# A REVIEW OF HIGH-SCHOOL MATHEMATICS

REEVE *and* SCHORLING

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# A REVIEW *of* HIGH-SCHOOL MATHEMATICS

BY

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AND

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TO THE PUPILS OF THE UNIVERSITY HIGH SCHOOL

whose interest in the study of mathematics and whose co-operation in the use of the material of the subsequent pages have inspired our efforts, this book is dedicated in the hope that they may receive further information from its pages

# PREFACE

The review material of the subsequent pages is practically the same as that used for the last three years by the authors in their high-school classes in mathematics. It has been very helpful to the pupils as a means of reviewing quickly the work under any subject or at the end of each semester. The review at the end of each semester has been productive of good results in the first two and one-half years of the high-school course in mathematics as a means of gathering up the loose ends and of giving the pupils a more connected idea of the semester's work.

This material has been invaluable also as a text in a fourth-year review class composed of pupils who are preparing to take college-entrance examinations in mathematics or who are going to colleges which require recommendation in mathematics.

The first-year review is intended to cover the work ordinarily done in the first-year mathematics course of the average high school. Similarly, the second-year outline furnishes a thorough review of the work usually given in that year and incidentally covers what is required by the *Harvard Syllabus*, the *New England Syllabus*, the *Report of the Committee of Fifteen*, and any good current text. The third-year review covers the ordinary course in intermediate algebra and, with what precedes, meets college-entrance requirements.

The material is published, in the first place, so that it may be in better form for our own class use. It is believed, however, that the book furnishes abundant material for an intelligent review of the mathematics given in a large number of high schools, and it is the hope of the authors that other teachers may find the book useful in the hands of their pupils.

It is not asserted that a maximum is reached in any of the courses herein represented, but that there is at least what may be expected as a minimum. On the other hand, it is thought that there is an advantage in such a review in having very little besides the essential ideas in order that these ideas may be made predominant.

The authors have placed at the end of the book an outline of a tentative minimum course for one and one-half years of mathematics in the effort to initiate a plan for the standardization of high-school mathematics.

We shall appreciate at any time suggestions as to improvement in the subject-matter or in its arrangement.

Our thanks are due to Messrs. E. R. Breslich and H. C. Wright of the mathematics department of the University High School for many valuable suggestions and to the University Press for its interest and aid in the publication of this material.

THE AUTHORS

UNIVERSITY HIGH SCHOOL
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# FIRST-YEAR REVIEW

## I. The Equation

1. What is an equation?
2. What is its chief use?
3. When is an equation containing one unknown said to be solved?
4. Typical introductory examples:
   *a*) $3x-2=12$.
   *b*) $\frac{x}{2}+4=8$.
   *c*) $18x+x-3x=18$.
   *d*) Find three consecutive numbers whose sum is 474.
   *e*) Three men, A, B, and C, divide 1,584 shares of stock among themselves so that A shall have 25 more than B, and C shall have 50 more than B. How many shares does each get?
5. How do you check a verbal problem like *e* above?

## II. The Equation as Applied to Perimeters and Areas

1. What is the perimeter of a triangle whose three sides are $2x$, $6x$, and $7x$? Of a triangle whose sides are $2x$, $3y$, and $5z$? How many terms are there in the first result? In the second? What special names are used for each? Define a binomial; monomial; trinomial; polynomial.
2. What is the area of a rectangle whose base is $2x$ and whose altitude is $3y$? What is the product of the three following numbers: $4x$, $3y$, $2z$? What is the factor 24 in $24xyz$ called? Define "coefficient."

3. What is the area of a rectangle whose base is $2x$ and whose altitude is $x$? The area of a square each of whose sides is $3x$ units?
4. What does an exponent show? What is the meaning of $x^5$? of $2x^3$? Define "exponent."
5. What is the area of a square each of whose sides is $(c+d)$ units? Draw a figure which will show the product of $(c+d)\,(c+d)$.
6. What is the product of $(x+3)$ and $(y+5)$? Illustrate by drawing a rectangle.
7. If $p$ is used for perimeter, of what plane figures may the following equations express the perimeter:

   $p=3x$; $p=4x$; $p=2x+2x+3x$; $p=5x$; $p=6x$?

8. If $A$ is used for areas, of what figures may the following equations express the areas:

   $A=x^4$; $A=9x^2$; $A=2(x+3)$; $A=x(y+2x)$?

9. Find the area and perimeter of the figures in problems 7 and 8 if $x=4$; if $y=5$; and $a=3$.
10. If $a=3$; $b=4$; and $c=5$, find the value of the following numbers:

    $(2a+b)\,(c+3)$; $ab(c+5)$; $2c(5a-2c+b)$; $4a^2+3ac$.

11. State the four fundamental laws used in solving equations and illustrate each.
12. Typical problems:
    *a*) The width of a rectangle is 12 ft. less than its length. Its perimeter is 288 ft. Find the dimensions.
    *b*) The perimeter of a square is 60 ft. Find a side; the area.
13. If an equation contains fractions, as for example $\frac{2x}{3}-\frac{x}{2}+\frac{x}{5}=11$, how may a second equivalent equation be

obtained from the given equation which will not contain fractions?

14. Solve the following equations:

*a*) $\frac{4y}{3}-\frac{2y}{5}+\frac{y}{6}=33$

*b*) $\frac{8x}{7}+2=26$

*c*) $\frac{3(x-1)}{6}=8$

*d*) $\frac{y}{7}-\frac{y}{8}=1$

## III. The Equation as Applied to Angles

1. Show how an angle may be formed by a rotating line. Define an acute angle; obtuse angle; right angle; straight angle; reflex angle; perigon.
2. What is a practical way for measuring an angle? What is the most accurate way of measuring an angle?
3. By the use of a protractor, draw an angle of any required number of degrees, as for example $35^\circ$; $52^\circ$; $68^\circ$.
4. What is the sum of all the angles about a point on one side of a straight line? Of all the angles about a point in a plane?
5. Find the value of $x$, then each angle, and make an accurate drawing of a figure in which the angular space about a point is divided into four angles designated as follows: $x+20$, $3x$, $2x-35$, and $4x+27$.
6. Define adjacent angles.
7. Define supplementary angles:
   *a*) One of two supplementary angles is $84^\circ$ larger than the other; find the two angles.
   *b*) What is the supplement of $10^\circ$; of $45^\circ$; of $120\frac{1}{2}^\circ$; of $400^\circ$; of $A^\circ$; of $x^\circ$; of $(2b)^\circ$?
   *c*) Find $x$ and the angles of the following pairs of supplementary angles:

   (1) $\frac{2x}{3}+14,\ 82-\frac{x}{3}$ (2) $\frac{5x}{7}+29,\ 97-\frac{2x}{7}$

Draw the figures with protractor.

8. Define complementary angles.
   *a*) What is the complement of 42°; of 28°; of 1°; of 100°; of $x°$?
   *b*) If an angle is doubled, and its complement increased by 40°, the sum of the angles obtained is 160°. Find the angles.
   *c*) The sum of an angle and half its complement is 75°. Find the angle.
9. How are vertical angles formed? What relation exists between them?
   *a*) Find the value of $x$ and the two vertical angles in the following pairs:

      (1) $3x-17$; and $x+103$

      (2) $x-\frac{x}{7}$; and $\frac{3x}{4}+90$

10. What is the sum of the three angles of a triangle?
    *a*) Draw a triangle and by rotating a pencil verify the answer to the above question.
    *b*) What is an exterior angle of a triangle?
    *c*) Find the angles of a triangle if the first is $\frac{1}{4}$ of the second, and the third is $\frac{1}{7}$ of the first plus 18°.
    *d*) Find the angles of a triangle if the first is 6 times the second plus 18; and the third is $\frac{1}{2}$ the first minus 7.
    *e*) If one angle of a triangle is a right angle, what is the sum of the other two?
    *f*) The acute angles of a right triangle are $\frac{x}{2}$ and $\frac{x}{3}$. Find the angles.
11. Solve the following equations:

    *a*) $\frac{4a}{5}-\frac{2}{3}=7$

    *b*) $\frac{7t}{6}-5\frac{1}{2}=8-\frac{13t}{12}$

    *c*) $6+3y+\frac{3(5+2y)}{4}=9\frac{3}{4}$

    *d*) $L+\frac{3L}{2}=\frac{12-7L}{2}$

## IV. Positive and Negative Numbers

1. Draw the temperature curve for the following twenty days: $+8$, 0, $-10$, 5, $+7$, $+9$, 4, $-2$, $+2$, $-5$, $+15$, $+20$, 0, 0, $+5$, $-3$, $-8$, $-7$, $+1$, $+4$.

2. Illustrate positive and negative numbers by the idea of direction; interpret the meaning as applied to debit and credit; as to forces; meaning on the thermometer scale; the use children make of it in playing games.

3. Add:

| | | | |
|---|---|---|---|
| $+8$ | $+7$ | $-8$ | $-12$ |
| $-9$ | $+4$ | $+7$ | $-4$ |

State the rule for addition.

4. Subtract the lower number from the upper in the following:

| | | | | |
|---|---|---|---|---|
| $+8$ | $-7$ | $-9$ | 15 | 4 |
| $-4$ | $-12$ | $+4$ | 4 | 15 |

State the rule for subtraction.

5. Write the product of the following:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $-5$ | $+3$ | $+5$ | $-8$ | $-4x$ | $-4x^2$ | $+3x^2$ | $+5xy^2x$ |
| $+2$ | $-7$ | $+4$ | $-7$ | $-3y$ | $+3y$ | $+2x^5$ | $-3x^2yz^5$ |

How is the sign of the product determined?

6. Write the quotients:

| | |
|---|---|
| $-15 \div -3 =$ | $-21x^2 \div 7x =$ |
| $-15 \div +3 =$ | $-45x^2y^2 \div 15xy =$ |
| $+15 \div -3 =$ | $23ax^2 \div ax^2 =$ |
| $+12 \div +3 =$ | $12axz^4 \div az^3 =$ |

How is the sign of the quotient determined?

7. Graph the following equations:

$y=3x$; $y=2x-4$; $x=y+3$

## V. Fundamental Operations as Applied to Polynomials

1. Add the following numbers and simplify the sum:

   $2y^3-4y^2+4y-1$, $8y^3-y^2+3y-15$, $3y-7+11y^2-15y^3$, $4y^2+12y^3+6+y$, $11-y^3+y^2-8y$.

2. Add the following numbers and combine terms where possible:

   $\frac{2x^2}{2}-5xy+4.2y$, $7xy-2\frac{1}{3}x^2+-7y^2$, $5\frac{1}{2}y^2-7x^2+3xy$.

3. Subtract $5x^2-3x+4y+3xy+7y^2$ from $7x+5y-12x^2+4xy-5y^2$.
4. Divide $a^4-4a^3b+6a^2b^2-4ab^3+b^4$ by $a^2-2ab+b^2$.
5. Divide $4x^6-9x^4+25-14x^3-x^2$ by $2x^3-x-5+3x^2$.

## VI. Scale Drawing; Ratio and Proportion

1. What is meant by scale drawing? Of what use is it?
2. What is meant by the ratio of one number to another?
3. Express the ratio of 3 to 7; 12 to 2; $a$ to $b$; $x+y$ to $z$.
4. How would you draw two triangles that have the same shape?
5. What relation do you discover between the pairs of corresponding sides? How do you choose the corresponding sides of two similar triangles?
6. What then are the two characteristics of similar triangles?
7. The sides of a triangle are 7, 11, 8 units, and the shortest side of a similar triangle is 13 units. Find the other sides.
8. What is a proportion? What is the test of proportionality?
9. If the product of two numbers such as $ad$ equals the product of two other numbers as $bc$, what proportions may be written?

What is the principle that guides you?

10. Solve for $x$:

$$\frac{4}{x}=\frac{132}{3};\quad \frac{x+2}{x+3}=\frac{x+3}{x+1}$$

11. What is meant by the bearing of point $B$ from point $A$?

12. A boy wishing to know the width of a stream flowing from east to west measures a line $AB$ on the south bank of the river and finds it to be 60 yards. At point $A$ he observes a point $C$ on the opposite bank to bear $55^\circ$ east of north, and the bearing of point $C$ from point $B$ is $25^\circ$ west of north. Find the width of the river.

## VII. Classified Equations in One Unknown; Solution of Formulas

1. Graph the linear equation $3x-2y=6$.
2. Solve the following concrete problems and note the type which each represents:
   *a*) Number problem: The sum of 3 numbers is 50. The first is twice the second, and the third is 16 less than 3 times the second. Find the numbers.
   *b*) Geometric problem: A tennis court is 42 ft. longer than it is wide. If a margin of 15 ft. on each end and of 10 ft. on each side is added, the area of the court is increased by 3,240 ft. Find the dimensions of the court.
   *c*) Age problem: A boy is 3 times as old as his brother. 5 years hence he will be only twice as old. Find the present age of each.
   *d*) Mixture problem: How many pounds of coffee worth 30 cents per lb. must be mixed with 12 lbs. of coffee worth 20 cents per lb. to make a mixture worth 34 cents per lb.?
   *e*) Motion problem: A boy starts walking at the rate of 3 miles per hour; 3 hours later, another boy starts

after him on a bicycle at the rate of 5 miles an hour. When will the first boy be overtaken?

*f*) Clock problem: At what time between 7 and 8 are the hands of a clock 20 minute spaces apart?

*g*) Alloy problem: An alloy of copper and silver weighing 50 oz. contains 5 oz. of copper. How much silver must be added so that 10 oz. of the new alloy contains $\frac{1}{4}$ oz. of copper?

*h*) Interest and percentage problem: A man invests part of \$6,400 at 5 per cent and the rest at 6 per cent. His income from this investment is \$350. How much is invested at each rate?

*i*) Beam problem: $AB$ is a crowbar, $6\frac{1}{2}$ ft. long, supported at $F$, $\frac{1}{2}$ ft. from $A$. A stone presses down at $A$ with a force of 1,800 lbs. How many pounds of force must be exerted by a man pressing down at $B$ to raise the stone?

3. Solve $I=prt$, for $p$; for $r$; for $t$. State the result of each solution in the form of an interest problem; build a problem showing its practical use.

4. Solve $C=\frac{E}{R+r}$ for $r$.

5. Solve $V=\frac{bh}{3}$ for $h$. State result in the form of a complete sentence. Do the same for $b$.

6. Solve for $x$:

*a*) $\frac{x}{a}-\frac{x}{b}=\frac{b^2-a^2}{ab}$

*b*) $\frac{cx-d}{dx}+\frac{dx-c}{cx}=2+\frac{c-d}{cdx}$

*c*) $.374x-.53+1.2x+.06=.8+1.32x$

*d*) $\frac{.4x+.39}{.7}-\frac{.2x-.66}{.9}=\frac{.08x+3.8}{.2}$

7. Write the equation of total turning tendency, and find what the unknown weight $w$ must be for balance for the following loadings of a lever whose support is at the center.

| | Weight | Arm | Weight | Arm |
|---|---|---|---|---|
| *a*) | $-2$ | $5$ | $W$ | $-6$ |
| *b*) | $2W$ | $-3$ | $-2$ | $-4$ |
| *c*) | $12$ | $3$ | $-W$ | $-8$ |

Interpret the signs before the above numbers.

## VIII. Theory and Application of Simultaneous Linear Equations in Two and Three Unknowns

1. Solve by the graphic method the following set of equations:

*a*) $x+7y=-11$
$x-3y+1=0$

2. Solve by the method of addition and subtraction:

*a*) $4x-6y+1=0$
*b*) $5x-7y=-1$

3. Solve by the substitution method:

*a*) $2x+3y=1$
*b*) $3x+4y=2$

4. Solve by comparison method:

*a*) $2x-3y=23$
*b*) $5x+2y=29$

5. Solve the following set of equations for the three unknowns:

*a*) $3x+4y-5z=32$
*b*) $4x-5y+3z=18$
*c*) $5x-3y-4z=2$

6. Solve:

$$2x+3y=7$$
$$3y+4z=9$$
$$5x+6z=15$$

7. Formulate a rule as to the method of solving a set of equations involving three unknowns.
8. Solve for $x$ and $y$:

$$\frac{5}{x}+\frac{13}{y}=59; \quad \frac{7}{x}+\frac{3}{y}=23$$

What rule is to be remembered?

9. Solve for $x$ and $y$:

*a*) $2ax+3by=4ab$ $\quad$ *b*) $ax+by=c$
$\quad 5ax+4by=3ab$ $\quad\quad dx+ey=f$

10. How are the numbers $a$, $b$, $n$, and $m$ to be treated in problems 9 and 10? In solving the above problems, would you expect to get arithmetic numbers for your results?
11. How many types of linear equations involving two unknowns can you illustrate? What is meant by a system of linear simultaneous equations? What are identical equations? What are inconsistent equations? What kinds of graphs would each set produce? When the types of problems studied under the topic "Equations of One Unknown" become complicated, they may often be more easily solved by the use of two or more unknowns.
12. Solve the following *mixture* problem by using two unknowns: How many pounds of 20-cent coffee and how many pounds of 32-cent coffee must be mixed to make 60 lbs. worth 28 cents a pound?
13. Solve the following *interest* problem: A man invests $5,050, part at 4 per cent and the rest at 5 per cent. His income on this investment is $220. How much has he invested at each rate?

14. Solve the following *number-relation* problem: If 1 is added to the numerator of a certain fraction, the value of the fraction becomes $\frac{1}{3}$; but if 1 is subtracted from the denominator, the value of the fraction becomes $\frac{1}{4}$. Find the fraction.
15. If a baseball team should play two more games and win both, it will have won $\frac{2}{3}$ of all the games played. If it should play 7 more and win 4 of them, it will have won $\frac{3}{5}$ of the games played. How many games has it so far played and how many has it won?
16. A tank can be filled by two pipes, one running 4 hours, the other 5, or by the same two pipes if the first runs 3 hours and the second 8 hours. In what time could each pipe fill it alone?
17. A bridge 20 ft. long weighs 2,400 lbs. and supports two loads, one of 600 lbs. 4 ft. from the left end, and the other of 800 lbs. 15 ft. from the left. What are the loads borne by the supports?
18. How would you write a number whose digits in order from left to right are $l$, $m$, and $n$? Why may you not express the number by the term $lmn$? A number consists of two digits whose sum is 13. If 4 be subtracted from double the number, the order of the digits is reversed. Find the number.
19. A man rows down the river 20 miles and back in 8 hours; he can row 5 miles down while he rows 3 miles up the river. Find the rate of the man's rowing in still water and the rate of the stream.
20. A train ran a certain distance at a uniform rate. If the rate had been 8 hours greater, the time would have been 2 hours less; if the rate had been 10 miles less, the time would have been 4 hours more. Find the distance and the rate.

## IX. Factoring

Factor the following and recognize the type to which each belongs:

a) $5x^2y-75x^2my^2+15x^3n^2y^3$
b) $.2x^3+.4ax^2$
c) $8a^2y-40axy+50x^2y$
d) $xy^2+2xy+x$
e) $225x^4-81y^2$
f) $m-169m^3$
g) $.25b^2-\frac{y^2}{25}$
h) $x^2-22x-48$
i) $x^5-25x^3+144x$
j) $2x^2+x-10$
k) $20-9x-20x^2$
l) $64y^3-27$
m) $64y^3+27$
n) $8x^6+y^3$
o) $250x-2x^7$
p) $2x^4-2x^3+2a^2x^2-2a^2x$
q) $3a^2+3ab-5am-5bm$
r) $4a^4-13a^2x^2+x^4$
s) $16r^4-9r^2+1$

## X. Quadratic Equations; Solution of the More Difficult Types of Formulas

1. Solve by the graphic method the quadratic equations:

a) $x^2+5x-24=0$
b) $y^2-7y=-12$
c) $m^2+6m+2=0$

2. Solve by the method of factoring the equations:

a) $3x^2+8=14x$
b) $8a^2+2a=15$
c) $20-9x-20x^2=0$

3. Solve by the method of completing the square:

a) $x^2-7x+12=0$
b) $3x^2-4x=4$
c) $5x^2+24x=5$
d) $x^2+6x+4=0$
e) $x^2+12x+29=0$

4. The base of a triangle exceeds the altitude by 4 in., and the area is 30 square in. Find the base and altitude.
5. A rectangular field is twice as long as wide. If it were 20 rods longer and 24 rods wider, the area would be doubled. What are the dimensions?
6. A tree standing on level ground was broken over so that the top touched the ground 50 ft. from the stump. The stump was 20 ft. more than two-fifths of the height of the tree. What was the height of the tree?
7. Write the equations whose roots are:

   *a*) $-3, -7$ *c*) $0, 2$

   *b*) $-5, 2$ *d*) $2, 0, 3$

8. *a*) $a^2K-b^2K=an+bn$. Solve for $K$.

   *b*) $3xt+6yh^3+5zt=9xh^3+2yt+15zh^3$. Solve for $t$.

   *c*) $\frac{x^2}{t}=\frac{x}{g}+\frac{y^2}{t}+\frac{y}{g}$. Solve for $g$; then for $t$.

## XI. Fundamental Operations as Applied to Fractions

1. Reduce to lowest terms:

   *a*) $\frac{2km^2-6bm^2}{3kn^3-9bn^3}=$ *b*) $\frac{25c^2+10cd+d^2}{5ac+ad+5bc+bd}=$

   *c*) $\frac{2a^2+17a+21}{3a^2+24a+21}=$

2. Add and subtract as indicated:

   *a*) $\frac{a-b}{ab}+\frac{b-c}{bc}+\frac{c-a}{ac}=$ *b*) $\frac{1}{2x-3y}+\frac{x+y}{4x^2-6xy}=$

   *c*) $\frac{(a+b)^2}{4ab}+2=$ *d*) $\frac{2}{xy}-\frac{4x}{3}+\frac{xy}{xyz}=$

   *e*) $\frac{x}{x^2-1}+\frac{x+3}{x-1}-\frac{x-3}{x+1}=$

3. Multiply and divide as indicated:

a) $\frac{21xy^2}{13z^2} \div \frac{28x^2}{39z^4} =$  b) $\frac{a^2b^2+3ab}{4a^2-1} \div \frac{ab+3}{2a+1} =$

c) $\frac{x^2+2x-3}{x^2+x-12} \cdot \frac{7(x^2-1)}{9(x-1)^3} \div \frac{14x}{8(x-1)} =$

4. In three hours a boatman rowed 10 miles up a stream and 4 miles back. If the velocity of the current was 2 miles an hour, what was his rate of rowing?

## XII. Introduction to Constructions Involved in Parallel Lines Cut by a Transversal; Application of Quadratic Equations and Simultaneous Linear Equations to the Various Angle Relations

1. As an introduction to important topics of geometry, it is convenient to think of parallel lines as lines having the same direction; e.g., *AB* and *CD* in Fig. 1 are parallel because they have had the same amount of angular rotation from the reference line *EF*. What angle determines the direction of *CD*? Of *AB*? What relation exists between the angles $x$ and $y$? What special name is applied to this pair of angles?

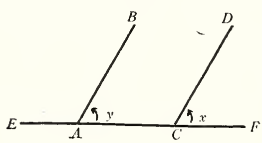

Fig. 1

2 If in the above figure one of the corresponding angles is designated $8y$ and the other by $y^2+12$, what is the value of $y$ and the value of the unknown angles?

3. In Fig. 1, what relation exists between angle *BAE* and angle *DCA*? Prove.

4. If, in Fig. 2, *AB* and *CD* are parallel (which means that $x=y$), show that $y=r$. What name is applied to this

pair of angles? Name a similar pair. What is the line *EF* called?

5. If in Fig. 2, one of the alternate interior angles is designated as $x^2+24$ and the other is designated by $10x$, find the value of $x$ and each angle in the figure.

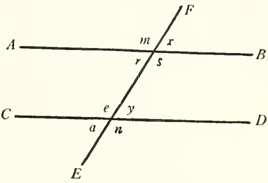


Fig. 2

6. Remembering the fact that $x=y$ in Fig. 2, prove that $m=n$. What special name is given to this pair of angles? Read a similar set in Fig. 2.

7. If in Fig. 2, one of the alternate exterior angles is designated as $42-t$ and if the other is designated by $t^2$, find the value of $t$ and of the unknown angles.

8. How many angles are made by the rails of two intersecting railroads? How many of these angles would you need to know in order to be able to calculate all the rest? Sketch a figure and prove your answer.


Fig. 3

9. What kind of a geometric figure is formed when two parallel lines cross another pair?

10. In the parallelogram *ABCD* of Fig. 3, prove the following equations:

*a*) $b=d$

*b*) $b+c=180°$

*c*) $a+b+c+d=360°$

11. State the equation of problem 10 in sentence form.

12. Draw a trapezoid, letter the angles as in Fig. 3, and prove as many principles of problem 10 as possible.

13. Determine how many of these principles you can prove as applying to a quadrilateral.

14. If one angle of a parallelogram is $w^2-11$ and the opposite angle is $6w+80$, find the angles of the parallelogram.
15. Reproduce Fig. 4 in notebook or on the blackboard, and by means of a protractor draw a line parallel to *CD* which shall pass through *P*.

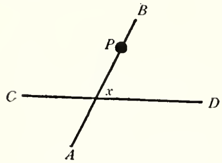


FIG. 4

16. The most accurate way to draw the required line is to transfer the angle $x$ with a pair of compasses to *P*. Describe this construction in precise terms.
17. In Fig. 5, *DE* is perpendicular to *AB*. Draw a line parallel to *DE* which shall pass through *C*.
18. Describe two methods of constructing a line perpendicular to a given line at a given point.

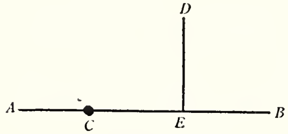


FIG. 5

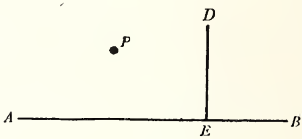


FIG. 6

19. In Fig. 6, *DE* is perpendicular to *AB*. Construct a line parallel to *DE* which shall pass through *P*.
20. Describe how to draw a line perpendicular to a given line from a given point outside the line.
21. Draw a triangle of which all angles are acute. From the vertex of each angle, construct a line perpendicular to the opposite side.
22. Draw a right triangle and construct perpendiculars as in problem 21.
23. Draw a triangle having an obtuse angle and construct perpendiculars as in problem 21.
24. What interesting geometrical fact do you observe as you compare the results of problems 21, 22, and 23?

25. In problems 19–24, what line-segments were bisected? Discuss the method of bisecting a given line-segment.

26. A line drawn from the vertex of a triangle to the middle of the opposite side is a median. Draw a triangle on squared paper and construct the three medians. What interesting fact do you observe?

27. Study carefully the comparative lengths of the two segments of each median and try to discover a very remarkable geometrical fact.

28. Draw a triangle and construct the perpendicular bisectors of the sides. What fact do you observe?

29. Why is not the line *CD* parallel to *AB* in Fig. 7? What would be an easy way of drawing a line through *C* parallel to *AB*?

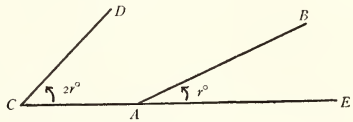


FIG. 7

30. Bisect a given angle and describe the construction in precise terms.

31. Draw a triangle and bisect the angles. What fact do you observe?

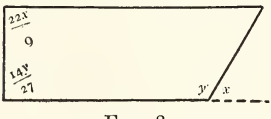


FIG. 8

32. In the trapezoid, Fig. 8, find $x$, $y$, and the four interior angles.

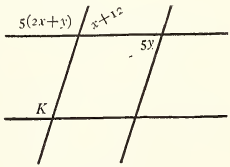


FIG. 9

33. The angles made by two pairs of parallels intersecting as in Fig. 9 are designated as shown. Find $x$, $y$, and all four angles about the crossing point *K*.

34. With parallels, transversal, and angles as shown in Fig. 10, find $x$, $y$, and all the eight angles.

35. If $a$ and $b$ are a pair of corresponding angles formed by parallel lines, cut by a transversal, determine the size of the angles in each of the following:

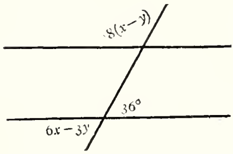


FIG. 10

*a*) if $a=h^2+20$, and $b=13h-20$

*b*) if $a=\frac{x^2}{24}$, and $b=\frac{x}{3}-\frac{1}{2}$

*c*) if $a=\frac{2t+3}{t+8}$, and $b=\frac{2t+9}{3t+4}$

*d*) if $a=\frac{k+1}{k+2}$, and $b=\frac{k+3}{k+4}+\frac{8}{3}$

*e*) if $a=\frac{k^2-85}{2}$, and $b=6k$

## SECOND-YEAR REVIEW

### I. Preliminary Theorems

The following theorems are sometimes stated as assumptions and are sometimes proved informally:

1. All straight angles are equal.
2. All right angles are equal.
3. Complements of equal angles are equal.
4. Supplements of equal angles are equal.
5. Vertical angles are equal.
6. The sum of two adjacent angles whose exterior sides lie in the same straight line is a straight angle, and conversely, if the sum of two adjacent angles is a straight angle, their exterior sides lie in the same straight line.
7. Only one perpendicular can be erected at a given point in a given line.
8. The bisectors of vertical angles lie in the same straight line.
9. A diameter bisects a circle.
10. Only one perpendicular can be drawn from a point without a line to the line.
11. Circles having equal radii are equal; and conversely.

### II. Congruence of Triangles and Applications

Two triangles $ABC$ and $A'B'C'$* are congruent if:

1. $a=a'$; $b=b'$; $C=C'$
2. $a=a'$; $B=B'$; $C=C'$
3. $a=a'$; $b=b'$; $c=c'$

* $A,B,C$ and $A',B',C'$ denote the angles; $a,b,c$ and $a',b',c'$ the sides opposite these angles respectively.

4. $a=a'$; $c=c'$; $C=C'=90°$

5. $c=c'$ $\begin{cases} A=A' \quad C=C'=90° \\ \text{or} \\ B=B' \end{cases}$

A triangle is determined when the following parts are given:

6. $a, b, C$
7. $a, B, C$
8. $a, b, c$
9. $a, c, C=90°$
10. $c$, $A$ or $B$, $C=90°$
11. If two oblique lines $o$ and $o'$ be drawn from a point in a line $p$ to a line $AA'$, cutting off distances $d$ and $d'$, then any three of the equalities $o=o'$, $y=y'$, $d=d'$, $A=A'$, $x=x'$, are consequences of the other two taken as hypotheses.

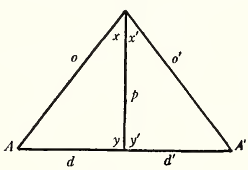

FIG. 11

12. If in a triangle $ABC$, $a=b$, the perpendicular from $C$ to $c$ divides the triangle into two congruent triangles.
13. In a triangle $ABC$, either of the equations $a=b$, $A=B$, is a consequence of the other.
14. The locus of all points equidistant from the end-points of a line-segment is the perpendicular bisector of that line-segment.
15. If in a triangle $ABC$, $a=b$, the perpendicular erected at the mid-point of $c$ passes through the vertex $C$.
16. If each of two points of one line is equally distant from two points of another line, the lines are perpendicular.
17. The shortest distance from a point to a line is the length of the perpendicular from the point to the line.

18. If one of the acute angles of a right triangle is equal to 30°, the side opposite that angle is equal to one-half the hypotenuse.

### III. Theorems Used to Prove Lines Parallel

1. Two lines parallel to the same line are parallel to each other.
2. Two lines perpendicular to the same line are parallel to each other.
3. If two straight lines are cut by a transversal, making
   *a*) the alternate-interior angles equal, or
   *b*) the alternate-exterior angles equal, or
   *c*) the corresponding angles equal, or
   *d*) the interior angles on the same side of the transversal supplementary,

   then the lines are parallel.

### IV. Theorems on Parallel Lines

1. If two lines cut by a transversal are parallel, the alternate interior angles are equal.
2. A perpendicular to one of two parallels is perpendicular to the other also.
3. If the sides of two angles are parallel right to right and left to left, the angles are equal; if the sides are parallel right to left and left to right, the angles are supplementary.
4. If the sides of two angles are respectively perpendicular right to right and left to left, the angles are equal; if the sides are perpendicular right to left and left to right, the angles are supplementary.

### V. Angle Sums

1. The sum of the interior angles of any triangle is equal to a straight angle, or two right angles.

2. If two angles of one triangle are equal respectively to two angles of another triangle, the third angles are equal.
3. The sum of the interior angles of a quadrilateral is two straight angles.
4. The sum of the interior angles of an $N$-gon is $(n-2)$ straight angles or $(2n-4)$ right angles.
5. Any interior angle of a regular $N$-gon is $\frac{n-2}{n}$ straight angles or $\frac{2n-4}{n}$ right angles.
6. The sum of the exterior angles of a triangle is four right angles.
7. The sum of the exterior angles of a quadrilateral is four right angles.
8. The sum of the exterior angles of an $N$-gon is four right angles.
9. An exterior angle of a triangle is equal to the sum of the two non-adjacent interior angles.

## VI. Parallelograms

1. In a parallelogram:
   *a*) a diagonal divides it into two congruent triangles;
   *b*) the consecutive angles are supplementary;
   *c*) the opposite angles are equal;
   *d*) the opposite sides are equal;
   *e*) the diagonals bisect each other.
2. If in a quadrilateral
   *a*) the opposite sides are equal, or
   *b*) the opposite angles are equal, or
   *c*) two sides are equal and parallel, or
   *d*) the diagonals bisect each other,
   then the quadrilateral is a parallelogram.

3. The diagonals of a square
   *a*) are equal;
   *b*) bisect each other;
   *c*) are perpendicular to each other;
   *d*) bisect the angles of the square.

4. The diagonals of a rhombus
   *a*) are not equal (give informal proof);
   *b*) bisect each other;
   *c*) are perpendicular to each other;
   *d*) bisect the angles of the rhombus.

5. The diagonals of a rectangle that is not a square
   *a*) are equal;
   *b*) bisect each other;
   *c*) are not perpendicular to each other;
   *d*) do not bisect the angles of the rectangle.

6. If two angles at the ends of one base of a trapezoid are equal, the trapezoid is isosceles; and conversely.
7. If two adjacent sides and the included angle of one parallelogram are respectively equal to two adjacent sides and the included angle of another parallelogram, the parallelograms are congruent.

## VII. Circles

1. A straight line can intersect a circle in only two points. (Informal proof).
2. The diameter of a circle bisects the circle.
3. The diameter of a circle is greater than any other chord.
4. In the same circle, or in equal circles, radii that form equal angles at the center intercept equal arcs; and conversely.
5. In the same circle, or in equal circles, equal arcs are subtended by equal chords; and conversely.

6. Prove the ten propositions that can be formulated from the following data, using two of the five conditions for the hypothesis, and the remaining three for the conclusion:

   *a*) a line passing through the center of a circle;
   *b*) a line perpendicular to a chord;
   *c*) a line bisecting a chord;
   *d*) a line bisecting a major arc of a chord;
   *e*) a line bisecting a minor arc of a chord.

7. In the same circle, or in equal circles, equal chords are equally distant from the center; and conversely.
8. Only one circle can be passed through three points not collinear.
9. Two circles can intersect in only two points. (Informal proof).
10. A tangent to a circle is perpendicular to the radius drawn to the point of contact; and conversely.
11. The arcs included between two parallel secants are equal; and conversely.
12. If two circles are tangent to each other, the centers and the point of tangency lie in a straight line.
13. The line joining the centers of two intersecting circles bisects the common chord perpendicularly.

## VIII. Problems of Construction

1. At a given point in a given line, to construct an angle equal to a given angle.
2. To bisect a given angle.
3. To draw a perpendicular bisector of a line segment.
4. To erect a perpendicular at a given point on a given line.
5. To drop a perpendicular from a given point to a given line.

6. To draw a line through a given point parallel to a given line.
7. To construct a parallelogram having given two adjacent sides and the angle included between them.
8. Given a side of a square, to construct the square.
9. Given three consecutive sides and the two included angles of a quadrilateral, to construct the quadrilateral.
10. Given the diagonals of a rhombus, to construct the rhombus.
11. To construct the complement and supplement of a given angle.
12. Given a circle, to find the center.
13. To bisect a given arc.
14. To draw a common tangent to two tangent circles.
15. To construct three circles, having given radii, tangent to each other externally.

## IX. Ratio and Proportion

1. If four quantities are in proportion then—
   *a*) The product of the means is equal to the product of the extremes.
   *b*) They are in proportion by alternation.
   *c*) " " " " " inversion.
   *d*) " " " " " addition.
   *e*) " " " " " subtraction.
   *f*) " " " " " addition and subtraction.
   *g*) Like powers of those quantities are in proportion.
   *h*) " roots " " " " " " .
2. If the product of two factors is equal to the product of two other factors, a proportion may be formed by taking either pair as means, and the other pair as extremes.
3. If two or more ratios are equal, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

4. The mean proportional between two quantities is equal to the square root of their product.
5. A line drawn from the mid-point of one side of a triangle parallel to a second side bisects the third side and is equal to one-half the second side.
6. If three or more parallel lines cut two transversals, and if the segments intercepted on one of the transversals are equal, the segments intercepted on the other are equal also.
7. A line drawn through the mid-point of one of the sides of a trapezoid parallel to the bases bisects the other side.
8. If two parallel lines cut two intersecting transversals, the segments of one transversal are proportional to the corresponding segments of the other; and conversely.
9. A line parallel to one side of a triangle divides the other two sides proportionally.
10. If any number of lines cut two transversals, the segments of one transversal are proportional to the corresponding segments of the other.
11. The line joining the mid-points of two sides of a triangle is parallel to the third side.
12. The line through the mid-points of the non-parallel sides of a trapezoid is parallel to the bases.
13. The medians of a triangle are concurrent in a point two-thirds of the distance from each vertex to the mid-point of the opposite side.
14. The bisector of an interior angle of a triangle divides the opposite side into segments whose ratio is equal to the ratio of the adjoining sides of the triangle; and conversely.
15. The bisector of an exterior angle of a triangle divides the opposite side externally into segments whose ratio is equal to the ratio of the adjoining sides of the triangle; and conversely.

16. Construct a fourth proportional to $a$, $b$, and $c$.
17. Construct a third proportional to $a$ and $b$.
18. To divide a line internally and externally in the ratio of $\frac{m}{n}$. (This divides the line harmonically.)
19. To divide a given line-segment into parts proportional to several given segments.
20. To divide a line-segment into five equal parts.
21. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a^2+c^2}{a^2-c^2}=\frac{ab+cd}{ab-cd}$. (Use the method of analysis.)

## X. Similar Triangles

1. Two triangles are similar if two angles of the one are respectively equal to two angles of the other.
2. Two triangles are similar if the ratio of two sides of one equals the ratio of two sides of the other and the angles included between these sides are equal.
3. Two triangles are similar if the corresponding sides are in proportion.
4. Triangles having the corresponding sides parallel or perpendicular are similar.
5. If lines are drawn joining the mid-points of the sides of a triangle, another triangle is formed which is similar to the first triangle.
6. The perimeters of similar triangles are to each other as any two homologous lines.
7. The perpendicular to the hypotenuse from the vertex of a right triangle divides the triangle into triangles similar to each other, and to the given triangle.
8. In a right triangle, the perpendicular from the vertex of the right angle to the hypotenuse is a mean proportional between the sects of the hypotenuse.

9. In a right triangle either side including the right angle is a mean proportional between the hypotenuse and the projection of that side upon the hypotenuse.
10. Construct a mean proportional between two given line-segments.
11. If two polygons are made up of the same number of triangles similar each to each and similarly placed, the polygons are similar.
12. Homologous diagonals divide similar polygons into similar triangles.
13. The perimeters of similar polygons are to each other as two homologous sides.
14. Prove algebraically that the square of the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides.

## XI. Problems of Computation

1. Find $x$ and the two equal sides of an isosceles triangle, the equal sides being denoted by $3x(10x-3)$ and $20(2x-1)$.
2. Find $x$ and $y$ and the length of a side of an equilateral triangle, the sides being designated by $2(5-x)$, $3y-x$, and $2y$.
3. One of two complementary angles is $x^2$ and the other is $x$. Find each angle.
4. One of two supplementary angles is $y^2$ and the other is $8y$. Find each angle.
5. The angles $x$ and $y$ are complementary and their difference is $10°$. Find each angle.
6. Find the angle between the bisectors of the acute angles of a right triangle.
7. The angles $x$ and $3y$ are supplementary and $x-y=20°$. Find the angles.

8. The two acute angles of a right triangle have the values $2x$ and $3x$. Find the third angle.
9. One pair of opposite sides of a parallelogram is denoted by $x^2+x$ and $6(3-x)$, and the other pair by $y^2-y$ and $3(5-y)$. Find the lengths of the sides.
10. The diagonals of a rectangle are denoted by $x^2-x$ and $2(2x+7)$. Find the diagonals.
11. The diagonals of a parallelogram divide each other so that the segments of one are $s^2+s$ and $2(5s-7)$, and of the other $t^2+2t$ and $8(3-t)$. Find the lengths of the diagonals.
12. One of the four angles that the diagonals of a rhombus make with each other is $x^2-10$. Find all the four angles.
13. Find a common unit of measure between two lines, 120 and 35 in. respectively.
14. Find $x$ in $\frac{7}{21}=\frac{14}{x}$.
15. The geometrical mean between two numbers is 12. One of the numbers is 16. Find the other.
16. Find a mean proportional between $a^2+2ab+b^2$ and $a^2-2ab+b^2$.
17. Four numbers, $a$, $b$, $c$, and $x$, are in proportion. What is the fourth proportional? (Solve algebraically.)
18. Solve for $x$ by applying addition and subtraction to:

*a)* $\frac{x-a+b}{x+a-b}=\frac{a-b-x}{a+b+x}$  *b)* $\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}=3$

*c)* $\frac{2+\sqrt{x}}{2-\sqrt{x}}=\frac{\sqrt{x+5}+\sqrt{x}}{\sqrt{x+5}-\sqrt{x}}$

19. The sides of a triangle $ABC$ are $AC=6$ in., $AB=10$ in., and $BC=12$ in. Find the lengths of the segments of $AC$ made by the bisector of angle $B$.

20. A man at a window sees a point on the ground in line with the top of a post 2 ft. 8 in. from the foot of the post. The post is 3 ft. high, and $24\frac{1}{2}$ ft. from a given point just under the window. Find the height of the window.

## XII. Similar Right Triangles; Trigonometry

1. Show that all right triangles having a given acute angle are similar.
2. Define the sine, cosine, and tangent of an acute angle.
3. Show that the sine, cosine, and tangent of a given acute angle are the same for all right triangles that contain that angle.
4. Show that the sine or cosine of an acute angle is each less than 1.
5. How large may the tangent of an acute angle become?
6. Draw the angle whose sine is $\frac{1}{2}$; $\frac{7}{8}$; .75. Find the cosine and tangent of each angle and the number of degrees in each. (Use cross-section paper.)
7. Compute the values of the sine, cosine, and tangent of $30°$; $45°$; $60°$.
8. Show by means of a geometric figure that

   *a*) $\sin^2 A + \cos^2 A = 1$

   *b*) $\tan A = \dfrac{\sin A}{\cos A}$

9. Draw with a protractor (on cross-section paper) angles of $25°$; $40°$; $50°$; and $65°$. Compute the sine, cosine, and tangent of each. Check the results obtained by means of a trigonometric table.
10. Find by an algebraic method the values of the sine, cosine, and tangent of angle $B$ where

    *a*) $\sin B = \frac{1}{4}$

    *b*) $\cos B = 0.6$

    *c*) $\tan B = \frac{4}{3}$

11. Prove $\cos^4 A - \sin^4 A + 1 = 2 \cos^2 A$
12. From the top of a cliff 150 ft. high the angle of depression of a boat is 25°. How far is the boat from the top of the cliff?
13. From a point $A$ on a horizontal plane the angle of elevation of the top of a tower is 65°. From a point $B$, which is in the horizontal plane at $A$ and 100 ft. farther away from the tower, the angle of elevation is 35°. Find the height of the tower.

## XIII. Measurement of Angles by Arcs of the Circle

1. In the same circle or in equal circles, two central angles have the same ratio as the arcs intercepted by their sides.
2. A central angle is measured by the arc intercepted by its sides.
3. An inscribed angle is measured by one-half the arc intercepted by its sides.
4. Prove that all angles inscribed in:
   *a*) the same segment of a circle are equal;
   *b*) a semicircle are right angles;
   *c*) a segment smaller than a semicircle are greater than a right angle;
   *d*) a segment greater than a semicircle are less than a right angle.
5. If two chords intersect within a circle and the end points of the sides of two vertical angles be joined by two chords, the triangles formed are mutually equiangular.
6. If two secants meet without a circle, the angle they form is measured by one-half the difference of the arcs included between them.
7. If two chords intersect within a circle, either angle they form is measured by one-half the sum of the intercepted arcs.

8. An angle formed by a tangent and a chord is measured by one-half the arc intercepted by its sides.
9. An angle formed by a tangent and a secant meeting outside a circle is measured by one-half the difference of the intercepted arcs.
10. An angle formed by two tangents to a circle is measured by one-half the difference of the intercepted arcs.
11. Two tangents drawn from a point outside a circle to the circle are equal.
12. If a quadrilateral is inscribed in a circle, the sum of two opposite angles is two right angles; and conversely.

## XIV. Construction Problems

1. To divide a circle into three arcs in the ratio of 1:2:3 by means of ruler and compasses only.
2. Upon a given line-segment to construct a segment of a circle in which an angle may be inscribed equal to a given angle of 60°; 30°; 120°, by means of ruler and compass only.
3. To draw a tangent to a given circle from a point without the circle.
4. To construct a square equivalent to a given rectangle.

## XV. Computation Problems

1. The area of a circle is 616 sq. in. How many degrees are there in an angle at the center that intercepts an arc 11 in. long? ($A=\pi R^2$; $C=2\pi R$; $\pi=\frac{22}{7}$.)
2. Two tangents to a circle intercept two arcs one of which is four times the other. How many degrees are there in the angle formed by the two tangents?
3. Two tangents to a circle from an outside point form an angle of 70°. What part of the circle is the larger arc

included by the points of tangency? If a race track is constructed upon the plan of the track going around the larger arc of the circle, how much farther than his competitor must a rider go whose path is 4 ft. farther from the rail ($C=2\pi R$)?

4. Two secants $AD$ and $AE$ are drawn from a point outside a circle forming an angle of 30°, and intersecting the circle at $B$ and $C$ respectively. The number of degrees in arc $DE$ is represented by

$$\frac{6x^2+29x+20}{2x+3},$$

in arc $BC$ by

$$\frac{2x^2-7x-15}{x-5}.$$

Find $x$ and the number of degrees in each of the two arcs.

## XVI. Similarity and Proportionality in Circles

1. A perpendicular to a diameter of a circle at any point, extended to the circumference, is a mean proportional between the sects of the diameter.
2. If 2 chords of a circle intersect, the product of the sects of one is equal to the product of the sects of the other.
3. If from a point without a circle 2 secants be drawn to the concave arc, the product of one secant and its external sect is equal to the product of the other secant and its external sect.
4. If from a point without a circle a tangent and a secant be drawn, the tangent is a mean proportional between the entire secant and its external sect.
5. To draw a common tangent to two circles exterior to each other.

6. The area of any triangle is equal to the product of the three sides divided by twice the diameter of the circumscribed circle.
7. The sects of two intersecting chords are $x+5$ and $x-6$ in the one, $x+2$ and $x-5$ in the other. Find $x$ and the length of each chord.
8. Two secants to the same circle from an external point are cut by the circle into chords of the circle and external sects in the ratio of 5 to 3, and 5 to 1 respectively. The first secant is 8 ft. long. Find the length of the second secant.
9. Construct a circle passing through two given points and tangent to a given line. (Discuss all the possibilities.)

## XVII. Quadratic Equations; Solution by Formula

1. Solve by the graph: $m^2+6m+2=0$.
2. In a circle of radius 20, a chord is drawn at a distance 8 from the center. Find the radius of a circle that is tangent to the given circle, to the chord, and to a diameter perpendicular to it.
3. What methods were used in solving problems 1 and 2 above? Which do you prefer? Why?
4. Explain how a quadratic equation may be solved by the method of completing the square. Solve an example to illustrate.
5. By using the method of completing the square on the quadratic equation $ax^2+bx+c=0$, derive a formula for $x$ in terms of $a$, $b$, and $c$.
6. If $x_1$ and $x_2$ represent the two roots of the quadratic equation $ax^2+bx+c=0$, calculate

   *a*) $x_1+x_2$
   
   *b*) $x_1 \cdot x_2$

7. Find the sum and product of the roots of:

   *a*) $3x^2+12x-18=0$

   *b*) $9x-\frac{3x-2}{x}=\frac{4-2x}{-x}$

8. Solve the following quadratic equations by the formula:

   *a*) $2x^2+5x+2=0$

   *b*) $6x^2-11x+5=0$

   *c*) $12x^2-16ax-3a^2=0$

   *d*) $2z^2-(2a+b)z+ab=0$

9. The sects of intersecting chords in a circle are as follows:

   | First Chord | Second Chord |
   | --- | --- |
   | $3x-\frac{2}{3}$ and $3x-3$ | $2x+1$ and $8x-4$ |

   Find $x$ and the length of each chord.

10. Show how we may determine the nature of the roots of a quadratic equation without actually solving it.

11. Determine by means of the *discriminant* the nature of the roots of the following quadratics:

    *a*) $x^2+6x+9=0$

    *b*) $4y^2-9y+2=0$

    *c*) $9y^2-3y+5=0$

    *d*) $x^2-3abx+2a^2b^2=0$

## XVIII. Inequalities in Triangles and Circles

1. The sum of any two sides of a triangle is greater than the third side, and their difference is less than the third side.
2. If, from a point within a triangle, line-segments are drawn to the end-points of one side, the sum of these line-segments is less than the sum of the other two sides.
3. The sum of the 3 line-segments joining a point inside a triangle with the vertices is less than the perimeter of the triangle but greater than the semi-perimeter.
4. If two angles of a triangle are unequal, the sides opposite them are unequal, the greater side lying opposite the greater angle.

5. If two sides of a triangle are unequal, the angles opposite them are also unequal, the greater angle lying opposite the greater side.
6. The perpendicular from a point to a line is shorter than any other line connecting the point with the line.
7. If two sides of one triangle are equal to two sides of another triangle, but the angle included between the two sides in the first is greater than the angle included by the corresponding sides in the second, then the third side in the first triangle is greater than the third side in the second.
8. If two sides of one triangle are equal to two sides of another triangle, the third side of the first triangle being greater than the third side of the second, then the angle opposite the third side of the first triangle is greater than the angle opposite the third side of the second triangle.
9. The locus of all points equally distant from the sides of an angle is the bisector of that angle.
10. The perpendicular bisectors of the three sides of a triangle pass through a common point.
11. The bisectors of the three angles of a triangle meet in a common point.
12. The three altitudes of a triangle pass through a common point.
13. In the same circle or in equal circles, the arcs subtended by unequal chords are unequal in the same order as the chords; and, conversely, chords subtending unequal arcs are unequal in the same order as the arcs.
14. In the same circle or in equal circles, unequal chords are unequally distant from the center of the circle, the shorter chord lying at a greater distance; and conversely.
15. On a circle whose center is $P$ the points $A$, $B$, and $C$ are taken so that the chord $AB$ equals $4t+14$, the chord

$BC$ equals $10t-2$. The distance from $P$ to $AB$ is 6 and from $P$ to $BC$ is 3. Find the length of each chord.

16. If in problem 15 $AB=6x-8$, $BC=2x+6$, the distance from $P$ to $AB$ is $2m$ and from $P$ to $BC$ is 1, find the value of $m$ for which the chords actually exist.
17. Construct a triangle $ABC$, the sides $a$ and $b$ and the angle $A$ being given. (Discuss all possibilities.)
18. In a triangle $ABC$, $AB=7$, $AC=8$, $BC=6$. A point $P$ within the triangle is joined to $A$, $B$, and $C$. If $PA=x+5$, $PB=x+3$, and $PC=x+4$, determine between what limits $x$ must lie. What values could $x$ have if it were required to be an integer?

## XIX. Areas of Polygons

1. Two parallelograms having equal bases and equal altitudes are equal.
2. The area of a parallelogram equals that of a rectangle having the same base and altitude.
3. The area of a rectangle is equal to the product of its base and altitude.
4. The area of a parallelogram is equal to the product of its base and altitude.
5. Compare the area of a parallelogram with the area of a triangle having the same base and altitude.
6. The area of a triangle is equal to one-half the product of the base and altitude.
7. (1) Two parallelograms are to each other as the product of their bases and altitudes. (2) Two parallelograms having equal bases are to each other as their altitudes. (3) Two parallelograms having equal altitudes are to each other as their bases. (4) Prove the three corresponding theorems for two triangles.

8. Divide a triangle into three equal triangles by lines from any vertex to the opposite side.
9. The area of a triangle is equal to one-half the perimeter times the radius of the inscribed circle.
10. The area of any triangle is equal to the product of the three sides divided by twice the diameter of the circumscribed circle.
11. The area of any triangle is equal to the product of the three sides divided by four times the radius of the circumscribed circle.
12. In a right triangle the square on the hypotenuse is equal to the sum of the squares of the other two sides.
13. Prove Hero's formula for the area of a triangle, viz.: $A=\sqrt{s(s-a)\ (s-b)\ (s-c)}$.
14. Find the area and altitudes of the triangles whose sides are as follows: (1) 13, 14, 15. (2) 70, 58, 16. (Use Hero's formula.)
15. The altitude of an equilateral triangle is equal to $\frac{1}{2}a\sqrt{3}$ and the area is equal to $\frac{a^2}{4}\sqrt{3}$. ($a$ is one of the sides.)
16. In a triangle the square on a side opposite an acute angle is equal to the sum of the squares of the other two sides minus two times the product of one of these two sides and the projection of the other side upon it.
17. In a triangle the square on a side opposite an obtuse angle is equal to the sum of the squares of the other two sides plus two times the product of one of these sides and the projection of the other side upon it.
18. The diagonal of a rectangle is 20 and one side is 16. If $x^2+4x$ denotes the area, what does $x$ equal?
19. The area of a rectangle is $16y^3-4y$, and the altitude is $4y$. The diagonal is 10. Find the sides of the rectangle.

20. In a triangle the sum of the squares of two sides is equal to twice the square of one-half the third side plus twice the square on the median to the third side. Solve for median.
21. The areas of similar triangles are to each other as the squares of the homologous sides.
22. The areas of similar polygons are to each other as the squares of the homologous sides.
23. The areas of two triangles that have an angle in one equal to an angle in the other are in the same ratio as the product of the sides including the equal angles.
24. There are two similar polygons whose homologous sides are 9 and 12. Find the side of a third similar polygon that is equivalent to their sum; one that is equivalent to their difference.
25. The area of a trapezoid is equal to the altitude times one-half the sum of the bases.
26. Find the side of a square equivalent to the sum of any number of given squares.
27. Show how $\sqrt{5}$ may be found in four different ways.
28. Transform a polygon into (1) an equivalent triangle, (2) an equivalent square.
29. Construct (1) a right triangle, (2) an isosceles triangle, (3) an obtuse-angled triangle, (4) an equilateral triangle, each equivalent to a given triangle.
30. Find the area and the altitude of the triangle whose sides are $a=45$, $b=40$, $c=13$.
31. Find the altitude and area of the following equilateral triangles: *a*) $a=12$; *b*) $a=2mn$.
32. Find the side of an equilateral triangle whose area is

    *a*) $\frac{121}{4}\sqrt{3}$; *b*) $10\sqrt{3}$.

## XX. Regular Polygons Inscribed in, and Circumscribed about, a Circle

1. If a circle be divided into equal parts and if the successive points of division are joined by chords, the figure thus formed is a regular inscribed polygon.
2. If a circle is divided into equal parts and if tangents are drawn at the points of division, the figure thus formed is a regular circumscribed polygon.
3. If the vertices of a regular inscribed polygon are joined with the middle points of the arcs subtended by the sides of the polygon, the figure thus formed will be a regular inscribed polygon of double the number of sides.
4. If at the middle points of the arcs joining adjacent points of contact of the sides of a regular circumscribed polygon tangents are drawn, the figure thus formed will be a regular circumscribed polygon of double the number of sides.
5. An equilateral polygon inscribed in a circle is a regular polygon.
6. A circle may be circumscribed about any regular polygon.
7. A circle may be inscribed in any regular polygon.
8. If at the mid-point of the arcs subtended by the sides of a given regular inscribed polygon, tangents are drawn to the circle, they are parallel to the sides of the given polygon and form a regular circumscribed polygon.
9. Regular polygons of the same number of sides are similar.
10. The perimeters of regular polygons of the same number of sides are to each other as the radii of the circumscribed circles, or as the radii of the inscribed circles; and their areas are to each other as the squares of these radii.
11. The perimeter of a regular inscribed $2n$-side is greater than the perimeter of the regular $n$-side inscribed in the same circle.

12. Divide a given line-segment in mean and extreme ratio.
13. Given a circle, to inscribe and circumscribe the following regular polygons:
    *a*) Triangle.
    *b*) Square.
    *c*) Pentagon.
    *d*) Hexagon.
    *e*) Decagon.
    *f*) 15-side.
14. Express in terms of the radius of a given circle the sides of the following regular polygons:
    *a*) Inscribed and circumscribed triangle.
    *b*) " " " square.
    *c*) " pentagon.
    *d*) " and " hexagon.
    *e*) " decagon.
15. The perimeter of a regular circumscribed $2n$-gon is less than the perimeter of the regular $n$-gon circumscribed about the same circle.
16. Compute the side of the regular circumscribed $2n$-gon in terms of the sides of the regular inscribed and circumscribed $n$-gon.
17. Compute the side of the regular inscribed polygon of $2n$-sides in terms of the side of a regular circumscribed polygon of $2n$-sides, and of the side of the regular inscribed polygon of $n$-sides.
18. The area of a regular inscribed polygon is equal to one-half the product of its perimeter and its apothem.
19. The area of a regular circumscribed polygon is equal to one-half the product of its perimeter and the radius of the circle.
20. The circumference of a circle is equal to the product of the radius and twice the constant number $\pi$.

21. The area of a circle is equal to one-half the product of its circumference and its radius, or to the square of its radius multiplied by the constant number $\pi$.
22. The circumferences of two circles are to each other as the radii, and the areas as the squares of the radii.
23. The area of a sector of a circle is equal to one-half the product of the length of the arc of the sector, and the radius of the circle.
24. The radii of two circles are 6 cm. and 1 cm. respectively. Find the ratio of the areas of the inscribed squares; of the circumscribed squares. How does the ratio of the areas compare with the ratio of the squares of the radii?
25. The radius of a circle is 10. Find the area of the regular inscribed hexagon.
26. A man has a round table top which he wishes to change into a pentagon. The diameter of the top is $2\frac{1}{2}$ ft. What is the length of the cut required?
27. The radius of a circle is 8 in. Find the side of a regular inscribed decagon.
28. The radii of two circles have the ratio $\frac{3}{5}$ and their combined area is 3,850. Find the radii of the two circles.
29. The radius of a circle is 8 in. Find the area of a sector with an arc of 36°.
30. The area of a circle is 15,400 sq. in. Find the area of a segment whose arc is 60°.
31. A goat is tethered by a rope 24 ft. long, one end of which is fastened to the middle point of one side of a shed 12 ft. square. How many square feet of pasturage are accessible to him? (Harvard.)
32. A point $C$ lies between two points $A$ and $B$. Three semicircles are constructed on $AB$, $AC$, and $BC$, as diameters, all on the same side of $AB$, and a perpendicular is erected at $C$ cutting the largest semi-circum-

ference at $D$. Prove that the area of the figure bounded by the three semi-circumferences is equal to that of the circle on $CD$ as diameter. (Harvard.)

33. Find the area that remains after the inscribed circle is removed from an equilateral triangle, a side of which is 6 inches. (Yale.)

## THIRD-YEAR REVIEW

### I. Simple Functions: Their Transformation; the Solution of Simple Equations

1. What is a formula? A constant in a formula? What is a variable? A function of a variable? What kinds of functions have you studied? Illustrate each.
2. Why is the cost of 5 yds. of silk a function of the price per yard? Of what is the area of a square a function? The circumference of a circle? The area of a circle?
3. Give five examples of functions.
4. If $y$ represents the area and $x$ the side of a square, then $y=x^2$ is the equation which expresses the relation between the area and the side. Graph this equation and tell how the graph shows that $y$ is a function of $x$.
5. Graph $S=16t^2$. Do you know the meaning which the physicist gives to this formula?
6. In problem 4 above, how did $y$ change as different values were given to $x$? In problem 5 above, how did $S$ vary with $t$?
7. Define direct variation. Express this definition algebraically by means of an equation. Give five concrete examples illustrating direct variation. If $x$ varies directly as $y$ and $x=6$ when $y=2$, find $x$ when $y=8$.
8. If $x$ and $y$ represent the dimensions of a rectangle whose area is 16 sq. ft., then $xy=16$ is the equation which expresses the relation between the dimensions and the area. Graph the equation $xy=16$. How many rectangles satisfy the condition that the equation sets forth? How does the width vary as the length varies? How do $x$ and $y$ vary with respect to each other?

9. If $r$ is the rate of an automobile and $t$ is the number of hours it takes the automobile to travel 50 miles, the equation $rt=50$ expresses the relation between the rate, time, and distance. Graph the equation $rt=50$ and show how $r$ and $t$ vary with respect to each other.
10. Define inverse variation. Express this definition by means of an algebraic equation. Give three concrete examples of inverse variation.
11. If $x$ varies inversely as $y$ and $x=5$ when $y=2$, what is the value of $x$ when $y=4$?
12. Define joint variation and express this definition by means of an algebraic equation.
13. If $x$ varies jointly with $y$ and $z$ and if $x=24$ when $y=2$ and $z=3$, find the value of $x$ when $y=3\frac{1}{2}$ and $z=7$.
14. If the circumference of a circle varies directly as the radius, write the equation that expresses the relation between the circumferences of two circles.
15. What special name is given to an equation made up of two ratios, as, $\frac{c_1}{c_2}=\frac{r_1}{r_2}$?
16. Review the properties of proportion by proving them, and give arithmetical and algebraic examples of each.
17. Find a fourth proportional to $x^3+y^3$, $x^2-xy+y^2$, and $x+y$.
18. Find a mean proportional between $(a+b)^2$ and $(a-b)^2$.
19. Find a third proportional to $\frac{x}{y}+\frac{y}{x}$ and $\frac{x}{y}$.
20. If $\frac{a}{b}=\frac{c}{d}$ show that

    *a*) $\frac{ma+nb}{ma-nb}=\frac{mc+nd}{mc-nd}$. (M.I.T.)

    *b*) $\frac{a}{b}=\frac{c}{d}=\frac{\sqrt{a^2+c^2}}{\sqrt{b^2+d^2}}$ (M.I.T.)

21. Solve by using principles of proportion:

$$\frac{\sqrt{y+2a}-\sqrt{y-2a}}{\sqrt{y-2a}+\sqrt{y+2a}}=\frac{y}{2a}$$

22. A large number of practical problems from the field of science can be solved either by the methods of variation or by those of proportion. Solve the following problems by both methods:

    *a*) Direct variation problem: The time of oscillation of a pendulum varies directly as the square root of its length. What is the length of a pendulum which makes an oscillation in 4 seconds, if a two-second pendulum is 156.8 in. long?

    *b*) Inverse variation problem: In pumping air into an automobile tire the pressure varies inversely as the volume. If the pressure is 25 lbs. when the volume is 125 cu. in., what is the pressure when the volume is 140 cu. in.?

    *c*) Joint variation problem: In beams of the same width and thickness the deflection due to a central load varies jointly as the load and the cube of the length. If a beam 10 ft. long is bent $\frac{1}{2}$ in. by a load of 1,000 lbs., how much will a load of 5,000 lbs. bend a 32-ft. beam?

## II. Linear Equations in Two or More Unknowns

1. Name five methods used to solve a system of linear equations in two unknowns.
2. Solve the following system by all of the above methods:

$$\begin{cases} 2x-3y=-4 \\ 3x-2y=9 \end{cases}$$

3. Develop the determinant formula for solving a system of linear equations in two unknowns.

4. Solve by determinants:

 a) $\begin{cases} 2m-3n=4 \\ 3m+2n=32 \end{cases}$

 b) $\begin{cases} y-4.5x=5.5 \\ 3y-4x=7 \end{cases}$

 c) $\begin{cases} m\ \ -2n=4 \\ 3m-6n=5 \end{cases}$

5. Interpret the zero in the denominator of the solution of one of the above problems.
6. What names are given to linear systems of equations in two unknowns that have no solution? Illustrate each.
7. Solve by determinants:

 a) $\begin{cases} \frac{3}{x}+\frac{4}{y}=3 \\ \frac{9}{x}-\frac{8}{y}=-1 \end{cases}$  b) $\begin{cases} \frac{a}{y}+\frac{b}{z}=c \\ \frac{b}{y}+\frac{a}{z}=d \end{cases}$

 c) $\begin{cases} \frac{x-y}{2}=\frac{25}{6}-\frac{x+y}{3} \\ \frac{x+y-9}{2}-\frac{x-y-6}{3}=0 \end{cases}$

8. Explain how a system of linear equations in three unknowns may be solved.
9. Write the determinant formula for a solution of a system of linear equations in three unknowns.
10. Solve by means of determinants:

 a) $\begin{cases} 5x+2y-4z=-3 \\ 3x-3y+5z=12 \\ 4x+5y+2z=20 \end{cases}$  b) $\begin{cases} \frac{1}{x}+\frac{1}{y}=2 \\ \frac{1}{x}+\frac{1}{z}=3 \\ \frac{1}{y}+\frac{1}{z}=4 \end{cases}$

11. Solve the following system:

$$\begin{cases} A \quad + C \quad = 2 \\ -A+B+C+D=1 \\ 2A-B+2C+D=5 \\ B \quad +D=1 \end{cases}$$

12. Solve the following problems:

*a*) Motion problem: *A* and *B* run a race of 450 ft. The first heat *A* gives *B* a start of 135 ft. and is beaten by 4 seconds; the second heat *A* gives *B* a start of 30 ft. and beats him by 3 seconds. How many feet can each run in a second?

*b*) Digit problem: The middle digit of a number of three digits is 2. The sum of the digits is 6. If the digits are written in reverse order the number will be multiplied by $\frac{107}{41}$. Find the number.

*c*) Work problem: *A* and *B* can do a piece of work in 35 days, B and C can do it in $17\frac{1}{2}$ days, C and A can do it in 21 days. How long will it take each alone to do it?

## III. Radicals

1. Define rational and irrational numbers, radical, surd, principal root, radical equation.
2. Illustrate the following properties of radicals:

*a*) $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

*b*) $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$

3. Arrange in order of magnitude:

$$3\sqrt{\frac{5}{2}},\ 7\sqrt{\frac{2}{5}},\ \sqrt{40},\ \frac{18}{\sqrt{10}}$$ (Yale)

4. Express as a single radical:

$$\frac{\sqrt{2a}\sqrt[4]{4a}\sqrt[3]{3a}}{\sqrt[6]{64a}}$$ (Yale)

5. Simplify the following expressions:

*a*) $\sqrt{\frac{2}{27}a^3}+\sqrt{\frac{8}{3}a}$

*b*) $\sqrt{2}\times\sqrt[3]{3}$

*c*) $\sqrt[3]{81}\div\sqrt[6]{9}$

*d*) $3\sqrt{3}+\sqrt{27}-11\cdot\sqrt{48}$

*e*) $\sqrt[3]{40x^4}-\sqrt[3]{320x^4}+\sqrt[3]{135x^4}$

*f*) $\frac{1}{3}\sqrt{45}+4\sqrt{\frac{5}{4}}-\sqrt{125}$

6. Multiply $\sqrt{2}+\sqrt{3}-\sqrt{7}$ by $\sqrt{2}-\sqrt{3}+\sqrt{7}$.

7. Divide $4\sqrt{105}+8\sqrt{40}-15\sqrt{12}$ by $2\sqrt{15}$.

8. Rationalize the denominator of

*a*) $\frac{2\sqrt{7}-2\sqrt{343}+7\sqrt{28}}{6\sqrt{63}}$ *b*) $\frac{2+\sqrt{-3}}{7+\sqrt{-3}}$ (Yale)

*c*) $\frac{1}{2+\sqrt{5}-\sqrt{2}}$ (Yale) *d*) $\frac{(\sqrt{2}+3)(\sqrt{5}-1)}{(3-\sqrt{2})(1+\sqrt{5})}$ (Yale)

9. Divide $\frac{1}{\sqrt{x-1}}$ to four terms and extract the square root of the quotient to 3 terms.

10. Solve $\sqrt{21+x}=\sqrt{28+2x}-1$. Check your solution.

11. Solve $\frac{2x-3}{\sqrt{x-2}}=2\sqrt{x-2}-1$.

## IV. Exponents

1. Prove informally the following laws for positive integral exponents:

*a*) $a^m\cdot a^n=a^{m+n}$

*b)* $\frac{a^m}{a^n}=a^{m-n} \quad (m>n)$

*c)* $(a^m)^n=a^{mn}$

*d)* $(abc)^n=a^n b^n c^n$

*e)* $\left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}$

*f)* $\sqrt[n]{a^{mn}}=a^m$

2. Reduce to simplest form $(x^3y^4z^5)^7$; $\left(\frac{7a^4b^{11}}{13yz^6}\right)^3$; $\frac{a^{2n}}{a^2}$; $(2^{a+b}\cdot 3^c\cdot 5^b)^{a-b}$; $\frac{(a^2-ab)^7}{(a-b)^5}$.

3. The 2d power of $x$ is multiplied by the 7th power of the cube of $a$. Divide this product by the 4th power of $c$. Find the square of this quotient.

4 Give the meaning of fractional exponent, zero exponent, and negative exponent.

5. Express with radical signs $a^{\frac{1}{2}}b^{\frac{3}{4}}$; $\left(\frac{a^3}{b^6}\right)^{\frac{1}{3}}$; $\left(\frac{27}{64}\right)^{-\frac{1}{3}}$.

6. Express as powers with fractional exponents $\sqrt[2]{a^4}$; $\sqrt[3]{x^2}\sqrt{x^6}$; $\sqrt[5]{\frac{ab^2}{c^3d^4}}$; $\sqrt[4]{\frac{16x^3y}{2^6}\,\frac{2^6}{\sqrt{x^2y^4}}}$.

7. Simplify the following:

*a)* $(ayx^{-1})^{\frac{1}{2}}\times(bxy^{-2})^{\frac{1}{3}}\times(y^2a^{-2}b^{-2})^{\frac{1}{4}}$

*b)* $a^{\frac{1}{3}}\cdot a^{-\frac{3}{4}}\cdot\sqrt{a^4}\cdot a^{\frac{1}{2}}\cdot\sqrt[3]{a^{\frac{25}{4}}}\cdot(a^{-\frac{7}{4}})^{\frac{7}{6}}$

*c)* $\left[(x^m)^{m-\frac{n^2}{m}}\right]^{\frac{1}{m-n}}$

*d)* $\left[(a^{x+y})^{x-y}\cdot(a^{y^3})^{\frac{1}{y}}\right]^{\frac{1}{y^2}}$

*e)* $\sqrt[7]{a^2b^{12}}\cdot\left(\frac{1}{ab}\right)^{\frac{1}{7}}\cdot\left(\frac{y^2}{x^3}\right)^{-\frac{2}{7}}$

*f*) $[\{(a^2-x^{-2})^{-1}\}^{-2}]^5$

*g*) $\left(x^{a+1}\div x^{\frac{a-1}{a+1}}\right)^{\frac{a-1}{2a}}$

*h*) $\dfrac{\dfrac{x-y}{2y+x}+\frac{1}{2}}{3\frac{1}{2}-\left(\dfrac{x+2y}{5x+7y}\right)^{-1}}$

## V. The Quadratic Equation

1. Evaluate the quadratic function $x^2-8x+12$ for values of $x$ from 0 to 8. Plot these values on cross-section paper and trace the curve that represents this function.
2. Define quadratic function; quadratic equation; pure quadratic; an affected quadratic.
3. Name four methods of solving quadratic equations.
4. Solve by factoring $2x^2-x-6=0$.
5. Solve by completing the square $6x^2-7x-4=0$.
6. Solve the quadratic equation $ax^2+bx+c=0$ by the method of completing the square and thus derive the quadratic formula $x=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$. Show that this number gives a solution for any quadratic equation.
7. Solve by the formula:

   *a*) $6x^2-7x-4=0$

   *b*) $x^2-1.6x-0.23=0$

   *c*) $x^2-2x(m-n)+n^2=2mn$

   *d*) $abcx^2-(a^2b^2+c^2)x+abc=0$

   *e*) $\sqrt{5x}-\sqrt{3x+1}=\dfrac{4}{\sqrt{3x+1}}$

   *f*) $\dfrac{5x+2}{2x+5}+\dfrac{1}{20}=\dfrac{2x+5}{3(x+2)}$

   *g*) $x^4-5x^2+4=0$

   *h*) $3x^{\frac{3}{2}}-4x^{\frac{3}{4}}=7$

*i*) $1-4x^{-\frac{1}{6}}+3x^{-\frac{1}{3}}=0$

*j*) $(5x-3)^2-9(5x-3)=286$

*k*) $x^2-4x+2\sqrt{x^2-4x+7}=1$

*l*) $\sqrt{x^2-2x+9}-\frac{x^2}{2}=3-x$

8. Verbal problems involving quadratics:

   *a*) A man standing on the edge of a cliff throws a stone vertically downward with a velocity of 16 ft. per second. After $t$ seconds the distance in feet of the stone below the starting-point is $16\ (t+t^2)$. Find correct to hundredths of a second when the stone will be 20 ft. below the starting-point.

   *b*) Two launches race over a course of 12 miles. The first steams $7\frac{1}{2}$ miles per hour. The other has a start of 10 minutes, runs over the first half of the course with a certain speed, but increases its speed over the second half of the course by 2 miles per hour, winning the race by a minute. What is the speed of the of the second launch?

   *c*) The hypotenuse of a right triangle is 17 ft. and one of the sides is 7 ft. longer than the other. Find the length of the sides.

9. Let $r_1$ and $r_2$ be the two roots of the quadratic equation $ax^2+bx+c=0$. Show that—

   *a*) $r_1+r_2=-\frac{b}{a}$

   *b*) $r_1 \cdot r_2=\frac{c}{a}$

10. Write the sum and product of the roots of the following quadratic equations:

    *a*) $3x^2-17x-18=0$.

    *b*) $x^2+x+3x\sqrt{3}+4=0$

    *c*) $(a-b)x^2-2ax-bx+3ab=0$

11. Write the quadratic equation whose roots are 6 and 4; 8 and $\frac{11}{3}$; $\frac{-1+\sqrt{-3}}{2}$; $\frac{-1-\sqrt{-3}}{2}$; $a$ and $\frac{b}{3}$.

12. What is the discriminant of the quadratic equation $ax^2+bx+c=0$?

By means of the discriminant discuss the nature of the roots of the following equations:

*a*) $x^2-6x+9=0$
*b*) $2x^2-5x+2=0$
*c*) $3x^2-7x+8=0$

13. Discuss the general theory concerning the nature of the roots of the quadratic equation $ax^2+bx+c=0$.

14. For what value of $k$ do the following equations have equal roots? Unequal real roots? Imaginary roots?

*a*) $9x^2+6x+k=0$
*b*) $kx^2+2kx-3x+2=0$

## VI. Simultaneous Quadratics

1. What is the form of the general type of a quadratic equation in two unknowns?
2. Graph the following equations (in different colors if possible) on the same sheet of cross-section paper and with a common origin:

*a*) $x^2=4y$
*b*) $x^2+y^2=49$
*c*) $4x^2+9y^2=288$
*d*) $xy=5$
*e*) $x-y-10=0$

Is it possible to tell from the equations anything about the shape of the curves?

3. From the set of curves drawn in problem 2 above choose by pairs the curves which intersect. Note the $x$ and $y$ values of these points of intersection and see if they satisfy the equations whose curves seem to cross at that point.
4. How may these graphic solutions be verified?
5. Solve algebraically the systems chosen in problem 3 above.
6. Are there any curves in the graph of problem 2 which do not intersect? How does the algebraic solution show that the equations of such a system are not simultaneous?
7. It is not possible to formulate a unique method for solving all systems of simultaneous quadratic equations. However, it is helpful to keep in mind four cases that are most common. Note each carefully and be able to solve each system.

   *a*) Case I: In which each equation is of the form $ax^2+by^2=c$:

$$\begin{cases} 16x^2+27y^2=576 \\ x^2+y^2=25 \end{cases}$$

   *b*) Case II: When one equation is quadratic and the other is linear (or when the system can easily be reduced to this case):

$$\begin{cases} 3x^2+4x+2y=89 \\ 2x^2-3x-4y=47 \end{cases}$$

$$\begin{cases} x-3y=2 \\ x^2-9y^2=8 \end{cases}$$

   *c*) Case III: When each equation is of the second degree and homogeneous in the terms containing the unknowns:

$$\begin{cases} x^2+3xy=28 \\ xy+4y^2=8 \end{cases}$$

*d*) Case IV: When the equations are symmetrical or symmetrical with respect for sign:

$$\begin{cases} x^3+y^3=35 \\ x+y=5 \end{cases}$$

$$\begin{cases} x^3-y^3=19 \\ x-y=1 \end{cases}$$

## VII. The Binomial Theorem

1. Write out the expansions indicated below:

$(a+b)^0=$
$(a+b)^1=$
$(a+b)^2=$
$(a+b)^3=$
$(a+b)^4=$

2. Show that the results of problem 1 above agree with the binomial formula

$$(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3+\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}a^{n-4}b^4+\ldots\ldots+\frac{n(n-1)(n-2)\ldots\ldots(n-k+1)}{1\cdot 2\cdot 3\cdot 4\ldots\ldots k}a^{n-k}b^k+\ldots\ldots+\frac{n(n-1)}{1\cdot 2}a^2b^{n-2}+nab^{n-1}+b^n$$

for $n=0, 1, 2, 3, 4$.

3. Show by examining the general term of the binomial formula and by comparing it with the expansions in problem 1 above that the formula for the general term is $(k+1)$st term $=\frac{n(n-1)(n-2)\ldots\ldots(n-k+1)}{1\cdot 2\cdot 3\cdot 4\ldots\ldots k}a^{n-k}b^k$.

4. Expand the following:

*a*) $\left(\frac{2}{x}-\frac{3x^2}{4}\right)^5$ *b*) $\left(\sqrt{a}+\sqrt[3]{b}\right)^4$

*c*) $(1+2x+x^2)^4$ *d*) $\sqrt[3]{9}=(8+1)^{\frac{1}{3}}$ (to 3 terms)

5. Find the first 5 terms of $\left(\frac{x}{\sqrt{2}}-y\sqrt{2}\right)^{12}$.

6. If the middle of the expansion of $\left(3x-\frac{1}{2\sqrt{x}}\right)^4$ is equal to the 4th term of the expansion of $\left(2\sqrt{x}+\frac{1}{2x}\right)^7$, find the value of $x$.

7. In the expansion $\left(x-\frac{1}{x}\right)^{12}$, what is the term that does not contain $x$?

## VIII. Logarithms

1. How may $a^x$ be defined if $x$ is an irrational number?
2. Show that if your definition of $a^x$ be adopted the index laws may be advantageously used for irrational exponents.
3. It can be proved that if $a$ is a positive number, not unity, there exists one and only one $x$ (positive negative, or zero) such that $a^x=N$ where $N$ is any positive number. Assuming the truth of this theorem, give a definition of a logarithm.
4. What are the logarithms of

   *a*) 2, 4, 18, 16, 64, 128 with respect to the base 2?
   *b*) 3, 9, 27, 81, 241 with respect to the base 3?
   *c*) 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$; $\frac{1}{64}$ with respect to the base 2?

5. Evaluate $1^x$, $2^x$, $3^x$, $5^x$ when $x=0$. What is the logarithm of 1 with respect to the bases 1, 2, 3, 5?
6. Show that the following properties of logarithms are consequences of the index laws and the definition of logarithms:

*a*) $\log_a(uv)=\log_a u+\log_a v$

*b*) $\log_a\left(\frac{u}{v}\right)=\log_a u-\log_a v$

*c*) $\log_a u^p=p\log u$

*d*) $\log_a\sqrt[n]{u}=\frac{\log_a u}{n}$

*e*) $\log_u u=1$

*f*) $\log_a 1=0.$

Give a word-statement of each.

7. If $\log_{10} 2=0.30103$, $\log_{10} 3=0.47712$, find $\log_{10} 12$, $\log_{10}\frac{3}{2}$, $\log_{10}\left(\frac{27}{4}\right)$, $\log_{10}\sqrt[3]{6}$.

8. Express, in terms of $\log_a m$ and $\log_a n$ the following:

$\log_a(m^2n^3)$, $\log_a\left(\frac{m^3}{n^2}\right)$, $\log_a\sqrt{\frac{m^{-4}}{n^{-7}}}$

9. Define "characteristic" and "mantissa" and show how each is determined for the logarithm of any number.
10. What is a cologarithm? When is it useful?
11. Evaluate to four significant figures by means of logarithms.

*a*) $9.307\times0.008769$

*b*) $-7984\times59.87$

*c*) $(0.8267)^{\frac{2}{3}}\times(0.7628)^{\frac{3}{4}}$

*d*) $\sqrt[3]{\frac{528\times0.05736}{0.6234\times(512.1)^2}}$

12. What is the volume of a hemispherical dome if its diameter is 252.42 ft.? (The volume of a sphere of radius $r$ is $\frac{4}{3}\pi r^3$.)

13. Prove

$$\log_a \frac{y+\sqrt{y^2-1}}{y-\sqrt{x^2-1}}=2 \log_a (y+\sqrt{y^2-1})$$

14. Solve $5^x=354$ by logarithms. (Harvard.)

## IX. Arithmetical Progression

1. Define an arithmetical progression.
2. Derive the four chief formulas of arithmetical progression given below:

   a) $l=a+(n-1)d$

   b) $S=\frac{n}{2}(a+l)$

   c) $S=\frac{n}{2}[2a+(n-1)d]$

   d) $A=\frac{a+l}{2}$

3. Find the 200th odd number.
4. A city with a population of 20,000 increased 500 persons per year for 5 years. What was the population at the end of 5 years?
5. Find the sum of the first 20 even numbers.
6. If a body falls approximately 16 ft. the first second and 32 ft. farther in each succeeding second, how far does it fall in 10 seconds?
7. Insert 10 arithmetic means between $-7$ and 144.
8. There are $m$ arithmetic means between 1 and 31, and the seventh mean is to the $(m-1)$st mean as 5 is to 9. What is the number of means? (M.I.T.)
9. If $a, b, c$ are in arithmetic progression, show that $a^2(b+c)$, $b^2(c+a)$, $c^2(a+b)$ are also in arithmetic progression.

## X. Geometrical Progression

1. Define a geometric progression.
2. Derive the five chief formulas of geometrical progression given below:

a) $l=ar^{n-1}$

b) $S=\frac{rl-a}{r-1}$

c) $S=\frac{ar^n-a}{r-1}$

d) $S_\infty=\frac{a}{1-r}$

e) $G=\sqrt{ab}$

3. Write the 25th term of the progression 1, 3, 9, 27, 81. . . . .
4. Find the sum of the first 20 multiples of 4.
5. A rubber ball falls from a height of 40 in., and on each rebound rises 40 per cent of the previous height. Find by formula how far it falls on its eighth descent. (Yale.)
6. Insert 4 geometric means between $-\frac{1}{10}$ and $3\frac{1}{5}$.
7. Find the limit of the sum of the following infinite series:

   a) $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+. . . . .$

   b) $.3+.03+.003+. . . . .$

## XI. On the Recognition of Form

A great many problems are hard to solve because they are not recognized as more complicated examples of a well-known type. They often become open to a rather simple treatment when they can be readily recognized or can be shown to be in the form of functions or equations already treated. A ready recognition of such forms is necessary in gaining mathematic power, and the problems which follow are examples that will furnish a drill toward this end.

1. Evaluation:

   a) Find the value of
   $a-\{5b-[a-(3c-3b)+2c-3(a-2b-c)]\}$
   where $a=-3$, $b=4$, $c=-5$. (Yale.)

*b*) Find to two decimal places the value of

$\sqrt{4a^{-\frac{2}{5}}+b^{0}\sqrt{ab^{-1}}}$

when $a=-32$, $b=-8$. (Vassar.)

*c*) If $x=2\pm\sqrt{-3}$, find the value of $x^2-4x+7$. (Wellesley.)

*d*) Find the value of the fraction

$$\frac{3x-1}{3x+2}$$

where $x$ is the positive root of the equation $9x^2+6x-19=0$. Compute the value of the above fraction to 3 significant figures. (Harvard.)

2. Type Products, Multiplication and Division, Roots:

*a*) Complete the following: $72\cdot 68=(70-2)(70+2)=$

*b*) $$\left[\frac{l^x+l^{-x}}{2}+\sqrt{\left(\frac{l^x+l^{-x}}{2}\right)^2-1}\right]\left[\frac{l^x+l^{-x}}{2}-\sqrt{\left(\frac{l^x+l^{-x}}{2}\right)^2-1}\right]$$ (Dartmouth)

*c*) Complete the following: $79=(80-1)^2=$

*d*) Expand $(a+b-c).^2$

*e*) Multiply $(4a^{-1}-5b^{-1}+6ab^{-2})$ by $(a^2b^{-1}+3a^3b^{-2})$.

*f*) Divide $6x^{-1}y^2+x^{-\frac{1}{2}}y-23+18xy^{-2}$ by $3x^{\frac{3}{4}}y^{-1}+x^{\frac{1}{4}}-2x^{-\frac{1}{4}}y$. (M. I. T.)

*g*) Divide 1 by $\sqrt{x}-1$ to four terms and extract the square root of the quotient to three terms. (Yale.)

*h*) Find the square root of:

(1) $4+2\sqrt{3}$ (3) $7+\sqrt{48}$

(2) $8-\sqrt{60}$ (4) $a+2b+\sqrt{ab}$

*i*) Find the square root of:

$$\frac{9a^2c^{2m}}{4b^{12}}-\frac{3ac^{m+n}}{b^3}+b^6c^{2m}-\frac{2^8ac^m}{b^6}+\frac{2^9b^3c^n}{3}+\frac{2^{16}}{9}.$$ (M. I. T.)

3. Factoring:

 *a*) Factor the following:

  (1) Binomials:

   (*a*) $64-2a^5$, $x^8-y^8$, $a^3-8b^{15}$, $a^4+4$, $x^{16}+a^6xy^9$, $8+2\sqrt{15}$, $64x^{6a}-y^{12a}$, $(x^2-y^2)^2-(x^2-xy)^2$, $4a^2b^2-(a^2b^2-c^2)^2$ (Yale)

   (*b*) $(a^3+8b^3)(a+b)-6ab(a^2-2ab+4b^2)$ (M.I.T.)

  (2) Trinomials:

   (*a*) $x^{10}-31x^5-32$, $8x^3-6xy(2x+3y)+27y^3$ (Dartmouth)

   (*b*) $x^{2p}-4x^p+4$, $\frac{x^2}{y^2}-\frac{3y^2}{x^2}+2$, $x^4-3x^2+9$, $6a^{-4}-2a^{-1}d^{\frac{1}{3}}-8a^2d^{\frac{2}{3}}$, $(a+b)^2-(a^2-b^2)+6(a-b)^2$

   (*c*) $x^3-27+3x(x-3)$, $(x+y)^3+3xy(1-x-y)-1$, $4x^4+9y^4-37x^2y^2$ (M. I. T.)

  (3) Four-termed expressions:

   (*a*) $a^2+ac-4b^2-2bc$, $x^2-6ax-9b^2-18ab$, $4x^2+2ax-y^2-ay$ (Princeton)

   (*b*) $x^3-7x^2+14x-8$, $x^2-x-y^2-y$ (Smith)

   (*c*) $p^3q^3-p^2q^2-pq+1$, $x^3-x^2-x-2$, $x^3-2x^2-2x-3$ (Williams)

  (4) Polynomials of more than four terms:

   (*a*) $x^2+y^2-z^2-n^2-2zn+2xy$, $ax^2-3ax+2a+bx^2-3bx+2b$ (Dartmouth)

   (*b*) $x^2-a^2+y^2-4-2xy+4a$, $a^3x-a^2c+a^2by-ab^2x-b^3y+cb^2$ (Yale)

   (*c*) $x^4+2x^3+3x^2+2x+1$, $a^2x+abx+ac+b^2y+aby+bc$

  (5) State and prove the Factor Theorem.

4. Highest Common Factor and Lowest Common Multiple:

 *a*) Find the H.C.F. and the L.C.M. of the following:

  (1) $(x^3+a^3)(x^2+a^2)$; $(x^2+ax+a^2)(3x-a)$ and $3x^2+2ax-a^2$ (Harvard)

(2) $3x^3+3$; $6x^2+36x+54$; and $x^3+2x^2-2x+3$ (Cornell)

(3) $8a^3-18ab^2$, $8a^3+8a^2b-6ab^2$ and $4a^2-8ab+3b^2$ (M.I.T.)

(4) $x^5-2x^4+x^2$ and $2x^4-4x^3-4x+6$ (Yale)

5. Reduction of fractions; solution of fractional equations:

*a*) Perform the indicated operations and simplify the result as far as possible:

(1) $$\frac{x^3+x^2-x+2}{x^4-x^3-x^2+2x-2}$$ (reduce to lowest terms) (Bryn Mawr)

(2) $$\frac{(a+b)^2-(c+d)^2}{(a+c)^2-(b+d)^2}$$ (M.I.T.)

(3) $$\frac{3x+2}{x^2-5x+6}-\frac{x}{x^2+2x-15}+\frac{4-x}{x^3+3x^2-10x}$$ (M.I.T.)

(4) $$\left(\frac{x^3-8y^3}{x^2-y^2}\right)\left(\frac{x^2-xy-2y^2}{x^2-4xy+4y^2}\right)$$ (Yale)

(5) $$\left(\frac{x^4-y^4}{x^2-y^2}\div\frac{x+y}{x^2-xy}\right)\div\left(\frac{x^2+y^2}{x-y}\div\frac{x+y}{xy-y^2}\right)$$ (Sheffield)

(6) $$1-\frac{\left(1-\frac{x^4}{9}\right)-\left(1-\frac{x^4}{16}\right)}{1-\frac{7x^4}{144}}$$ (Wellesley)

(7) $$\frac{\frac{1}{a}+\frac{1}{b+c}}{\frac{1}{a}-\frac{1}{b+c}}\left[1+\frac{b^2+c^2-a^2}{2bc}\right]$$ (Harvard)

(8) $$\frac{\dfrac{x^{-5}(-2x^{-1})-(x^3+3)(-5x^{-4})}{(x^{-5})^2}}{\dfrac{x^{-2}+3}{x^{-5}}}$$ (Williams)

(9) Solve for $x$:

(a) $\frac{5}{x+1}+\frac{8}{x-2}=\frac{12}{40-2x}$ (Vassar)

(b) $\frac{2(x-7)}{x^2+3x-28}+\frac{2-x}{4-x}-\frac{x-3}{x+7}=0$ (Yale)

(c) $\frac{x-p+q}{x+p-q}=\frac{p-q-x}{p+q+x}$ (Princeton)

6. Exponents:

*a*) Simplify

(1) $\left[x^{\frac{a+b}{c}} \cdot x^{\frac{a-b}{c}} \div x^{\frac{2a-1}{c}}\right]^{-c}$

(2) $7x^0-(7x)^0-1^7+\frac{3}{4^{\frac{3}{2}}}$ (Yale)

*b*) Express as one term and then evaluate to two decimal places the expression

$$\sqrt{20}+3\sqrt{\frac{4}{5}}+3\left(\frac{5}{9}\right)^{-\frac{1}{2}}-\left(\frac{25}{81}\right)^{\frac{1}{4}}$$ (M.I.T.)

*c*) Solve for $x$ when

$$\frac{\frac{2b}{c}\sqrt[12]{\frac{a^8b^{18}}{c^3}}}{x}=\frac{\frac{2(3ac)^2}{\sqrt[15]{b^9c^5}}}{\frac{3\sqrt{a}}{4b}}$$ (M.I.T.)

*d*) Simplify the expression

$$\left[\frac{1+\frac{1}{4}(c^5-c^{-5})^2}{\frac{1}{2}(c^5+c^{-5})}\right]^{\frac{3}{2}}$$ (Harvard)

7. Binominal Theorem:

*a*) Expand $(1+2x+x^2)^4$ (M.I.T.)

*b*) Write the 6th term of

$$\left(\frac{4\sqrt[4]{b^3}}{3a^5}-\tfrac{1}{2}a\sqrt{ab^{-\frac{1}{5}}}\right)^9$$

Reduce the result so that it contains only one radical and no negative or fractional exponents. (Yale.)

*c*) Find the middle term of the expansion for $(1+x)^{16}$ when

$$x=(39)^{-\frac{1}{8}}\frac{\sqrt{a^3}}{\sqrt[3]{a^2}}b^{-\frac{3}{8}},$$

and reduce to its simplest form. (Harvard.)

*d*) Raise 98 to the 5th power by the Binomial Theorem. (Yale.)

8. Logarithms:

*a*) Solve the equation $3.142^x=2.718$ by logarithms. (Harvard.)

*b*) Find by means of logarithms the value of

$$\left[\frac{3.1416\times 0.0321^2}{0.0241}\right]^{-0.32}.$$ (Harvard)

*c*) Find the value of $\log_9 81+\log_2 8$. (Yale.)

*d*) Show by logarithms how many ciphers there are between the decimal point and the first significant figure of the value of $\left(\frac{2}{3}\right)^{100}$, given $\log 2=0.30103$, $\log 3=0.47712$. (Cornell.)

## A TENTATIVE MINIMUM COURSE IN FIRST-YEAR MATHEMATICS

### FIRST SEMESTER

I. The Equation.
  1. Definition.
  2. Its use as applied to simple number relation problems.
  3. As applied to perimeters and areas of rectangles, squares, and triangles.
  4. Solution of abstract problems involving fractions.
  5. Evaluation of functions.
  6. The equation as applied to angles.
    *a*) Definition of an angle.
    *b*) Measurement of an angle.
    *c*) Use of protractor.
    *d*) Sum of all the angles about a point in a plane.
    *e*) Sum of all the angles about a point on one side of a straight line.
    *f*) Adjacent angles.
    *g*) Complementary angles.
    *h*) Supplementary angles.
    *i*) Vertical angles.
    *j*) Sum of the interior angles of a triangle; of the exterior angles.

II. Positive and Negative Number.
  1. Drawing of temperature curves.
  2. Illustration of positive and negative number by the idea of direction; interpretation of meaning as applied to debit and credit; as to forces; meaning on thermometer scale; the use children make of it in playing games.
  3. Application of fundamental operations to simple algebraic monomial numbers.
  4. Graphing of simple equations in two unknowns.

III. Fundamental Operations as Applied to Polynomials.

IV. Further Work.
1. In graphing of linear equations.
2. Discussion of ratio.
3. Similarity of triangles.
4. Laws of proportion.

### SECOND SEMESTER

I. Concrete Problems to Be Solved by Equations in One Unknown.
1. Geometric problems.
2. Number problems.
3. Mixture problems.
4. Motion problems.
5. Clock problems.
6. Weight and alloy problems.
7. Percentage problems.
8. Beam problems.

II. Solution of Equations Involving Two Unknowns.
1. Graphical solution.
2. Algebraic methods of elimination.
   *a*) Addition and subtraction.
   *b*) Comparison.
   *c*) Substitution.
3. Study of consistent and inconsistent equations.
4. Systems of three linear equations.
5. Solution of equations in which unknown numbers occur in denominators of fractions, and which reduce to linear equations.
6. Solution of concrete problems by means of two unknowns.
7. Emphasis on beam problems.

III. Factoring.
1. Monomial factors.
2. Grouping so as to discover a binomial factor.

3. Trinomial squares.
4. Solution of quadratic equations by factoring method and by method of completing square. A graphical study of simple quadratic equations introduces the above methods. Interpretation of irrational numbers: e.g., $\sqrt{7}$, $\sqrt{5}$, and $\sqrt{3}$.
5. Difference of two squares.
6. Trinomials of the form $x^2+ax+b$.
7. Given: the roots of a quadratic equation, to write the equation.
8. Trinomials of the form $ax^2+bx+c$.
9. Sum and difference of two cubes.
10. Solution of equations involving fractions which may be simplified by factoring.
11. Study of the simpler types of verbal problems leading to quadratic equations.

IV. Fractions.

1. Reduction.
2 Simplification.
3. Fundamental operations as applied to fractions.

V. The construction of figures by the use of compasses and unmarked straight edges only.

1. To construct a perpendicular to a line at a point on the line.
2. To construct a perpendicular to a line from a point outside the line.
3. To bisect a line.
4. To bisect an angle.
5. To construct an angle equal to a given angle.
6. Simple applications of constructions 1 to 5 above.

VI. Propaedeutical study of geometric relations of parallel lines cut by a transversal; of parallelograms, trapezoids, and various angle relations; solution of problems by means of simultaneous linear equations.

# A TENTATIVE MINIMUM COURSE FOR INTERMEDIATE ALGEBRA

I. Simple Functions: Their Transformation; the Solution of Simple Equations.

1. Notions of variable (dependent and independent), constant, and function of a variable.
2. Relation of two variables as shown by the graph.
3. Concrete problems in variation leading up to direct, inverse, and joint variation.
4. More problems in variation leading up to ratio and proportion.
   *a*) Simple treatment of properties.
5. Simple transformations. (Usual type-products as review.)
6. Solution of simple equations. (Formal and verbal.)

II. Equations in Two, Three, or More Unknowns: Methods of Solution.

1. Equations in two unknowns. Solution by
   *a*) Graph.
   *b*) Addition and subtraction.
   *c*) Comparison.
   *d*) Substitution.
   *e*) Determinants.
2. Verbal problems involving two unknowns.
3. Equations in three or more unknowns. Solution by
   *a*) Ordinary methods of elimination.
   *b*) Determinants.

III. Radicals.

1. Introduction.
   *a*) Solution of problems in variation.
   *b*) " " " " mean proportionals that give rise to radicals.

2. Properties of radicals.
3. Transformation of radicals.
4. Operations on radicals.
5. Radical equations.

IV. Exponents and the Theory of Logarithms.

1. Laws for positive integral exponents.
2. Meaning of fractional, negative, and zero exponent.
3. Problems involving the use of the above.
4. Definition of a logarithm.
5. Properties of logarithms.
6. Problems whose solution involve the use of logarithms.

V. The Quadratic Equation.

1. Study of the graphs of the quadratic function $y=ax^2+bx+c$.
2. Reduction of expressions to the quadratic form.
3. Solution of the general quadratic equation $ax^2+bx+c=0$ by
   *a*) Factoring.
   *b*) Completing the square.
   *c*) Formula.
   *d*) Graph.
4. Maximum and minimum value.
5. Verbal problems.
6. Theory of quadratic equations.
   *a*) Incomplete quadratics.
   *b*) Nature of the roots.
   *c*) Sum and product formulae.
   *d*) Factor theorem on number of roots.
7. Problems involving a knowledge of the above theory.

VI. Simultaneous Quadratic Equations.

1. One equation quadratic and the other linear.
2. Both equations of type $ax^2+by^2=c$.
3. Both equations homogeneous.

4. Both equations symmetrical.
5. Methods of solution for above.

VII. The Binomial Formula.
1. Proof by mathematical induction.
2. Development of formula for $(k+1)$st term.
3. Problems.

VIII. Progressions.
1. Arithmetical.
   *a*) Formulae for
      (1) Last term.
      (2) Sum of $n$-terms.
      (3) Arithmetic mean.
2. Geometric.
   *a*) Formulae for
      (1) Last term.
      (2) Sum of $n$-terms.
      (3) Geometric mean.
3. Problems.

IX. On the Recognition of Form.
1. Type products and their application to arithmetic.
2. Factoring of complicated cases.
3. Taking of square root of binomial radical expressions.
4. Solution of simultaneous equations in $\frac{1}{x}$ and $\frac{1}{y}$.
5. Irrational equations reducible to quadratic form.