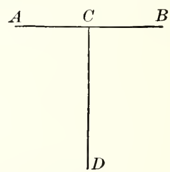
10. Construct an isosceles triangle having its base equal to twice one of the two equal sides.

At any time that a construction seems to be impossible the student should state the fact in writing and give what seems to be the reason.

39. Judging by Appearances. In looking at geometric figures we often find that if we judge simply by appearances, we make mistakes. It is partly for this reason that we shall take up the proof of certain statements in geometry.

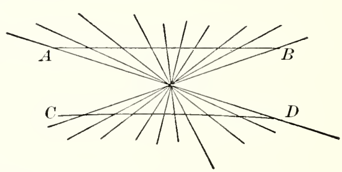
Exercise 5. Judging by Appearances

1. Estimate which is the longer line, *AB* or *CD*, and how many sixteenths of an inch longer. Then test the result of your estimate by measuring with the compasses or with a carefully marked piece of paper.

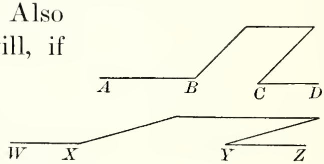


2. Estimate which is the longer of these two lines, *AB* or *XY*, and test the result as in Ex. 1.

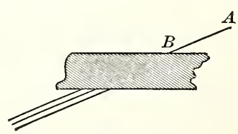
3. Look at this figure and state whether *AB* and *CD* are both straight lines. If one of them is not a straight line, which one is it? Test your answer by using a ruler or the folded edge of a piece of paper.



4. Look at this figure and state whether the line *AB* will, if prolonged, lie on *CD*. Also state whether the line *WX* will, if prolonged, lie on *YZ*. Test your answers by laying a ruler along the lines.



5. Look at this figure and state which of the three lower lines is *AB* prolonged. Then test your answer by laying a ruler along *AB*.



40. Bases for Proofs. Since we cannot trust to appearances when absolute accuracy of statement is demanded, and since instruments for measuring can never be absolutely exact, we must depend upon our reasoning powers to prove most of our statements in geometry.

There are, however, certain statements that are so evident that human reason and common sense always accept them as bases for proofs. For example, it would be unreasonable to ask you to prove such a simple statement as that two numbers must be equal if they are both equal to a certain other number.

There are two kinds of simple statements that we shall assume as bases upon which to build our geometry. The first is used generally in all mathematics, and we have already met with it in algebra (page 16); the second is used only in some particular part of mathematics; in this case, geometry. These will now be defined.

41. Axiom. A general statement admitted without proof to be true is called an *axiom.*

For example, it is stated in algebra that "if equals are added to equals the sums are equal." This is so simple that it is generally accepted without proof. It is therefore an axiom.

42. Postulate. In geometry a geometric statement admitted without proof to be true is called a *postulate.*

For example, it is so evident that all straight angles are equal, that this statement is a postulate. It is also evident that a straight line may be drawn and that a circle may be described, and these statements are therefore postulates of geometry.

Axioms are therefore general mathematical assumptions, while geometric postulates are the assumptions peculiar to geometry. Postulates and axioms are the assumptions upon which the whole science of mathematics rests, and some teachers prefer to use the word "assumptions" to include both.

43. List of Axioms. The following are some of the axioms used in geometry. They have already been studied in algebra (p. 16) and are given here for the purpose of review. Other axioms will be assumed later when needed.

1. *If equals are added to equals, the sums are equal.*

For example, since $9 = 5 + 4$

and $5 = 3 + 2$

we see at once that $\overline{9 + 5} = \overline{5 + 4 + 3 + 2}$

or $14 = 14$

Likewise, if $a = 3$ and $b = 7$, then $a + b = 3 + 7 = 10$.

2. *If equals are subtracted from equals, the remainders are equal.*

For example, since $9 = 5 + 4$

and $3 = 2 + 1$

we see at once that $\overline{9 - 3} = \overline{5 + 4 - 2 - 1}$

or $6 = 6$

Likewise, if $a = 10$ and $x = 3$, then $a - x = 10 - 3 = 7$.

3. *If equals are multiplied by equals, the products are equal.*

For example, since $12 = 15 - 3$

and $2 = 2$

we see at once that $\overline{2 \times 12} = \overline{2 \times 15 - 2 \times 3}$

or $24 = 30 - 6$

or $24 = 24$

Likewise, if $\frac{1}{2}x = 7$, then $x = 2 \times 7 = 14$.

4. *If equals are divided by equals, the quotients are equal.*

For example, since $16 = 9 + 7$

we see at once that $16 \div 4 = (9 + 7) \div 4$

or $4 = \frac{9}{4} + \frac{7}{4}$

or $4 = \frac{16}{4} = 4$

The divisor is never zero, division by zero having no meaning.

44. List of Postulates. The following are among the more important postulates used in geometry. Other postulates will be introduced as needed.

These postulates have already been stated. They should now be memorized, the first three by number.

1. *One straight line and only one can be drawn through two given points.*

Two points determine a straight line.

Two straight lines cannot intersect in more than one point.

These are three different ways of expressing the same idea.

2. *A straight line is the shortest path between two points.*

Since distance in a plane is measured on a straight line, this postulate is often stated: *A straight line is the shortest distance between two points.*

3. *All straight angles are equal.*

All right angles are equal.

The second statement follows from the first by Ax. 4.

4. *A straight line can be bisected.*

5. *An angle can be bisected.*

6. *At a given point in a given line one perpendicular and only one can be constructed to the line.*

7. *From a given point outside a given line one perpendicular and only one can be constructed to the line.*

8. *The shortest path from a given point to a given line is the perpendicular from the point to the line.*

9. *The sum of two adjacent angles which one straight line makes with another is equal to a straight angle.*

10. *If the sum of two adjacent angles is a straight angle, their outer arms are in the same straight line.*

Teachers will recognize that several of these postulates admit of easy proof and that such postulates as those of drawing lines and circles are omitted. This arrangement is desirable for beginners.

45. Theorem. A statement which is to be proved is called a *theorem.*

For example, it is stated in arithmetic that the square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides. This statement is one of the most important theorems of geometry.

46. Problem. A construction which is to be made so that it shall satisfy certain given conditions is called a *problem.*

For example, required to construct an angle equal to a given angle. This construction was made in § 19, and later it can easily be proved that the construction is correct.

47. Proposition. A statement of a theorem to be proved or a problem to be solved is called a *proposition.*

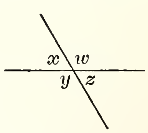
In geometry, therefore, a proposition is either a theorem or a problem. We shall find that most of the propositions at first are theorems. After we have proved a number of theorems we shall prove that some of the constructions already made in problems are correct.

48. Corollary. A truth that follows from another with little or no proof is called a *corollary.*

For example, since we admit that all straight angles are equal, it follows as a corollary that all right angles are equal, since a right angle is half of a straight angle.

49. How Propositions are Proved. We have said that we are now about to prove our statements in geometry, and we shall now see what is meant by a proof. For this purpose we shall take a simple proposition concerning vertical angles, a term which we must first define.

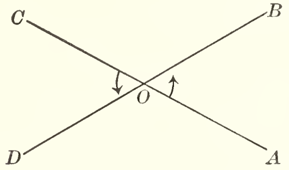
50. Vertical Angles. When two angles have the same vertex, and the sides of the one are prolongations of the sides of the other, these angles are called *vertical angles.*



In the figure the angles x and z are vertical angles, as are also the angles w and y.

THEOREM. VERTICAL ANGLES

51. *If two lines intersect, the vertical angles are equal.*



Given the lines *AC* and *BD* intersecting at *O*.

To prove that $\angle AOB = \angle COD$.

Proof. $\angle AOB + \angle BOC =$ a st. $\angle$. Post. 9

(*The sum of two adjacent angles which one straight line makes with another is equal to a straight angle.*)

Likewise $\angle BOC + \angle COD =$ a st. $\angle$. Post. 9

Therefore $\angle AOB + \angle BOC = \angle BOC + \angle COD$. Post. 3

(*All straight angles are equal.*)

Therefore $\angle AOB = \angle COD$. Ax. 2

(*For we have subtracted* $\angle BOC$ *from equals, leaving* $\angle AOB = \angle COD$.)

52. Nature of a Proof. From an examination of the above proof it may be inferred that in the treatment of a theorem there are three things to be considered:

1. Certain things are *given*; in the above case, that *AC* and *BD* intersect at *O*.

2. A definite thing is stated as the proposition which is *to be proved*; in the above case, that $\angle AOB = \angle COD$.

3. There is a *proof*, consisting of definite statements, each supported by the authority of a definition, an axiom, a postulate, or a proposition previously proved.

Exercise 6. Statements to be Memorized

All work oral

Define the following terms:

1. Circle.
2. Equal angles.
3. Straight angle.
4. Right angle.
5. Perpendicular.
6. Acute angle.
7. Obtuse angle.
8. Adjacent angles.
9. Complement.
10. Supplement.

The definitions of the above terms are needed in certain proofs and hence they should now be memorized.

11. State the postulates by number and state the ground for assuming each.

The postulates are so often used that it is convenient to refer to them by number. If desired, however, only the first three need be learned by number, the others being given in full as required. The ground for assuming each may be illustrated at the blackboard.

State and illustrate the following axioms:

12. Axiom 1. **13.** Axiom 2. **14.** Axiom 3. **15.** Axiom 4.

Define and illustrate the following:

16. Theorem. **17.** Problem. **18.** Corollary.

19. Define and illustrate the terms *axiom* and *postulate.*

20. Define vertical angles.

21. State the theorem relating to vertical angles.

Geometry is not a science to be memorized; it is a science to be understood. There are a few definitions and statements, like those mentioned above, that should be memorized for convenience in future work, and the statements of the propositions should be memorized for the same reason; but in the case of many terms it is merely necessary that they should be used properly.

Exercise 7. Statements to be Explained

All work oral

Explain in your own language what you understand by the following terms, illustrating at the blackboard if necessary:

1. Surface.
2. Line.
3. Point.
4. Plane.
5. Straight line.
6. Broken line.
7. Curve.
8. Circumference.
9. Center.
10. Arc.
11. Radius.
12. Diameter.
13. Semicircle.
14. Bisect.
15. Mid-point.
16. Angle.
17. Vertex.
18. Arms of an angle.
19. Sum of angles.
20. Proposition.
21. Intersect.
22. Greater angle.
23. Reflex angle.
24. Horizontal.

Explain what is meant by the following statements, illustrating at the blackboard if necessary:

25. A surface has no thickness.

26. A line has position, shape, and size, its size consisting only of its length.

27. A point has position but not size.

28. A line turning about a point generates an angle.

29. The smallest of the angles formed by two lines through a point is called the angle between the lines.

30. The foot of a perpendicular is the point in which a perpendicular meets the line to which it is drawn.

Exercise 8. Review

1. Draw a line AB which shall be 2 in. long. Then find a point C such that $AC = 3$ in. and $BC = 2\frac{1}{2}$ in.

Notice that two such points can be found on the paper.

2. Mark on paper three points P, Q, R such that $PQ = 1$ in., $QR = 1\frac{1}{4}$ in., and $RP = 1\frac{1}{2}$ in.

3. Construct an equilateral triangle ABC with each side $1\frac{1}{2}$ in. Within this triangle find a point P such that $AP = 1$ in. and $BP = 1$ in.

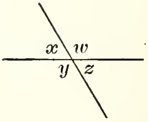
4. A point P is 2 in. from a line AB. Draw a figure showing how you would find two points on the line, each of which is 3 in. from P.

5. Draw a line and construct any point P that shall be 2 in. from the line.

6. Draw a line AB, mark any point P on it, and at P construct a line PQ perpendicular to AB. Then from Q construct a line perpendicular to AB.

Of course the two perpendiculars should coincide, and thus each construction forms a check upon the other.

7. In this figure, supposing that $\angle w$ is equal to 120°, find the number of degrees in each of the angles x, y, and z.



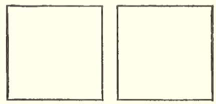
8. Consider Ex. 7 when $\angle w = 116° \, 30'$.

9. With a protractor draw an angle of 60°, bisect this angle by § 23, and measure each of the halves with the protractor, thus checking your work.

10. Draw any triangle, with a protractor measure each angle, and find the sum of the angles. Now cut the triangle from paper, cut off the three angles and fit them together, measure their sum, and thus check your work.

II. TRIANGLES

53. Congruent Figures. If two figures have exactly the same shape and size, they are called *congruent figures.*

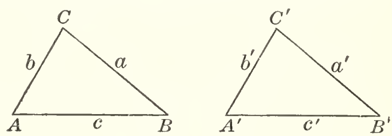


For example, the two squares here shown are congruent figures, two circles with equal radii are congruent figures, and two straight lines each of which is 1 in. long are congruent figures.

If two figures can be made to coincide in all their parts, they are congruent figures.

By the parts of a figure we mean the sides, angles, and surface.

54. Corresponding Parts. It is customary in geometry to letter the angles of a triangle by capitals arranged about the figure in counterclockwise order; that is, reading about the figure in the direction opposite to that in which the hands of a clock move. As we have already seen, it is customary to letter the sides of a triangle by using small letters placed opposite their respective capitals.



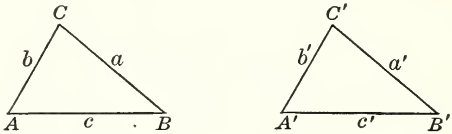
In the case of two congruent triangles it is convenient to use in one triangle the same letters that are used in the other, but with a prime (′) after each letter in one of the figures.

In the triangles shown above, A' (read "A-prime") *corresponds* to A, B' corresponds to B, C' corresponds to C, a' corresponds to a, and so on; that is, these pairs of parts are respectively equal.

It is therefore evident that

In two congruent figures the parts of one figure are respectively equal to the corresponding parts of the other figure.

55. Inferences as to Congruent Triangles. When we examine two triangles we easily infer certain facts relating to them. Consider, for example, the following questions



relating to the two triangles here shown, drawing the necessary figures to explain each answer:

1. If $\angle A = \angle A'$ and you are not sure about any of the other parts, are the triangles necessarily congruent?

If the triangles are not congruent, draw two triangles having $\angle A = \angle A'$ and yet evidently not congruent. Do the same in considering the other questions given below.

2. If $\angle A = \angle A'$ and $b = b'$, are the triangles necessarily congruent?

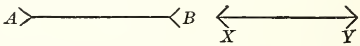
3. If $\angle A = \angle A'$, $b = b'$, and $c = c'$, are the triangles necessarily congruent?

4. If $\angle A = \angle A'$, $\angle B = \angle B'$, and $c = c'$, are the triangles necessarily congruent?

5. If $\angle A = \angle A'$, $\angle B = \angle B'$, and $\angle C = \angle C'$, are the triangles necessarily congruent?

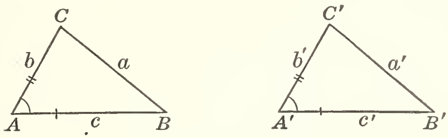
6. If $a = a'$, $b = b'$, and $c = c'$, are the triangles necessarily congruent?

56. Doubts as to the Inferences. None of the inferences in § 55 may be correct. When we look at these lines we may think that they are not equal, but they are.



To be certain of any inference we must find some way of proving it. Proving correct inferences or disproving incorrect ones is one of the main purposes of geometry.

57. Examination of an Inference. Let us consider the inference of § 55, 3, that if $\angle A = \angle A'$, $b = b'$, and $c = c'$, the two triangles are necessarily congruent.



It aids the eye if we mark the equal corresponding parts in some such way as in the above figures. On a blackboard we may use colored crayons, c and c' being in red, for example, and b and b' in blue, with $\angle A$ and $\angle A'$ designated by green arcs.

Teachers will see the objections to the use of colored crayons except in the case of a few propositions at the most. The student should early become familiar with the tools that he will actually use, the black lead pencil and the white crayon.

To prove that the two triangles are congruent let us see if one triangle can be placed on the other so as to exactly coincide. To help us see this clearly we may, if we wish, cut two triangles out of paper.

Suppose that $\triangle ABC$ is placed upon $\triangle A'B'C'$ so that the point A lies on the point A', and c lies along c'; then where does the point B lie, and why?

On what line does b then lie, and why?

Then where must C lie, and why?

Having found where B and C lie, where does a lie?

What have we now shown with respect to $\triangle ABC$ coinciding with $\triangle A'B'C'$? Are the triangles congruent?

Complete the following statement: *Two triangles are congruent if two sides and the included angle of one are equal respectively to* . . . .

The statement and formal proof will now be given on page 168.

Theorem. Two Sides and Included Angle

58. *Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*

Given the triangles *ABC* and *A′B′C′*, with side *b* equal to side *b′*, side *c* equal to side *c′*, and angle *A* equal to angle *A′*.

To prove that $\triangle$*s ABC and A′B′C′ are congruent.*

Proof. Taking the figure on the opposite page, place $\triangle ABC$ upon $\triangle A'B'C'$ so that point A lies on point A', and c lies along c', C and C' lying on the same side of c'.

Then B lies on B',

(*For c is given equal to c′.*)

b lies along b',

(*For* $\angle A$ *is given equal to* $\angle A'$.)

and C lies on C'.

(*For b is given equal to b′.*)

Hence a coincides with a'. Post. 1

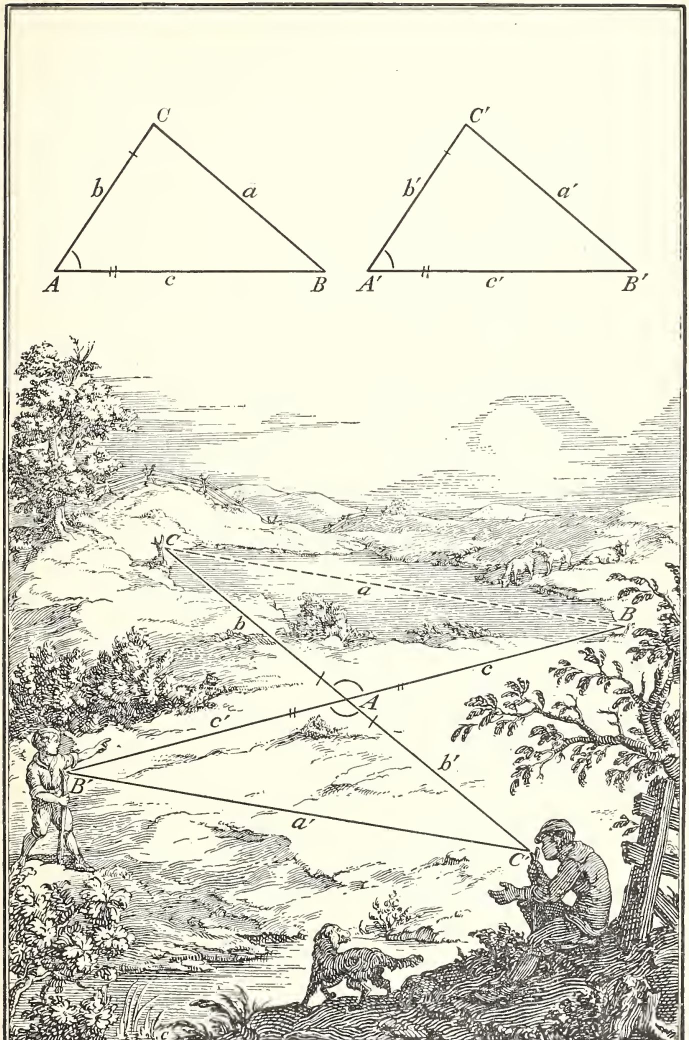
(*One straight line and only one can be drawn through two given points.*)

Therefore $\triangle$s ABC and $A'B'C'$ are congruent. § 53

59. Application. In the lower figure on the opposite page show how $\triangle AB'C'$ can be laid out so as to be congruent to $\triangle ABC$. Then show how to find the length of a certain line by measuring the length of a certain other line. Explain the method in full.

In all such cases the whole figure must be laid out in the same plane; that is, the points A, B, C, B' and C' must either be at the same level or at least be in the same plane.

While the purpose of intuitive geometry is largely practical and that of demonstrative geometry largely intellectual, a few applications of the propositions demonstrated add materially to the interest.


C
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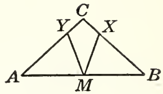
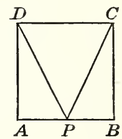
Exercise 9. Congruence of Triangles

1. In this figure, $\angle A = \angle B$, M is the mid-point of AB, and $AY = BX$. Prove that $MY = MX$.

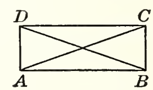


2. In Ex. 1 what other equalities may be proved by the congruence of the triangles?

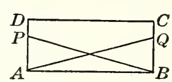
3. In this figure, $ABCD$ is a square and P is the mid-point of AB. Prove that $\triangle APD$ is congruent to $\triangle BPC$. Since the triangles are congruent, what other parts of the figure are respectively equal?



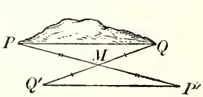
4. In this figure, $AD = BC$ and each is perpendicular to AB. What do you infer as to the relation of AC to BD? Prove the correctness of your inference.



5. In this figure, $AD = BC$ and each is perpendicular to AB; also $DP = CQ$. What do you infer as to the relation of $\angle APB$ to $\angle BQA$? Prove the correctness of your inference.



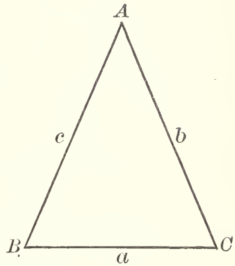
6. Show how to find the distance from a point P west of a hill to a point Q east of the hill, using the figure here shown.



7. Suppose that it is known that a machine will run satisfactorily if three wheels properly gear into three other wheels. Suppose also that it is given that wheel a gears into wheel a', that it can be shown that wheel b gears into wheel b', and that it can then be shown that wheel c gears into wheel c'. What follows as to the running of the machine?

The reasoning is identical, in its main points, with that of § 58. Such exercises are, of course, not geometric, but they give training in transferring geometric reasoning to other lines.

60. Inferences as to Isosceles Triangles. If we examine an isosceles triangle, we find that it has other qualities besides having two equal sides. Using this figure, in which $b = c$, we shall now consider a few of these qualities.



1. If a is also equal to b and c, the triangle is not only isosceles, but what other name may be given to it?

2. If $\angle A$ is a right angle, the triangle is not only isosceles, but what other name may be given to it?

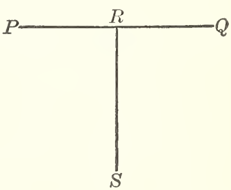
3. If $b = c$, as stated, it looks as if $\angle B$ and $\angle C$ must each be smaller than what kind of angle?

4. It looks as if there were a certain relation with respect to size between $\angle B$ and $\angle C$. What does this relation appear to be?

5. It looks as if the vertex A were directly above a certain point on BC. What point does this seem to be?

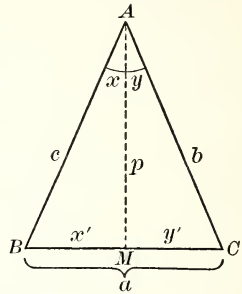
6. It looks as if a perpendicular from A to BC would divide BC into what kind of segments, with respect to size?

7. The perpendicular from A to BC divides the triangle ABC into two triangles. What relation apparently exists between these two triangles?



No one of inferences 3–7 may be correct. When we look at this figure the line PQ seems to be about equal to the line RS, but when we measure their lengths we find that they are not equal. As stated in § 56, we must find some way of proving or disproving our inferences before we can be certain of their truth, and this constitutes the important part of demonstrative geometry.

61. Further Inferences. There are other inferences that we may easily draw from a study of the isosceles triangle. Consider, for example, this figure, *AM* being drawn so as to bisect $\angle A$, thus making $\angle x$ equal to $\angle y$.



1. Since $\angle x = \angle y$, what seems to be the relation of x' to y'?

It is often convenient to use a prime (′) to designate a quantity which has some definite relation to another quantity. We have seen this in connection with congruence, and here we have another example of its use. It is also convenient to use a dotted line to represent a line like *AM* that is an auxiliary line and is drawn merely to aid us in a discussion.

2. What seems to be the relation of the two angles at *M*? Then what name can be given to each of the angles?

3. What kind of line does *AM*, or p, seem to be with respect to *BC*, or a?

4. If we draw the line p so as to bisect a instead of bisecting $\angle A$, that is, so as to make x' equal to y', what seems to be the relation as to size between $\angle x$ and $\angle y$?

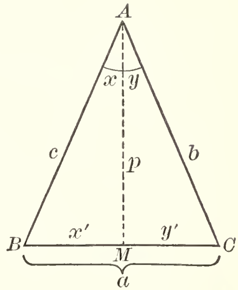
5. If we draw the line p so as to make x' equal to y', what kind of line does it seem to be with respect to being oblique or perpendicular to a?

6. If a perpendicular is drawn to a at its mid-point *M*, do you think it will pass through *A* or not? What else can you infer, say with respect to $\angle A$?

As stated in § 60, no one of these inferences may be correct, and if we wish to be certain as to any one of them we must prove the truth of that inference.

We shall now examine one of the most important of the inferences of §§ 60 and 61.

62. Examination of an Inference. In § 60, 4, you probably drew the inference that $\angle B = \angle C$. We shall now examine this inference and see how we can prove that it is correct; that is, how we can prove that $\angle B = \angle C$ if we know that $b = c$.



We have already proved one proposition about equal angles (§ 51), but since that referred to vertical angles it does not help us in this case.

We have also proved a proposition about congruent triangles (§ 58), and congruent triangles have equal angles. Possibly we may be able to prove that $\angle B = \angle C$ if we can divide $\triangle ABC$ into two congruent triangles.

In order to use § 58 we must have two sides and the included angle of one triangle equal respectively to two sides and the included angle of another triangle, so in order to get two equal angles, let us suppose AM to be the bisector of $\angle A$ (§ 44, 5).

Then in ⊿ ABM and ACM, what is the relation of b to c with respect to size? How do you know this?

What is the relation of $\angle x$ to $\angle y$ with respect to size? How do you know this?

What line is the same in ⊿ ABM and ACM; that is, what line is *common* to the two triangles?

Then what parts of one triangle have you shown to be equal to what parts of the other triangle?

What can you say as to congruence of the triangles?

What can you say as to the relation of $\angle B$ to $\angle C$?

Complete the following statement:

In an isosceles triangle the angles opposite the equal · · ·.

The statement and formal proof will now be given on page 174.

Theorem. Isosceles Triangle

63. *In an isosceles triangle the angles opposite the equal sides are equal.*

Given the isosceles triangle *ABC* with side *b* equal to side *c*.

To prove that $\angle B = \angle C$.

Proof. In the figure suppose that p bisects $\angle A$, so that $\angle x = \angle y$, and suppose that p meets BC at M.

Then in $\triangle ABM$ and ACM we have it given that

$$b = c.$$

Further, $\angle x = \angle y$,

(*For p bisects* $\angle A$.)

and side p is common to both triangles.

Therefore $\triangle ABM$ and ACM are congruent. § 58

(*Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*)

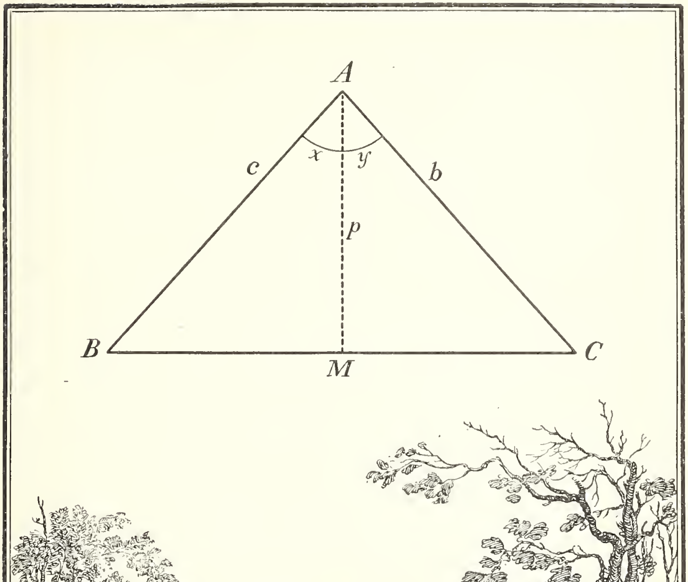
Therefore $\angle B = \angle C$. § 54

64. Corollary. *If a triangle has two equal angles, the sides opposite these angles are equal.*

That is, in the figure, if $\angle B = \angle C$, then $b = c$. This is called the *converse* of the theorem of § 63, what is given in one being that which is to be proved in the other. Not all converse propositions are true, and each case must be considered by itself for the present.

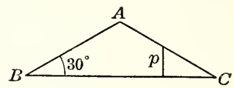
For the present we shall assume that this corollary is true, as seems evidently to be the case. It will be assigned later as an exercise to be proved. We shall not use this corollary in the proof of any proposition that will be needed to prove it when it is given later as an exercise, so that the proof when given will be complete.

65. Application. In the figure of the bridge on the opposite page, state your inferences as to the various equal parts and how these inferences are related to §§ 63 and 64.

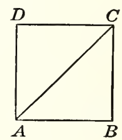

A
x
y
c
b
p
B
M
C

Exercise 10. Isosceles Triangles

1. In this figure, which represents the cross-section of the attic of a house, it is known that the rafters AB and AC are equal in length. Suppose that we find by measuring that $\angle B = 30°$ but that we cannot conveniently pass the partition p so as to measure $\angle C$. If we are told that $\angle C = 28°$, is the information correct? Why?

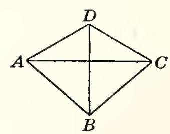


2. This figure represents a square $ABCD$ separated into two triangles by the diagonal AC. State what angles are equal by § 63.



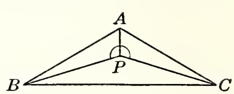
3. In the same figure state what triangles are congruent by § 58, and hence show what other angles are equal besides those found in Ex. 2.

4. In this figure it is given that $AB = BC$ and $\angle DBA = \angle DBC$. Prove that $\triangle ACD$ is isosceles.



5. The diagonals AC and BD of a square $ABCD$ intersect at P. Prove that $\triangle ABP$ is isosceles.

6. In this figure it is given that $PB = PC$ and $\angle APB = \angle APC$. Prove that $\triangle ABC$ is isosceles.



7. It is known that B will pay C a certain sum if T will pay T' a certain sum. But T will pay T' if b will sell his house to c, if x will pay y what he owes him, and if p has a certain sum in the bank. If b now sells his house to c, if x pays y, and if p has the certain sum in the bank, what conclusion do you draw as to B's paying C the sum specified?

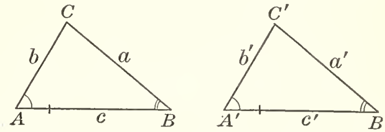
It should be observed that the reasoning is practically identical with the reasoning of § 63.

66. Another Inference. Suppose that these two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other; that is, suppose that

$$\angle A = \angle A',$$

$$\angle B = \angle B',$$

and $$c = c'.$$



From the general appearance of the triangles, what do you infer as to their congruence?

67. Examination of the Inference. Let us see if one of the triangles can be placed on the other, as in § 58, so as to coincide with it; in other words, let us make certain that all the parts of one triangle fit perfectly the respective parts of the other.

Suppose that $\triangle ABC$ is placed upon $\triangle A'B'C'$ so that A lies on A' and c lies along c', C and C' lying on the same side of c'. Then where does B lie? How do you know that it lies there?

On what line does b then lie? How do you know that it lies there?

On what line does a then lie? How do you know that it lies there?

Because C is on both a and b, at what point does it lie on a' and b'?

What have you now shown with respect to the triangles?

Have you fully proved this statement about the congruence of the triangles, or do you merely infer from the appearance of the figures that it is probably true?

Complete the following statement:

Two triangles are congruent if two angles and the included · · ·.

The statement and formal proof will now be given on page 178.

Theorem. Two Angles and Included Side

68. *Two triangles are congruent if two angles and the included side of one are equal respectively to two angles and the included side of the other.*

Given the triangles ABC and $A'B'C'$ with angle A equal to angle A', angle C equal to angle C', and side b equal to side b'.

To prove that $\triangle ABC$ *and* $A'B'C'$ *are congruent.*

Proof. Taking the figure on the opposite page, place $\triangle ABC$ upon $\triangle A'B'C'$ so that A lies on A' and b lies along b', B and B' lying on the same side of b'.

Then C lies on C',

(*For b is given equal to b'.*)

c lies along c',

(*For $\angle A$ is given equal to $\angle A'$.*)

and a lies along a'.

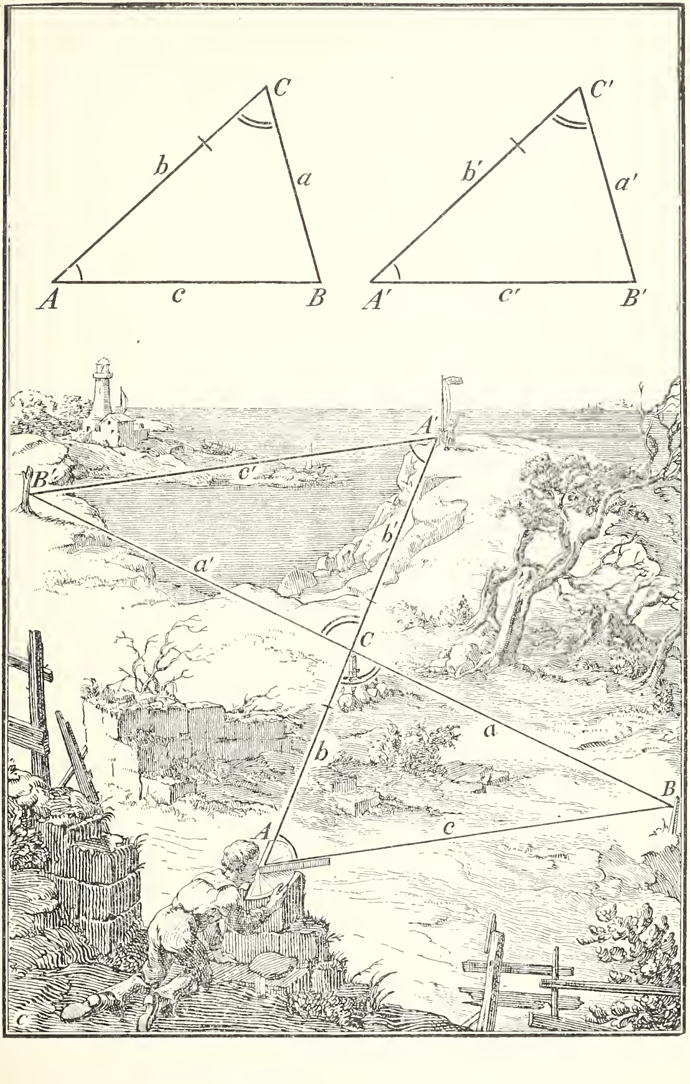
(*For $\angle C$ is given equal to $\angle C'$.*)

Since B is on a and on c, it lies on both a' and c', and hence B lies on B', the only point common to both a and c.

Therefore $\triangle ABC$ and $A'B'C'$ are congruent. § 53

69. Application. The lower figure on the opposite page shows how $\triangle ABC$ can be laid out so as to be congruent to $\triangle A'B'C$ and how a certain line can then be measured so as to find the distance c' across a harbor. Explain the operation in full.

From any convenient point C draw the line CA'. From A' sight through C and make b equal to b'. From B' sight through C and lay a taut string along a. Then at A, using a protractor and ruler, make an angle equal to angle A'. Sight along c, thus fixing the position of B, and measure c.


C
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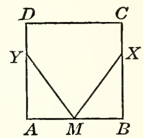
c

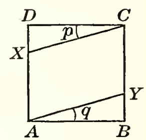
Exercise 11. Congruence of Triangles

1. In this figure, $ABCD$ is a square, M is the mid-point of AB, and the lines MX and MY make equal angles with AB. Prove that $\triangle$s AMY and BMX are congruent. What other angles in the congruent triangles are equal, and why?



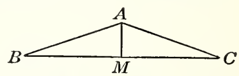
2. In the figure of Ex. 1 what angles of $MXCDY$ are equal, and why?

3. In this figure, $ABCD$ is a square and $\angle p$ is equal to $\angle q$. What other angles in the two triangles are equal? What lines are equal? Give the necessary proofs.

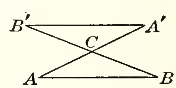


4. In this figure if AM bisects $\angle A$ and is also perpendicular to BC, $\triangle ABC$ is isosceles.

Evidently this must be done by § 68.

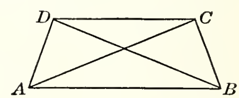


5. In this figure it is given that $\angle A = \angle A'$, $\angle B = \angle B'$, and $AB = A'B'$. Find the other equal lines and equal angles and prove that they are equal.



6. A perpendicular to the bisector of an angle forms an isosceles triangle with the arms of the angle.

7. In this figure it is given that $\angle DCB = \angle CDA$, $\angle CBD = \angle DAC$, and $BC = AD$. Find the other equal lines and equal angles and prove that they are equal.



8. C promises to go into business with C' if A goes into business with A' and if B goes into business with B'. If A does go into business with A', and if B goes into business with B', what follows?

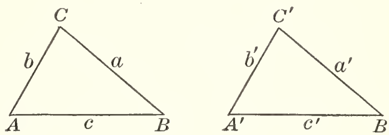
Notice that the reasoning is analogous to that of § 68.

70. Another Inference. Suppose that these two triangles have the three sides of one equal respectively to the three sides of the other; that is, suppose that

$a = a'$,

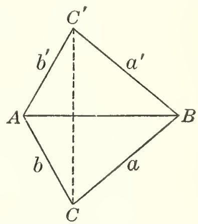
$b = b'$,

and $c = c'$.



From the appearance of the triangles, what do you infer as to their congruence? Would you draw the same inference if the three angles of one were equal respectively to the three angles of the other? Draw figures to illustrate your answer to this second question.

71. Examination of the Inference. In the case in which the three sides of one are equal respectively to the three sides of the other, see if you can give a satisfactory proof by placing $\triangle ABC$ on $\triangle A'B'C'$ as in §§ 58 and 68. If not, try placing them as here shown.



Because $a = a'$, what kind of triangle is $\triangle BC'C$? Therefore what two angles of $\triangle BC'C$ are equal?

Because $b = b'$, what kind of triangle is $\triangle ACC'$?

Therefore what two angles of $\triangle ACC'$ are equal?

Adding two pairs of equal angles, what can now be said as to the equality of $\angle C$ and $\angle C'$?

Can you now prove that ⚠ ABC and $A'B'C'$ are congruent by using § 58? Try it.

Complete the following statement:

Two triangles are congruent if the three · · ·.

The statement and formal proof will now be given on page 182.

Theorem. Three Sides

72. *Two triangles are congruent if the three sides of one are equal respectively to the three sides of the other.*

Given the triangles *ABC* and *A'B'C'* with side *a* equal to side *a'*, side *b* equal to side *b'*, and side *c* equal to side *c'*.

To prove that $\triangle ABC$ *and* $A'B'C'$ *are congruent.*

Proof. Suppose that there are no sides longer than c and c'. Then place $\triangle ABC$ so that A lies on A', c lies along c', C and C' lying on opposite sides of $A'B'$.

Then B lies on B'.

(*For c is given equal to c'.*)

Draw CC'.

Since $b = b'$,

we have $\angle AC'C = \angle ACC'$. § 63

(*In an isosceles triangle the angles opposite the equal sides are equal.*)

Since $a = a'$,

we have $\angle CC'B = \angle C'CB$. § 63

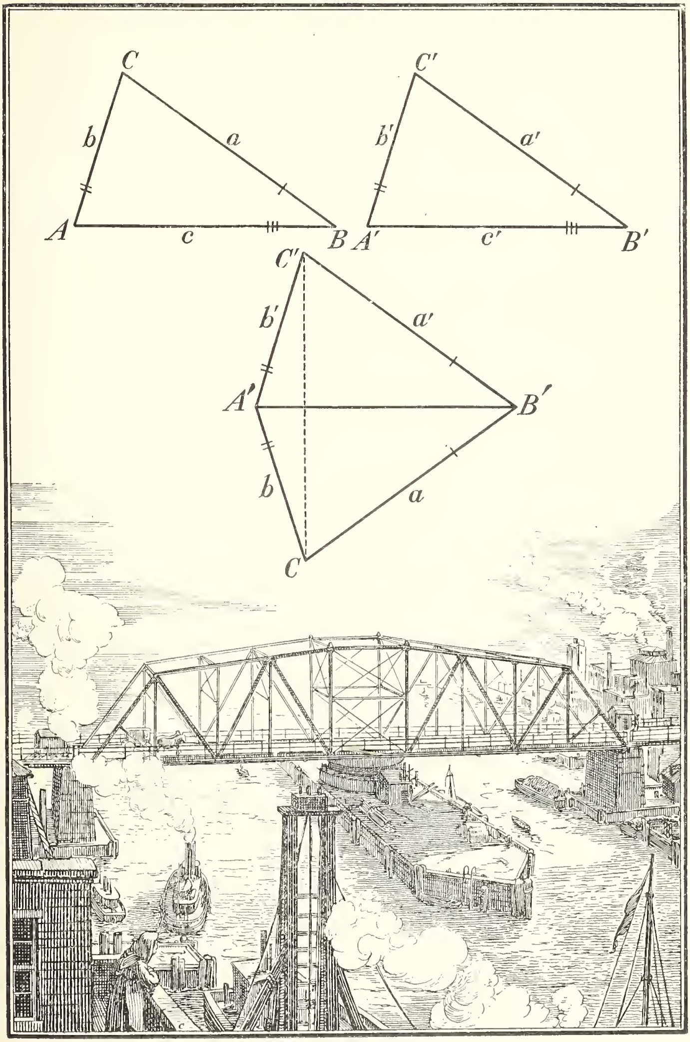
Adding, $\angle AC'C + \angle CC'B = \angle ACC' + \angle C'CB$; Ax. 1

that is $\angle AC'B = \angle ACB$.

Therefore $\triangle ABC$ and $A'B'C'$ are congruent. § 58

(*Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*)

73. Application. In the lower figure on the opposite page a bridge is shown supported on a single pier. The bridge is made up of triangles, and although these triangles are jointed at the vertices, they are rigid; that is, their shape is fixed if their sides are fixed. How does § 72 apply?


C
b
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c
B
C′
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a′
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C′
b′
a′
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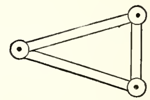
B′
b
a
C

Exercise 12. Congruence of Triangles

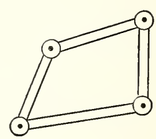
1. Using three rods of different lengths placed end to end so as to form a triangle, can you form triangles of different shape and size? State the reason for your answer, drawing a figure to illustrate.

2. Three iron rods are hinged at their extremities as shown in this figure. Is the figure rigid; that is, can its shape be changed? State the reason.



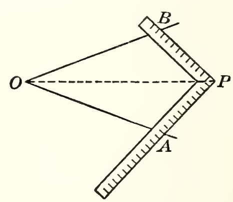
This explains the statement that *a triangle is determined by its three sides.* It also explains why the triangle is called a *unit of rigidity* in bridge building and in steel construction generally.

3. Four iron rods are hinged at their extremities as shown in this figure. Is the figure rigid? If not, state two ways in which, by the addition of a single rod in each case, it can be made rigid. Upon what theorem does this depend?



4. Draw a rough figure of the framework of a bicycle. State the reason or reasons for its rigidity.

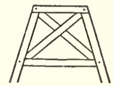
5. The following method is sometimes used for bisecting an angle by the aid of a carpenter's square: Place the square as here shown so that the edges shall pass through A and B, two points equidistant from O on the arms of the given angle AOB, and so that $AP = BP$. Draw OP and show that it bisects $\angle AOB$.



6. If from any vertex of a square there are drawn line-segments to the mid-points of the two sides not adjacent to the vertex, these line-segments are equal.

7. Prove that either diagonal of a square bisects two angles of the square.

8. In this section of a support for a heavy tank, are both cross braces necessary for rigidity? State the reason.

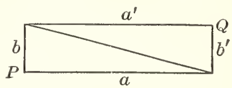


9. In an equilateral triangle a line is drawn from one vertex to the mid-point of the opposite side, thus cutting the equilateral triangle into two triangles T and T'. What can be said as to the congruence of T and T'? Prove your statement.

10. Two isosceles triangles of different heights are constructed on the same base and on the same side of the base. Prove that the line through their vertices bisects the angles at the vertices.

11. In Ex. 10 suppose the two isosceles triangles to be on opposite sides of the base.

12. In this figure $a = a'$ and $b = b'$. Prove that $\angle P = \angle Q$.



13. Two isosceles triangles, OAB and PAB, are constructed on the same base AB. Prove that $\angle PAO = \angle PBO$.

14. The line from the vertex of an isosceles triangle to the mid-point of the base is perpendicular to the base.

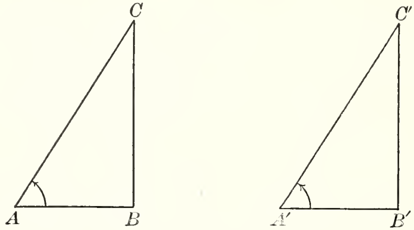
15. A lawyer wishes to prove that T signed a certain paper T'. He can prove it if he can show that c was at c' at a certain time. He can prove that c was at c' at that time if he can prove two statements. But he can prove these statements by reliable witnesses. What follows, and why?

Notice that the reasoning is analogous to that of § 72.

The student has now reached the point where he may profitably read the proofs without any assistance. These proofs should not be memorized, but the student should read the statements and try to work out the proofs for himself before reading those given in the book.

Theorem. Congruence of Right Triangles

74. *Two right triangles are congruent if the hypotenuse and an acute angle of one are equal respectively to the hypotenuse and an acute angle of the other.*



Given the right triangles *ABC* and *A′B′C′*, with hypotenuse *AC* equal to hypotenuse *A′C′* and with angle *A* equal to angle *A′*.

To prove that △ ABC and A′B′C′ are congruent.

Proof. Place △ *ABC* on △ *A′B′C′* so that *A* lies on *A′* and *AC* lies along *A′C′*.

Then *C* lies on *C′*,

(*For AC is given equal to A′C′.*)

and *AB* lies along *A′B′*.

(*For ∠A is given equal to ∠A′.*)

Then because *C* lies on *C′* Proved

(*For it has been shown that C lies on C′.*)

and ∠s *B* and *B′* are rt. ∠s, Given

(*For the △ are right △.*)

CB must coincide with *C′B′*. Post. 7

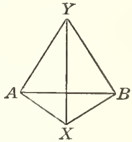
(*Only one ⊥ can be constructed from C′ to A′B′.*)

Therefore △ *ABC* and △ *A′B′C′* are congruent. § 53

(*For the two △ have been made to coincide.*)

Exercise 13. Congruence of Right Triangles

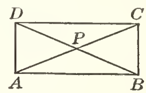
1. The perpendicular from the vertex of an isosceles triangle to the base bisects the base.



2. In this figure, $\angle XAY = \angle XBY = 90°$ and $\angle BXA$ is bisected by XY. Prove that $\triangle ABY$ is isosceles.

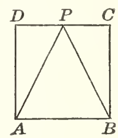
3. In Ex. 2 prove that XY is perpendicular to AB and bisects it, and also prove that $\triangle AXB$ is isosceles.

4. In this figure, $AC = BD$, $\angle BAD = \angle ABC = 90°$, and $\angle ADB = \angle BCA$. Prove that $BC = AD$ and that $AB = DC$.



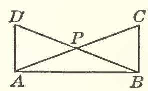
5. In Ex. 4 prove that $\triangle ABP$ is isosceles.

6. In this figure, $\angle C = \angle D = 90°$, $\angle DPA = \angle CPB$, and $AP = BP$. Draw AC and BD and prove the following statements:



1. $AD = BC$.
2. $PD = PC$.
3. $AC = BD$.
4. $\angle BAP = \angle ABP$.
5. $\angle PAD = \angle PBC$.
6. $\angle BAD = \angle ABC$.

7. In this figure, $\angle BAD = \angle ABC = 90°$, $AC = BD$, and $\angle ABD = \angle BAC$. Write all the other equalities in the figure and then prove each statement concerning these equalities.

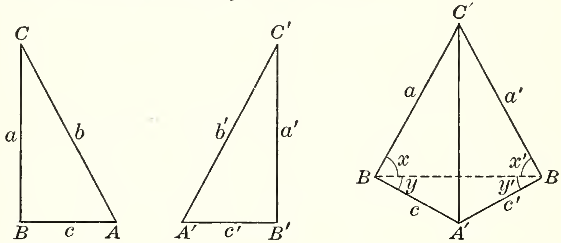


8. A physician's test for a certain disease is that an invalid has a temperature C and a certain symptom A, and that a blood test reveals a certain condition B'. He finds that the invalid has the temperature C, the symptom A is apparent, and a blood test reveals the condition B'. What is the physician's decision as to whether the invalid has the disease?

Notice that the reasoning is analogous to that of § 74.

Theorem. Congruence of Right Triangles

75. *Two right triangles are congruent if the hypotenuse and a side of one are equal respectively to the hypotenuse and a side of the other.*



Given the right triangles *ABC* and *A'B'C'*, with hypotenuse *b* equal to hypotenuse *b'* and with side *a* equal to side *a'*.

To prove that △ *ABC and A'B'C' are congruent.*

Proof. Place △ *ABC* beside △ *A'B'C'* so that hypotenuse *b* coincides with hypotenuse *b'*, *A* lying on *A'*, *C* on *C'*, and *B* on the opposite side of *AC* with respect to *B'*.

Draw *BB'*.

Then in the figure at the right,

$a = a'$, Given

and therefore $x = x'$. § 63

(*For they are angles opposite the equal sides of an isosceles* △.)

Since $\angle B = \angle B'$, Post. 3

we have $y = y'$. Ax. 2

(*For we have subtracted equals, x and x', from equals.*)

Therefore $c = c'$. § 64

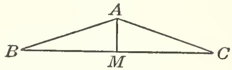
(*For they are sides opposite equal angles of a triangle.*)

Therefore △ *A'BC'* and *A'B'C'* are congruent. § 58

That is, △ *ABC* and *A'B'C'* are congruent.

Exercise 14. Congruence of Right Triangles

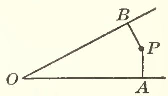
1. In this figure $AB = AC$ and AM is perpendicular to BC. Prove that $BM = CM$.



2. In Ex. 1 prove that AM bisects $\angle BAC$.

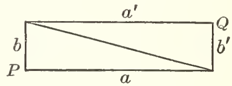
3. If from any point on a given perpendicular to a given line two equal oblique lines are drawn to the line, they meet the line at equal distances from the perpendicular.

4. In this figure $AP = BP$ and $\angle A = \angle B = 90°$. Prove that $OA = OB$.



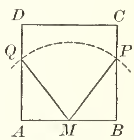
5. In Ex. 4 prove that a line drawn through O and P bisects both $\angle O$ and $\angle P$.

6. In this figure $\angle P = \angle Q = 90°$ and $b = b'$. What other statement can you prove concerning the figure? Prove it.



7. If the perpendiculars from the mid-point of one side of a triangle upon the other two sides are equal, the triangle is isosceles.

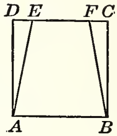
8. $ABCD$ is a square and M is the mid-point of AB. With M as center an arc is drawn cutting BC at P and AD at Q, and the lines MP and MQ are drawn. Name the pairs of equal angles in the figure besides the right angles and prove your statements.



9. A man decides to go into a certain business if conditions A and b are favorable and if he can secure the consent of a and of a' to the undertaking. He finds that the condition A is favorable if b is, and he further finds that condition b is favorable. He then secures the consent of a and of a'. What is his conclusion?

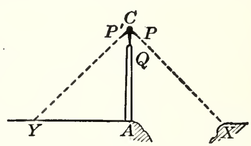
Exercise 15. Review of Congruence

1. In this figure, $ABCD$ is a square and the lines AE and BF are so drawn that $\angle EAD = \angle FBC$. Write all the other equalities in the figure and prove each statement.

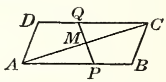


2. If the mid-points of the sides of an equilateral triangle are joined by lines, the resulting triangle is also equilateral.

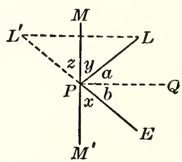
3. Wishing to measure the distance AX in this figure, a boy placed a pair of compasses QCP at the top of a post AQ so that the arm CP pointed to X. He then turned the compasses around, keeping the angle fixed, and sighted along the arm to Y. He then measured AY and thus found the distance AX. Explain the principle involved.



4. In this figure, $AB = DC$, $BC = AD$, and $AM = MC$. Prove that $PM = MQ$.



5. In the figure if a ray of light LP is reflected from a mirror MM', the angle a, known as the *angle of incidence*, is equal to the angle b, known as the *angle of reflection*, the line PQ being perpendicular to MM'. The light from L is reflected at P and strikes the eye at E. The line LL' is perpendicular to MM', and EPL' is a straight line. Prove that $\angle x = \angle y$ and $\angle z = \angle y$, and explain why the light appears to be at the same distance behind the mirror, at L', that it really is in front of it, at L.



6. Lines from the mid-points of the equal sides of an isosceles triangle to the mid-point of the base are equal.

76. How to Prove an Original. Thus far special suggestions have been given so freely that the student has probably found little serious difficulty with the exercises. When he has met with new theorems, or *originals* as they are often called, he has found that he can easily prove them by the aid of one or two propositions immediately preceding them. It is now, however, desirable to consider certain general suggestions that will be of assistance.

1. *Draw the figure as you read the proposition, making the figure general, clear, neat, and accurate.*

That is, if the proposition relates to a triangle, do not draw an equilateral triangle, or a right triangle, or an isosceles triangle, but draw a general triangle without special peculiarities. Draw rapidly, but make the figure clear, neat, and accurate.

2. *Write down exactly what is given, and then write down exactly what is to be proved.*

Failure to do this is the cause of much of the difficulty found.

3. *Analyze the proposition.*

This means that you should proceed somewhat as follows: "I have to prove this, and I can prove it if I can prove X; I can prove X if I can prove Y; I can prove Y if I can prove Z; but I *can* prove Z, so I can reverse this reasoning and thus prove my theorem."

Another form of analysis consists in assuming the proposition proved, seeing what conclusion follows, reasoning from this until a known truth is reached, and then retracing the steps.

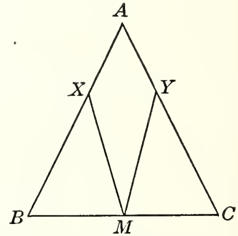
In all this work it is well to recall the *conditions of congruence of triangles* as thus far proved:

1. *Two sides and the included angle* (§ 58).
2. *Two angles and the included side* (§ 68).
3. *Three sides* (§ 72).
4. *Hypotenuse and an acute angle of a right triangle* (§ 74).
5. *Hypotenuse and side of a right triangle* (§ 75).

77. Attacking an Original. Suppose that the following original is given to be proved:

Two lines drawn from the mid-point of the base of an isosceles triangle making equal angles with the base meet the equal sides at points equidistant from the vertex.

Following the suggestions given in § 76 we proceed as follows:



1. *Draw the figure.*

It is desirable to take as general an isosceles triangle as we can, and in particular to avoid an equilateral triangle lest our eye should be deceived by such a special figure.

It is convenient to use M for mid-point because it is an initial, but any other letter, say the letter P, will serve the purpose. It is well to use X and Y for the special points, or some letters not likely to be confused with A, B, and C, although this is not absolutely necessary.

The figure need not be constructed, since this would take too much time, but it should be drawn neatly and should be accurate enough for the purposes.

2. *Write down exactly what is given, and then write down exactly what is to be proved.*

That is:

Given $AB = AC$, $BM = CM$, and $\angle XMB = \angle YMC$.

To prove that $AX = AY$.

3. *Then analyze the proposition.*

For example: I can prove that $AX = AY$ if I can prove that $BX = CY$, because I already know that $AB = AC$.

I can prove that $BX = CY$ if I can prove that △ MBX and MCY are congruent.

I can prove this if I can bring it under the case of two sides and the included angle, or the case of two angles and the included side.

But I can do this, for $\angle B = \angle C$, $\angle XMB = \angle YMC$, and $BM = CM$.

I can now reverse my reasoning and prove the theorem.

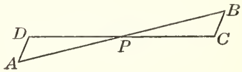
Exercise 16. Review

1. *Two lines drawn to a given line from any point in a perpendicular to the line, cutting off on the given line equal segments from the foot of the perpendicular, are equal and make equal angles with the perpendicular.*

Exercises which are printed in italics in this book are often given as basal propositions in textbooks on geometry and should, therefore, be given to all classes. They are not, however, essential to the logical sequence of the propositions in this book.

2. *Two lines drawn to a given line from any point in a perpendicular to the line, making equal angles with the given line, cut off equal segments from the foot of the perpendicular.*

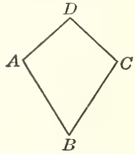
3. In this figure state what must be known in order that the two triangles may be proved congruent. Give as many answers as you can, but let no answer include any unnecessary condition.



4. If two lines, AB and CD, bisect each other at O, the line joining A and C is equal to the line joining B and D.

5. State another proposition relating to the figure described in Ex. 4, and prove it.

6. Two triangles, ABC and ABD, are constructed on AB so that $BC = AD$ and $BD = AC$, the vertices C and D lying on the same side of the base AB. State all the pairs of equal angles in the figure and prove each statement.



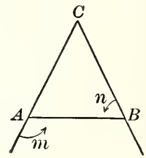
7. In the figure of Ex. 6 prove that a certain triangle is isosceles.

8. In this figure, $AB = BC$ and $CD = DA$. Prove that $\angle A = \angle C$.

There are two simple methods of proving this exercise, and the student should discover both methods.

9. In the equilateral triangle ABC the points P and Q are taken on AB so that $AP = BQ$. Draw CP and CQ, state all the other equalities that you can in relation to the figure, and prove each statement.

10. The points M and N are the mid-points of the respective sides BC and AD of the square $ABCD$. Draw AM, BN, MN, MD, and NC, find as many isosceles triangles as you can in the figure, and prove that each of these triangles is isosceles.



11. In this figure angles m and n are supplementary. Prove that $\triangle ABC$ is isosceles.

12. On two sides of an equilateral triangle ABC two congruent isosceles triangles BCP and CQA are constructed so as to lie outside of $\triangle ABC$. Prove that the distance from A to P is equal to the distance from B to Q.

13. The perpendicular bisectors of the sides AB and BC of the equilateral triangle ABC meet in the point P. Compare the distances of P from A, B, and C.

14. In Ex. 13 prove that AP, BP, and CP bisect the angles at A, B, and C respectively.

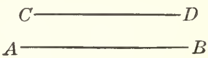
15. In Ex. 13 prove that the perpendiculars from P to the three sides of the triangle are equal.

16. From Exs. 13–15, where is P situated with respect to the three perpendicular bisectors of the three sides? to the three bisectors of the three angles?

17. In the isosceles triangle ABC, $AB = AC$, M is the mid-point of the base BC, and X and Y are taken on AB and AC respectively so that $BX = CY$. State four other relations of equality in the figure, and prove each of the four statements.

III. PARALLEL LINES

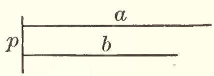
78. Parallel Lines. Lines which lie in the same plane and cannot meet, however far they may be produced, are called *parallel lines*, or simply *parallels*.



For example, AB and CD are parallel lines. Since the student is already familiar with such lines, further illustrations are not necessary.

It should be observed that in the above definition the words "in the same plane" are essential.

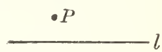
79. Inferences as to Parallels. If two lines, a and b, are both perpendicular to the same line p, what do you infer as to their being parallel? Give several illustrations of two such lines in the schoolroom, using for each illustration two lines on the walls, floor, ceiling, or desks.



Complete the following statement:

Two lines in the same plane perpendicular to the . . . .

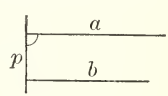
This figure shows a line l and a point P not on l. How many lines do you think can be drawn through P parallel to l? Give an illustration of such a case in the schoolroom, preferably with respect to lines on the walls, ceiling, or floor.



Complete the following statement:

Through a given point only one line can . . . .

In this figure suppose that a is parallel to b and that p is perpendicular to a; what do you infer as to the relation of p to b with respect to perpendicularity? Give an illustration of such a case in the schoolroom.



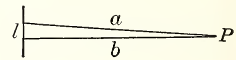
Complete the following statement:

If a line is perpendicular to one of two parallel lines, it is . . . .

80. Three Postulates of Parallels. From the statements made on page 195 you have inferred three facts concerning parallel lines. Two of these facts are easily proved, but all three seem so evident that, for the present, we may assume that the statements are true; that is, we may take them as postulates (§ 42).

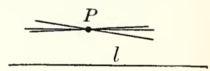
In beginning the study of geometry it is desirable to avoid proving various propositions whose truth is easily inferred. In a second course in geometry these may be considered more scientifically.

1. *Two lines in the same plane perpendicular to the same line are parallel.*



If a and b are supposed to be perpendicular to l they cannot meet, as at P; for if they did meet we should have two perpendiculars from P to l, which is contrary to Post. 7.

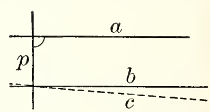
2. *Through a given point only one line can be drawn parallel to a given line.*



From this figure it seems quite evident that only one of the lines that can be drawn through P can be parallel to l. While this is no proof for the statement, we are probably as convinced of the simple truth as we would be if a proof could be given.

3. *If a line is perpendicular to one of two parallel lines, it is also perpendicular to the other.*

That is, if a and b are parallel lines, and if p is perpendicular to a, then p is also perpendicular to b.



This is easily seen, for if p is not perpendicular to b, suppose that it is perpendicular to some other line, such as c, drawn through the point of intersection of p and b.

If this supposition were correct, c would be parallel to a by the first of these three postulates.

But this is impossible, for b and c cannot both be parallel to a, by the second of these postulates.

81. Transversal. A line which cuts two or more lines is called a *transversal* of those lines.

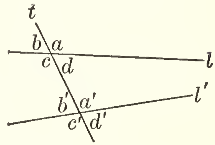
For example, in the figure below, the line t is a transversal of the lines l and l'.

82. Angles made by a Transversal. In this figure it is customary to give special names to certain angles:

a, b, c', d' are called *exterior angles*;

a', b', c, d are called *interior angles*;

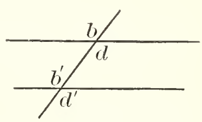
d and b' are called *alternate angles*, and similarly for c and a';



a and a' are called *corresponding angles*, and similarly for b and b', for c and c', and for d and d'.

The angles a and c' are called *alternate exterior angles*, and similarly for b and d'; but when alternate angles are mentioned we ordinarily mean alternate interior angles; that is, we ordinarily mean d and b', or c and a'.

83. Inferences as to Parallels. In this figure, representing two parallel lines cut by a transversal, d and b' are alternate angles. From the appearance of the figure, what do you infer as to the relative size of d and b'?



What do you infer as to the relative size of b and b'?

Write the statements of these two inferences of geometric theorems relating to parallel lines, beginning each statement thus:

If two parallel lines are cut by a transversal, ...

The student should have no difficulty in making further correct inferences as to equality of angles in the figure given. He will find it interesting to letter the other angles and to state every equality which exists between any two of them. The most important inferences, however, are the two referred to above.

THEOREM. PARALLELS CUT BY A TRANSVERSAL

84. *If two parallel lines are cut by a transversal, the alternate angles are equal.*

Given l and l', two parallel lines, and the transversal t forming with them the alternate angles a and a'.

To prove that $a = a'$.

Proof. Taking the figure as shown on the opposite page, through M, the mid-point of AA', suppose that BB' is drawn perpendicular to l', meeting l at B.

Then BB' is likewise $\perp$ to l. § 80, 3

(*If a line is $\perp$ to one of two $\parallel$ lines, it is also $\perp$ to the other.*)

Since b and b' are rt. ∠s, ABM and $A'B'M$ are rt. △s.

Furthermore, $m = m'$, § 51

(*If two lines intersect, the vertical ∠s are equal.*)

and $AM = A'M$. Const.

(*For M is the mid-point of AA'.*)

Therefore △s ABM and $A'B'M$ are congruent. § 74

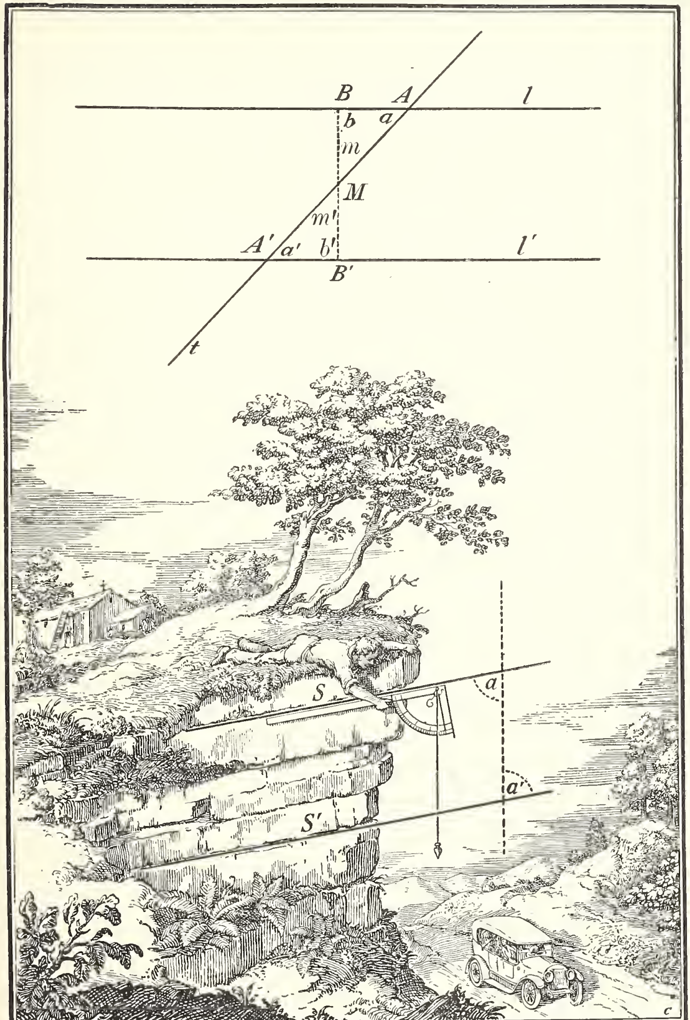
(*Two right triangles are congruent if the hypotenuse and an acute angle of one are equal respectively to the hypotenuse and an acute angle of the other.*)

Therefore $a = a'$. § 54

(*For they are corresponding parts of congruent triangles.*)

85. Application. In the lower figure on the opposite page two strata, S and S', of rock are known to be parallel. It is also known that the upper stratum makes an $\angle a$ with a vertical line. Without measuring $\angle a'$, which the lower stratum makes with the vertical line, how does it compare in size with $\angle a$? State the reason for your opinion.

What other angles in the figure can you show to be equal? State the reason in the case of each pair of angles.


B
A
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B'
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S
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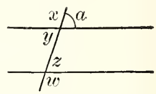
S'

Exercise 17. Parallel Lines

1. *If two parallel lines are cut by a transversal, each exterior angle is equal to its corresponding interior angle.*

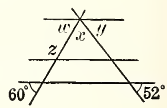
2. *If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary.*

3. This figure represents two parallel lines cut by a transversal. Find the values of x, y, z, and w, given that $a = 70°$; given that $a = 82°$.



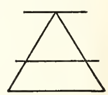
4. The crosspieces supporting electric wires are usually at right angles to the poles. What postulate of parallels is illustrated by several such crosspieces on one pole?

5. In this figure three parallel lines are cut by two transversals, and certain angles are formed as shown. Find the values of w, y, z, and x.



6. A man walking northward changes his direction to southwest. Through how many degrees does he turn? If he wishes to walk northward again, through how many degrees must he turn? Draw a figure and state the proposition on which your second answer depends.

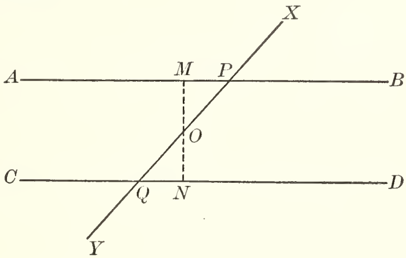
7. In this figure each angle of the triangle is 60°, and two lines have been drawn parallel to the base. What can you discover as to the number of degrees in the other angles?



8. A man wishes to show that the cost of manufacturing a articles is b dollars. He can show this if he can show that the cost of manufacturing c parts of each article is d dollars. To show this depends upon knowing three facts, p, q, and r. The fact p is admitted and the facts q and r are easily shown. What follows?

THEOREM. ALTERNATE ANGLES GIVEN EQUAL

86. *When two lines in the same plane are cut by a transversal, the two lines are parallel if the alternate angles are equal.*



Given the lines *AB* and *CD* cut by the transversal *XY* at *P* and *Q* respectively, with the angles *APQ* and *DQP* equal.

To prove that AB is parallel to CD.

Proof. Through O, the mid-point of PQ, suppose MN drawn $\perp$ to CD, meeting AB at M.

Then $\angle MPO = \angle NQO$, Given

$\angle POM = \angle QON$, § 51

(*If two lines intersect, the vertical angles are equal.*)

and $OP = OQ$. Const.

Therefore $\triangle$ OPM and OQN are congruent, § 68

(*Two triangles are congruent if two angles and the included side of one are equal respectively to two angles and the included side of the other.*)

and hence $\angle OMP = \angle ONQ$. § 54

But $\angle ONQ$ is a rt. angle, Const.

and so $\angle OMP$ is also a rt. angle.

Therefore AB is $\parallel$ to CD. § 80, 1

(*For each is $\perp$ to the line MN.*)

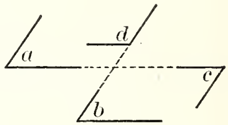
Exercise 18. Parallel Lines

1. *When two lines in the same plane are cut by a transversal, the two lines are parallel if an exterior angle is equal to the corresponding interior angle.*

2. *When two lines in the same plane are cut by a transversal, the two lines are parallel if two interior angles on the same side of the transversal are supplementary.*

3. *Two lines parallel to the same line are parallel to each other.*

4. *Two angles whose sides are parallel each to each are either equal or supplementary.*

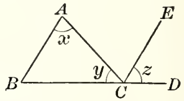


Show that a and b are both equal to the same angle.

Show that c is equal to a.

Show that d is supplementary to an angle that is equal to a.

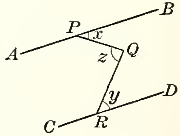
5. In this figure, B, C, and D are in a straight line. If $x = 74°$, $y = 48°$, and $z = 58°$, prove that CE is parallel to BA and find the number of degrees in $\angle CBA$ and in $\angle ECA$.



6. In the figure of Ex. 5 suppose that $x = 142°$, $y = 12°$, and $z = 26°$, prove that CE is parallel to BA, and find the number of degrees in $\angle CBA$.

The student should sketch a new figure, making the angles conform approximately to the new measurements.

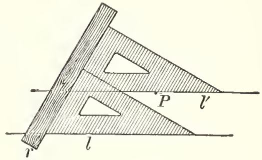
7. In this figure suppose that $x = 35°$, $y = 48°$, and $z = 83°$. Prove that AB and CD are parallel.



Through Q draw a line parallel to AB.

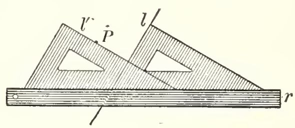
8. In the figure of Ex. 7 prove that if AB and CD are given parallel, $z = x + y$.

9. In order to draw a line parallel to a given line l and passing through a given point P, a draftsman uses a celluloid or wooden triangle, as here shown. He lays the hypotenuse along the given line l, places a ruler r along one of the sides, and slides the triangle along the ruler until the hypotenuse passes through P. He then draws a line l' along the hypotenuse. Using this construction, draw a line through a given point and parallel to a given line. State the authority on which this construction depends.



10. Draw a figure showing that the construction in Ex. 9 can be made about as easily by using an equilateral triangle as by using a right triangle.

11. In order to draw a line perpendicular to a given line l and passing through a given point P, a draftsman lays one side of his triangle along the given line l, places a ruler r along the hypotenuse, and slides the triangle along the ruler until the other side passes through P. He then draws a line l' along this side. Using this construction, draw a line through a given point perpendicular to a given line. State the authority on which this construction depends.

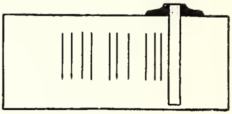


12. Draw a line AB and through a point P outside AB draw a line parallel to AB, using the method of Ex. 9.

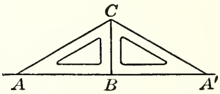
13. A man wishing to know if a certain statement A is true finds that if A is true, then B is true. He can show that B is true. Does it follow that A is true? State the reason for your answer.

Exercise 19. Review

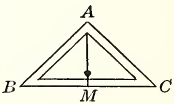
1. A draftsman draws a series of parallel lines by means of a T-square as here shown. What is the geometric authority for stating that the lines are parallel?



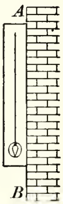
2. The accuracy of the right angle of a triangle may be tested by first drawing a perpendicular BC to the line AA', the triangle being on the left, at ABC, and by then drawing a perpendicular with the triangle on the right, at $A'BC$. State the geometric principle involved.



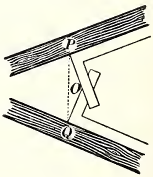
3. The ancient kind of leveling instrument here shown consists of an isosceles right triangle. When the plumb line cuts the mid-point M of the base BC, the line BC is level. State the geometric principle involved.



4. A bricklayer often uses the instrument here shown for determining if a wall is vertical. When the plumb line lies along a line that is parallel to the edge AB, he knows that AB is vertical. State the geometric principle involved.



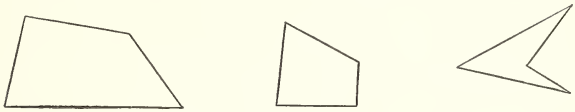
5. In order to put in a brace joining two converging beams and making equal angles with them, a carpenter places two steel squares as here shown, so that $OP = OQ$. Show that PQ makes equal angles with the beams.



6. A certain kind of cloth C is worth n dollars a yard if it is all wool. A certain kind of cloth D is worth n dollars a yard. What is your conclusion as to D being all wool?

IV. PARALLELOGRAMS

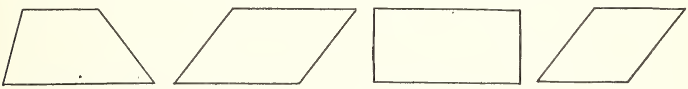
87. Quadrilateral. A rectilinear figure formed by four straight lines is called a *quadrilateral.*



For the terms *sides*, *perimeter*, *vertices*, *angles*, and *base*, see § 35.

88. Kinds of Quadrilaterals. A quadrilateral may be general in shape, with no special features, or it may be

a *trapezoid*, having two sides parallel;
a *parallelogram*, having the opposite sides parallel.



TRAPEZOID PARALLELOGRAM RECTANGLE RHOMBUS

If the nonparallel sides of a trapezoid are equal, the trapezoid is said to be *isosceles*, but the isosceles trapezoid is not often used.

89. Kinds of Parallelograms. A parallelogram may be

a *rectangle*, having all its angles right angles;
a *rhombus*, having all its sides equal.

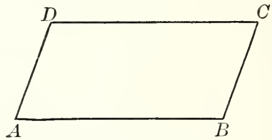
In a parallelogram or a trapezoid the side parallel to the base is called the *upper base*, the base itself being then called the *lower base*.

90. Height or Altitude. The perpendicular distance between the bases of a parallelogram or a trapezoid is called the *height* or the *altitude*.

The perpendicular distance from the vertex of a triangle to the base is called the *height* or the *altitude* of the triangle.

91. Diagonal. In any rectilinear figure the line joining any two vertices not consecutive is called a *diagonal.*

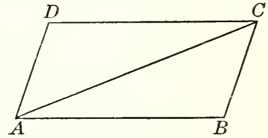
92. Inferences as to Parallelograms. If we look at a parallelogram we infer from its appearance certain facts which seem to be worth proving.



1. How do AB and DC seem to be related as to length? How do BC and AD seem to be related as to length? Write a statement about the relative lengths of the opposite sides of a parallelogram.

2. How do $\angle A$ and $\angle C$ seem to be related as to size? Do $\angle B$ and $\angle D$ seem to be equal, or do they seem to be unequal? Write a statement about the relative sizes of the opposite angles of a parallelogram.

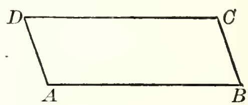
3. Can you judge, by the aid of a protractor if necessary, the number of degrees or the number of right angles in the sum of $\angle A$ and $\angle B$? of $\angle B$ and $\angle C$? Write a statement concerning the sum of two consecutive angles of a parallelogram.



4. Suppose that a diagonal AC is drawn as here shown; what do you infer as to the triangles ABC and CDA?

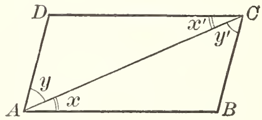
If the parallelogram is cut from paper and is then cut into two triangles, the correctness of the inference is more apparent.

5. It is not known that this figure is a parallelogram, but it is known that $AB = DC$ and that $BC = AD$. Do you infer that $ABCD$ must necessarily be a parallelogram? Would drawing a diagonal AC help to decide the question? Would it enable you to prove your statement? Write a statement about the kind of quadrilateral that has its opposite sides equal.



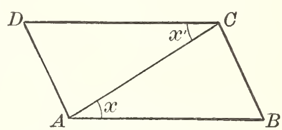
93. Examination of the Inferences. We shall now consider the inferences suggested on page 206.

1. If this figure is a parallelogram and the diagonal AC is drawn, how does x compare in size with x', and why? How does y compare in size with y', and why? What reason is there for knowing that $\triangle ABC$ and CDA are congruent? What is then true as to $\angle B$ and $\angle D$? as to AB and CD? as to BC and AD? as to $\angle A$ and $\angle C$?



While we have used the symbol $\angle x$, the student should use merely x when no confusion will arise. Similarly, he should speak of $\angle A$ instead of $\angle BAD$, although there are several angles at A, because common sense shows that $\angle BAD$ is meant. When we ask what is true as to $\angle B$ and $\angle D$, the student's judgment should suggest that we mean with respect to size.

2. In this figure suppose that it is known only that $AB = DC$ and that $BC = AD$. What reason is there for knowing that $\triangle ABC$ and CDA are congruent? If this is known, what is then known as to x and x'? If this is known, what is then true about the parallelism of AB and DC? Can you show the same thing for BC and AD? What kind of quadrilateral is $ABCD$?



3. In the preceding figure suppose that it is known that AB is both equal to and parallel to DC, nothing being known as to BC and AD. Because AB is parallel to DC, what is known about x and x'? In $\triangle ABC$ and CDA, what parts are now known to be equal? What can then be said about the congruency of the triangles? What can then be said about BC and AD? What kind of quadrilateral is $ABCD$? Why?

Theorem. Equal Parts of a Parallelogram

94. *The opposite sides of a parallelogram are equal, the opposite angles are equal, either diagonal divides the parallelogram into two congruent triangles, and the two diagonals bisect each other.*

Given the parallelogram *ABCD* with diagonals *AC* and *BD* intersecting at *O*.

To prove that $BC = AD$, $AB = DC$, $\angle A = \angle C$, $\angle B = \angle D$, *△ ABC and CDA are congruent, △ ABD and BCD are congruent,* $AO = OC$, *and* $BO = OD$.

Proof. $\angle BAC = \angle DCA$, § 84

(*For AB is ∥ to DC, and these are alternate △.*)

and $\angle ACB = \angle DAC$. § 84

Also $AC = AC$.

Therefore △ ABC and CDA are congruent. § 68

Hence $BC = AD$, $AB = DC$, § 54

and $\angle B = \angle D$. § 54

In the same way it can be proved that △ ABD and DCB are congruent and hence that $\angle A = \angle C$.

Then in △ ABO and CDO we have

$AB = DC$, Proved

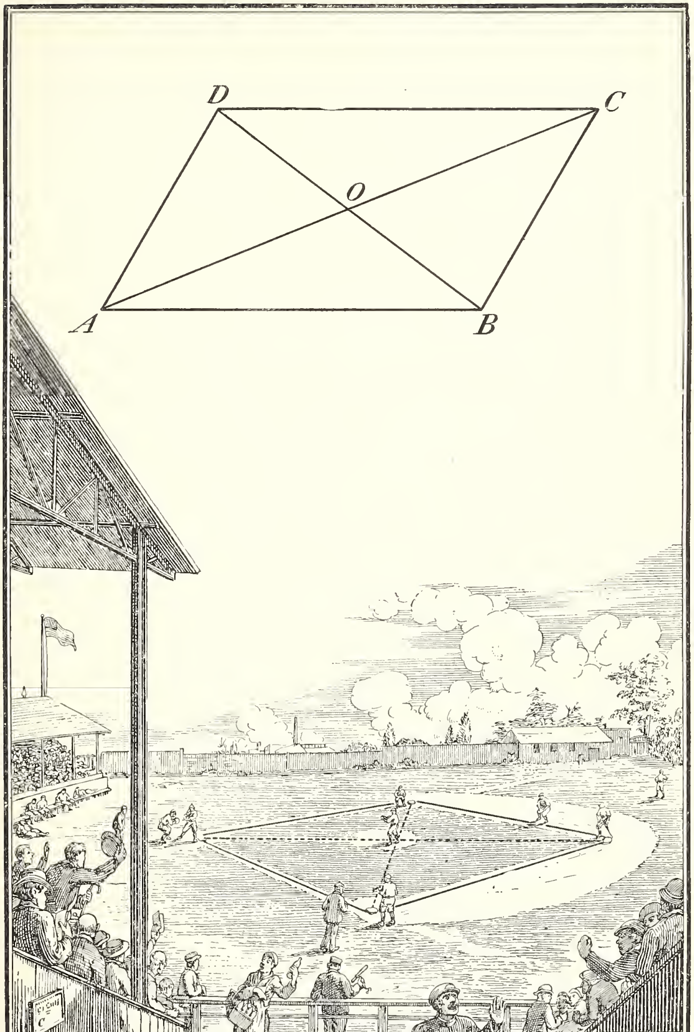
$\angle BAO = \angle DCO$,

and $\angle OBA = \angle ODC$. § 84

Therefore △ ABO and CDO are congruent. § 68

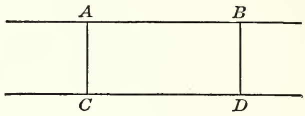
Therefore $AO = OC$ and $BO = OD$. § 54

95. Application. Show how the proposition applies with respect to the baseball diamond shown on the opposite page.


D
C
O
A
B

Exercise 20. Parallelograms

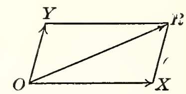
1. *Two parallel lines are everywhere equally distant from each other.*



If AB is $\parallel$ to CD, and AC and BD are $\perp$ to CD, prove that AC is $\parallel$ to BD. Then it follows that $ACDB$ is a parallelogram.

2. *A parallelogram having one of its angles a right angle is a rectangle.*

3. It is proved in physics that two forces acting on an object O have the same effect as a single force known as their *resultant*. If one force of 100 lb. is pulling in the direction OX, and the other of 50 lb. is pulling in the direction OY, the resultant will be represented by the diagonal OR of the parallelogram $OXRY$. By taking OX to represent 100 lb. and OY to represent 50 lb., and measuring OR and $\angle XOR$, we can find the magnitude and direction of the resultant. Using a protractor and ruler, find these in the above case.



4. Two forces at right angles to each other are exerted upon an object. One force is 300 lb. to the right and the other is 400 lb. upwards. Find the resultant as in Ex. 3.

5. An airplane moving horizontally at the rate of 176 ft./sec. drops a bomb which starts falling at the rate of 16 ft./sec. Find the direction and rate of the bomb.

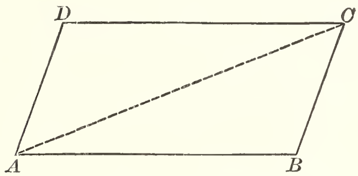
The expression ft./sec. means feet per second.
Proceed as with the *parallelogram of forces* in Exs. 3 and 4.

6. The diagonals of a rhombus form four right angles.

7. The perpendiculars from two opposite vertices of a parallelogram, drawn to the diagonal determined by the other vertices, are equal.

THEOREM. OPPOSITE SIDES EQUAL

96. *If both pairs of opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



Given the quadrilateral *ABCD*, having *BC* equal to *AD*, and *AB* equal to *DC*.

To prove that ABCD is a parallelogram.

Proof. Draw the diagonal AC.

In the △ ABC and CDA,

$BC = DA$, Given

$AB = DC$, Given

and $AC = AC$.

Therefore △ ABC and CDA are congruent. § 72

(*Two triangles are congruent if the three sides of one are equal respectively to the three sides of the other.*)

Therefore $\angle BAC = \angle DCA$, § 54

and so AB is ∥ to DC. § 86

(*For ∠s BAC and DCA are equal alternate ∠s made by the transversal AC with the lines AB and DC.*)

Similarly, $\angle ACB = \angle CAD$, § 54

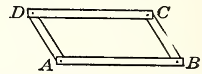
and so BC is ∥ to AD. § 86

Therefore $ABCD$ is a parallelogram. § 88

(*For both pairs of opposite sides are parallel.*)

Exercise 21. Criterion of a Parallelogram

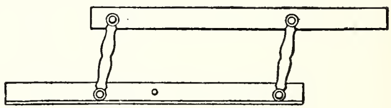
1. This figure represents four hinged rods, AB being equal to DC, and BC being equal to AD. As the angles change, does the figure continue to be a parallelogram? Upon what theorem does your answer depend?



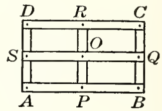
2. In the figure of Ex. 1, if $\angle A$ is 110°, how large is $\angle B$? $\angle C$? $\angle D$?

3. In the figure of Ex. 1, if $\angle D$ is 10°, how large is $\angle A$? $\angle B$? $\angle C$?

4. This figure represents a parallel ruler. Explain its construction and use and state the theorem on which its principle depends.



5. This figure represents six hinged rods, all the angles being right angles and P, Q, R, S bisecting AB, BC, CD, DA respectively. Prove that the figure can be pulled into different shapes, the angles then ceasing to be right angles, but that all the quadrilaterals remain parallelograms.

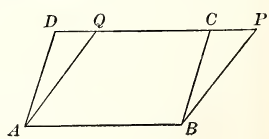


6. If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.

7. If the diagonals of a quadrilateral bisect each other at right angles, the quadrilateral is a rhombus.

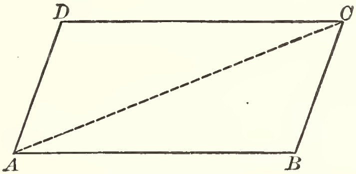
8. In this figure the quadrilaterals $ABCD$ and $ABPQ$ are both parallelograms. Prove that the triangles AQD and BPC are congruent.

Observe that CD is equal to PQ, for each is equal to a certain other line.



THEOREM. TWO SIDES EQUAL AND PARALLEL

97. *If two sides of a quadrilateral are equal and parallel, then the other two sides are equal and parallel, and the figure is a parallelogram.*



Given the quadrilateral *ABCD*, having side *AB* equal and parallel to side *DC*.

To prove that ABCD is a parallelogram.

Proof. Draw the diagonal AC.

In the $\triangle ABC$ and CDA,

$AC = AC$,

$AB = DC$, Given

and $\angle BAC = \angle DCA$. § 84

(*For AB is given* ∥ *to DC, and these are alternate* $\angle$s.)

Therefore $\triangle ABC$ and CDA are congruent. § 58

Therefore $BC = DA$,

and $\angle ACB = \angle CAD$. § 54

Therefore BC is ∥ to AD. § 86

(*For* $\angle$s *ACB and CAD are alternate* $\angle$s *made by a transversal and have been proved to be equal.*)

But AB is ∥ to DC. Given

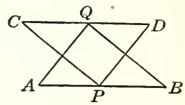
Therefore $ABCD$ is a parallelogram. § 88

(*For both pairs of opposite sides are parallel.*)

It will be seen that § 94 gives *properties of parallelograms,* while §§ 96 and 97 give *tests for finding if figures are parallelograms.*

Exercise 22. Quadrilaterals as Parallelograms

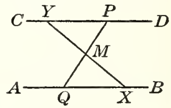
1. It is given that AB is equal and parallel to CD, that P is the mid-point of AB, and that Q is the mid-point of CD. Prove that $PBQC$ is a parallelogram.



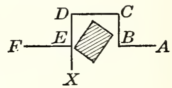
2. In Ex. 1 prove that $APDQ$ also is a parallelogram.

3. In the above figure for Exs. 1 and 2 find another parallelogram and prove the truth of your statement concerning it.

4. In laying out a tennis court it is desired to draw a line through a point P parallel to a line AB already fixed. To do this, a line is drawn from P to Q, any point on AB, and the mid-point M of PQ is found. Then from any other point X on AB a tape is stretched through M, and a point Y is found such that MY is equal to XM. A line CD is then drawn through Y and P. Prove that CD is parallel to AB.



5. In surveying it often becomes necessary to run a straight line beyond an object through which it is impossible to sight and over which it is impossible to pass. One of the methods, applied to the adjoining figure, is as follows: Suppose the surveyor runs the line AB to B; he then runs a line BC at right angles to AB; at C he runs a line CD at right angles to BC; at D he runs a line DX at right angles to CD; on DX he lays off DE equal to CB, and at E he runs a line EF at right angles to DE. Prove that EF is part of the straight line AB prolonged.



6. If two adjacent angles of a parallelogram are equal, the parallelogram is a rectangle.

7. *If the diagonals of a parallelogram are equal, the parallelogram is a rectangle.*

8. If two equal lines bisect each other at right angles, they form the diagonals of a certain square. Draw the square and prove the theorem as stated.

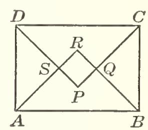
9. In the trapezoid $ABCD$, AB being parallel to DC, if CE is drawn to meet AB at E so as to make $\angle BEC$ the supplement of $\angle D$, the figure $AECD$ is a parallelogram.

10. The line joining the mid-points of two opposite sides of a parallelogram divides the figure into two congruent parallelograms.

Prove the congruence by superposing one parallelogram on the other and proving, as in § 58, that the two coincide.

11. The mid-points of the sides a, b, c of the equilateral triangle ABC are X, Y, Z respectively. Prove that A, Z, X, and Y are the vertices of a parallelogram.

12. In this rectangle AR, BR, CP, and DP are the respective bisectors of the angles. Prove that $PQRS$ is a rectangle. Consider whether or not it is also a square.



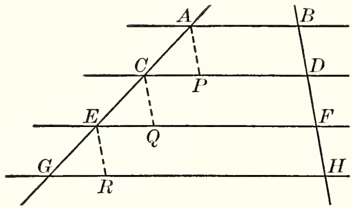
13. Consider Ex. 12 when $ABCD$ is a parallelogram but not a rectangle.

14. A certain figure can be proved to be a parallelogram if it can be shown that certain angles are right angles. If it can be shown that these angles are right angles, does it follow that the figure is a parallelogram? If the figure is a parallelogram, does it follow that these angles are right angles? Explain your answers.

15. Certain goods can be proved to be all wool if it can be shown that they were bought from a certain dealer. The goods are all wool. What conclusion do you draw as to where they were bought? Explain your answer.

Theorem. Transversals and Parallels

98. *If three or more parallels intercept equal segments on one transversal, they intercept equal segments on every transversal.*



Given the parallels *AB*, *CD*, *EF*, *GH*, intercepting equal segments *BD*, *DF*, *FH* on the transversal *BH*, and intercepting segments *AC*, *CE*, *EG* on another transversal.

To prove that $AC = CE = EG$.

Proof. Suppose AP, CQ, and ER drawn ∥ to BH.

Since $\angle DPA = \angle PDF$ (§ 84), $\angle APC = \angle BDC$. § 32

Similarly, $\angle CQE = \angle DFE$ and $\angle ERG = \angle FHG$.

Also, $\angle BDC = \angle DFE = \angle FHG$.

Therefore $\angle APC = \angle CQE = \angle ERG$,

and each of these is equal to $\triangle QCP$ and REQ. § 84

Therefore AP, CQ, ER are parallel. § 86

As above, $\angle CAP = \angle ECQ = \angle GER$.

Now $AP = BD$, $CQ = DF$, and $ER = FH$. § 94

But $BD = DF = FH$. Given

Therefore $AP = CQ = ER$.

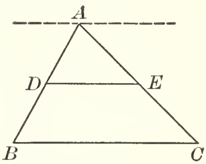
(*For they are equal to equal quantities.*)

Therefore $\triangle CPA$, EQC, and GRE are congruent. § 68

Therefore $AC = CE = EG$. § 54

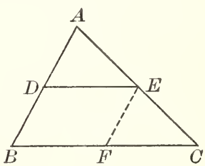
99. Corollary 1. *If a line is parallel to one side of a triangle and bisects another side, it bisects the third side also.*

Let DE be parallel to BC and bisect AB. Suppose a line drawn through A parallel to BC. Then how do we know this line to be parallel to DE? Since it is given that the three parallels intercept equal segments on the transversal AB, what can we say of the segments intercepted on AC? What can we then say that DE does to AC?



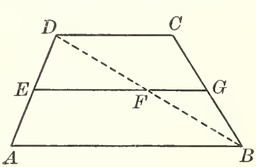
100. Corollary 2. *The line which joins the mid-points of two sides of a triangle is parallel to the third side and is equal to half the third side.*

A line DE drawn through the mid-point of AB, $\parallel$ to BC, divides AC in what way (§ 99)? Therefore the line joining the mid-points of AB and AC coincides with this parallel and is $\parallel$ to BC. Also, since EF drawn $\parallel$ to AB bisects AC, how does it divide BC? What does this prove as to the relation of BF, FC, and BC? How do we know that $BFED$ is a parallelogram? What do we know as to the equality of DE, BF, and $\frac{1}{2}BC$?



101. Corollary 3. *The line joining the mid-points of the non-parallel sides of a trapezoid is parallel to the bases and is equal to half their sum.*

Draw the diagonal DB. In the $\triangle ABD$ join E, the mid-point of AD, to F, the mid-point of DB. By § 100, what relations exist between EF and AB? In the $\triangle DBC$ join F to G, the mid-point of BC. What relations exist between FG and DC? Since these relations exist, what relation exists between AB and FG? But only one line can be drawn through $F \parallel$ to AB (§ 80, 2). Therefore FG is the prolongation of EF. Hence EFG is $\parallel$ to AB and CD, and is equal to $\frac{1}{2}(AB + DC)$.



Write out proofs of Corollaries 1, 2, and 3 in full, as in § 98.

Exercise 23. Review

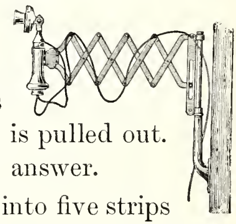
1. *Segments of parallel lines cut off by parallel lines are equal.*

2. *Two rectangles having equal bases and equal heights are congruent.*

Prove by *superposition*, that is, by placing one figure on the other and showing that the two figures coincide, as in §§ 58 and 68.

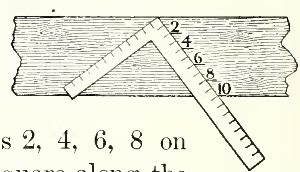
3. *Two parallelograms are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*

Prove by superposition, as in Ex. 2.

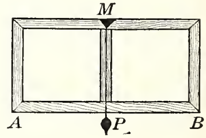


4. Explain geometrically why this telephone extends horizontally when it is pulled out. State each proposition involved in the answer.

5. A board 8 in. wide is to be sawed into five strips of equal width. In order to draw the lines for sawing, a carpenter lays his steel square as here shown, placing the corner on one edge and the 10-inch mark on the other. He then marks the board at the divisions 2, 4, 6, 8 on the square. He then moves the square along the board and repeats the process of marking. Through the respective marks he now draws straight lines. Prove that these lines satisfy the requirements.

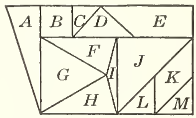


6. The rectangular frame here shown has a plumb line MP hung from M, the mid-point of the upper strip of wood. Show geometrically, stating each proposition used, that when the line crosses the mid-point of the base AB this base is level.



7. Write four independent conditions which make a quadrilateral a parallelogram and draw the figure illustrating each condition.

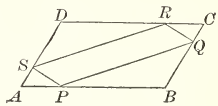
8. Describe by name each of the lettered figures included in this complete figure.



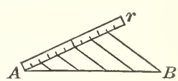
9. If the mid-points of the sides of a parallelogram are joined in order by straight lines, the resulting figure is also a parallelogram.

10. In this parallelogram *ABCD* it is given that $PB = DR$ and $AS = CQ$. Prove that *PQRS* is a parallelogram.

Show first that $AP = CR$.

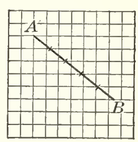


11. A draftsman placed a ruler *r* as shown in the figure, making an acute angle with *AB*, and from points on *r* that were 1, 2, 3, 4, and 5 inches from *A* he drew parallel lines cutting *AB*. Prove that these lines divide *AB* into five equal segments.



12. The line joining the mid-points of two opposite sides of a parallelogram passes through the intersection of the diagonals.

13. In Ex. 11 show that the draftsman might have divided *AB* into five equal segments by transferring this line to a piece of squared paper, as here shown.

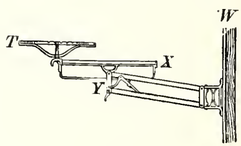


14. In certain of the following-named figures the diagonals bisect each other: a square, a rhombus, a trapezoid, a rectangle, a parallelogram, any other quadrilateral. State in which figures this is the case, and prove that, in general, it is not the case with the other figures.

15. Two rectangles are congruent under certain of the following cases: (1) Two sides of one equal respectively to two sides of the other; (2) Four sides of one equal respectively to four sides of the other; (3) Two adjacent sides of one equal respectively to two adjacent sides of the other; (4) A diagonal of one equal to a diagonal of the other. State in what cases they are congruent and draw figures to show they are not congruent in the other cases.

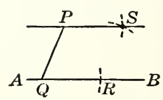
16. In Ex. 15 substitute parallelograms for rectangles and then consider the four cases mentioned.

17. A dentist's working table is adjusted as shown in this figure. State in full the geometric proof that table T is always horizontal provided the apparatus is properly made and is fastened to a vertical wall W.



The bar X is fastened to Y at right angles.

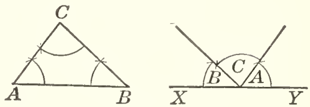
18. In laying out a tennis court it is desired to run a line through P parallel to AB. This is a convenient method: stretch a tape from P to any point Q on AB; then with Q as center swing the tape to cut AB at R; with P and R as centers and the same radius as before mark arcs intersecting at S; and draw a line through P and S. Prove that PS is the line required.



19. A student believes he can prove a certain theorem if he can prove that $B = B'$ and $O = O'$. Another student proves the theorem. Can any definite conclusion be drawn as to how he did it? A mechanic believes that he can make a machine run properly if he tightens a certain bolt and oils a certain bearing. Another mechanic makes the machine run properly. Can any definite conclusion be drawn as to how he did it?

V. ANGLE SUMS

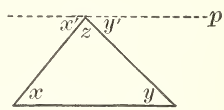
102. Inference as to Angles of a Triangle. If you cut several kinds of triangles from paper and then cut off the three angles of each triangle and fit them together as here shown, what kind of angle does the sum of each set of angles appear to be?



State your inference, beginning thus:

The sum of the angles of a triangle is equal to . . . .

103. Examination of the Inference. Although the above inference may seem to be correct for several triangles, it does not follow that it is correct for all triangles. Even in the case of any triangle we examine, our eyes may be deceived, and the sum of the angles may really be 179° 40′, or 180° 25′, or some other angle that is approximately 180°. We therefore proceed to see if there is any way of proving that the inference is always true.



Suppose that the line p is drawn through the vertex of $\angle z$ of a triangle, and parallel to the opposite side, as here shown. Then in the figure, to what angle is x equal, and why?

To what angle is y equal, and why?

How does $x + z + y$ compare with $x' + z + y'$, and why?

But $x' + z + y'$ is equal to what kind of angle?

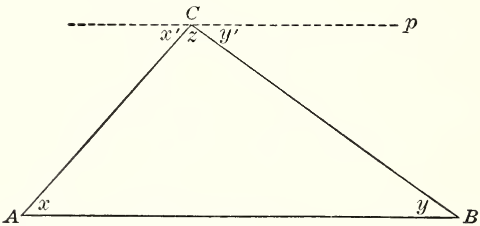
Then $x + z + y$ is equal to what kind of angle?

Does the proof depend in any way upon the kind of triangle taken? That is, is the proof equally valid for an equilateral, isosceles, or right triangle as for any other kind?

What is your conclusion as to the sum of the angles of any triangle?

THEOREM. ANGLES OF A TRIANGLE

104. *The sum of the three angles of a triangle is equal to a straight angle.*



Given the triangle ABC, with angles x, y, and z.

To prove that $x + y + z =$ *a straight angle.*

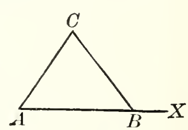
Proof. Suppose p to be drawn through C parallel to AB making angles x' and y' with sides AC and BC respectively.

Then $x' + z + y' =$ a straight angle. § 20

But $x' = x$ and $y' = y$. § 84

Substituting, $x + z + y =$ a straight angle.

105. Exterior Angle. The angle included by one side of a plane figure and an adjacent side produced is called an *exterior angle* of the figure.



For example, $\angle XBC$ is an exterior angle of $\triangle ABC$, and $\angle A$ and $\angle C$ are called the *opposite interior angles.*

106. COROLLARY. *An exterior angle of a triangle is equal to the sum of the two opposite interior angles.*

For $\angle XBC + \angle CBA =$ a st. $\angle$,

and $\angle A + \angle C + \angle CBA =$ a st. $\angle$.

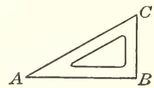
Therefore $\angle A + \angle C + \angle CBA = \angle XBC + \angle CBA$.

Subtracting $\angle CBA$, $\angle A + \angle C = \angle XBC$.

Exercise 24. Angles of a Triangle

1. If two angles of one triangle are equal respectively to two angles of another triangle, how do the third angles of the triangles compare in size? Prove your statement.

2. In a draftsman's triangle, $\angle B$ is often a right angle, as shown in the figure, and $\angle A$ is 30°. In such a triangle how many degrees are there in $\angle C$?



3. Prove that the sum of the two acute angles of a right triangle is equal to 90°.

Given these two angles of a triangle, find the third angle:

4. 27°, 30°.

5. 42°, 18°.

6. 68°, 29°.

7. 56°, 37°.

8. 82°, 41° 27′.

9. 78°, 53° 45′.

10. 120°, 40° 30′.

11. 132°, 16° 50′.

12. 128° 43′, 26°.

It being asserted that the angles of a triangle are as given below, find the impossible cases:

13. 48°, 37°, 95°.

14. 39°, 62°, 87°.

15. $56\frac{1}{2}°$, $48\frac{1}{2}°$, 75°.

16. $27\frac{1}{4}°$, $62\frac{3}{4}°$, 90°.

17. 59° 6′, 47° 54′, 73°.

18. 42° 10′, 21° 18′, 98° 32′.

19. How many degrees are there in each angle of an equilateral triangle? Prove your statement.

20. If the vertical angle of an isosceles triangle is 40°, how many degrees are there in each of the other angles?

21. If one of the base angles of an isosceles triangle is 40°, how many degrees are there in each of the other angles?

22. If one angle of a triangle is 48°, find the sum of the other two angles. Find also the exterior angle at the vertex of the given angle. Compare the results.

23. If one angle of a triangle is $29° 30'$, what is the sum of the other two angles?

24. If the sum of two angles of a triangle is $29° 30'$, how many degrees are there in the other angle?

25. An exterior angle at the base of an isosceles triangle being 100°, find the number of degrees in each angle of the triangle.

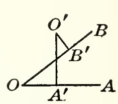
26. The exterior angle at the vertex of an isosceles triangle being 100°, find the number of degrees in each angle of the triangle.

27. An exterior angle of an isosceles triangle being 120°, what other special name can be given to the triangle? Prove your statement.

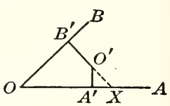
28. The sum of the two exterior angles at the base of an isosceles triangle being 270°, what other special name can be given to the triangle? Prove your statement.

29. If one angle of a triangle is right or obtuse, each of the other angles is what kind of angle? Prove it.

30. In this figure, $O'A'$ is perpendicular to OA, and $O'B'$ to OB. Name all the pairs of equal angles in the figure and prove each statement.

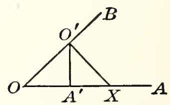


31. In Ex. 30 suppose that O' lies within $\angle AOB$, as shown in this figure.



Produce $B'O'$ to meet OA, as at X. Show that the angles of $\triangle XO'A'$ are respectively equal to the angles of $\triangle XOB'$.

32. In Ex. 31 suppose that O' lies on OB, as shown in this figure.



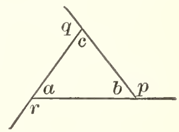
Show that the angles of $\triangle XO'A'$ are respectively equal to the angles of $\triangle XOO'$.

33. In Ex. 31 show $\angle B'O'A'$ to be supplementary to $\angle O$.

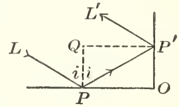
34. *Two angles whose sides are perpendicular to each other are either equal or supplementary.*

35. Of the angles of a triangle the second is twice as large as the first and the third is twice as large as the second. Find the number of degrees in each angle.

36. In this triangle p is the sum of what two angles? q is the sum of what two? r is the sum of what two? Compare $a + b + c$ with $p + q + r$. How many degrees are there in $p + q + r$?



37. If a ray of light LP strikes a mirror OP at P and is reflected to a mirror OP' perpendicular to the first mirror, striking it at P', the ray is reflected back in a line $P'L'$. The angle of incidence QPL is equal to the angle of reflection $P'PQ$. Find all the angles in the figure in terms of i, and show that $P'L'$ is parallel to PL.

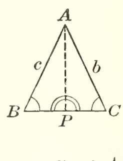


38. *Two triangles are congruent if two angles and a corresponding side of one are equal respectively to two angles and a corresponding side of the other.*

Ex. 38 adds a new condition of congruence to the list given in § 76.

39. *If a triangle has two equal angles, the sides opposite these angles are equal.*

This corollary was for the time being assumed to be true in § 64. It was not used thereafter in any proof except in § 75, and that theorem has not since been used. It would have been possible to prove § 64 when it was given, and it would be possible to prove § 75 without using § 64. For an elementary treatment, however, the arrangement given is the most satisfactory. We can now prove § 64.



Suppose that $\angle B = \angle C$. We have then to prove that $b = c$. Suppose a perpendicular AP to be drawn from A to BC and prove that $\triangle$s ABP and ACP are congruent.

107. Polygon. A closed plane figure formed by joining a succession of points, each to the following one and the last to the first, is called a *polygon*.

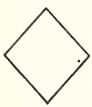
The terms *sides*, *perimeter*, *angles*, *vertices*, and *diagonals* are employed as usual.

108. Polygons classified as to Sides. A polygon is called a *triangle* if it has three sides, a *quadrilateral* if it has four, a *pentagon* if it has five, a *hexagon* if it has six.

These names are sufficient for most cases. The next few names in order are *heptagon, octagon, nonagon, decagon, undecagon, dodecagon.*

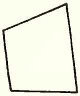
A polygon is *equilateral* if all of its sides are equal.

109. Polygons classified as to Angles. A polygon is *equiangular* if all of its angles are equal; *convex* if each of its angles is less than a straight angle; *concave* if it has one or more angles greater than a straight angle.

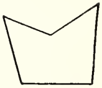


EQUILATERAL

EQUIANGULAR



CONVEX



CONCAVE

An angle of a polygon greater than a straight angle is called a *re-entrant angle*. When the term *polygon* is used without restriction, a convex polygon is always understood.

110. Regular Polygon. A polygon which is both equiangular and equilateral is called a *regular polygon.*

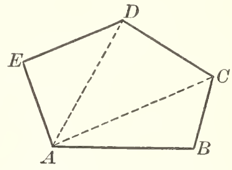
111. Relation of Two Polygons. Two polygons are:

Mutually equiangular if the angles of one are equal to the angles of the other respectively, taken in the same order;

Mutually equilateral if the sides of one are equal to the sides of the other respectively, taken in the same order;

Congruent if one can be made to coincide with the other.

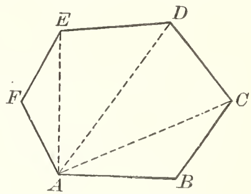
112. Inference as to a Polygon. Having considered the sum of the angles of a triangle, we shall now see if there is an equally important relation as to the sum of the angles of a polygon.



Considering the pentagon *ABCDE*, it is natural to divide it into triangles because we have already found the sum of the angles of a triangle.

What is the sum of the angles of each triangle? What is the sum of the angles of the pentagon?

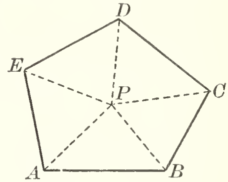
In the case of a hexagon, how many triangles are there? What is the sum of all the angles?



How many triangles would there be in the case of a figure of seven sides? What would be the sum of all the angles?

Using figures divided as shown above, the number of triangles is always how many less than the number of sides? Then the number of straight angles in the sum of the angles of a polygon is always how many less than the number of sides?

Let us try another way of dividing the polygon into triangles, connecting any point *P*, within the polygon, with the vertices.



What is the sum of the angles of each triangle? of all the triangles? How many straight angles must be subtracted because of the angles about *P* which are not angles of the polygon? Then how many straight angles are there in the sum of the angles of the polygon?

Write the statement as to the sum of the angles of any polygon.

Theorem. Angles of a Polygon

113. *The sum of the interior angles of a polygon is equal to as many straight angles as the figure has sides, less two.*

Given the polygon *ABCDEF*, having *n* sides.

To prove that the sum of the interior angles is equal to $(n-2)$ straight angles.

Proof. From A draw the diagonals AC, AD, AE.

The sum of the angles of the triangles is equal to the sum of the angles of the polygon.

There are $(n-2)$ triangles.

(*For there is one △ for each side except the two sides adjacent to* A.)

The sum of the angles of each of these triangles is equal to one straight angle. § 104

Therefore the sum of the angles of the $(n-2)$ triangles is equal to $(n-2)$ straight angles.

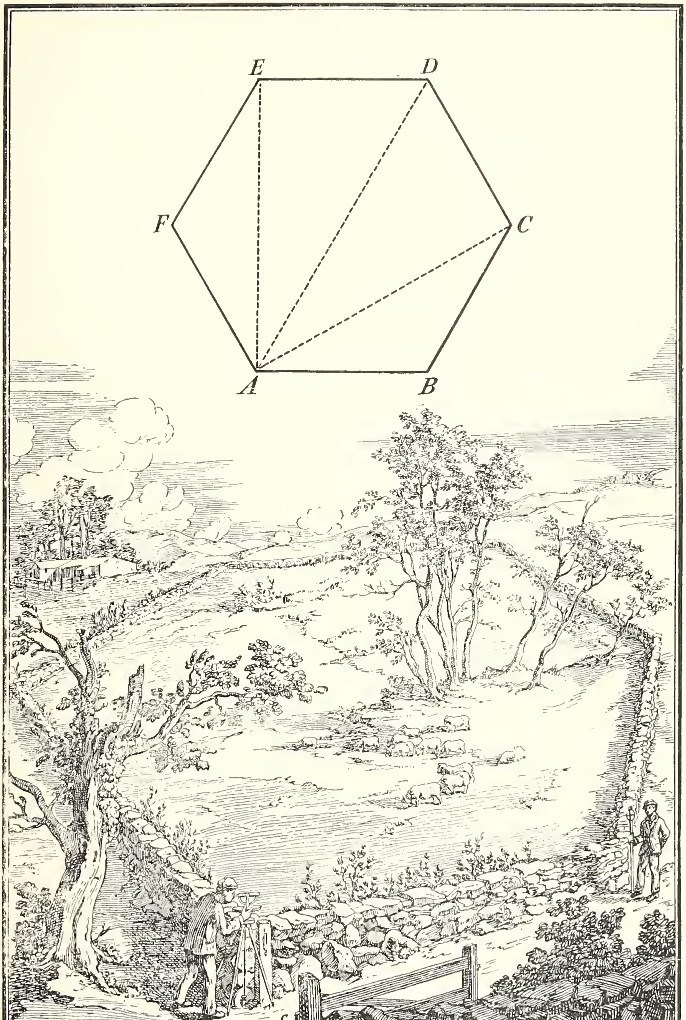
Therefore the sum of the angles of the polygon is equal to $(n-2)$ straight angles.

We may, of course, express the sum as $(n-2)\,180°$.

In a regular polygon of n sides, each angle is $\frac{1}{n}\cdot(n-2)\,180°$.

114. Application. In the lower figure on the opposite page two surveyors are measuring a field of six sides. They have an instrument for measuring the angles, and they find these angles to be 128° 30′, 125° 15′, 87° 45′, 132° 30′, 85° 45′, and 159° 45′. Is this result correct? If not, what is the error in the sum of the angles?

Surveyors actually make use of this proposition or one similar to it to check the work in measuring the angles of a field. In case the work does not check the error is due to carelessness or to the inaccuracy of the instruments used.


E
D
F
C
A
B

Exercise 25. Angles of a Polygon

1. Show that the theorem of § 113 holds for a triangle.

2. What is the sum of the angles of a pentagon? of an octagon? of a decagon?

3. Find the number of degrees in each angle of an equilateral triangle, of a regular hexagon, and of a regular polygon of 16 sides.

4. Five angles of a hexagon are $115°$, $130°$, $98° 30'$, $100°$, and $102° 15'$. Find the sixth angle.

5. How many sides has a regular polygon, each angle of which is $1\frac{3}{4}$ right angles? each angle of which is $108°$? each angle of which is $140°$?

6. The angles of a triangle are a, $2a$, and $1.5a$ degrees. Find the number of degrees in each angle.

7. The angles of a quadrilateral are n, $n+1$, $n+2$, and $n+3$ degrees. Find the number of degrees in each angle.

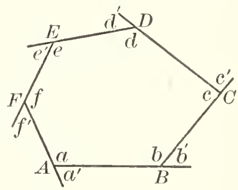
8. The angles of a certain pentagon are all equal except one, and that is an angle of $80°$. Find the number of degrees in each of the four equal angles.

9. In surveying a four-sided field it was found that the angles were respectively $109° 30'$, $80° 30'$, $110° 15'$, and $59° 45'$. Check the work by the aid of § 113.

10. Find the number of degrees in the vertical angle of an isosceles triangle, given that one of the equal angles of the triangle is $72° 15' 45''$.

11. A manufacturer carries a certain number of lines of goods on his list, and he has a salesman for every line except two. The salary of each salesman is $180 a month. If the manufacturer carries n lines of goods, how much is his pay roll per month for his agents?

115. Inference as to Exterior Angles. If we produce the sides of a polygon as here shown, we shall have the exterior angles a', b', c', d', e', and f'. In speaking of the exterior angles of a polygon, the sides are supposed to be produced in the same order about the figure, as in this case.



We may easily infer what the sum of these angles is. Suppose that you are walking along the line FA, from F towards A, and suppose that you turn at A and walk from A towards B. You turn at A through the angle a'. Similarly, at B you turn through b'; at C, through c'; at D, through d'; at E, through e'; and at F, through f'. You now notice that you have turned completely around and are again facing in the direction FA.

Through how many straight angles, or through how many right angles, do you turn when you turn completely around? What is then the sum of the exterior angles a', b', $\cdots$, f'?

The series of dots in this case means "and so on to."

116. Examination of the Inference. What is the sum of a and a'? of b and b'? of c and c'? of each interior angle and its corresponding exterior angle?

If the figure has n sides, it has how many vertices? It has how many such sums as $a + a'$?

Then the sum of all the interior and all the exterior angles is how many straight angles?

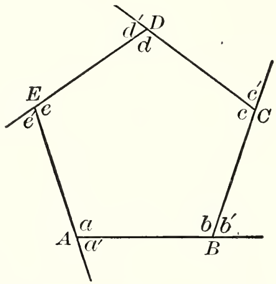
But what is the sum of all the interior angles?

Subtracting this sum of the interior angles, what is the sum of the exterior angles?

State the result of this proof as a theorem.

THEOREM. SUM OF THE EXTERIOR ANGLES

117. *The sum of the exterior angles of a polygon is equal to two straight angles.*



Given the polygon *ABCDE*, having its *n* sides produced in succession.

To prove that the sum of the exterior angles is equal to two straight angles.

Proof. Denote the interior angles by a, b, c, d, e, and the corresponding exterior angles by a', b', c', d', e'.

Then, considering the angles at A,

$$\angle a + \angle a' = \text{a st.} \angle. \qquad \text{Post. 9}$$

Similarly, the sum of the interior angle and the exterior angle at each of the n vertices is a straight angle.

Therefore the sum of all the interior and all the exterior angles of the polygon is equal to n st. ∠s.

But the sum of all the interior angles is

$$(n-2) \text{ st. } \angle\text{s}. \qquad \S\ 113$$

(*The sum of the interior angles of a polygon is equal to as many st. ∠s as the figure has sides, less two.*)

Subtracting, we see that the sum of all the exterior angles is $n-(n-2)$ st. ∠s, or 2 st. ∠s.

Exercise 26. Exterior Angles

1. Two of the exterior angles of a triangle are 110° and 100° respectively. Find the number of degrees in each of the three angles of the triangle.

By the angles of a polygon is meant the interior angles.

2. Two of the angles of a triangle are 88° and 72° respectively. Find the number of degrees in each exterior angle of the triangle.

3. Find the number of degrees in each exterior angle of a regular pentagon.

4. In a certain right triangle one acute angle is three times the other. Find the number of degrees in each exterior angle of the triangle.

5. Each exterior angle of a certain regular polygon is 36°. Find the number of sides of the polygon.

6. Make out a table showing the number of degrees in each interior angle and each exterior angle of regular polygons of three, four, five, six, · · ·, ten sides.

7. Five of the exterior angles of a hexagon are 56°, 63°, 57°, 70°, 49°. Find the number of degrees in each angle of the hexagon.

8. Seven of the angles of an octagon are 150°, 112°, 120°, 130°, 140°, 132°, 125°. Find the number of degrees in each exterior angle of the octagon.

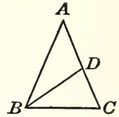
9. Each exterior angle of a certain regular polygon is 20°. Find the sum of the angles of the polygon.

10. The total expenses for work done inside and outside a certain shop in a day is n dollars. The expenses for work done inside the shop is $(n-2)$ dollars. Find the expenses for work done outside the shop.

Exercise 27. Review

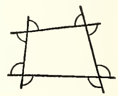
1. The two base angles of an isosceles triangle are bisected by lines meeting at *P*. The vertical angle of the isosceles triangle is 40°. Find the number of degrees in $\angle P$.

2. In the isosceles triangle *ABC* each base angle is twice the angle at the vertex. If the line *BD* bisects $\angle B$, what kind of triangle does *BCD* seem to be? Prove that your inference is correct.



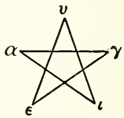
3. Two consecutive angles of a rectangle are bisected by lines meeting at *P*. Draw the figure carefully, study it particularly with respect to $\angle P$, write a theorem concerning it, and prove this theorem.

4. Study this figure with respect to the sum of the marked angles, write a theorem concerning it, and prove this theorem.



5. In a certain right triangle one of the angles is 45°. The two acute angles are bisected by lines meeting at *P*. Draw the figure, study it, write a theorem concerning the bisectors, and prove this theorem.

6. Find the sum of the angles at the five points of the usual form of the five-pointed star.

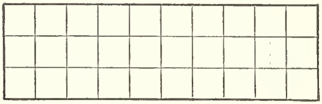


Such a star is sometimes called a *pentagram*. It was used as a badge by the followers of Pythagoras, one of the greatest of the Greek mathematicians, about 525 B.C. At the five points were the Greek letters υ, γ, ι, ε, α, the word ὑγίεια (hygieia) meaning "health," the single letter ε being used for ει.

7. A lawyer knows that if A has told the truth in a certain case, then B has also told the truth. He also knows that B has actually told the truth. What can he or can he not infer with respect to A?

VI. AREAS

118. Area of a Rectangle. The *area* of any surface is the number of square units that the surface contains. The number of square units of area of a rectangle is equal to the product of the number of units in the base and the number of units in the height.



For example, in this figure there are 10 units in the base and 3 units in the height, the number of square units being 3×10, or 30.

In measuring a rectangle we may take any units we wish, such as inches and square inches, feet and square feet, tenths of an inch and hundredths of a square inch, or centimeters and square centimeters.

It should be noticed that although the area of a square that is 1 in. on a side is 1 sq. in., we may have an area of 1 sq. in. that is triangular, circular, or oblong, or of any other shape.

In a small rectangle we can expect to find the lengths of the sides correct to 0.05 in. and to compute the area correct to 0.05 sq. in. If our measurements are correct only to two decimal places, the results will probably be correct only to two (not four) decimal places.

For brevity it is usually stated that *the area of a rectangle is equal to the product of the base and height.*

The rectangle of the lines AB and BC is usually indicated by $AB \cdot BC$ and the square on AB is indicated by $\overline{AB}^2$.

By the product of lines we mean the product of their numerical measures considered as abstract numbers.

A rectangle is often named by using the letters of two opposite vertices.

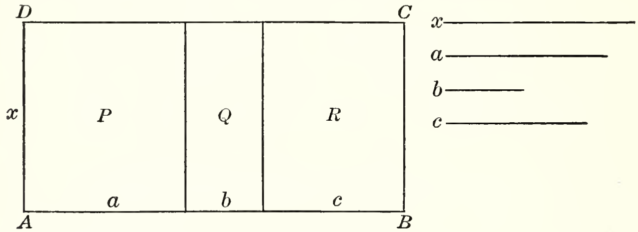
Thus a rectangle $ABCD$ may be named AC or BD.

119. Equivalent Polygons. Polygons which have equal areas are called *equivalent polygons.*

It is customary to say that two polygons are *equal* when they are equal in area, unless some misunderstanding is likely to arise.

THEOREM. RECTANGLES

120. *The rectangle contained by one line and the sum of several other lines is equal to the sum of the rectangles contained by the first line and each of the other lines.*



Given the line x and the lines a, b, and c.

To prove that the rectangle $x(a+b+c)$ is equal to the sum of the rectangles xa, xb, and xc.

Proof. Place a, b, and c in one line so that their sum is AB, and place x at right angles to AB at the point A. Complete the rectangle $ABCD$ and from the ends of a and b draw parallels to x as shown in the figure.

Then P is the rectangle of x and a, Q is the rectangle of x and b, and R is the rectangle of x and c. § 89

(*For their opposite sides are parallel by construction or by* § 80, 3, *and the angles are right angles.*)

Also $P = xa$,

$Q = xb$,

and $R = xc$. § 118

But $AC = P + Q + R = xa + xb + xc.$

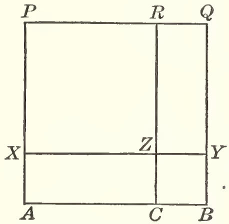
Also $AC = x(a+b+c).$

Hence $x(a+b+c) = xa + xb + xc.$

Thus we see that a certain algebraic product is represented in geometry by a rectangle.

THEOREM. SQUARE ON THE SUM

121. *The square on the sum of two lines is equal to the sum of the squares on the two lines, together with twice the rectangle contained by the lines.*



Given the two lines *AC* and *CB*.

To prove that $(AC+CB)^2=\overline{AC}^2+\overline{CB}^2+2\,AC\cdot CB$.

Proof. Place AC and CB so that A, C, and B, in this order, are in one straight line.

Suppose squares constructed on AB and CB as shown and lettered in the figure, and CZ produced to R, and YZ to X. Then CX and YR are rectangles.

(*For their opposite sides are* ∥, *and one* ∠ *of each is a rt.* ∠.)

Also $ZP=\overline{AC}^2$, § 118

(*For* $XZ=AC$, *and* $ZR=XP=AP-AX=AB-CB=AC$.)

and $BZ=\overline{CB}^2$. § 118

Also $CX=AC\cdot CB$,

and $YR=AC\cdot CB$. § 118

(*For* $CZ=CB$, $YZ=CB$, *and* $YQ=XP=AC$, *as above*.)

Adding and remembering that $ZP+BZ+CX+YR=BP$,

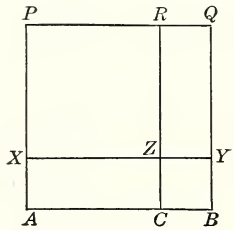
$$\overline{AB}^2=\overline{AC}^2+\overline{CB}^2+2\,AC\cdot CB.$$

Thus we see the geometric form of the algebraic identity

$$(a+b)^2=a^2+b^2+2\,ab.$$

Theorem. Square on the Difference

122. *The square on the difference between two lines is equal to the sum of the squares on the two lines, less twice the rectangle of the lines.*



Given the two lines *AB* and *BC*, *AB* being the longer.

To prove that $(AB - BC)^2 = \overline{AB}^2 + \overline{BC}^2 - 2\,AB \cdot BC.$

Proof. Place AB and BC so that A, C, and B, in this order, are in one straight line.

Suppose squares constructed on AB and CB as shown and lettered in the figure, and CZ produced to R, and YZ to X. Then CX and YR are rectangles.

(For their opposite sides are ∥, and one ∠ of each is a rt. ∠.)

Also, $XR = \overline{AC}^2$, § 118

(For $XZ = AC$, and $XP = AP - AX = AB - CZ = AB - BC = AC$.)

$AY = AB \cdot BC,$

and $CQ = AB \cdot BC.$ § 118

(For $BY = BC$, and $BQ = AB$.)

But $XR = AQ - AY - ZQ$

$= AQ - AY + CY - CQ.$

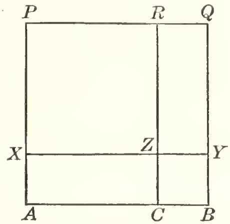
Therefore $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 - 2\,AB \cdot BC.$

Thus we see the geometric form of the algebraic identity

$$(a - b)^2 = a^2 + b^2 - 2\,ab.$$

THEOREM. RECTANGLE OF SUM AND DIFFERENCE

123. *The rectangle of the sum and the difference of two lines is equal to the difference of the squares on the lines.*



Given the two lines AB and BC, AB being the longer.

To prove that $(AB+BC)\cdot(AB-BC)=\overline{AB}^2-\overline{BC}^2$.

Proof. Place the lines AB and BC, as in the figure, so that C lies between A and B.

Suppose the rest of the figure to be constructed as in § 122, each separate figure then being a rectangle.

Then $\overline{AB}^2-\overline{BC}^2=AQ-CY=AR+ZQ.$

(*For $AQ-CY$ and $AR+ZQ$ are each equal to the figure $ACZYQP$.*)

But $AR=AC\cdot AP$ § 118

$=AC\cdot AB$

and $ZQ=ZR\cdot ZY$ § 118

$=AC\cdot BC.$

Hence $\overline{AB}^2-\overline{BC}^2=AC\cdot AB+AC\cdot BC$

$=(AB+BC)AC$ § 120

$=(AB+BC)(AB-BC).$

Thus we see the geometric form of the algebraic identity

$$a^2-b^2=(a+b)(a-b).$$

Exercise 28. Areas of Rectangles

1. The side of a square is found to be 2.85 in. Find the area to the proper number of decimal places (§ 118).

2. The sides of a rectangle are 4.35 in. and 2.65 in. Find the area to the proper number of decimal places.

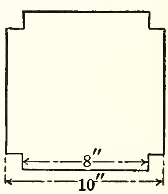
3. The perimeter of a square is 21.2 in. Find the area.

4. A rectangle is 8 in. long and has an area of 27.2 sq. in. Find the width.

5. A square and a rectangle have equal perimeters, 84 in., and the length of the rectangle is twice the width. Find the difference in the areas of the figures.

6. A square and a rectangle have equal areas, 100 sq. in., and the length of the rectangle is four times the width. Find the difference in the perimeters of the figures.

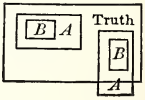
7. From a square piece of tin which is 10 in. on a side there are cut at the corners four squares, each 1 in. on a side. The tin is then folded up so as to form a rectangular box with base 8 in. on a side and with height 1 in. Find the volume of this box. Also find the volume if the small squares cut out at the corners are 3 in. on a side.



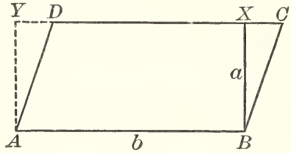
8. The perimeter of a rectangle is 21 in. and one side is 6.3 in. Find the area.

9. Illustrate the fact that a square has a greater area than any other rectangle having the same perimeter.

10. If A is true, then it is known that B is also true. But it is known that B is true. Does it necessarily follow that A is true? Explain your answer.



124. Inference as to Parallelograms. Consider the parallelogram *ABCD* and the rectangle *ABXY*, each of which is shown in this figure. As you look at the rectangle and then at the parallelogram, how do the areas of the two figures seem to compare?



If you should cut the figure out of paper, is there any part of the parallelogram that you could cut off and move to another position so as to show that the parallelogram has the same area as the rectangle?

Could this be done with a parallelogram of different shape? Draw two parallelograms of different shape on the blackboard, or cut two such parallelograms from paper, to illustrate your answer.

What do you infer as to the area of a parallelogram compared with the area of a rectangle of the same base and the same height? Write out this statement.

How do you find the area of a rectangle?

How would you proceed to find the area of a parallelogram? State the reason.

Write a complete statement of the method of finding the area of a parallelogram.

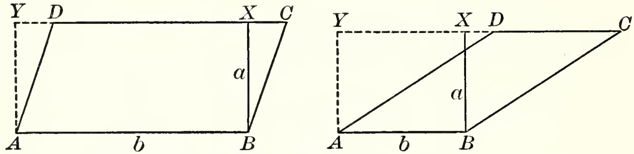
What do you infer as to the comparative areas of two parallelograms having equal bases and equal heights? Draw a figure to explain your answer.

Are two parallelograms necessarily congruent if they have equal bases and equal heights? Are two such parallelograms necessarily equivalent?

Are two rectangles necessarily congruent if they have equal bases and equal heights? Are they necessarily equivalent?

Theorem. Area of a Parallelogram

125. *The area of a parallelogram is equal to the product of its base and its height.*



Given the parallelogram *ABCD*, with base *b* and altitude *a*.

To prove that the area of parallelogram $ABCD = ab$.

Proof. Suppose BX drawn from $B \perp$ to CD or to CD produced, and AY drawn from $A \perp$ to CD or to CD produced.

Then $ABXY$ is a rectangle, with base b and height a.

Now $AD = BC$, and $AY = BX$. § 94

Therefore rt. $\triangle s$ ADY and BCX are congruent. § 75

From $ABCY$ take $\triangle BCX$; then $\square ABXY$ is left.

From $ABCY$ take $\triangle ADY$; then $\square ABCD$ is left.

Therefore $\square ABXY = \square ABCD$. Ax. 2

But the area of $\square ABXY = ab$. § 118

Therefore the area of $\square ABCD = ab$.

126. Corollary 1. *Parallelograms having equal bases and equal altitudes are equivalent.*

127. Corollary 2. *A parallelogram is equivalent to a rectangle having its base and height equal respectively to the base and height of the parallelogram.*

This theorem was regarded as very interesting by the ancients, since it might at first be thought impossible that the areas of two parallelograms could remain the same although their perimeters differed without limit.

Exercise 29. Areas of Parallelograms

Find the areas of parallelograms whose bases and heights are respectively as follows:

1. 3.2 in., 0.5 in. **3.** 6.2 ft., 8 in. **5.** 3 ft. 8 in., 11 in.

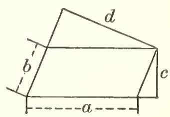
2. 4.8 in., 2.7 in. **4.** 5.8 ft., 2.4 ft. **6.** 4 ft. 6 in., 3 ft.

Find the heights of parallelograms whose areas and bases are respectively as follows:

7. 7 sq. in., 2 in. **9.** 9 sq. ft., $1\frac{1}{2}$ ft. **11.** 8 sq. in., 9 in.

8. 8.4 sq. in., 2 in. **10.** 8 sq. in., 16 in. **12.** 9 sq. in., 8 in.

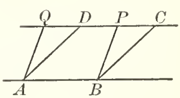
In this figure it being known that d is perpendicular to b produced and that c is perpendicular to a produced, prove that:



13. $ac = bd$.

14. The perimeter of the rectangle with sides a and c is less than the perimeter of the parallelogram with sides a and b.

15. Without reference to a rectangle as in § 125, prove that parallelograms on the same base and between the same parallels are equivalent.

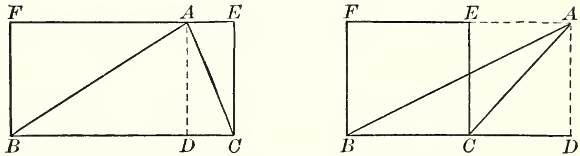


By parallelograms between the same parallels we mean parallelograms having each a pair of opposite sides on the same two parallel lines. Prove that $QD = PC$ and that $\triangle s\ ADQ$ and BCP are congruent. Consider the case in which the point D falls to the right of the point P.

16. A rectangle is a parallelogram, a square is a parallelogram, therefore a rectangle is a square. Is the reasoning correct? If not, where is the error? Does it invalidate a statement that is often given as an axiom, that quantities equal to the same quantity are equal to each other?

Theorem. Area of a Triangle

128. *The area of a triangle is equal to half the product of its base and its height.*



Given the triangle *ABC*, with base *BC* and height *AD*.

To prove that the area of $\triangle ABC = \frac{1}{2} BC \cdot AD$.

Proof. Suppose that a line is drawn through A parallel to BC, and that BF and CE are drawn perpendicular to this parallel.

Then $BDAF$, $DCEA$, and $BCEF$ are rectangles. § 89

(*For the sides are* ∥ *by const. or by* § 80, 1, *and the* ∠s *are right angles by const. or by* § 94.)

Also $\triangle ABD = \frac{1}{2} FD$. § 94

(*A diagonal divides the* ▱ *into two congruent* △s.)

Similarly, $\triangle ADC = \frac{1}{2} DE$.

Therefore, in the left-hand figure,

$$\triangle ABD + \triangle ADC = \tfrac{1}{2} FD + \tfrac{1}{2} DE, \qquad \text{Ax. 1}$$

and therefore $\triangle ABC = \frac{1}{2} BE$.

Similarly, in the right-hand figure,

$$\triangle ABD - \triangle ADC = \tfrac{1}{2} FD - \tfrac{1}{2} DE, \qquad \text{Ax. 2}$$

and therefore $\triangle ABC = \frac{1}{2} BE$.

Again, $BE = BC \cdot AD$. § 118

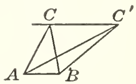
Therefore, in either case,

$$\triangle ABC = \tfrac{1}{2} BC \cdot AD.$$

Exercise 30. Areas of Triangles

1. The product of the two sides of a right triangle is equal to the product of the hypotenuse by the perpendicular from the vertex of the right angle to the hypotenuse.

2. What can be said as to the comparative areas of the two triangles ABC and ABC' if CC' is parallel to AB? Prove your statement.



3. The base and height of a triangle are 14.75 in. and 7.25 in. respectively. What is the area?

In Exs. 3–9 the lengths are supposed to be found by measurement and to be correct only to the number of decimal places stated. Hence the areas are subject to the limitation mentioned in § 118.

Find the areas of triangles whose bases and heights are respectively as follows:

4. 2.3 in., 1.7 in.

5. 4.8 in., 2.6 in.

6. 7.75 in., 9.35 in.

7. 17.45 in., 18.65 in.

8. 3 ft., 2 ft. 4 in.

9. 4 ft. $6\frac{1}{2}$ in., $11\frac{3}{4}$ in.

Find the heights of triangles whose areas and bases are respectively as follows:

10. 144 sq. in., 12 in.

11. 350 sq. ft., 70 ft.

12. 684 sq. in., 18 in.

13. 476 sq. ft., 28 ft.

14. 320 sq. yd., 40 yd.

15. 480 sq. mi., 96 mi.

16. A triangle, a rectangle, and a parallelogram have each a base 14.6 in. and a height 7.3 in., and a square has a side 7.3 in. Compare the areas of the figures.

17. The three sides of a right triangle are 21 in., 28 in., and 35 in. Find the area of the triangle.

In all cases in which a problem gives more than is required for the solution, use only what is necessary.

18. Show how a line should be drawn through one vertex of a triangle so as to bisect the area.

19. Find the area of the parallelogram $ABCD$ if AC is 17.4 in. and the perpendicular from B on AC is 9.3 in.

20. Draw a triangle and a parallelogram on the same base and between the same parallels, write a statement as to the comparative areas, and prove the statement.

A triangle and a parallelogram are said to be between the same parallels when their bases are segments of the same line and the vertex of the triangle is in line with the upper base of the parallelogram.

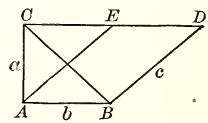
21. *Triangles having equal bases and equal altitudes are equivalent.*

22. *A triangle is equivalent to half of a parallelogram of the same base and the same altitude.*

This theorem is a formal statement of Ex. 20.

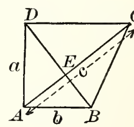
23. In the right triangle ABC, given that $AB = 10$ in., $BC = 8$ in., and $CA = 6$ in. A line is drawn from the mid-point of BC to the mid-point of AB, thus cutting off a small triangle. Find the area of the small triangle.

24. In this figure, given that BAC is a right angle, that CD is parallel to AB, and that AE is parallel to BD, find a formula for the distance from E to BD.



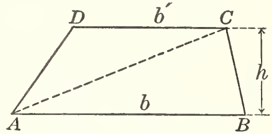
25. In the figure of Ex. 24 find the distance of the point E from the line BD when $b = 4$ in., $a = 6$ in., and $c = 8$ in.

26. In the figure below, given that BAD is a right angle, that DC is parallel to AB, and that AC is perpendicular to BD, find a formula for the length of BE.



27. In this figure, using the formula of Ex. 26, find the length of BE when $a = 11$ in., $b = 7$ in., and $c = 15$ in.

129. Area of a Trapezoid. If we consider the trapezoid $ABCD$, we see that it can be divided into two triangles by drawing either diagonal. Since we have learned how to find the area of a triangle, we can see at once a simple method of finding the area of a trapezoid.



Suppose that the bases of the trapezoid are b and b' and that the height is h. What is then the base of $\triangle ABC$? the height of $\triangle ABC$? the area of $\triangle ABC$?

What is the base of $\triangle ACD$?

What is the height of $\triangle ACD$?

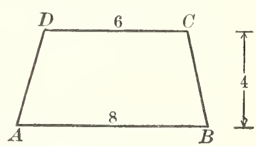
Evidently any side may be taken as base of $\triangle ACD$, but since we know only the side b', and since h is the distance from A to the line through C and D, we find it desirable to take b' as the base, even though $\triangle ACD$ is below it.

What is the area of $\triangle ACD$?

Since you now have the area of $\triangle ABC$ and the area of $\triangle ACD$, how can you find the area of the trapezoid?

Let us now apply this method to finding the area of a trapezoid whose bases are 8 in. and 6 in. and whose height is 4 in.

Suppose the diagonal AC to be drawn dividing the trapezoid into the two triangles ABC and ACD.



What is the area of $\triangle ABC$?

What is the area of $\triangle ACD$?

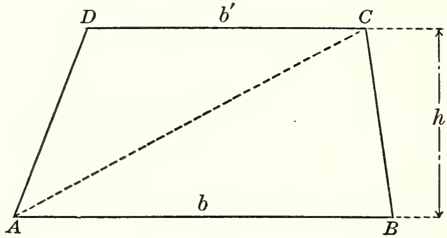
What is the sum of these areas; that is, what is the area of the trapezoid?

Write a statement of the method discovered above for finding the area of a trapezoid.

Write a formula for the area of a trapezoid with bases b and b' and with height h.

Theorem. Area of a Trapezoid

130. *The area of a trapezoid is equal to half the product of the sum of its bases by its height.*



Given the trapezoid $ABCD$, with bases b and b' and height h.

To prove that the area of $ABCD = \frac{1}{2}h(b + b')$.

Proof. Draw the diagonal AC.

Then the area of $\triangle ABC = \frac{1}{2}hb$,

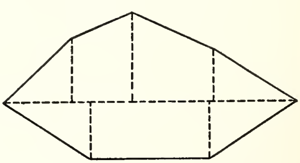
and the area of $\triangle ACD = \frac{1}{2}hb'$. § 128

$\therefore$ the area of $ABCD = \frac{1}{2}hb + \frac{1}{2}hb'$

$= \frac{1}{2}h(b + b')$. § 120

131. Corollary. *The area of a trapezoid is equal to the product of the line joining the mid-points of its nonparallel sides by its altitude.*

132. Area of an Irregular Polygon. We can find the area of an irregular polygon by dividing the polygon into triangles and finding the area of each triangle, or by using the method suggested by this figure.



Draw the longest diagonal and perpendiculars to it from the vertices. The sum of the areas of the right triangles, rectangles, and trapezoids is equal to the area of the polygon.

Exercise 31. Areas of Trapezoids

1. An excavation for a railway track is 24 ft. deep, 84 ft. wide at the top, and 68 ft. wide at the bottom. Find the area of the cross section.

2. How much leather will just cover a flat window seat, the two parallel sides being 4 ft. and $3\frac{1}{2}$ ft., and the distance between the parallel sides being 2 ft. 1 in.?

3. In the quadrilateral $ABCD$ it is given that $AB = 11$ in., $CD = 8$ in., $\angle B = 80°$, $\angle C = 100°$, and the distance from C to AB is $7\frac{1}{2}$ in. Find the area.

Use only such data as are necessary.

4. A surveyor measures a lot $ABCD$ and finds that B and C are right angles, that $AB = 120$ ft., $BC = 68$ ft. 6 in., and $CD = 74$ ft. Find the area.

Find the areas of trapezoids whose parallel sides are the first two of these numbers and heights the third numbers:

5. 3 ft., 2 ft.; 8 in.

6. $4\frac{1}{2}$ ft., $3\frac{1}{2}$ ft.; $1\frac{1}{4}$ ft.

7. 3 ft. 7 in., 2 ft.; 3 ft.

8. 6 ft., 4 ft. 2 in.; 3 ft.

Find approximate areas of these polygons by drawing the figures to scale and finding the heights of the triangles by measuring:

9. A field $ABCD$, of four sides, having $AB = 20$ rd., $BC = 30$ rd., $CD = 40$ rd., $DA = 40$ rd., and $AC = 25$ rd.

10. A field $ABCDE$, of five sides, having $AB = 30$ rd., $BC = 40$ rd., $CD = 50$ rd., $DE = 80$ rd., $EA = AB + CD$, $AC = 60$ rd., and $AD = 90$ rd.

11. A field $ABCDEF$, of six sides, having $AB = 60$ rd., $BC = DE = 50$ rd., $CD = FA = \frac{1}{2}BD = \frac{1}{2}CE = 40$ rd., and $CA = BF - 20 = 70$ rd.

Exercise 32. Review of Areas

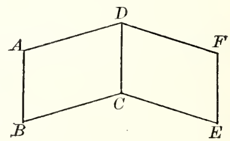
1. Find the side of a square whose area is equal to the area of a triangle of base 24 in. and height 12 in.

2. Find the height of a parallelogram of base 38.2 in. and area 267.4 sq. in.

3. Find the height of a trapezoid whose bases are 16 in. and 12 in. respectively and whose area is 154 sq. in.

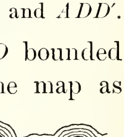
4. The height of a trapezoid is 10 in., one base is 20 in., and the area is 170 sq. in. Find the other base.

5. In this figure, given that the parallelograms *BCDA* and *CEFD* are equivalent, prove that the triangle formed by joining *F* to *A* and *B* is equivalent to either parallelogram.



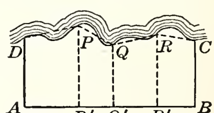
6. In the same figure draw *AC* and *FC*. Then prove quadrilateral *ACFD* equivalent to either parallelogram.

7. In surveying the field *ABCD* the surveyor takes a north and south line through *A*, draws the perpendiculars BB', CC', and DD' as here shown, and then finds that $BB'=19$ rd., $CC'=17.5$ rd., $DD'=7$ rd., $B'A=14$ rd., $B'C'=21$ rd., and $AD'=13$ rd. Find the area of the field.

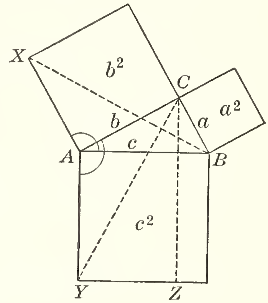


From the sum of the areas of trapezoids $B'BCC'$ and $C'CDD'$ subtract the sum of the areas of △ $B'BA$ and ADD'.

8. Wishing to find the area of a field *ABCD* bounded on one side by a river, a surveyor plotted the map as here shown. He found that $AP'=13$ rd., $P'Q'=9$ rd., $Q'R'=11.5$ rd., $R'B=9$ rd., $AD=17.5$ rd., $PP'=21$ rd., $QQ'=16$ rd., $RR'=19$ rd., and $BC=17.5$ rd. Find the approximate area of the field.



133. Pythagorean Theorem. You have learned from arithmetic or from Book I that the square on the hypotenuse of a right triangle is equivalent to the sum of the squares on the other two sides, but the fact was not then proved. We shall now prove the statement true for all kinds of right triangles.



In this figure can you see any reason for believing that $\triangle AYC$ is congruent to $\triangle ABX$?

Study the figure carefully and make certain that the reason for this statement as to congruence is clear.

Can you see any reason for believing that $\triangle AYC$ is equivalent to half the rectangle AZ? State the reason.

Can you see any reason for believing that $\triangle ABX$ is equivalent to half of b^2? State the reason.

Then do you see any reason for believing that b^2 is equivalent to rectangle AZ? State the reason.

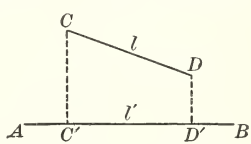
How would you show that $a^2 = BZ$?

It is not necessary to show this at present, the proof being given only roughly in outline. Simply state how you would proceed.

If $a^2 = BZ$ and $b^2 = AZ$, to what square is $a^2 + b^2$ equal?

The first proof of this theorem is attributed to Pythagoras (about 525 B.C.), although the truth of the proposition was known earlier.

134. Projection. If from the ends of a line segment perpendiculars to another line are drawn, the segment cut off on the latter is called the *projection* of the first line upon the second.



Here l' is the projection of l upon AB and in § 133 YZ is the projection of AC upon the base of BY.

THEOREM. PYTHAGOREAN THEOREM

135. *The square on the hypotenuse of a right triangle is equivalent to the sum of the squares on the sides.*

Given the right triangle *ABC* with sides *a* and *b* and hypotenuse *c*.

To prove that $c^2 = a^2 + b^2$.

Proof. In the figure suppose CZ drawn parallel to AY.

Draw BX and CY.

Since p and r are rt. $\angle$s,

$\angle BCW$ is a st. $\angle$, § 21

and line BCW is a straight line. § 20

Similarly, line ACV is a straight line.

Also $AY = AB$ and $AC = AX$. § 34

Furthermore, $\angle YAC = \angle BAX$.

(*For each is the sum of a rt.* $\angle$ *and the* $\angle BAC$.)

Therefore $\triangle$s AYC and ABX are congruent. § 58

Also $\triangle ABX = \frac{1}{2} b^2$, §§ 128, 118

(*For they have the same base* AX *and equal heights* AC.)

and similarly, $\triangle AYC = \frac{1}{2} AZ$.

Therefore $b^2 = AZ$.

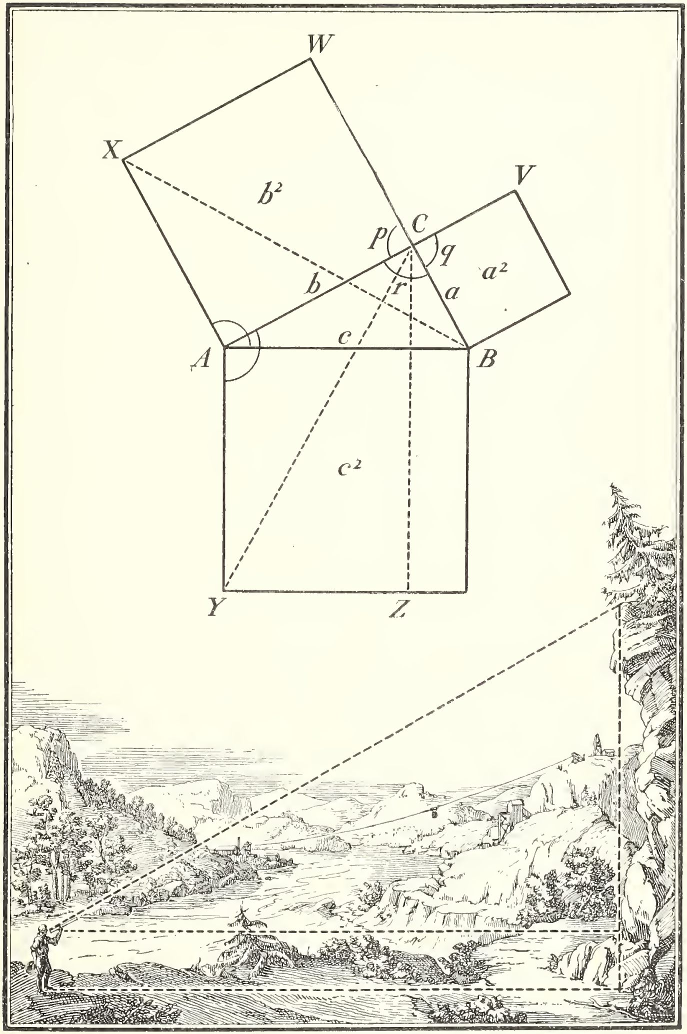
(*For their halves are equal* $\triangle$s.)

Similarly, $a^2 = BZ$.

Therefore $AZ + BZ = a^2 + b^2$, Ax. 1

or $c^2 = a^2 + b^2$.

136. Application. In the lower figure on the opposite page the man's eye is 5 ft. from the ground and 48 ft. from the cliff, and the cliff is 33 ft. high. How can you find the distance of the man's eye from the top of the cliff?


W
X
V
b²
C
p
q
a²
b
r
a
c
A
B
c²
Y
Z

Exercise 33. Pythagorean Theorem

1. A ladder 55 ft. long is placed against a wall, the foot of the ladder being 33 ft. from the base of the wall. How far up the wall does the ladder extend?

Carry approximate square roots to two decimals and use the table on page 280 for square roots of whole numbers from 1 to 100.

2. A schoolroom is 40 ft. long and 30 ft. wide. Find the length of a diagonal of the floor.

3. Find the height of an isosceles triangle whose base is 8 in. and whose other sides are each 15 in.

4. A man walking directly east at the rate of 4 mi. an hour leaves a station just as a train passes north at the rate of 40 mi. an hour. If their respective rates and directions are maintained, how far apart are they in 15 min.?

Given the sides of a right triangle as follows, find the length of the hypotenuse:

5. 20 ft., 30 ft.

6. 15 ft., 19 ft.

7. 18 yd., 23 yd.

8. 1.7 in., 2.3 in.

9. 6 mi., $4\frac{1}{2}$ mi.

10. 1 ft. 6 in., 2 ft.

11. 3 ft., 4 ft. 2 in.

12. 32 in., 1 yd. 2 in.

Given the hypotenuse and one side of a right triangle as follows, find the length of the other side:

13. 35 in., 28 in.

14. 40 ft., 30 ft.

15. 25 ft., 15 ft.

16. 2.3 in., 0.9 in.

17. 2 ft. 3 in., 1 ft.

18. 4 ft., 2 ft. 3 in.

19. A baseball diamond is a square 90 ft. on a side. Find to the nearest tenth of a foot the distance from the home plate to second base.

20. Show that the three sides of a right triangle may be represented by $2n$, n^2-1, and n^2+1.

We here assume that a triangle is a right triangle if the sum of the squares on two sides is equivalent to the square on the third side. This is a converse proposition that can easily be proved, but since it is not needed in subsequent propositions we may assume its truth for the purposes of these exercises.

21. In Ex. 20 give to n four values and thus find in integers the sides of four different right triangles.

22. Show that the three sides of a right triangle may be represented by p^2+q^2, p^2-q^2, and $2pq$.

23. From Ex. 22 find in integers the sides of four different right triangles.

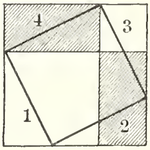
24. A room is 25 ft. long, 20 ft. wide, and 10 ft. high. How far is it from an upper corner diagonally through the room to the opposite lower corner?

First make a sketch; then find a diagonal of the floor, and find the hypotenuse which is the required diagonal.

25. As in Ex. 24, find the length of the diagonal of a cube whose volume is 27 cu. in.

26. In the quadrilateral $ABCD$ the angles A and B are right angles and $AD = 6$ in., $AB = 2$ in., $BC = 8$ in. Find the length of CD.

27. In the quadrilateral $ABCD$ the angles A and B are right angles and $AB = 10$ in., $BC = 30$ in., and $CD = 26$ in. Find the two possible lengths of AD.



28. Prove the Pythagorean Theorem by the aid of this figure.

By taking away the four numbered triangles there is left the square on the hypotenuse. By taking away the two shaded rectangles, each being twice one of the numbered triangles, there is left the sum of the squares.

Show that triangles with sides as follows are right triangles:

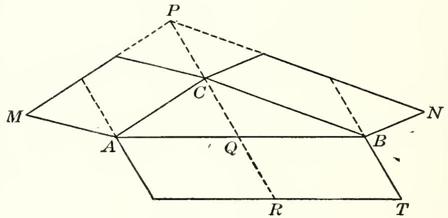
29. $\sqrt{2}$, 1, 1. **31.** $\sqrt{5}$, 2, 1. **33.** x^2+1, $2x$, x^2-1.

30. $\frac{\sqrt{12}}{2}$, 2, 1. **32.** $\frac{\sqrt{3}}{2}$, 1, $\frac{1}{2}$. **34.** $x-\frac{1}{x}$, $x+\frac{1}{x}$, 2.

35. A parallelogram $ABXY$ and a rectangle $ABCD$ stand on the same base AB, and Y is the mid-point of CD. If $AB=7$ in. and $BC=3$ in., what is the length of BX?

36. In Ex. 35 find the length of the diagonal AX.

37. Upon any two sides AC and BC of any triangle ABC two parallelograms CM and CN are constructed. Two sides of these parallelograms are produced to meet at P as here shown. PC is produced so that $QR=PC$, and the parallelogram AT is constructed with BT equal to and parallel to QR. Prove that $CM+CN=AT$.



This interesting generalization of the Pythagorean Theorem is due to a Greek geometer, Pappus, about A.D. 300. It is not difficult to derive the Pythagorean Theorem from it by starting with a right triangle and by making CM and CN squares.

38. A lawyer wishes to corroborate the testimony of a witness C. He can corroborate part of the testimony by A if he can prove that A was at X at a certain time. He can corroborate the rest of the testimony by B if he can prove that B made a certain statement to Y. What should he now attempt to do in order to complete his corroboration of C's testimony?

Notice that the argument is substantially like that used in § 135.

VII. PROBLEMS OF CONSTRUCTION

137. Problems of Construction. In Book I, and also on pages 140–147 of this book, directions were given for making simple constructions. It was not proved at either time that these constructions were correct, because no theorems had been studied on which proofs could be based. It is now purposed to review these constructions, to prove that they are correct, and to apply the methods employed to the solution of other problems.

138. Nature of a Solution. A solution of a problem has one requirement that a proof of a theorem does not have.

In a theorem we have three general steps: (1) *Given*, (2) *To prove*, (3) *Proof*.

In a problem we have four steps: (1) *Given*, (2) *Required* (to do some definite thing), (3) *Construction* (showing how to do it), (4) *Proof* (that the construction is correct).

We *prove* a theorem, but we *solve* a problem, and then prove that our solution is a correct one.

In the figures of this text the given lines are shown as full, black lines; construction lines are shown as dotted lines.

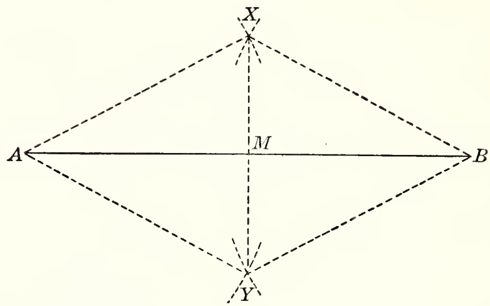
139. Discussion of a Problem. Besides the four necessary general steps in treating a problem, there is another desirable step to be taken in many cases. This step is the discussion of the problem, in which is considered whether there is more than one solution, under what conditions the construction is impossible, and other similar questions.

In like manner we may have the discussion of a theorem.

It often happens that a full statement of all the exceptional cases of a proposition would be too long to be readily understood. The discussion of the proposition allows all these cases to be brought up and explained.

Problem. Bisecting a Line

140. *Bisect a given line.*



Given the line *AB*.

Required to bisect AB.

Construction. With A and B as centers and radius greater than $\frac{1}{2}AB$, draw arcs.

Call the points of intersection of the arcs X and Y, and draw XY.

Then XY bisects AB.

Proof. Draw AX, BX, AY, and BY.

Let the intersection of AB and XY be called M.

Then △ AYX and BYX are congruent, § 72

(*Two △ are congruent if the three sides of one are equal respectively to the three sides of the other.*)

and hence $\angle AXY = \angle BXY$. § 54

Hence △ AXM and BXM are congruent, § 58

(*Two △ are congruent if two sides and the included ∠ of one are equal respectively to two sides and the included ∠ of the other.*)

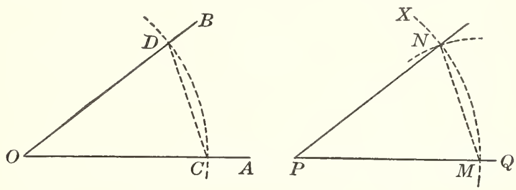
and $AM = BM$. § 54

That is, AB is bisected at M. § 14

This proves the accuracy of the construction given in § 15.

Problem. Equal Angles

141. *From a given point in a given line, construct a line making an angle equal to a given angle.*



Given the angle AOB, the line PQ, and the point P in PQ.

Required from P to construct a line making with the line PQ an angle equal to $\angle AOB$.

Construction. With O as center and any radius draw an arc cutting OA at C and OB at D, and draw CD.

With P as center and OC as radius draw an arc cutting PQ at M.

With M as center and CD as radius draw an arc cutting at N the arc which was just drawn.

Draw PN.

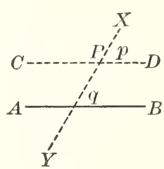
Then $\angle QPN = \angle AOB$.

Proof. Draw MN.

Then $\triangle PMN$ and OCD are congruent, § 72

and hence $\angle QPN = \angle AOB$. § 54

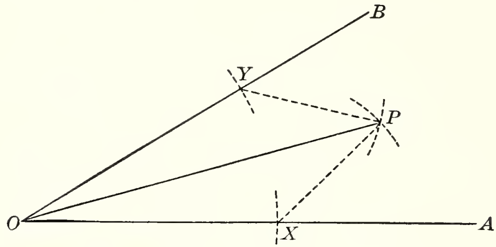
This proves the accuracy of the construction given in § 19.



In particular we can now draw a line CD through a point P parallel to a given line AB, since, in this figure, we can draw any line XY through P and cutting AB, and can then make $\angle p$ equal to $\angle q$.

Problem. Bisecting an Angle

142. *Bisect a given angle.*



Given the angle *AOB*.

Required to bisect $\angle AOB$.

Construction. With O as center and any convenient radius draw arcs intersecting OA at X and OB at Y.

With X and Y as centers and radius greater than half the distance from X to Y draw arcs.

Call the point of intersection of these arcs P.

Draw OP. Post. 1

Then OP bisects $\angle AOB$.

Proof. Draw PX and PY.

Then $\triangle OXP$ and OYP are congruent, § 72

and hence $\angle AOP = \angle POB$. § 54

Therefore, by definition of the bisection of an angle,

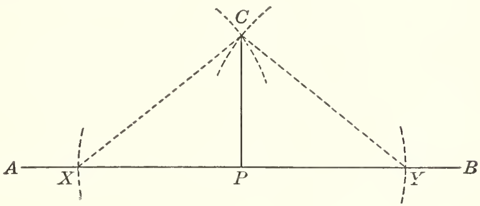
$\angle AOB$ has been bisected. § 23

This proves the accuracy of the construction given in § 23, an accuracy that was inferred when § 23 was studied but that is now proved beyond all question.

In particular we can now bisect a straight angle, which would mean constructing a line perpendicular to a given line, but this is more conveniently done by the method of § 143.

Problem. Perpendicular at a Point on a Line

143. *Construct a perpendicular to a given line at a given point in the line.*



Given the point *P* in the line *AB*.

Required to construct a perpendicular to AB at P.

Construction. With P as center and any convenient radius draw arcs cutting AB.

Call the points of intersection X and Y.

With X as center and radius greater than half XY draw an arc and with Y as center and the same radius draw another arc. Call one intersection of the arcs C.

Draw CP.

Then CP is $\perp$ to AB.

Proof. Draw CX and CY.

Then $\triangle XPC$ and YPC are congruent, § 72

and hence $\angle CPX = \angle CPY$. § 54

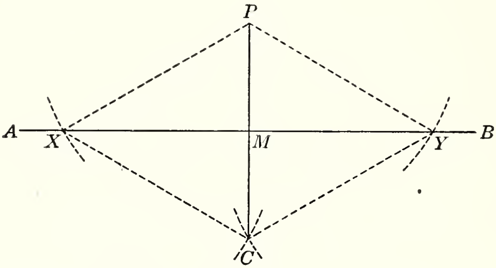
Hence CP is $\perp$ to AB. § 22

In this case the point P was not at the end of the line AB. If it is at the end of the line, we may produce the line in ordinary cases and proceed as above.

Suppose, however, that AB is the edge of a beam between two walls, so that it cannot be produced. We may then construct a perpendicular at some other point and construct a parallel to this perpendicular through the point required.

PROBLEM. PERPENDICULAR FROM A POINT OUTSIDE

144. *Construct a perpendicular to a given line through a given point outside the line.*



Given the line *AB* and the point *P* outside the line.

Required through P to construct a line perpendicular to AB.

Construction. With P as center and any convenient radius draw arcs cutting AB.

Call the points of intersection X and Y.

With X and Y as centers and any convenient radius draw arcs which shall intersect.

Call one point of intersection C.

Draw PC.

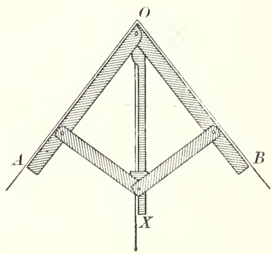
Then PC is the required perpendicular.

Proof. Draw PX, PY, CX, and CY.

Then	△ PCX and PCY are congruent,	§ 72
and hence	$\angle XPC = \angle YPC$.	§ 54
Hence	△ PXM and PYM are congruent,	§ 58
and hence	$\angle PMX = \angle PMY$.	§ 54
Therefore	PM is $\perp$ to AB.	§ 22

Exercise 34. Problems of Construction

1. Four forts are so situated that B is 16 mi. west of A, C is 12 mi. north of A, and D is 12 mi. south of A. Construct the plan to the scale of 1 in. to 4 mi., prove that B is equidistant from C and D, and find the distance BC.



2. Show how this instrument can be used to bisect any given angle, as $\angle AOB$.

On account of the joints and other mechanical features of such an instrument this method of bisection is not nearly as accurate as the one in § 142.

3. Construct the bisector of the vertical angle of an isosceles triangle and prove that it is the perpendicular bisector of the base.

Using only ruler and compasses, construct these angles:

4. 45°. **6.** 30°. **8.** 150°. **10.** 105°. **12.** 75°.

5. 60°. **7.** 15°. **9.** 120°. **11.** 135°. **13.** 22° 30′.

If the student checks the constructions in Exs. 4–13 with a protractor, he will see that the geometric method of ruler and compasses is much more nearly accurate than that with the protractor alone.

Part of this work is in the nature of review, as in Exs. 4–7, these cases being very important.

14. Construct a parallelogram such that one of the four angles is one third of a right angle.

All such constructions are to be made with ruler and compasses alone unless, as in Ex. 15, a certain measure of length is required.

15. Construct a square whose diagonal is 1 in.

16. Construct a rectangle such that one diagonal is 2 in. and one side is 1 in.

17. Draw any triangle and construct the bisectors of the three angles. Write your inference as to the way in which the three bisectors intersect.

If the figure is carefully drawn, a correct inference will probably be made. The proof of this inference may be attempted after a study of loci if desired, and similarly for Exs. 18–22.

18. Draw any acute triangle and construct the perpendicular bisectors of the three sides. Write your inference as in Ex. 17.

19. Consider Ex. 18 for a right triangle and also for an obtuse triangle.

20. Draw any acute triangle and construct the perpendiculars from the three vertices on the opposite sides. Write your inference as in Ex. 17.

21. Consider Ex. 20 for a right triangle and also for an obtuse triangle.

22. Draw any triangle and from the three vertices construct lines bisecting the opposite sides. Write your inference as in Ex. 17.

23. Given a line equal to the perimeter of a square, construct the square.

24. Given two of the angles of a triangle, construct the triangle. Discuss the question whether more than one such triangle can be constructed and give the reason.

25. Given two sides and the included angle of a parallelogram, construct the parallelogram.

26. Make a list of all the theorems given and proved in full in this book, stating under each the preceding theorems used in its proof.

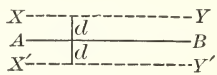
In this way the dependence of one theorem on another is clearly seen, and the significance of a logical sequence becomes apparent.

VIII. LOCI

145. Locus. In the geometry in Book I we considered three questions relating to an object: What is its shape? How large is it? Where is it? We shall now consider more fully the geometry involved in this third question.

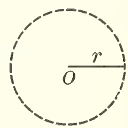
The path of a point that moves according to certain given geometric conditions is called the *locus* of the point.

Thus, considering for the present only figures in a plane, a point at a given distance from a given line of indefinite length is evidently in one of two lines parallel to the given line and at the given distance from it. Thus, if AB is the given line and d the given distance, the locus is evidently the pair of parallel lines XY and $X'Y'$.



The locus of a point in a plane at a given distance r from a given point O is evidently the circle described with center O and radius r.

The plural of *locus* (a Latin word meaning "place") is *loci* (usually pronounced lō-sī).



We may think of the locus as the *place* of all points that satisfy certain given geometric conditions, and speak of the locus of points. Both expressions, *locus of a point* and *locus of points*, are used.

Exercise 35. Loci

State without proof the loci of these points in a plane:

1. A point 2 in. from the right-hand edge of this page.

2. A point 1 in. from this dot (·).

3. A point at the center of the hub of an automobile wheel of diameter 38 in., moving straight on a level road.

4. A point that moves so as always to be equidistant from the lower edge and the right-hand edge of this page.

5. A point equidistant from two given parallel lines.

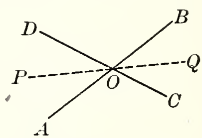
6. The lowest point of the pendulum of a clock.

146. Proof of a Locus. To prove that a certain line or group of lines is the locus of a point that fulfills a given condition, it is necessary and sufficient to prove two things:

1. *That any point in the supposed locus satisfies the given condition.*

2. *That any point outside the supposed locus does not satisfy the given condition.*

For example, if we wish to find the locus of a point equidistant from the intersecting lines AB and CD, shown in the figure, it is not sufficient to prove that any point on the angle-bisector PQ is equidistant from AB and CD, because the line PQ may be only part of the locus. It is necessary to prove that no point outside of PQ satisfies the condition. In fact, in this case there is another line in the locus, the bisector of the $\angle BOD$, as will be shown in § 148.



Exercise 36. Drawing Loci

Draw the loci of these points in a plane, giving no proofs:

1. A point $\frac{1}{2}$ in. from the vertex of a given isosceles triangle.

2. A point $\frac{1}{2}$ in. below the base of the triangle in Ex. 1.

3. A point $\frac{1}{2}$ in. from the base of the triangle in Ex. 1.

4. A point $\frac{1}{2}$ in. from either of the two equal sides of the triangle in Ex. 1.

5. A point $\frac{1}{2}$ in. within the circle described with radius 1 in. about a given point O.

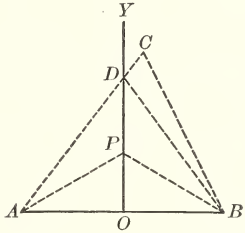
6. A point $\frac{1}{2}$ in. from the circle in Ex. 5.

7. A point less than 1 in. from a given point.

From Ex. 7 we see that a locus need not always be a line or a group of lines. In this case the locus is a portion of the plane.

Theorem. Points Equidistant from Points

147. *The locus of a point equidistant from the extremities of a given line is the perpendicular bisector of the line.*



Given YO, the perpendicular bisector of the line AB.

To prove that YO is the locus of a point equidistant from A and B.

Proof. Let P be any point in YO; C any point not in YO.

Draw the lines PA, PB, CA, and CB.

Now $AO = BO$, Given

$OP = OP$,

and $\angle AOP = \angle BOP$. § 21

Therefore △ AOP and BOP are congruent, § 58

and hence $PA = PB$. § 54

Let CA cut the perpendicular at D, and draw DB.

Then, as above, $DA = DB$.

But $CB < CD + DB$. Post. 2

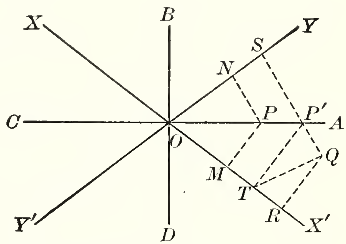
Therefore $CB < CD + DA$,

or $CB < CA$.

Therefore YO is the required locus. § 146

Theorem. Points Equidistant from Lines

148. *The locus of a point equidistant from two given intersecting lines is the pair of lines bisecting the angles formed by the lines.*



Given XX' and YY' intersecting at O, AC the bisector of angle $X'OY$, and BD the bisector of angle YOX.

To prove that the pair of lines AC and BD is the locus of a point equidistant from XX' and YY'.

Proof. Suppose that P is any point on AC or BD, and that Q is any point not on either AC or BD.

Construct ⊥s PM and QR from P and Q upon XX', and ⊥s PN and QS from P and Q upon YY'. § 144

Now $\angle MOP = \angle NOP$, Given

and $OP = OP$.

Also ∠s M and N are rt. ∠s.

(*For PM is ⊥ to XX' and PN is ⊥ to YY', by construction.*)

Hence △s OMP and ONP are congruent, § 74

(*For we have the case of the hypotenuse and an acute* ∠.)

and hence $PM = PN$. § 54

Therefore every point on AC is on the locus. Similarly every point on BD is on the locus.

Since the point Q is so taken that it is not on AC, it must lie either above AC or below AC, and hence either QR or QS must cut AC.

Suppose that QS cuts AC at P'.

Draw $P'T \perp$ to XX', and draw QT. § 144

Then, as before, $P'T = P'S$.

But $P'T + P'Q > QT$, Post. 2

and $QT > QR$. Post. 8

Therefore $P'T + P'Q > QR$.

(For, if $P'T + P'Q > QT$ and $QT > QR$, so much the more is $P'T + P'Q > QR$.)

In this inequality, substituting $P'S$ for its equal, $P'T$,

we have $P'S + P'Q > QR$, or $QS > QR$.

Hence any point not on AC or BD is not on the locus.

Therefore the pair of lines AC and BD is the locus. § 146

(For it has been proved that all points on AC and BD, and no other points, are equidistant from XX' and YY'.)

149. Method of Finding a Locus. To find the locus of a point, first find several possible positions of the point, so as to fix in mind the nature of the locus.

If the locus appears to be a circle, try to prove that the point is always at the same distance from a fixed point; if it appears to be a straight line, or two or more straight lines, try to prove that the point is always equidistant from two fixed points or two fixed lines, or that it is always at the same distance from a fixed line, or that the line joining this point to some fixed point always makes the same angle with some fixed line.

If a particular point is required, try to show that it lies on each of two intersecting lines, in which case it is the point of intersection.

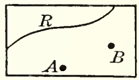
If the locus of a point is required, the distance of this point from a fixed line being less than $\frac{1}{4}$ in., the locus is evidently a strip of the plane bounded by lines $\frac{1}{4}$ in. on either side of the fixed line.

Exercise 37. Loci

1. *A*, *B*, and *C* are three villages located as here shown. It is decided to establish a union school for these three villages, locating the building so as to be equidistant from them. Show by construction the location of the building.

In a practical case of course the distance along established roads would have to be considered, and similarly for Ex. 2.

2. A map is here shown, *R* representing a railway and *A* and *B* two villages. It is proposed to build a station *S* on the railway at a point equidistant from *A* and *B*. Construct the position of *S*.



3. Find the locus of the vertices of all triangles having a common base and the same area.

In every such case limit the locus to one plane and give the proof.

4. Find the locus of points less than $1\frac{1}{2}$ in. from a given point.

In such a case the locus is part of the plane.

5. Find the locus of points $\frac{1}{2}$ in. from a given line passing through a given point and less than 1 in. from that point.

6. Find the locus of points within a given inch square and more than $\frac{1}{4}$ in. from the intersection of the diagonals.

7. Given the triangle *ABC*, find a point that is equidistant from *AB* and *BC* and also equidistant from *AB* and *AC*. If it is equidistant from all three sides, where does it lie with respect to the bisector of $\angle C$?

8. Given the triangle *ABC*, find a point that is equidistant from *A* and *B* and also equidistant from *A* and *C*. If it is equidistant from all three points, where does it lie with respect to the perpendicular bisector of *BC*?

IX. REVIEW

150. General Suggestions. Having now proved some of the most important theorems upon which the science of geometry rests, and having solved some of the basal problems of construction, we shall review the work accomplished, doing this by means of original exercises involving the propositions which have been studied.

As soon as it could profitably be done we gave a number of directions (§ 77) for attacking an original exercise, but the time has now come for adding further directions based upon the leading propositions which have been given. This is particularly necessary at this time because the exercises no longer follow some particular theorem or problem, and hence there is no hint as to the propositions by which they can probably be proved or solved. Part of the interest of the student in geometry, however, comes from the necessity for discovering the way for himself.

The student should first review § 77 and then consider the following additional suggestions:

1. *If two lines are to be proved equal, try to prove them corresponding sides of congruent triangles, or sides of an isosceles triangle, or opposite sides of a parallelogram, or segments between parallels that cut equal segments from a transversal.*

2. *If two angles are to be proved equal, try to prove them alternate angles of parallel lines, or corresponding angles of congruent triangles, or base angles of an isosceles triangle, or opposite angles of a parallelogram.*

3. *If no other method seems to promise good results, assume that the theorem is false and then show that this assumption leads to an absurdity.*

This is called the *indirect method* or the *reductio ad absurdum*. It will be illustrated in the exercises on page 276.

Exercise 38. Equal Lines

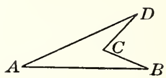
Prove the following theorems relating to equal lines:

1. Given any point P on the bisector of an angle AOB, OA being equal to OB, prove that $BP = AP$.

2. In Ex. 1 prove that the perpendiculars to the line OP from the points A and B are equal.

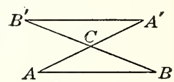
3. In Ex. 1 prove that the perpendiculars from the point P to the lines OA and OB are equal.

4. In this figure $AB = AD$ and $BC = DC$. Prove that C lies on the bisector of $\angle A$.



5. In Ex. 4 prove that A lies on the bisector of $\angle BCD$.

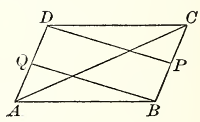
6. In this figure, AC and BC have been produced to A' and B' respectively so that $CA' = AC$ and $CB' = BC$. State four other pairs of equals and prove the statements.



7. If $AB = AC$ in the $\triangle ABC$, and on AB and AC there are laid off from A the equal segments AM and AN respectively, prove that $BN = CM$.

8. In a parallelogram $ABCD$ perpendiculars are drawn from A and C upon the diagonal BD. Find three pairs of congruent triangles and prove the congruence in each case.

9. In a parallelogram $ABCD$, if BQ bisects AD and DP bisects BC, then BQ and DP divide AC into three equal segments.



10. The perpendiculars from the three vertices of an equilateral triangle to the opposite sides are equal.

11. From the vertex of the right angle of a right triangle a line is drawn to the mid-point of the hypotenuse. Prove that this line is equal to half the hypotenuse.

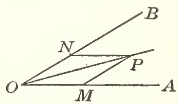
Exercise 39. Equal Angles

Prove the following theorems relating to equal angles:

1. A line drawn parallel to any side of an equilateral triangle makes an equilateral triangle with the other sides, or with those sides produced.

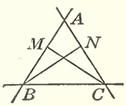
2. The bisector of the exterior angle at the vertex of an isosceles triangle is parallel to the base.

3. In this figure, OP is the bisector of $\angle AOB$, PM is parallel to BO, and PN to AO. Prove that OP bisects $\angle NPM$.



4. Lines drawn perpendicular to the bisector of an angle make equal angles with the arms of the angle.

5. In this figure, $AB = AC$, and $AM = AN$. Name all the pairs of equal angles in the figure and prove your statements.



6. If a line drawn perpendicular to BC, the base of an isosceles triangle ABC, cuts AB at P, and cuts CA produced at Q, prove that $\angle APQ = \angle PQA$.

7. The bisectors of any two angles of an equilateral triangle form at their point of intersection an angle equal to any exterior angle of the triangle.

8. The bisectors of the equal angles of an isosceles triangle form with the base an isosceles triangle.

9. The bisectors of two adjacent angles of a rectangle are perpendicular to each other.

10. Two lines perpendicular respectively to the two arms of a given angle form an angle which is either equal to or supplementary to the given angle. Consider the two cases, prove each, and state in which case the two angles are equal and in which case they are supplementary.

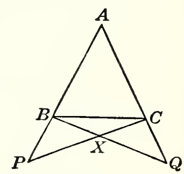
Exercise 40. Congruent Triangles

Prove the following theorems by showing that two triangles are congruent:

1. A perpendicular to the bisector of an angle forms with the arms an isosceles triangle.

2. If two lines bisect each other at right angles, any point in either is equidistant from the ends of the other.

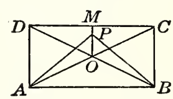
3. If the equal sides AB and AC of an isosceles triangle are produced to P and Q respectively so as to make $AP = AQ$, then $BQ = CP$.



4. In Ex. 3 state two pairs of equalities of angles and prove the two statements.

5. In Ex. 4 prove that $PX = QX$.

6. If M is the mid-point of the side CD of a square $ABCD$, the $\triangle ABM$ is isosceles.



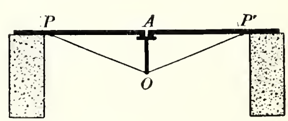
7. If M is the mid-point of the side CD of the rectangle $ABCD$, then $AP = BP$.

8. In Ex. 7 show that MO produced bisects the side AB.

9. If either diagonal of a parallelogram bisects one of the angles, the sides of the parallelogram are all equal.

10. The diagonals of a square are perpendicular to each other and bisect the angles of the square.

11. In bridge building the simple construction here shown is occasionally used. The beams AP and AP' rest loosely on the perpendicular plate of the support OA. The rods OP and OP' are fastened at O, P, and P'. Show by means of the congruence of certain triangles that the bridge will support a weight at A.



Exercise 41. Parallels and Angle Sums

Prove the following theorems relating to parallels or to angle sums:

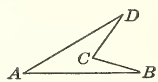
1. The bisectors of any two angles of a parallelogram are either perpendicular to or parallel to each other.

The discussion should show that, in special cases, the bisectors coincide. Theorems are usually stated in a general form like this theorem, and it is left to the student to consider the special cases.

2. The line joining the mid-points of the equal sides of an isosceles triangle is parallel to the base.

3. From any point P on the side AB of a pentagon $ABCDE$ draw PC, PD, PE and prove that the sum of the interior angles of the pentagon is three straight angles.

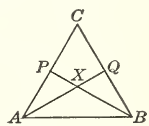
4. In the figure here shown prove that $\angle A + \angle B + \angle D = \angle BCD$.



5. Discuss Ex. 4 when $\angle BCD$ is a straight angle; when $\angle BCD$ is greater than a straight angle.

6. Find the sum of the angles of the figure in Ex. 4.

7. In the figure here shown, ABC is an equilateral triangle and P and Q are the mid-points of AC and BC respectively. Prove that $\angle AXB$ is the supplement of $\angle C$.



8. The opposite angles of the quadrilateral formed by the bisectors of the interior angles of any quadrilateral are supplementary.

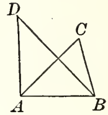
9. Find the sum of the angles in a figure of 17 sides.

10. What is the number of degrees in each angle of a regular polygon of 24 sides?

11. If each angle of a regular polygon is 156°, how many sides has the polygon?

Exercise 42. Indirect Method of Proof

1. Given ABC and ABD, two triangles on the same base AB, and on the same side of it, the vertex of each triangle being outside the other. If AC is equal to AD, prove that BC cannot be equal to BD.

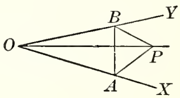


Assume that $BC = BD$ and show that the result is absurd, since the equality of BC and AD would make D fall on C, which is contrary to the given conditions.

If the result is absurd, the assumption that $BC = BD$ must be false. But if BC cannot be equal to BD, then the theorem is proved.

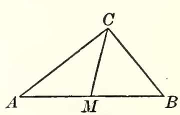
This illustrates the indirect method referred to in § 150.

2. On the sides of the angle XOY two equal segments OA and OB are taken. On AB a triangle APB is constructed with AP greater than BP. Prove that OP cannot bisect the angle XOY.



Assume that OP does bisect $\angle XOY$. What is the result? Is this result possible?

3. From M, the mid-point of a line AB, MC is drawn oblique to AB. Prove that CA cannot be equal to CB.



Assume that CA does equal CB. What is the result? Is this result possible?

4. If perpendiculars are drawn to the sides of an acute angle from a point within the angle, they cannot inclose a right angle or an acute angle.

Assume that they inclose a right angle and show that this leads to an absurdity. Similarly for an acute angle.

5. One of the equal angles of an isosceles triangle is five ninths of a right angle. Prove that the angle at the vertex cannot be a right angle.

Assume that the angle at the vertex is a right angle. Is the result possible? If not, what is your conclusion?

Exercise 43. Miscellaneous Exercises

1. Write the theorems relating to parallel lines, and under each one write the theorems used in the proof.

Exs. 1–3 refer to the basal theorems proved in the book.

2. Write the theorems relating to congruent triangles, and under each write the theorems used in the proof.

3. Write the theorems relating to equal angles, and under each write the theorems used in the proof.

Write a list of the more important special properties of each of the following figures:

4. Isosceles triangle.

5. Equilateral triangle.

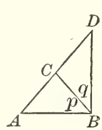
6. Right triangle.

7. Parallelogram.

8. If the opposite sides of a hexagon $ABCDEF$ are equal, and if one pair of opposite sides are parallel, then the opposite angles of the hexagon are equal.

Let AB and DE be the parallel sides. Draw AE and BD.

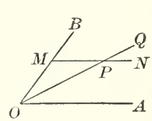
9. If one of the equal sides of an isosceles triangle is produced through the vertex by its own length, the line joining the end of the side produced to the nearer end of the base is perpendicular to the base.



$\angle DBA$ is a rt. $\angle$ if it is equal to the sum of what angles of $\triangle ABD$? It is equal to this sum if $\angle p$ is equal to what angle and $\angle q$ is equal to what other angle?

10. If the line drawn from the vertex of a triangle to the mid-point of the base is equal to half the base, the angle at the vertex is a right angle.

11. If through any point in the bisector of an angle a line is drawn parallel to either arm, the triangle thus formed is isosceles.



12. The bisectors of the angles of a triangle are concurrent in a point equidistant from the sides of the triangle.

Lines are *concurrent* when they all meet in the same point.

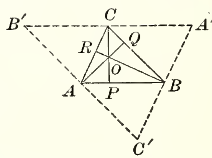
Given the bisectors of ∠s B and C meeting at O. Draw OA, and from O draw OD, OE, and $OF \perp$ to BC, CA, and AB respectively. We must prove that OA bisects $\angle A$. First prove $OE = OD$, and then $OF = OD$. What follows? Use § 148.

13. The bisectors of two exterior angles of a triangle intersect on the bisector of the third interior angle.

14. The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the vertices.

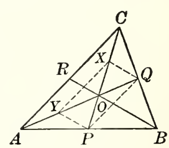
Given two of the ⊥s meeting at O. Join O to the mid-point of the third side. What must be proved about this joining line? Where does O lie with respect to the $\perp$ bisector of the third side?

15. The perpendiculars from the vertices of a triangle to the opposite sides are concurrent.



Given the ⊥s AQ, BR, and CP. Through A, B, C draw $B'C'$, $C'A'$, and $A'B'$ parallel to CB, AC, and BA respectively. Now show that $C'A = BC = AB'$. In the same way, what are the mid-points of $C'A'$ and $A'B'$? How does this prove that AQ, BR, and CP are the $\perp$ bisectors of the sides of the $\triangle A'B'C'$? Proceed as in Ex. 14.

16. The lines joining the vertices of a triangle to the mid-points of the opposite sides are concurrent in a point two thirds of the distance from each vertex to the mid-point of the opposite side.



Two such lines, as AQ and CP, meet as at O. If Y is the mid-point of AO, and X the mid-point of CO, show that YX and PQ are $\parallel$ to AC and equal to $\frac{1}{2}AC$. Then show that $AY = YO = OQ$ and that $CX = XO = OP$. Hence *any* such line cuts off on *any* other such line what fraction of the distance from the vertex to the mid-point of the opposite side?

TABLES FOR REFERENCE

Length

12 inches (in.)	= 1 foot (ft.)
3 feet	= 1 yard (yd.)
$5\frac{1}{2}$ yards, or $16\frac{1}{2}$ feet	= 1 rod (rd.)
320 rods, or 5280 feet	= 1 mile (mi.)

Square Measure

144 square inches (sq. in.)	= 1 square foot (sq. ft.)
9 square feet	= 1 square yard (sq. yd.)
$30\frac{1}{4}$ square yards	= 1 square rod (sq. rd.)
160 square rods	= 1 acre (A.)
640 acres	= 1 square mile (sq. mi.)

Equivalents

231 cu. in. = 1 gal.   2150.42 cu. in. = 1 bu.

Metric Length

10 millimeters (mm.)	= 1 centimeter (cm.)
10 centimeters	= 1 decimeter (dm.)
10 decimeters	= 1 meter (m.)
1000 meters	= 1 kilometer (km.)

Metric Weight

1000 grams (g.)	= 1 kilogram (kg.)
1000 kilograms	= 1 metric ton (t.)

Metric Capacity

100 liters (l.) = 1 hektoliter (hl.)

Powers and Roots

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.260	52	2,704	140,608	7.211	3.733
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.780
5	25	125	2.236	1.710	55	3,025	166,375	7.416	3.803
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.826
7	49	343	2.646	1.913	57	3,249	185,193	7.550	3.849
8	64	512	2.828	2.000	58	3,364	195,112	7.616	3.871
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.893
10	100	1,000	3.162	2.154	60	3,600	216,000	7.746	3.915
11	121	1,331	3.317	2.224	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.958
13	169	2,197	3.606	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.742	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.873	2.466	65	4,225	274,625	8.062	4.021
16	256	4,096	4.000	2.520	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.062
18	324	5,832	4.243	2.621	68	4,624	314,432	8.246	4.082
19	361	6,859	4.359	2.668	69	4,761	328,509	8.307	4.102
20	400	8,000	4.472	2.714	70	4,900	343,000	8.367	4.121
21	441	9,261	4.583	2.759	71	5,041	357,911	8.426	4.141
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.796	2.844	73	5,329	389,017	8.544	4.179
24	576	13,824	4.899	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.718	4.236
27	729	19,683	5.196	3.000	77	5,929	456,533	8.775	4.254
28	784	21,952	5.292	3.037	78	6,084	474,552	8.832	4.273
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.291
30	900	27,000	5.477	3.107	80	6,400	512,000	8.944	4.309
31	961	29,791	5.568	3.141	81	6,561	531,441	9.000	4.327
32	1,024	32,768	5.657	3.175	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.745	3.208	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.831	3.240	84	7,056	592,704	9.165	4.380
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.220	4.397
36	1,296	46,656	6.000	3.302	86	7,396	636,056	9.274	4.414
37	1,369	50,653	6.083	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.362	88	7,744	681,472	9.381	4.448
39	1,521	59,319	6.245	3.391	89	7,921	704,969	9.434	4.465
40	1,600	64,000	6.325	3.420	90	8,100	729,000	9.487	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.498
42	1,764	74,088	6.481	3.476	92	8,464	778,688	9.592	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.644	4.531
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.547
45	2,025	91,125	6.708	3.557	95	9,025	857,375	9.747	4.563
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.798	4.579
47	2,209	103,823	6.856	3.609	97	9,409	912,673	9.849	4.595
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.950	4.626
50	2,500	125,000	7.071	3.684	100	10,000	1,000,000	10.000	4.642

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