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UNIFIED MATHEMATICS

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PREFACE

This text presents a course in elementary mathematics adapted to the needs of students in the freshman year of an ordinary college or technical school course, and of students in the first year of a junior college. The material of the text includes the essential and vital features of the work commonly covered in the past in separate courses in college algebra, trigonometry, and analytical geometry.

The fundamental idea of the development is to emphasize the fact that mathematics cannot be artificially divided into compartments with separate labels, as we have been in the habit of doing, and to show the essential unity and harmony and interplay between the two great fields into which mathematics may properly be divided; *viz.*, analysis and geometry.

A further fundamental feature of this work is the insistence upon illustrations drawn from fields with which the ordinary student has real experience. The authors believe that an illustration taken from life adds to the cultural value of the course in mathematics in which this illustration is discussed. Mathematics is essentially a mental discipline, but it is also a powerful tool of science, playing a wonderful part in the development of civilization. Both of these facts are continually emphasized in this text and from different points of approach.

The student who has in any sense mastered the material which is presented will at the same time, and without great effort, have acquired a real appreciation of the mathematical problems of physics, of engineering, of the science of statistics, and of science in general.

A distinctly new feature of the work is the introduction of series of "timing exercises" in types of problems in which the student may be expected to develop an almost mechanical ability. The time which is given in the problems is wholly tentative; it is hoped, in the interest of definite and scientific knowledge concerning what may be expected of a freshman, that institutions using this text will keep a somewhat detailed record of the time actually made by groups of their students. The authors invite the coöperation of teachers of elementary college mathematics in the attempt to secure this valuable information. The authors will make every effort to put information thus secured at the service of the public interested.

In general, the diagrams are carefully drawn on paper with subdivisions of twentieths of an inch. It is expected that this kind of paper will be used as far as possible in the graphical work, as students will be found to acquire rapidly the ability to use intelligently this type of coördinate paper. Considerable attention should be paid by the teacher to the intelligent reading and interpretation of the diagrams which appear in the text, as the student will in this way gain power to handle his own diagrams, and appreciation of the vital importance of the method. The photographic illustrations should also be used in a somewhat similar manner.

The material can be covered without systematic omissions in a course which devotes five hours per week for one year to the study of mathematics. In a four-hour course there are certain omissions which can be made by the teacher at his own discretion; the three chapters on solid analytical geometry are not commonly presented in the ordinary four-hour course; the chapter on "Poles and Polars" may also be omitted. The exercises are so numerous that any teacher can make a selection, which can be varied, if desired, in succeeding years.

No attempt has been made to introduce the terminology of the calculus as it is found that there is ample material in the more elementary field which should be covered before the student embarks upon what may properly be called higher mathematics. However, the fundamental idea of the derivative is presented and utilized without the new terminology.

The authors are greatly indebted to a large number of their colleagues who have been most generous in furnishing real illustrations in various fields. Professor N. H. Williams of the Department of Physics at the University of Michigan has given very pertinent and valuable comment on numerous sections, in addition to furnishing the beautiful oscillograms of alternating currents. Professor W. J. Hussey of the Detroit Observatory furnished the temperature and barometer chart, and has given generously of his time in the discussion of astronomical problems adapted to an elementary text. Professors J. J. Cox, H. E. Riggs, A. F. Greiner, H. H. Higbie, J. C. Parker, Leon J. Makielski, E. M. Bragg, H. W. King, and L. M. Gram of the Department of Engineering, University of Michigan, have given valu-

able advice and suggestions. The diagram illustrating the use of the ellipse in determining the proper amounts of sand and gravel to use from given pits to obtain the best results was furnished by Professor Cox. To Professor Greiner we are indebted for the cut of the six cylinders of an automobile engine, and for criticising the piston-rod motion. To Mr. Makielski, the well-known artist, we are indebted for the drawing of a box which is reproduced. Professor James W. Glover of the Actuarial and Statistical Department, University of Michigan, has read and corrected the material relating to his field. To Professor C. L. Meader of the Department of Linguistics, and to Professors Pillsbury and Shepard of the Department of Psychology, University of Michigan, we are indebted for the tuning-fork records and for the vowel and consonant records. To Professor F. G. Novy of the Hygienic Laboratory we are indebted for certain information concerning bacterial growth. Captain Peter Field, Coast Artillery, U.S.A., has indicated to us certain simple problems connected with artillery work. To Mr. H. J. Karpinski we owe the photographs of the Rialto and the Colosseum. To the Albert Kahn Company of Detroit we are indebted for information concerning details of the Hill Auditorium, and to the Tyrrell Engineering Company of Detroit for permission to reproduce a number of photographs of bridges. We render to these gentlemen and to our colleagues who have been generous in giving time and thought to our inquiries our sincere appreciation for their friendly coöperation. In every field which we touch, we assume full responsibility for all errors, and we shall be grateful to teachers who will assist in removing the inevitable blemishes in a book of this size and character.

The proof has been carefully read by Professor E. V. Huntington of Harvard University and by Professor C. N. Moore of the University of Cincinnati. Many blemishes have been removed and many important additions and changes have been made on their advice. Professor J. W. Bradshaw of the University of Michigan and Professor J. D. Bond of the Texas Agricultural College have read the galley proof and given numerous and excellent suggestions. Professor W. W. Beman of the University of Michigan has read the page proof and has made numerous vital corrections and suggestions. Professor J. L. Markley has given advice on the early chapters. To all of these gentlemen we acknowledge our real indebtedness.

In putting the work through the press, the responsible editorship has been placed in the hands of Professor Karpinski, as the exigencies of time and space — Texas to Michigan to New York to Boston — would have delayed the book for a full year with a divided responsibility. Certain chapters, including the chapter on the applications of the conic sections, the chapters on the sine curve, on the growth curve and on complex numbers, the treatment of solid analytics, the tables and most of the problems, are due entirely to Professor Karpinski.

The drawings have been made at the University of Michigan, chiefly by Mr. E. T. Cranch, an engineer now in the service, U.S.A. Most of the photographs are by Miss F. J. Dunbar of the University of Michigan Lantern Slide Shop, and a few are by Mr. G. R. Swain.

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UNIFIED MATHEMATICS

CHAPTER I

NUMBERS OF ALGEBRA

1. Representation of points on a line. — With any given unit of length and a fixed point of reference, called the origin, the points upon a given line are located by numbers. The unit of length and parts thereof are laid off in both directions from the origin to locate further points. To each point corresponds one number (a symbol) and only one, and to each number corresponds one and only one point. We call this a one-to-one correspondence. To any point upon the line of reference corresponds evidently another point symmetrically placed with respect to the origin. This symmetry is indicated in the symbols by using the same set of symbols twice, distinguishing by two "quality" signs + and −. All points on one side of 0 have the + sign prefixed to the symbols designating them, while the corresponding points on the other side take the same symbols, prefixing the negative sign. Thus $+a$ and $-a$ represent symmetrically placed points on the scalar line. The line of reference is now called a *directed* line. Of two numbers represented by points on this line, the one represented by that one of the two points which lies to the right hand is called the greater. Such a line is the line on an ordinary thermometer; to each number, then, there corresponds further a certain temperature. Thousands of physical and material interpretations of the points upon such a line and the corresponding numbers are possible.

This scalar line, as the above is termed, is not necessarily a straight line. Thus the equator is a scalar line as it is represented upon any globe, with the zero at the intersection with the meridian of Greenwich, and distances given in degrees, each representing $\frac{1}{360}$ part of the equatorial circumference of the earth; + and − are represented on this line by E. and W.

PROBLEMS

1. Interpret a scalar line as representing distance upon the main line of the Michigan Central Railroad from Detroit, east and west.

2. What is the significance of points to the left of the origin when the line represents your bank account?

3. Interpret the line as representing weights.

4. Interpret the line as the prime meridian. What length is represented by 1° (circumference of earth is 25,000 miles)?

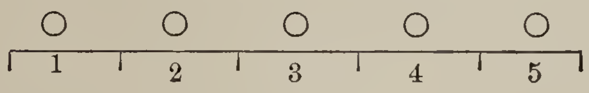
5. Interpret the scale as representing percentage of fat in foods.

6. Represent the Fahrenheit scale on one side of such a line and the Centigrade upon the other, making the zero and the 100° of the Centigrade scale fall upon the 32° and 212° of the Fahrenheit; note that for a convenient total length 20° Fahrenheit may be taken as corresponding to 1 centimeter or to one half an inch.

2. Real numbers; positive integers. — The symbols representing the points upon a line, as above, are called real numbers. Elementary algebra is largely a study of such numbers, combined according to certain rules. The rules of the game of algebra, as we may term it, can be studied entirely apart from any physical application, but the study is of fundamental importance because of the part which algebraic numbers play in the sciences. However, a knowledge of the laws of algebra, apart from the applications, is necessary to enable one to apply the numbers effectively to physical problems.

The real numbers are sub-divided into *positive* and *negative* numbers; another classification is into *rational* and *irrational* numbers, the rational numbers being further sub-divided into *integers* and *fractions.* Numbers are represented by the letters $a, b, c, \cdots x, y, z$, etc.

Integers were undoubtedly conceived long before man began to write. The idea of an integer involves the notion of a group of individual objects, and of one-to-one correspondence. The idea or notion which is common to all groups of objects which can be placed in one-to-one correspondence with the objects of a given group is called the number of the given group of objects. Thus the pennies ○ ○ ○ ○ ○ can be placed in one-to-one correspondence with some segments of our line, or with the group of symbols which correspond to these



segments, or with the individuals of any one of infinitely many other groups, of number *five*, which have the one common property that they can be placed in one-to-one correspondence with each other. The definition is recent; the idea is old. One-to-one correspondence appears frequently in physical problems, as in the one-to-one correspondence between degrees Centigrade and degrees Fahrenheit above.

Integers can be used to represent segments of our line of reference, from O as reference point, with some length as unit of measure (or as individual of the group). The extremity farthest from O is marked with the integer corresponding to the number of the segments between that point and O. Evidently certain groups of segments include as subgroups other groups of segments. The number of the including group is called greater than the number of any included group; the included group is smaller, and its number is less than the number of the including group. Thus the group called eight, 8, has the smaller sub-groups, 1, 2, 3, 4, 5, 6, and 7.

3. Positive integers; fundamental laws, definitions, assumptions, and theorems. — Given two positive integers, a and b, the single group composed of the individuals from two distinct groups of objects, represented by a and b respectively, is represented by another number x, the *sum* of a and b, which latter we term *summands*. The process of finding such a number is called *addition*, and is indicated by writing the sign + between the two given numbers a and b. By the sum of three numbers is meant the number obtained by adding the third to the sum of the first two, and similarly for more numbers than three. The following are assumptions and theorems concerning integers.

I. $x = a + b$, given two integers, the sum exists.

II. $a + b = b + a$, addition is commutative, *i.e.* the order of addition is immaterial.

III. $a + b + c = (a + b) + c = a + (b + c)$; the associative law for addition.

IV. $x + b = a$. Given the sum a and one of the summands, the other summand exists: x is the number which added to b gives a. This defines the operation of subtraction, which is represented by the sign $-$, to be placed between the sum and the given summand, as in $a - b = x$.

By definition, $(a - b) + b = a$, and for the present this has meaning only when b is less than a; a is termed *minuend*, b is termed *subtrahend*, and $a - b$ is the *remainder*.

Thus, given $x + 2 = 7$, x is evidently 5, as one remembers that in the operation of addition 5 added to 2 gives 7. Given $x + 2 = 2$, or $x + 2 = 1$, we have, at this stage of development, no number x which satisfies the given condition.

V. If $a + c = b + c$, $a = b$, and conversely.

The converse is equivalent to the axiom, if equals be added to equals the sums are equal.

VI. $x = a \cdot b$. Suppose that each individual of a group of a objects consists of b individuals of another type, *e.g.* 4 rows, each of 7 dots, then the single group consisting of all the second type of individuals involved is called the product of a and b. The operation is called *multiplication* and is represented by the sign × between a and b, or by a period (slightly elevated) between a and b, or by simple juxtaposition of the two numbers, a and b, called factors; a is termed *multiplier*, and b is *multiplicand.*

VII. $a \cdot b = b \cdot a$, the commutative law for multiplication, evident from the figure.

VIII. $a \cdot b \cdot c = (a \cdot b) \cdot c = a\,(b \cdot c)$, the associative law for multiplication.

IX. $a(b + c) = ab + ac$, multiplication is distributive with respect to addition. This corresponds precisely to our ordinary method of multiplication.

X. $b \cdot x = a$. Given the product a, and one factor b, x is defined by this relation as the number which multiplied by b gives a. This operation is limited *when dealing with integers* to numbers a and b, which are so related that a is one of the products obtained by multiplying b by an integer.

The process of finding x is called *division*, and is represented by the sign ÷, or by placing a over b, $b \cdot \frac{a}{b} = a$; b is termed the *divisor*, a is the *dividend*, and x is termed the *quotient.*

These laws concerning positive integers constitute simply a restatement of facts with which the student is familiar. The four fundamental operations to this point have been confined to the field of positive integers; evidently the operation of division when b is the divisor applies only to those positive integers which are multiples of b. Similarly the operation of subtraction of b from a is limited to integers so related that

$a > b$. We extend our field of numbers by removing these limitations. Thus if you wish to have a number x which multiplied by 8 gives 5 you do not find it among the positive integers; you may then decide to create such a number, calling it $\frac{5}{8}$, the two symbols indicating the definition and genesis of the new number. Such extensions of the number field are briefly indicated in the next section.

4. Rational numbers; zero, fractions, and negative integers. — These fundamental equalities and definitions from I to X are now extended by removing all limitations (except one, as noted below) upon the numbers, a, b, c, and x. Note that only the operations of addition, subtraction, multiplication, and division are included at this point.

Extension of IV. $x + b = a$, when $b = a$ defines zero, written 0. By definition then, $0 + a = a$. $x + b = 0$, defines a negative number which is written $-b$. The negative here is a sign of quality; by definition $-b$ is the result of subtracting b from 0, and $-b + b = 0$.

$x + b = a$, when $b > a$, defines the negative number $a - b$, which is the negative of $b - a$.

Subtracting a negative number can now be shown to be equivalent to adding a positive number, and similarly the other rules of elementary algebra relating to the addition and subtraction of positive and negative quantities. That the product of two numbers with like signs is positive and the product of two numbers with unlike signs is negative follows from the above development.

A negative number $-a$ is placed in our line of reference symmetrically to the corresponding positive number a, with respect to the origin; of two negatives, the one toward the right is called the greater.

Extension of X. $b \cdot \frac{a}{b} = a$, for all values of b except 0.

This extension of $\frac{a}{b}$ to mean a number which multiplied by

b gives a introduces new numbers of the type $\frac{a}{b}$, rational fractions in which a and b are positive or negative integers. Integers are included in this definition if b is a factor of a or if b is equal to one. Division by 0 is explicitly excluded.

A rational number is any number which can be expressed as the quotient of two integers, the denominator not to be zero.

All of our rules for operating with fractions follow from the definition of $\frac{a}{b}$ and from the preceding development. Thus $\frac{a}{-b} = -\frac{a}{b} = \frac{-a}{b}$. Further, by definition,

$\frac{a}{b} > \frac{c}{d}$, when both are positive if $ad > bc$;

$\frac{a}{b} = \frac{c}{d}$, when both are positive if $ad = bc$; and

$\frac{a}{b} < \frac{c}{d}$, when both are positive if $ad < bc$.

Positive fractions can thus be arranged in a determined order upon our line of reference; the value of the fraction determines the position and a graphical method of locating $\frac{a}{b}$ on the scalar line is indicated in the next section; negative fractions are placed symmetrically to the corresponding positive fractions, with respect to the origin.

Of any two rational numbers a and b, a is greater than b ($a > b$) if $a - b$ is positive; for when $a - b$ is positive, a positive length must be added to b to give a, and consequently a must lie to the right of b. If on the line of reference two points x_1 and x_2 are taken (fixed points), $x_2 - x_1$ gives the distance from the first point to the second; this expression is positive if $x_2 > x_1$, and negative if $x_2 < x_1$.

That there is a distinction between + and − used as signs of operation, as with positive integers in the preceding section, and + and − used as quality signs is apparent. Thus $-b$ may indicate that b is to be subtracted from some preceding number, or $-b$ may indicate that the distance b is taken on the negative side of the origin. The fact that $a+(-b)$, the addition to a of negative b, gives the same result as subtracting b from a, or $a-b$, is readily shown by the graphical method of section 9 below. This type of relationship obviates any need for careful distinction between the two possible meanings of these signs and makes separate symbols not necessary. When no sign is used with a number symbol the + sign is understood.

EXERCISES

1. Explain the distinction between $\frac{-3}{7}$ and $-\frac{3}{7}$; between $\frac{3}{-7}$ and $\frac{-3}{7}$.

2. Write $\frac{-5}{3-x}$ in the three forms corresponding to $\frac{-a}{b}$, $-\frac{a}{b}$, and $\frac{a}{-b}$.

3. Is $-3 > -2$? Which is greater, 0 or -3? Explain.

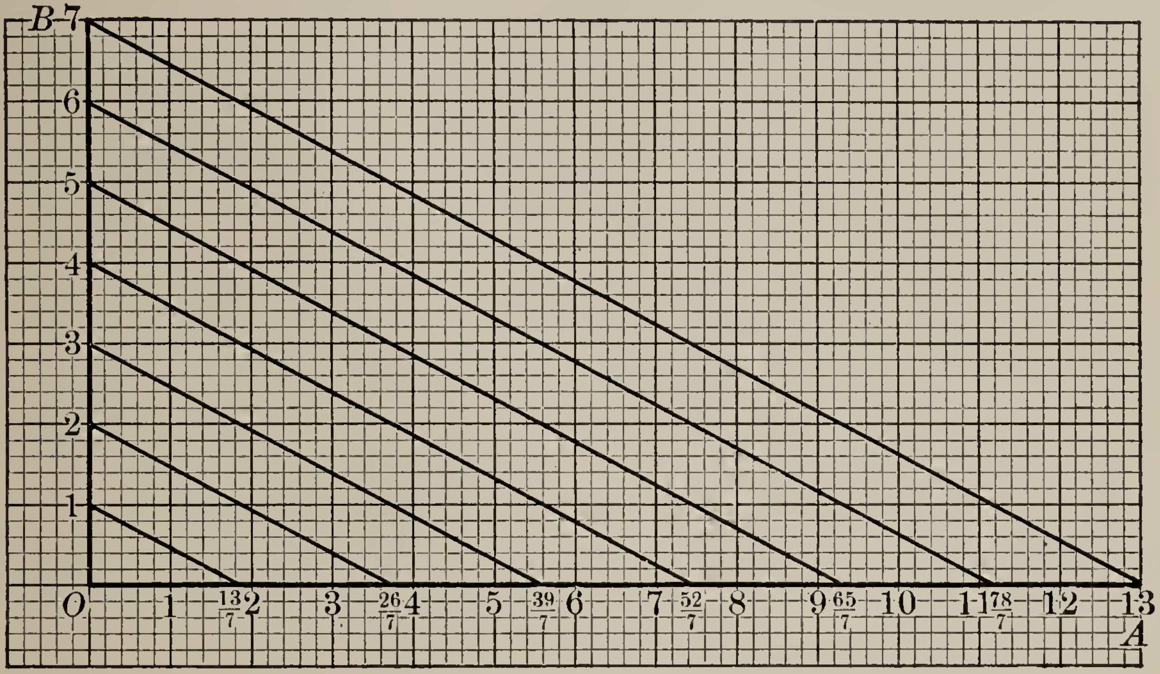
4. What is the difference between 4 and -3? 4 and 3? 4 and 11?

5. What fundamental law is assumed in the common process of multiplication, *e. g.* as in 325 by 239 and also $x-7$ by $x-2$? Is there a corresponding assumption in division?

6. Which is greater, $\frac{1}{2}$ or $-\frac{2}{3}$? Is $\frac{17}{19}$ greater or less than $\frac{33}{38}$? Explain.

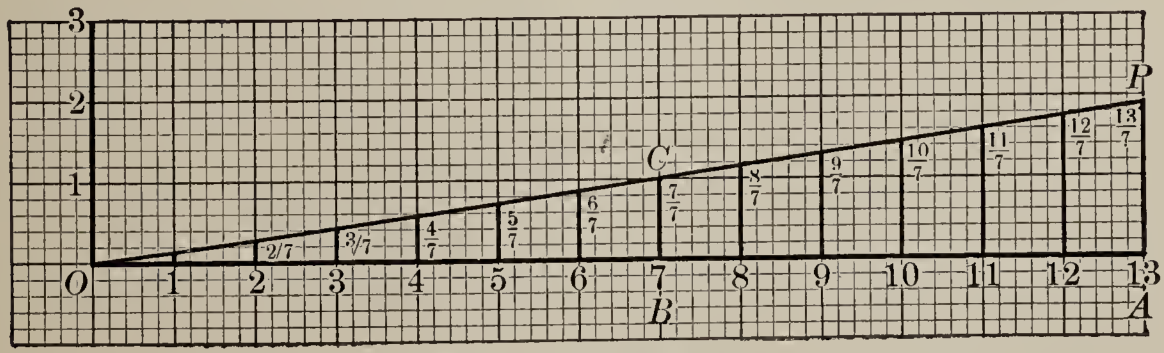
7. What is the product of 0 by 7; by -8; by 3; by $\frac{5}{17}$? If a product is zero, what limitation is imposed upon the factors?

5. Representation of a rational number, $\frac{a}{b}$. On cross-section paper any rational fraction can be represented, using ruler and compass. Using 5 divisions to represent unity, each division represents $\frac{1}{5}$ of a unit. To represent $\frac{13}{7}$, one measures off 13 units, OA, on the line of reference, and 7 units, OB, on a second line through the origin (for convenience, use cross-section paper). Connect the ends, AB, and through the point U, one unit from O on the second line, draw a line parallel to AB. The intersection point on the reference line represents the fraction $\frac{13}{7}$. Similarly any fraction $\frac{p}{q}$ can be represented. The series of parallels to AB through the first 7 unit points on the vertical axis will cut off (plane geometry theorem) 7 equal parts of 13 on the horizontal axis.



Graphical division

$\frac{1}{7}$ to $\frac{6}{7}$ of 13 represented on horizontal line of reference.

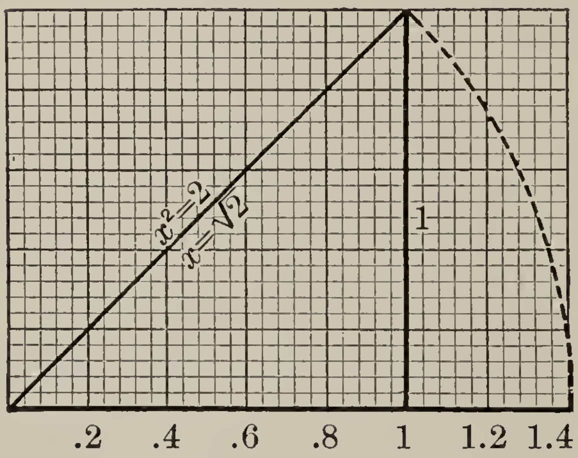


Graphical division

$\frac{1}{7}$ to $\frac{13}{7}$ expressed decimally.

On cross-section paper a somewhat better method of indicating any quotient $\frac{a}{b}$ is to move out on the line of reference b units and up 1 unit; connecting this point with the origin O gives a straight line which can be used to read the desired quotients. Thus, since $OB = 7$ units, $BC = 1$, and $OA = 13$, it follows that $\frac{AP}{BC} = \frac{OA}{OB}$, or $\frac{AP}{1} = \frac{13}{7}$, whence AP equals $\frac{13}{7}$. To obtain $\frac{8.5}{7}$, you find the point 8.5 units from O, and the vertical distance to the oblique line represents $\frac{8.5}{7}$, or 1.2.

6. Irrational numbers. $x^2 = 2$ is a simple and familiar illustration of a relation which is not satisfied by any rational number, $\frac{a}{b}$, with a and b integers; geometrically, the diagonal of a square with side unity is not represented by any rational number. If you wish the length of this diagonal for any practical purpose, you use $\frac{14}{10}$, or $\frac{113}{80}$, or $\frac{141}{100}$, or $\frac{17}{12}$, or $\frac{1414}{1000}$, or $\frac{14142}{10000}$. The carpenter uses 1 foot 5 inches, or $1\frac{5}{12}$ feet, in the diagonal for every foot of side, with an error of $\frac{3}{10}$ of one per cent. The series of rational numbers 1, $\frac{14}{10}$, $\frac{141}{100}$, $\frac{1414}{1000}$, $\frac{14142}{10000}$, which can be indefinitely extended always increasing, and the series, always decreasing, 2, $\frac{15}{10}$, $\frac{142}{100}$; $\frac{1415}{1000}$, $\frac{14143}{10000}$, with a constantly decreasing difference of limit 0 between corresponding terms, together define the irrational number called the square root of 2. No rational number satisfies the relation; no number $\frac{a}{b}$ is at the end of either series, but either series determines a definite point on

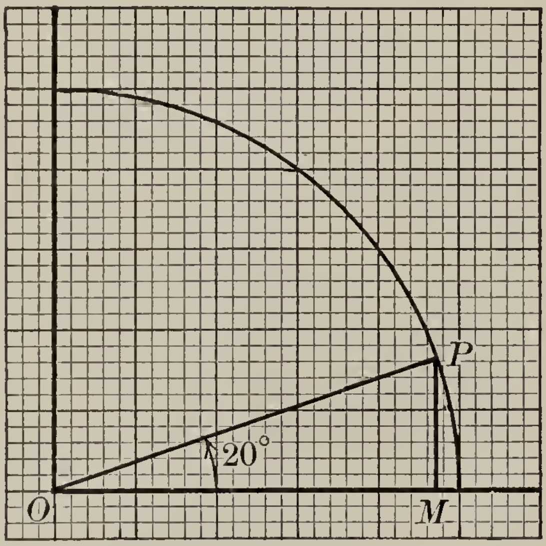


Graphical representation of $\sqrt{2}$

our line, and algebraically defines our number, which we will call the square root of 2.

Proof of the irrationality of $\sqrt{2}$. — Assume that $\sqrt{2}=\frac{p}{q}$, a rational fraction in lowest terms, with p and q integers. Both p and q cannot be even numbers, either p is odd or q is odd. If p is odd, squaring and clearing of fractions, $2q^2=p^2$; but p is odd and you have an even number equal to an odd number. Hence p cannot be an odd number. Now assume that p is even and that q is odd, and further let $p=2m$. Then $2q^2=4m^2$, $q^2=2m^2$, and again we have an odd number equal to an even number. Our assumption that $\sqrt{2}=\frac{p}{q}$ leads to an absurdity, that an odd number equals an even number.

Describe about the origin with a radius 10 a circle, and using a protractor measure an angle of 20 degrees. The length of the perpendicular and the part cut off by the perpendicular from the end of this line are definite and precise points which can be computed to any degree of accuracy desired. No rational numbers represent these lengths, which are trigonometric irrationalities. A series of constantly increasing rational numbers can be found, such that there is no greatest of the series, to represent lines which are always shorter than the given line; and another series of terms constantly decreasing, but approaching to the terms of the first series, can be found. No largest number can be found in the first series and no smallest in the second; both sequences together define, we may say, an irrational number.



A trigonometric irrational

EXERCISES

1. Write 6 terms of the decreasing series defining $\sqrt{2}$; $\sqrt{3}$.

Arrange in order of magnitude 6 numbers with squares greater than 2.

2. How is the series for defining the length of the circumference of a circle obtained? What assumption is made?

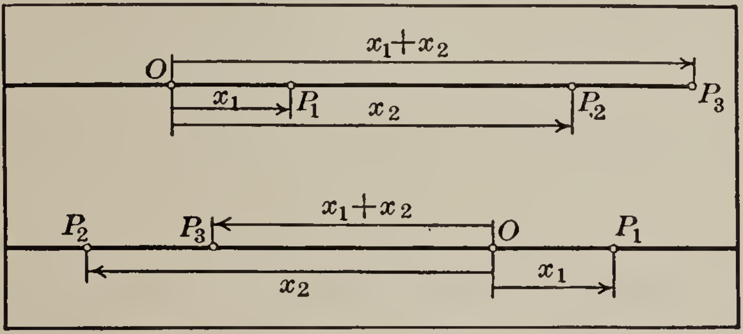
3. Find by actual multiplication six numbers between 60 and 63 whose squares are less than 3910.

4. Inscribe a square and an equilateral triangle in the circle of radius 10; find the sides.

7. Constants and variables. — Every fixed point on the line of reference is at a fixed distance from the point of reference, O; the distance is constant. We can think of a point as moving on the line OX in either direction. The distance from O then varies and we speak of the distance as a variable. Thus, also, the price of wheat during a term of years or in different parts of the world is a variable; the weight of an animal at different ages is a variable. We think of the variable quantity as taking a series of values under diverse conditions. We can represent the variable distance of a point from O on our line by the single letter x, which may then in the various possible positions on the line be thought of as positive, or negative, or zero, as rational or irrational. This letter x represents then a variable quantity and is essentially a number, subject to all the operations on algebraic quantities as noted above. In general, we designate the variable point by the single letter P; the distance from O is OP, of which the length and direction from O are indicated by the number or variable x. A point on the line $X'X$ is represented by a single letter x, called the *abscissa* of the point. The fixed points on the line are frequently represented by the letters $a, b, c, d, \cdots$ or by $x_1, x_2, x_3, \cdots$, each of which may represent any point upon the line.

8. Historical note. — Modern algebra with the systematic employment of literal coefficients, letters to represent general constants, was introduced by the great French mathematician and statesman, François Viète (1540–1603); Viète used the consonants to represent known quantities and vowels to represent unknowns, using capitals for both. The use of the symbols of operation in equations dates also from about the same time; our equality sign was introduced by Robert Recorde, an English physician and mathematician, whose "Whetstone of Witte," 1557, is the first treatise in the English language on algebra. The + and – symbols are due to a German, Widmann, and date from 1489.

9. Geometrical equivalents of the four fundamental operations. *a.* **Addition.** — The operation of addition of x_1 and x_2 is represented graphically by placing the length OP_2 upon the line



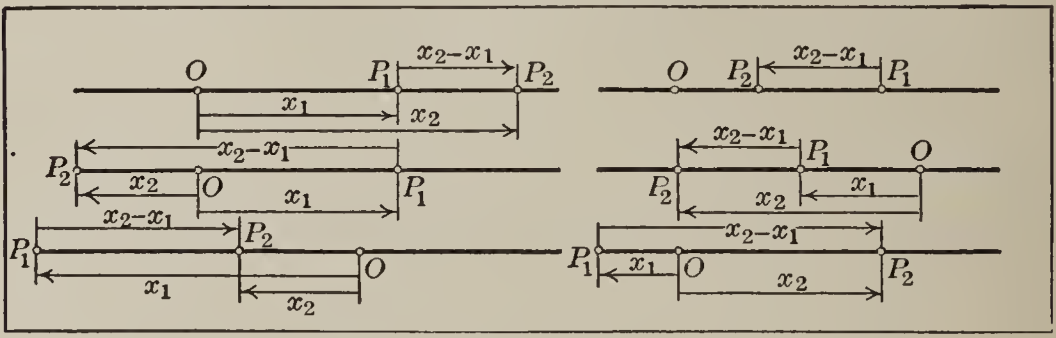
$$OP_3 = OP_1 + P_1P_3 = OP_1 + OP_2 = x_1 + x_2.$$

from the point P_1 in the direction of OP_2. Physically, addition is the result in general of two different causes. Thus a weight of 3 pounds + a weight of 5 pounds; a vertical velocity due to the action of gravity (on a falling body) + a vertical velocity due to some other force; a transportation (translation) from one point to another + another translation in the same direction; two successive rotations of a wheel about its axis; these are familiar examples of addition.

b. **Subtraction.** $OP_1 + P_1P_2 = OP_2$; $P_1P_2 = OP_2 - OP_1$. Whatever the relative positions on the line OX of P_1 and P_2, with respect to the position of O, $OP_1 + P_1P_2 = OP_2$, all of these representing directed line segments. In words, the equality $P_1P_2 = OP_2 - OP_1$ states that the distance from any point P_1 on a directed line to a second point on the line is given by the abscissa of the second point minus the abscissa of the first, with respect to any third point O, on the line, as origin.

If we represent the distance from P_1 to P_2 by the letter d, we have $x_1 + d = x_2$, or $d = x_2 - x_1$. Subtraction is represented by the distance from the first point to the second point, which equals $OP_2 - OP_1$. Since in physical problems the distance represents the change in numerical value of physical quantities, graphical subtraction is more frequently noted than addition. We say the temperature has risen 10 degrees or fallen 10 degrees, having in mind the original and the final reading; when we say that an iron bar has expanded an inch, the initial and the final length and the change in length are important.

DISTANCE BETWEEN TWO POINTS ON A SCALAR LINE



$OP_1 + P_1P_2 = OP_2$; $P_1P_2 = OP_2 - OP_1$; $d = x_2 - x_1$.

Fundamental property of any three points on a directed line

The formula,

$$d = x_2 - x_1,$$

gives the distance P_1P_2 from the point P_1, abscissa x_1, to the point P_2, abscissa x_2; the algebraic sign gives the direction. The formula gives then the change in value of a variable x in passing from the value x_1 to the value x_2.

c. **Multiplication and division.** — Graphical multiplication and division upon cross-section paper, involving theorems concerning similar triangles, are indicated by the diagrams on page 9.

PROBLEMS

1. What is the distance from 2 to 8 on a scalar line?

2. On a thermometer what is the change from $-3°$ to $+25°$? from $-10°$ to $+30°$? from $+30°$ to $-10°$? from $-10°$ to $-20°$?

3. Use the formula, $d = x_2 - x_1$, to find the distance from -2 to $+8$ on a scalar line, noting that $x_2 = 8$, and $x_1 = -2$; from -2 to -8; from -11.3 to $+24.7$.

4. Represent the square root of 2 geometrically, taking 10 quarter-inches as 1. Represent the square root of 3 on coördinate paper, using the same scale. Represent the square root of 5, 6, 7, and 8.

5. Draw a semicircle on a diameter of 10 half-inches. Note that the perpendicular at any point on this diameter is, by plane geometry, a mean proportional between the segments of the diameter. Read the mean proportional between 1 and 9, as the vertical line drawn at the point on the diameter 4 units out from the center. Read the mean proportional between 2 and 8, similarly; between 3 and 7; between 4 and 6.

6. Regard the preceding circle as having a radius 10 quarter-inches. Find approximately, from it, $\sqrt{19}$, $\sqrt{36}$, $\sqrt{51}$, $\sqrt{64}$, $\sqrt{75}$, $\sqrt{84}$, $\sqrt{91}$, $\sqrt{96}$, $\sqrt{99}$, and $\sqrt{100}$. These are the vertical lengths at the points dividing the diameter in the ratio 19 to 1; 18 to 2; 17 to 3; ⋯ 10 to 10.

7. Take a circle of diameter 12 half-inches; from it approximate $\sqrt{11}$, $\sqrt{20}$, $\sqrt{27}$, $\sqrt{32}$, $\sqrt{35}$, and $\sqrt{36}$. Note that $\sqrt{20} = 2\sqrt{5}$, $\sqrt{27} = 3\sqrt{3}$, and $\sqrt{32} = 4\sqrt{2}$; from these approximate $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$. See diagram on page 72.

8. On the preceding circle check the geometrical fact that either side of a right triangle is a mean proportional between the whole hypotenuse and the adjacent segment of the hypotenuse cut off the perpendicular from the vertex of the right angle.

CHAPTER II

FORMULAS OF ALGEBRA AND GEOMETRY WITH ARITHMETICAL APPLICATIONS

1. Algebraical formulas, products, and factors. —

$$(x+a)^2 = x^2 + 2ax + a^2.$$

$$(x-a)^2 = x^2 - 2ax + a^2.$$

$$(x+a)(x-a) = x^2 - a^2.$$

$$(x+a)(x+b) = x^2 + (a+b)x + ab.$$

$$(ax+b)(cx+d) = acx^2 + (ad+bc)x + bd.$$

$$mx + nx = (m+n)x.$$

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2).$$

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2).$$

These formulas may readily be derived by actual multiplication. They should be read and used both from left to right and from right to left. In the problems drill is given on the use of these formulas both in obtaining products and in the reverse operation of factoring.

The student would do well to state all of these formulas in words.

Thus, the square of a binomial equals the square of the first term plus the square of the second term plus twice the product of the first by the second.

PROBLEMS

1. Expand the following:

$(3x-a)^2$, $(3v-2a)^2$, $(3x+4a)^2$, $(x-10)^2$, $(10+2y)^2$, $(x+\frac{1}{2})^2$, $(x+\frac{7}{2})^2$, $(x-\frac{5}{3})^2$, $(2m+3n)^2$, $(-2m-3n)^2$.

2. Perform the operations indicated:

$(3x+2a)(3x-2a)$; $(x-2)(x^2+2x+4)$; $(2x+3)(4x^2-6x+9)$; $\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{a}+\frac{y}{b}\right)$; $(x-\frac{1}{2})(x+\frac{1}{2})$; $(-a+x)(-a-x)$; $(x+a+b)(x+a-b)$; $(10+a)(10+b)$; $(20+a)(20+b)$.

3. Factor the following:

a. x^2-4x+4.
b. $4x^2-4x+1$.
c. $x^2+x+\frac{1}{4}$.
d. $y^2-\frac{1}{4}$.
e. $8x^2-32$.
f. $27x^3-48x$.
g. v^3-8.
h. $8x^3+1$.
i. $ax+x$.
j. $ax-x$.

4. Factor the following:

a. $x^2+8x+15$.
b. x^2+8x+7.
c. x^2-8x+7.
d. $4x^2+16x+15$.
e. $4x^2+16x+7$.
f. $4x^2-9$.
g. x^2-3x+2.
h. $y^2-10y+16$.
i. v^2-3v-4.
j. $6x^2+29x+35$.

5. Factor:

a. x^3-8y^3.
b. x^3-64xy^2.
c. $27-8y^6$.
d. $8m^3+n^3$.
e. $100-49n^2$.
f. $p^2-v^2t^2$.
g. x^2-2y^2.
h. $m^2-4mn-21n^2$.
i. $(x+y)^2-z^2$.
j. $mx-my+nx-ny$.

6. Complete the following expressions, involving the squares of binomials:

a. $(x^2-4x+\quad)=(x-\quad)^2$.
b. $(y^2+8y\quad)=(y+\quad)^2$.
c. $(x^2-\frac{3}{2}x+\quad)=(x-\quad)^2$.
d. $(t^2-\frac{7}{5}t+\quad)=(t-\quad)^2$.
e. $3(x^2-4x+\quad)=3(x\quad)^2=3x^2-12x+\quad$.
f. $5(y^2-\frac{4}{5}y+\quad)=5(y-\quad)^2=5y^2-4y+\quad$.
g. $3t^2+7t+\quad=3(t^2+\frac{7}{3}t+\quad)=3(t+\frac{7}{6})^2$.

2. Division of polynomials. — An expression of the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots a_{n-1}x + a_n$$

wherein n is a positive integer, a_0 is not zero, and $a_0, a_1, a_2 \cdots a_n$ are real numbers is called a rational integral function of degree n in x, or a polynomial of degree n in x. For convenience the symbols $f(x)$, $Q(x)$, $F(x)$, $\cdots$ are used to represent such polynomials.

Given a polynomial in x and $x - a$, the process of finding a second polynomial which when multiplied by $x - a$ gives the first, or the first with the exception of a remainder, is called division; the polynomial found is called the quotient.

Illustrative Problem. — Divide $3x^3 - 2x^2 + 7x - 5$ by $x - 2$.

$$\begin{array}{r}
3x^2 + 4x + 15 \\
x - 2\overline{)3x^3 - 2x^2 + 7x - 5} \\
\underline{3x^3 - 6x^2} \\
+4x^2 \\
\underline{+4x^2 - 8x} \\
+15x \\
\underline{+15x - 30} \\
+25
\end{array}$$

$$3x^3 - 2x^2 + 7x - 5 \equiv (x - 2)(3x^2 + 4x + 15) + 25.$$

The three bars, $\equiv$, are used to indicate that the expression is an identity, true for all values of x and thus placing no limitation upon x.

PROBLEMS

In each problem express the result of the division in the form of an identity, as in the problem above.

1. Divide $3x^3 + 8x^2 - 6x - 5$ by $x - 1$; by $x + 2$.

2. Divide $2x^3 - 8x^2 + 3x - 5$ by $x - 2$; by $x + 1$.

3. Divide $x^3 - 2{,}000{,}000$ by $x - 120$; by $x - 126$.

4. Divide 3842 by 27, and express the result as a numerical identity.

5. Divide $3x^4 - 6x^2 + 10x - 7$ by $x - 1$.

6. Divide $x^4 - 10x^2 - 8$ by $x^2 - 2$.

7. Divide $6t^3 - 2t + 1$ by $2t + 1$.

3. Remainder theorem and factor theorem. — When a polynomial in x is divided by $x - a$, the division can be continued until the remainder is a constant. If $f(x)$ represents the polynomial, $Q(x)$ the quotient, and R the remainder, we have the identity,

$$f(x) \equiv (x - a)Q(x) + R.$$

Substituting a for x in this identity, we have

$$f(a) \equiv (a - a)Q(a) + R,$$

or,
$$R = f(a).$$

In words, the remainder obtained by dividing a polynomial in x by $x - a$ is the same polynomial in a, *i.e.* the dividend with a substituted for x. This is the remainder theorem.

ILLUSTRATION.

$$3x^3 - 2x^2 + 7x - 5 \equiv (x - 2)(3x^2 + 4x + 15) + 25;$$

substituting 2 for x, we have

$$3 \cdot 2^3 - 2 \cdot 2^2 + 7 \cdot 2 - 5 \equiv 25,$$

since $2 - 2 = 0$; the remainder 25 could have been obtained without actual division by $x - 2$, by simply substituting 2 for x in

$$3x^3 - 2x^2 + 7x - 5.$$

If $f(a) = 0$, a is called a **root** of $f(x) = 0$, or a **zero** of $f(x)$. In this case the remainder when $f(x)$ is divided by $x - a$ equals zero, or $x - a$ is a factor of $f(x)$.

$$\boldsymbol{f(x) \equiv (x - a)\,Q(x)}, \text{ when } f(a) = 0.$$

This is called the factor theorem.

ILLUSTRATIONS.

$3x^3 - 2x^2 + 7x - 30 = 0$, when $x = 2$; $x - 2$ is a factor.
$2x^2 - 7x + 5 = 0$, when $x = 1$; $x - 1$ is a factor.
$2x^2 + 7x + 5 = 0$, when $x = -1$; $x + 1$ is a factor.

The common "check by nines" may readily be proved by the remainder theorem:

When 12,738, which may be written

$$1 \times 10^4 + 2 \times 10^3 + 7 \times 10^2 + 3 \times 10 + 8,$$

in powers of 10, is divided by $10 - 1$, the remainder is equal to the original expression with 1 put for 10. Hence the remainder when 12,738 is divided by 9 is $1 + 2 + 7 + 3 + 8$, or 21, or 3 (since 21 divided by 9 gives 3 as remainder, or $2 + 1$).

Thus, $a \times 10^n + b \times 10^{n-1} + c \times 10^{n-2} + \cdots g \times 10 + h$ divided by $10 - 1$ gives $a + b + c + \cdots + g + h$ as remainder.

Division by 11, $10 + 1$, gives as remainder the sum of the odd coefficients less the sum of the even coefficients, counting from units' place; a sum to 11 can, of course, be dropped as it occurs, or 11 can be added to make the remainder a positive number.

PROBLEMS

1. By substituting $+1$ and -1, respectively, in the following expressions determine those in which either $x - 1$ or $x + 1$ is a factor; factor where possible.

a. $2x^2 - 6x + 4.$

b. $3x^2 + 5x + 2.$

c. $5x^2 + 6x - 11.$

d. $5x^2 - 5x.$

e. $x^3 - 1.$

f. $3x^2 - 5x - 2.$

g. $2x^3 - 7x^2 - 8x - 1.$

h. $2x^3 + 7x^2 - 4x - 9.$

2. For what value of a is $x^3 - 3x^2 - 7x - a$ exactly divisible by $x - 1$? by $x - 2$?

3. For what value of a will $ax^3 - 7x^2 - 3x - a$ be exactly divisible by $x + 1$?

4. Show that $x^n - y^n$ is always divisible by $x - y$ when n is an integer.

5. For what integral values of n is $x + y$ a factor of $x^n + y^n$?

6. Form the equation in x in which 3 and 4 are the roots, employing the factor theorem.

NOTE. If 3 is a root, $x - 3$ is a factor.

7. Form the equation whose roots are -3, 4, and 1.

3. What is the remainder when each of the following numbers is divided by 9? by 11?

a. 327.
b. 847.
c. 951.
d. 3276.
e. 8217.
f. 12,321.
g. 1001.
h. 3003.

4. Arithmetical application of algebraic formulas. — Algebraic formulas and methods can frequently be applied to arithmetical problems with a great saving of labor; practice with numerical examples is absolutely essential for success.

The four formulas of elementary algebra which enjoy the widest use are undoubtedly:

$$(x+a)^2 = x^2 + 2\,ax + a^2.$$
$$(x-a)^2 = x^2 - 2\,ax + a^2.$$
$$(x+a)(x-a) = x^2 - a^2.$$
$$(x+a)(x+b) = x^2 + (a+b)x + ab.$$

$(x+a)(x+b)$ gives a simple rule for the product of two "-teens," *e.g.* 19×17.

Thus, $(10+a)(10+b) = 10^2 + (a+b)10 + ab$, or $= 10(10+a+b) + ab$.

Put into words, this formula states that the product of two numbers between 10 and 20 is equal to the whole of one plus the units of the other; this sum is to be multiplied by 10; to this product is to be added the product of the units.

Rule. — *To find the product of two "-teens," add the whole of one to the units of the other and annex a zero; to this number add the product of the units.*

$$\begin{array}{r} 19 \\ 17 \\ \hline 260 \\ 63 \\ \hline 323 \end{array}$$

If x is taken as 20, 30, 40, 50, ⋯, the corresponding rule for the product of two two-place numbers having the same tens' digit is to add to the one of two numbers the units of the other; the sum is to be multiplied by the tens' digit, and a zero annexed to the product; to this number add the product of the units.

$$\text{Thus, } (37 \times 35) = 30 \times 42 + 35 = 1260 + 35$$
$$= 1295.$$

Such products are most easily found, evidently, if the two units' digits sum to 10.

$$87 \times 83 = 8 \times 9 \times 100 + 21$$
$$= 7221.$$
$$64 \times 66 = 6 \times 7 \times 100 + 24$$
$$= 4224.$$

In mental work with numbers work from *left to right,* and not from right to left, dealing first with the numbers of greater significance.

$(x + a)^2$ and $(x - a)^2$ are particularly useful in the computation of squares of numbers of three places beginning with 1 or 9.

$$(10.7)^2 = 100 + 2 \times 10 \times .7 + .49$$
$$= 114.49.$$
$$(11.3)^2 = 121 + 6.6 + .09 = 127.69,$$

or
$$= 100 + 26 + 1.69 = 127.69.$$
$$(1.57)^2 = 2.25 + .21 + .0049,$$

where .21 is obtained as $1.5 \times .14$ by the rule for the product of two "-teens."

$$(.97)^2 = (1.00 - .03)^2 = 1 - .06 + .0009$$
$$= .9409.$$
$$(8.70)^2 = (10 - 1.3)^2 = 100 - 26 + 1.69 = 75.69.$$

Frequently it is more convenient to use these formulas rearranged as follows:

$$(x + a)^2 = x(\overline{x + a} + a) + a^2.$$
$$(x - a)^2 = x(\overline{x - a} - a) + a^2.$$

Thus,
$$(84)^2 = 100(100 - 16 - 16) + 16^2$$
$$= 100(84 - 16) + 16^2$$
$$= 6800 + 256 = 7056.$$
$$(25.7)^2 = 20(25.7 + 5.7) + (5.7)^2$$
$$= 628 + 32.49$$
$$= 660.49.$$
$$(25.7)^2 = 25(25.7 + .7) + .49 = 25(26.4) + .49$$
$$= 660.0 \text{ (since } 25 = \tfrac{100}{4}) + .49.$$

The square of any number between 25 and 75 is obtained from $(x + a)^2$, as follows:

$$(50 \pm a)^2 = 2500 \pm 100\,a + a^2 = 100 \times (25 \pm a) + a^2.$$

Thus,
$$\begin{aligned}(37)^2 &= 2500 - 1300 + 169\\ &= 1369.\end{aligned}$$

Rule. — *To find the square of any number between 25 and 75; find the difference between the given number and 50; add, if the given number is greater than 50, or subtract, if the given number is less than 50, this difference from 25 and annex to this two zeros. Add to this number the square of the difference.*

Thus,
$$\begin{aligned}(65)^2 &= (25 + 15) \times 100 + 15^2\\ &= 4225.\end{aligned}$$

For numbers between 75 and 150 the squares may be obtained as $100(\overline{100 - a} - a) + a^2$ or $(100)(\overline{100 + a} + a) + a^2$, noting that $100 - a$ or $100 + a$ is your given number whose square is sought.

Thus,
$$112^2 = 12{,}400 + 144.$$
$$13.7^2 = 174.00 + 3.7^2 = 174.00 + 13.20 + .49 = 187.69.$$

Frequently, of course, only three or four significant figures are desired, and the methods mentioned give the significant figures first.

$(x + a)(x - a)$ may also be used for squares, thus:

$$\begin{aligned}x^2 &= (x + a)(x - a) + a^2.\\ (87)^2 &= (87 + 13)(87 - 13) + 13^2.\\ (2.33)^2 &= (2.33 + .17)(2.33 - .17) + .17^2\\ &= 5.4 + .0289.\\ (41.7)^2 &= (41.7 + 8.3)(41.7 - 8.3) + (8.3)^2\\ &= 50 \times 33.4 + 8.3^2\\ &= 1670 + 68.89\\ &= 1738.89.\\ (41.72)^2 &= 1738.89 + (.04)(41.7) + .0004\\ &= 1738.89 + 1.6684 = 1741 \text{ to units, or } 1740.6 \text{ to tenths.}\end{aligned}$$

PROBLEMS

1. Multiply mentally 19 × 18, 17 × 15, 18 × 14.

2. Use the rule given above to give the table of 18's from 18 × 11 to 18 × 19.

3. Multiply mentally 12 × 13, 36 × 34, 45 × 45, 82 × 88, 91 × 99.

4. Multiply mentally 27 × 25, 34 × 32, 54 × 58, 92 × 98.

5. What is the product of 44 × 36 or $(40+4) \times (40-4)$, 58 × 62, 44 × 37 or $(40+4) \times (40-3)$?

6. What are the first three figures of $(114)^2$, $(107)^2$, $(131)^2$, and $(118)^2$? Note $(114)^2$ is $12{,}800 + 14^2$, and the first three figures 129; in $(116)^2$ to 13,200 you must add $(16)^2$, which increases the first 13,200 to 13,400.

7. From the preceding answers in 6 write the first three figures of $(1.14)^2$, $(.107)^2$, $(1.31)^2$, $(1180)^2$.

8. Write the squares of 9.7, 88, 940, 8.7, and 9.2.

9. Approximately how much greater is $(9.71)^2$ than $(9.7)^2$? $(88.2)^2$ than $(88)^2$? $(941)^2$ than $(940)^2$? $(8.75)^2$ than $(8.7)^2$? $(9.26)^2$ than $(9.2)^2$?

Note that $(88.2)^2$ differs from $(88)^2$ first by .4 × 88, or by a little more than 35 units; the .04 is usually negligible.

10. Square 43, 47, 52, 63, and 62 by using the difference between these numbers and 50 according to the rule.

11. Using the preceding answers, square 4.3, .47, .052, 630, and 6.2.

Note. — Use common sense rules to determine the position of the decimal point.

12. Using the formulas for $(50 \pm a)^2$, $(x \pm a)^2$, $(100 \pm a)^2$, write the following 25 squares. Time yourself on writing simply the answers; the exercise should be completed in 6 minutes.

$\overline{57}^2 =$
$\overline{63}^2 =$
$\overline{87}^2 =$
$\overline{43}^2 =$
$\overline{98}^2 =$

$\overline{17}^2 =$
$\overline{42}^2 =$
$\overline{63}^2 =$
$\overline{10.8}^2 =$
$\overline{16}^2 =$

$\overline{24}^2 =$
$\overline{39}^2 =$
$\overline{59}^2 =$
$\overline{21}^2 =$
$\overline{92}^2 =$

$\overline{33}^2 =$
$\overline{52}^2 =$
$\overline{71}^2 =$
$\overline{66}^2 =$
$\overline{55}^2 =$

$\overline{62}^2 =$
$\overline{67}^2 =$
$\overline{82}^2 =$
$\overline{1.9}^2 =$
$\overline{49}^2 =$

13. Using the results of the preceding exercise, compute to four significant figures the following squares, timing yourself.

$\overline{57.1}^2 =$
$\overline{63.2}^2 =$
$\overline{87.4}^2 =$
$\overline{43.5}^2 =$
$\overline{98.6}^2 =$

$\overline{17.3}^2 =$
$\overline{42.5}^2 =$
$\overline{63.3}^2 =$
$\overline{10.82}^2 =$
$\overline{16.6}^2 =$

$\overline{24.5}^2 =$
$\overline{39.7}^2 =$
$\overline{59.2}^2 =$
$\overline{21.4}^2 =$
$\overline{92.6}^2 =$

$\overline{33.1}^2 =$
$\overline{52.9}^2 =$
$\overline{71.8}^2 =$
$\overline{66.7}^2 =$
$\overline{55.3}^2 =$

$\overline{62.4}^2 =$
$\overline{67.1}^2 =$
$\overline{82.9}^2 =$
$\overline{1.92}^2 =$
$\overline{49.8}^2 =$

14. Employing the formula for $(x+a)(x+b)$ write the following products; the exercise should be completed in 6 minutes.

16 × 19
15 × 14
13 × 18
17 × 12
16 × 18

22 × 24
23 × 26
24 × 29
28 × 28
36 × 33

32 × 38
43 × 42
46 × 44
54 × 59
82 × 87

51 × 52
33 × 31
27 × 24
24 × 22
17 × 13

66 × 64
88 × 82
97 × 93
57 × 53
79 × 71

5. Extraction of roots. — In extraction of square root, the method of successive approximation should frequently be employed.

Thus, $\sqrt{179.63} > 13$ and < 14.

$\sqrt{169 + 10.63} = 13 + a$, wherein a must be a number such that $2a \times 13$ equals approximately 10.6. A glance shows that $.4 \times 13$ equals 5.2, which doubled gives 10.4. Hence, $(13.4)^2 = 169 + 10.4 + .16$.

$$(13.4)^2 = 179.56, \text{ or } 179.63 - .07.$$

$$(13.4 + a)^2 = 179.56 + 2a \times 13.4 + a^2.$$

a now is less than .01; hence, a^2 is less than .0001; a is to be a number of hundredths or thousandths, evidently, so that $2a \times 13.4$ is approximately .07; a is roughly .003, slightly too large.

$$(13.403)^2 = 179.56 + .0804 + .000009$$
$$= 179.640409.$$

The rule is commonly given to take $x \pm \frac{a}{2x}$ as first approximation of $\sqrt{x^2 \pm a}$, wherein a is small as compared with x^2. The process illustrated follows this rule, but suggests thinking multiplication instead of division. Thus in the square root of 300, as $\sqrt{17^2 + 11}$, approximately $\frac{11}{34}$ is to be added to 17; however it is easier to think $34 \times a = 11$, whence $a = .3$, or not quite .33; trying .32 (since the a^2 term is to be added) gives 17.32 of which the square is

$$289 + 10.88 \text{ (or } .32 \times 34) + (.32)^2 = 299.9829.$$

Similarly, $\sqrt{3000} = \sqrt{3025 - 25} = \sqrt{55^2 - 25} = 55 - a$, wherein a must be a number such that $2 \times 55 \times a$ will give approximately 25; a is evidently .2 to one decimal place or .23 to two; 54.77 is correct to four significant figures as given.

6. Approximate roots. — Another method of approximating square root is to divide the given number by the first approximation, then to use the arithmetic mean of the two numbers as a second approximation. Thus, $179.63 \div 13 = 13.82$; taking $\frac{13 + 13.82}{2}$ as the approximate root gives 13.41 as a second approximation. $179.63 \div 13.41$ gives 13.3952 and the average 13.4026 is within .0001 of the correct value.

Similarly the cube root may be obtained. Thus in 179.63, 5 is the first approximation. $5^2 = 25$; $179.63 \div 25 = 7.2$ nearly. Taking the average of 5, 5, and 7.2 gives 5.7 as second approximation; $(5.7)^2 = 32.49$; $179.63 \div 32.49 = 5.529$; the average of 5.7, 5.7, and 5.529 gives 5.643, which is correct within .001.

PROBLEMS

1. What is the approximate square root of 1.26? 128?

2. Is the square root of 1.35 nearer to 1.16 or to 1.17?

NOTE. $(1.16)^2 = 1.32 + .0256$ and $(1.17)^2 = 1.34 + .0289$.

3. Find the square roots of the following numbers, employing the approximation $x \pm \frac{a}{2x}$, as the square root of $x^2 \pm a$:

a. 102.
b. 108.
c. 125.
d. 83.
e. 80.
f. 48.
g. 51.
h. 130.
i. 146.
j. 173.
k. 200.
l. 230.

4. By successive approximations find $\sqrt{1.26}$ to four decimal places and compare with ordinary process of extraction of root.

HINT. — Use 1.12 as first approximation.

5. Find the approximation to one decimal place of the square roots of 65, 63, 8.30, 8.76, and 27.32.

HINT. — Regard 8.30 as $(3 - x)^2$, whence x must be roughly .12.

6. Write the square roots of the following numbers, correct to 2 decimal places. Time yourself.

9.9	35	65	140	200
16.8	34	68	150	300
17.2	37.2	78	125	10.4
25.8	39.4	85	108	20.8
28	48	90	112	30.6

7. What is the remainder when $x^2 - 102$ is divided by $x - 10$? by $x - 10.1$?

8. What is the remainder when $x^3 - 1060$ is divided by $x - 10$? by $x - 10.2$? by $x - 10.3$?

9. Expand $(x + h)^3$; if h is small as compared with x, how do the four quantities involved in $(x + h)^3$ compare in value? What would be an approximate value for the cube root of $x^3 + a$, wherein a is small compared with x^3?

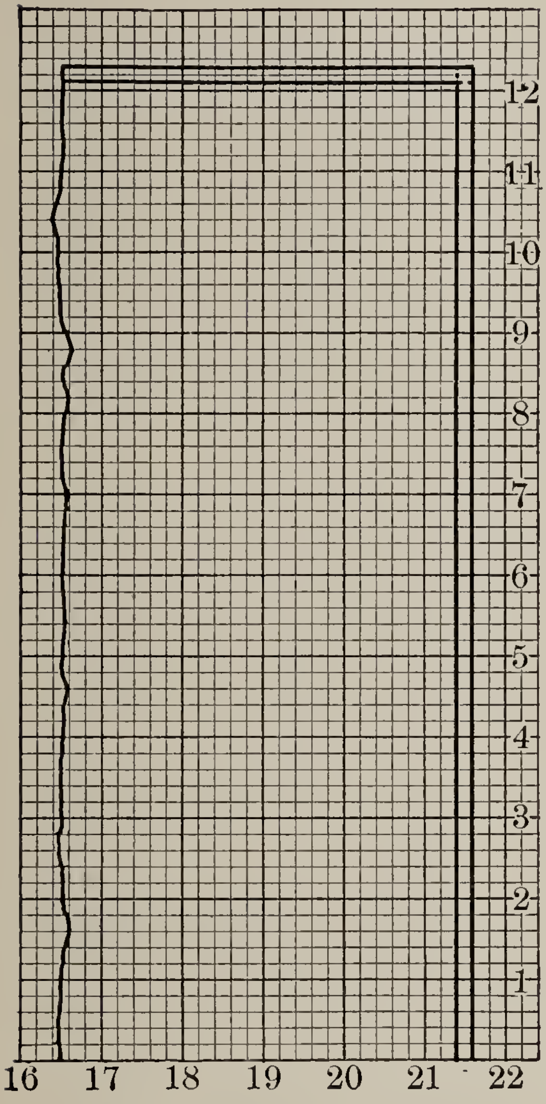
10. Find the approximate cube root of 1060; similarly the approximate cube root of 940.

7. Percentage of error. — When any measurement of physical objects is given, the measurement has a certain limit of accuracy, determined in part by the instruments and methods of measurement and in part by the very nature of the thing measured. In measuring the distance to the sun from the earth, at some fixed time, the measurement may be given as 93,000,000 miles ± 1,000,000 miles, or 93,000,000 miles, within a million miles; the thickness of a watch spring may be measured as .014 inch with a possible error of one thousandth of an inch, or .014 ± .001 inch. However, from the point of view of the physicist and mathematician, the distance to the sun is more accurately given than the thickness of the watch spring, for the percentage of error — ratio of possible error to measured value — in the case of the sun's distance is slightly more than 1 % of the distance, while in the other case it is more than 7 % of the thickness of the spring.

Every number which represents a measurement involves this type of error. Obviously, in any computations with such numbers, results are significant only within limits determined by the percentage of error.

8. Significant figures. — The significant figures in any number representing a measurement are those which are given by the measurement, and do not include those initial or terminal zeros which are determined by the unit in which the measurement is made. The terminal zeros in 93,000,000 are not significant figures, as the unit of measurement here is evidently a million miles; as the measurement can be made to one further place, the distance may be written, in "standard form," 9.3×10^7 miles or 9.30×10^7 miles in which only significant digits appear in the first factor combined with powers of 10. In the thickness .014 inch, the initial zero is not a significant figure, as it is apparent that the measurements are made in thousandths of an inch; in "standard form," this is 1.4×10^{-2} inches.

9. Measurement computations. Products. — If the length of a rectangle is measured with an error of less than 1 % of its true value, and if the breadth is given absolutely, the true area will be given with the same percentage of error as the length. But if the breadth is also only approximately measured, the possible error in the area obtained as the product will be greater than if only one factor involves a possible error. The graph represents the right-hand end of a rectangle whose length and breadth are measured as 21.5 cm. and 12.2 cm. respectively, where it is understood that the measurement only pretends to give these dimensions to within one millimeter, one tenth of one centimeter. The meaning of these figures then is that the length lies between 21.4 cm. and 21.6 cm., and the breadth between 12.1 cm. and 12.3 cm. The first is an error of less than $\frac{1}{2}$ of 1 % and the second of less than 1 %. The uncertainty in area due to the possible error in length is indicated by the areas at the right end with dimensions .1 cm. by 12.1 cm. or .1 cm. by 12.3 cm.; the uncertainty in area due to the breadth measurement is of dimensions .1 cm. by 21.4 cm. or .1 cm. by 21.6 cm. The area uncertainty is then at most .1 cm. by (21.6 + 12.2) cm. or 3.39 sq. cm. of area. This error may evidently affect the third figure in our computation of the area and hence in the product the figures beyond the third place are not significant, and give no real information concerning the actual area in question.



Measurement of an area

A rectangle measured as 21.5 cm. by 12.2 cm.

Note that the area as the product of 12.2 by 21.5 is 262.30

sq. cm., but the inaccuracy of measurement of the length means that there is an uncertainty of area at the right-hand end amounting to ± 1.22 sq. cm. (cm^2.), and similarly at the top an uncertainty of ± 2.16 cm^2.; the total uncertainty of area amounts to more than 3 cm^2., and should be given as ± 3.38 cm^2., or ± 3.4 cm^2. Commonly, of course, the measurements 12.2 cm. and 21.5 cm. mean that the area has been measured to one half of the last unit given; thus this area actually falls between rectangles of dimensions 12.25 by 21.55 and 12.15 by 21.45; even in this case the area uncertainty is greater, by precisely similar reasoning, than ± 1.5 cm^2., and is approximately 1.7 cm^2. To give 262.30 as the area of this measured rectangle is giving nonsense in the last two places; it should be given as 262 or 262.3 ± 1.7 cm^2.

Let k and k' represent measured quantities given with possible errors of i % and e % respectively, e and i being assumed as smaller than unity (in common practice); the absolute values of these measured quantities lie between $k\left(1+\frac{i}{100}\right)$ and $k\left(1-\frac{i}{100}\right)$ and between $k'\left(1+\frac{e}{100}\right)$ and $k'\left(1-\frac{e}{100}\right)$. The true product lies, then, between $kk'\left(1+\frac{i+e}{100}+\frac{ie}{10000}\right)$ and $kk'\left(1-\frac{i+e}{100}+\frac{ie}{10000}\right)$; in other words, the true product may vary by $\frac{i+e}{100}$ from the computed product; $\frac{ie}{10000}$ is disregarded, if i and e are less than 1, since the fraction is less than 1 % of 1 % of kk'. In the graph the product $\frac{ie}{10000}\times kk'$ is represented by one of the small corner squares with dimensions .1 cm. by .1 cm.

Illustrative example. — The product of 987 by 163 wherein each number is correct to within $\frac{1}{2}$ a unit need be computed only to the fourth significant figure as the percentage error may be as great as $\frac{1}{20}$ of 1 % + $\frac{3}{10}$ of 1 %, since $\frac{1}{2}$ in 987 parts is approximately $\frac{1}{2000}$ or $\frac{1}{20}$ of 1 %, and $\frac{1}{2}$ in

163 is greater than $\frac{3}{1000}$ or $\frac{3}{10}$ of 1 %. The error in the product may be as great as $(\frac{1}{20} + \frac{3}{10})$ of 1 % or $\frac{7}{20}$ of 1 % ; but $\frac{7}{20}$ of 1 % of any number certainly affects the fourth place and probably affects the third place in the number. Hence there is no point whatever in carrying this computation beyond four places,

987	987	
163	163	
98700	987 begin with 100 × 987.	
59220	592 take 6 × 98, carrying however the 4 from 6 × 7.	
2961	29 take 3 × 9, carrying the 2 of 3 × 8.	
160881	160800	
163	$987 + \frac{1}{2}$	$987 - \frac{1}{2}$
987	$163 + \frac{1}{2}$	$163 - \frac{1}{2}$
1467	$160881 + \frac{1}{2}(987 + 163) + \frac{1}{4}$	$160,881 - \frac{1}{2}(987 + 163) + \frac{1}{4}$
130	$= 161456\frac{1}{4}$	$= 160306\frac{1}{4}$
11		
160800 *ans.*		

The product of 987 × 163 is 160,881 ; $987\frac{1}{2} \times 163\frac{1}{2}$ gives $161,456\frac{1}{4}$; $986\frac{1}{2} \times 162\frac{1}{2}$ gives $160,306\frac{1}{4}$; the actual area, if these represent dimensions of a rectangle measured to three significant figures lies between $160,306\frac{1}{4}$ and $161,456\frac{1}{4}$. In practice we take the product 987 × 163 to four significant figures, which gives the area slightly more accurately than our measurements justify.

10. Abbreviated multiplication of decimals. — The abbreviated process of multiplication applies particularly well to decimal fractions, but the method can be extended to integers quite as well. To find .9873 × .1346 correct to four decimal places.

.9873	
.1846	
9873	begin with the highest digit of the multiplier ; first × fourth decimal place gives fifth decimal place.
7898	continue with 8 × 7 (second × third place), carrying the 2 from 8 × 3.
395	take 4 × 8 (third × second place) carrying 3 from 4 × 7, or 28, which is more than 2 units in the fifth place.
59	6 × 9, or 54, + 5 carried from the 6 × 8.
18225	It assists in the process to cross out the last upper digit as it is used ; thus here 3 would be crossed out first, then 7, then 8, and finally 9.

If a check is desired, multiply again, reversing the order of the factors; thus:

.1846	
.9873	
16614	begin with 9 × 6.
1477	take 8 × 4, adding 5 from the 8 × 6 product.
129	take 7 × 8, adding 3 from the 7 × 4 product.
5	take 3 × 1, adding 2 from the 3 × 8 product.
.18225	Read this as .1823 to four places.

Obviously, if these were integers, you could proceed in the same way, writing the final product with four zeros, as 18,230,000.

A similar abbreviation can be effected in division by dropping each time the last figure of the divisor used, and using the remaining part of the original divisor as new divisor.

Thus, to divide .18225 by .9873 or by .1846 you proceed as follows:

$$\begin{array}{r} .1846 \\ .9\cancel{8}\cancel{7}\cancel{3}\overline{).18225} \\ \underline{9873} \\ 8352 \\ \underline{7898} \\ 454 \\ \underline{395} \\ 59 \\ \underline{59} \end{array} \qquad\qquad \begin{array}{r} .9873 \\ .1846\overline{).18225} \\ \underline{16614} \\ 1611 \\ \underline{1477} \\ 134 \\ \underline{129} \\ 5 \\ \underline{5} \end{array}$$

Here 9873 is used as the first divisor; then 987 is used, but to the partial product, 8 × 987, is added the tens' digit of 8 × 3, the digit just crossed out; then 98 is taken as divisor and to the product is added the tens' digit of 4 × 7 (28 is taken as giving a tens' digit of 3); then 9 is used and 5 carried over from 6 × 8.

11. Percentage effect of errors in divisor. — If a divisor is known to be too large or too small by a definite percentage of itself, the quotient will be respectively smaller or larger than the correct quotient, for small per cents, by approximately the same per cent.

By division, $\frac{1}{1+i} = 1 - i + i^2 - i^3 + i^4 - \cdots.$

$$\frac{1}{1-i} = 1 + i + i^2 + i^3 + i^4 + \cdots.$$

For values of i less than .05, i^2 is less than .0025, or $\frac{1}{4}$ of 1 %; i^3, i^4, and i^5 are less than .000125, .00000625, and .0000003125, respectively. Hence an error of from 1 % to 5 % of excess in the divisor means an error of deficiency varying also from 1 % to 5 %, within $\frac{1}{4}$ of 1 %, or from .99 % to 4.75 %, or from .9901 % to 4.7625 % in the quotient. For values of i between $\frac{1}{2}$ of 1 % and 1 %, an error of deficiency in the divisor means the same error of excess in the quotient, within $\frac{1}{100}$ of this error. The meaning in physical measurements of these results is that when the divisor is correct only to the third significant figure, with a possible error of $\frac{1}{2}$ to 1 unit in the third place, the quotient will be correct to about the same degree of accuracy.

For three-place numbers the divisor may vary from 100 to 999. The possible error of $\frac{1}{2}$ unit, $\pm \frac{1}{2}$, means that 100 must be replaced by $(100 \pm .5)$ or $100(1 \pm .005)$ and 999 by $999 \pm .5$, or approximately $999(1 + .0005)$; the quotient will vary from $1 - .005$ to $1 - .0005$ times the obtained quotient. Hence the quotient obtained is valuable at most to the fourth place, and frequently not beyond the third place.

Illustration. — Given that 76,430 is divided by 180; what variation in the quotient would a change of 1 in the divisor produce?

$76{,}430 \div 180 = 424.6.$

Suppose that instead of 180, 179 should have been used. What is the error in the quotient? $180 - 1 = 180(1 - .006)$, the error in the divisor is more than .5 % and less than .6 % of the divisor; the error in the quotient is no more than 2.5 and no less than 2.1, since 1 % of 424.6 is 4.246 and .6 % and .5 % are respectively 2.5 and 2.1; the quotient may be taken as 427.1, whereas 426.9 is obtained by actual division. Even an error of $\frac{1}{2}$ a unit in the divisor 180 affects the third place in the quotient. In obtaining .5 % and .6 % of 424.6, there is no point in carrying the work beyond two places; the values show that the error is between 2.1

and 2.5, and further places add nothing to the accuracy. The fraction $\frac{1}{180}$, or in the case of the $\frac{1}{2}$ unit error of $\frac{1}{360}$, might just as well be used as per cents. This gives in the latter case $\frac{1}{360}$ of 424.6, or $+ 1.2$ as correction, giving 425.8 as quotient; the actual quotient is 425.794.

Do not carry divisions and multiplications beyond the degree of accuracy warranted by the data.

Illustrative examples. — You can multiply 3.14159 by 140.8 and obtain the result numerically correct to six decimal places. But if the 140.8 represents the diameter of a circle, measured correctly to the tenth of an inch (or of a foot, or of a meter) the product of 140.8 by 3.14159 gives a valuable result *only to the first decimal place;* the circumference cannot be computed correctly to any further percentage of accuracy than that with which the diameter is measured. The area can be computed here with any meaning only to four significant figures; in fact an error of ± .05 inch in the diameter makes a possible error of ± 10 square inches in the area. It is convenient to write the products from left to right, dropping work beyond the second decimal place.

$$\begin{array}{rl} 140.8 & \\ 3\frac{1}{7} & \\ \hline 422.4 & \\ 20.1 & \\ \hline 442.5 & \text{circumference} \end{array} \qquad \begin{array}{rl} 140.8 & \\ 3.14159 & \\ \hline 422.4 & \\ 14.1 & \\ 5.6 & \\ 1 & \\ \hline 442.2 & \text{circumference} \end{array}$$

the .00059 is of no use as it does not figure in the product.

For most practical purposes $3\frac{1}{7}$ is sufficiently accurate, as in finding this area, πr^2:

$$\begin{array}{rl} 70.4 & \\ 70.4 & \\ \hline 4928 & \\ 28 & \\ \hline 4956 & \text{only 4 places to be retained.} \end{array}$$

$$\begin{array}{rl} 4956 & \\ 3\frac{1}{7} & \\ \hline 14868 & \\ 708 & \\ \hline 15576 & \text{area.} \end{array} \qquad \begin{array}{rl} 4956 & \\ 3.14159 & \\ \hline 14868 & \\ 496 & \\ 198 & \\ 5 & \\ \hline 15567 & \text{area.} \end{array}$$

Area as found to correspond to data, **15570.**

PROBLEMS

1. The distance of the earth from the sun varies between 91.4×10^6 miles and 94.4×10^6 miles. The length of the earth's orbit lies between circles having these lengths as radii. Between what values does this orbit lie? What is the approximate orbital speed of the earth in miles per hour?

2. The mean distance of the earth from the sun is 92.9×10^6 miles. Compute the circumference and the mean speed and compare by percentages with the preceding.

3. Compute the speed of a point on the earth due to the rotation, taking that at latitude 45° the radius of the circle of latitude is 3050 miles. Compare the rotational speed with the revolutional speed.

4. What effect on the computed velocity would it have to take 365.25 instead of 365? How would you correct your division for 365.25 as divisor after having obtained the quotient, dividing by 365? what change would using 365.26 instead of 365.25 effect in the computed velocity?

5. The distance of the moon from the earth varies between 221,000 and 260,000 miles, mean 238,000; discuss the length of the path of the moon and the velocity of the moon which has a periodic time of 27.32 days.

6. A man whose salary is \$ 3000 pays \$ 480 for rent. What per cent is this of his salary? Suppose that he earns \$ 275 in addition to his salary, what per cent is the rent paid of his income? Compute only to tenths of one per cent.

7. If a man with an income \$ 3275 pays \$ 1100 per annum for food, \$ 630 for clothes, \$ 240 for life insurance, \$ 200 for "higher life," and saves the balance, compute his budget by per cents.

8. Given that a pendulum of length l cm. makes one beat, one oscillation, in t seconds, connected by the relation,

$$t = \pi\sqrt{\frac{l}{980}},$$

find the length l to two decimal places when $t = 1$.

9. What effect on l does a change from 980 to 981 produce? What decimal place in l would be affected?

10. What error would the use of $3\frac{1}{7}$ instead of 3.14159 introduce?

11. The number n of vibrations of a pendulum of length 99.39 cm. is 86400, when $g = 980.96$; g is the acceleration due to gravity, and the formula for the number of vibrations is given by the formula,

$$n = \frac{86400}{\pi}\sqrt{\frac{g}{l}},$$

or for the seconds' pendulum, when $l = 99.39$, $g = 980.96$, it is $n = 86,400$. Suppose that at the top of a mountain (g diminishes) this pendulum of length 99.39 loses 86 beats per day, what is the approximate percentage of loss in n? The percentage of loss in g is approximately double this since $\sqrt{1-i} = 1 - \frac{i}{2} - \frac{i^2}{8}$. What is the approximate loss in g? Take 86 as $\frac{1}{10}$ of 1 % of 86,400.

12. Given $g = 980$, $l = 50$, compute n in the formula of problem 11. What maximum effect on n would a change from $l = 50$ to $l = 50.5$ cm. produce?

13. Compute the weight of a table top, hardwood, dimensions correct to .05 foot, top $48.1 \times 36.4 \times 2.1$ inches. Weight of wood 48 pounds per cubic foot.

14. If a table top similar to the above weighs 97 pounds, compute the weight per cubic foot of the wood.

15. If water weighs 62.4 pounds per cubic foot, compute the specific gravity of each of the preceding woods.

$$S = \frac{\text{wt. of cubic foot of wood}}{\text{wt. of cubic foot of water}}.$$

16. The path of the earth is approximately a circle of radius 92.9×10^6 miles, of which the center is approximately $1\frac{1}{2}$ million miles from the sun. Compute this circumference and compare with the results in problems 1 and 2.

17. Factor the following, using wherever possible the factor theorem to determine factors:

a. $x^3 - 7x^2 + 10x$.

b. $3y^2 - 5y + 2$.

c. $3y^2 - 12$.

d. $x^4 - 8x^2 - 20$.

e. $v^6 - 7v^3 + 8$.

f. $xy^4 + x^4y$.

18. Give the approximate square root of the following:

a. 36.6.

b. 35.4.

c. 104.

d. 96.

e. 126.

f. 98.

g. 1.04.

h. 1.12.

i. 4.08.

j. 9.12.

19. Complete the following:

$(x - 2y)(\qquad\qquad) = x^2 - 4y^2.$
$(x - 2y)(\qquad\qquad) = x^2 - 4xy + 4y^2.$
$(x - 2y)(\qquad\qquad) = x^3 - 8y^3.$
$(x - 2y)(\qquad\qquad) = x^2 - 6xy + 8y^2.$
$(x - 2y)(\qquad\qquad) = x^2 + 10xy - 24y^2.$
$(x - 2y)(\qquad\qquad) = 3x^2 - 10xy + 8y^2.$

CHAPTER III

EXPONENTS AND LOGARITHMS

1. Exponent laws. — For convenience the product of a by itself, $a \times a$, is represented by a^2, $a \times a \times a$ by a^3, ..., and $a \cdot a \cdot a \cdot a$ to m factors by a^m. In this notation m is called the exponent and a the base. The following laws evidently hold:

$$\text{I.} \quad a^m \cdot a^n = a^{m+n}.$$

$$\text{II.} \quad \frac{a^m}{a^n} = a^{m-n}, \text{ when } m > n.$$

$$\text{III.} \quad (a^m)^n = a^{m \cdot n}.$$

$$\text{IV.} \quad (a \cdot b)^m = a^m \cdot b^m.$$

In the definition as given, m represents the number of factors and is assumed to be a positive integer. However, it is found possible to define a^m for all real values (fractional, negative, zero, irrational) of m so as to have the resulting numbers combine according to the four laws given above.

Thus, $a^0 \cdot a^m = a^{0+m} = a^m$, if Law I is to continue to hold; hence, a^0 must be defined to equal 1, since multiplying a number, a^m, it gives that number. To be justified in using a zero exponent with this meaning the other exponent laws must be shown to hold when either m or n is zero, but in II only n could be zero at this point.

For a negative integer, $-n$, if Law I is to hold, a^{-n} must be defined as such a number that $a^n \cdot a^{-n} = a^{-n+n} = a^0 = 1$; hence we define a^{-n} as the reciprocal of a^n, $a^{-n} = \frac{1}{a^n}$. All the laws I to IV can be shown to hold under this extension of the meaning of a^n.

Similarly, $a^{\frac{p}{q}}$, if Law III is to hold, must represent a number which raised to the qth power equals a^p; $a^{\frac{p}{q}}$ is thus defined as the qth root of the pth power of a. Taking this definition of $a^{\frac{p}{q}}$, Laws I to IV can be shown to hold with this extension in possible values of m and n; p and q are assumed to be integers.

For fractional exponents a fifth law is introduced, $a^{\frac{p}{q}} = a^{\frac{mp}{mq}}$.

For irrational values of n, a^n is defined by a limiting process. Thus, $a^{\sqrt{2}}$ is defined as the limit of the series $a^{1.4}$, $a^{1.41}$, $a^{1.414}$ ..., wherein the successive rational exponents define the square root of 2.

The operations of elementary algebra with radicals are made subject to the exponent laws.

Thus, $2\sqrt{3} = 2 \cdot 3^{\frac{1}{2}} = (2^2)^{\frac{1}{2}} \cdot 3^{\frac{1}{2}} = (2^2 \cdot 3)^{\frac{1}{2}} = 12^{\frac{1}{2}} = \sqrt{12}$, by successive application of V, III, and IV.

The operation of raising to a power indicated by a^m, with m integral, is called involution. The inverse operation of finding x when x^m is given equal to a is called evolution.

PROBLEMS

1. Write as ordinary numbers, without exponents:

$$10^2, \quad 10^{-2}, \quad 100^{\frac{1}{2}}, \quad 100^{\frac{3}{2}}, \quad 100^{-\frac{1}{2}}, \quad 1000^{\frac{1}{3}}, \quad 10^0.$$

2. Find the approximate numerical value:

$$10^{\frac{1}{2}}, \quad 10^{\frac{3}{2}}, \quad 10^{\frac{5}{2}}, \quad 10^{-\frac{1}{2}}, \quad 10^{-\frac{3}{2}}, \quad 5^{\frac{1}{2}}, \quad 2^{\frac{1}{2}}, \quad 5^{\frac{3}{2}}, \quad 5^{-3}.$$

3. Write the following expressions in the form, 10^n:

$$\sqrt{10}, \quad \sqrt[3]{10}, \quad \frac{1}{\sqrt{10}}, \quad \left(\frac{1}{\sqrt{10}}\right)^3, \quad 1, \quad \frac{1}{10}, \quad \frac{1}{10^5}.$$

4. Which of the exponent laws are applied in simplifying the following expressions:

$$\sqrt{200}, \quad \sqrt[3]{2000}, \quad \frac{2}{\sqrt{3}}?$$

5. What exponent must be applied to 10, as a base, to give 1000? to give .001? to give 1?

6. Simplify the following expressions:

$$(10^3)^5, \quad (10^3)(10^5); \quad \frac{10^3}{10^5}, \quad 2^3 \cdot 5^3; \quad \sqrt[8]{10^{12}}.$$

2. Logarithms. — *A logarithm is an exponent.*

The relation $x = a^m$ may be written $m = \log_a x$.
m is the exponent which applied to a gives x;
m is the logarithm of x to the base a.

3. Fundamental laws of logarithms.

a. **Logarithm of a product.**

$$\text{If } x = a^m \text{ and } y = a^n,$$
$$x \cdot y = a^m \cdot a^n = a^{m+n}.$$
$$\log_a(x \cdot y) = m + n = \log_a x + \log_a y.$$

In the language of logarithms and translated into ordinary language this theorem is as follows:

$$\text{I.} \quad \log_a(x \cdot y) = \log_a x + \log_a y;$$

in words, *the logarithm of a product is the sum of the logarithms of the factors.*

b. **Logarithm of a quotient.**

$$\frac{x}{y} = \frac{a^m}{a^n} = a^{m-n}, \quad \log_a \frac{x}{y} = m - n = \log_a x - \log_a y.$$

$$\text{II.} \quad \log_a \frac{x}{y} = \log_a x - \log_a y;$$

in words, *the logarithm of a quotient is the logarithm of the dividend minus the logarithm of the divisor.*

c. **Logarithm of a power.**

If $x = a^m$, $x^n = (a^m)^n = a^{mn}$; $\log_a x^n = m \cdot n = n \log_a x$.

$$\text{III.} \quad \log_a x^n = n \log_a x;$$

the logarithm of a power of a number is the index of the power times the logarithm of the number.

Since our exponent laws hold for all values of m and n, these theorems hold for all values of m and n.

x and y are assumed to be positive numbers and for computation purposes 10 is commonly taken as the base.

We assume that as the logarithm increases the number increases. This can be readily proved from the fact that $10^m \cdot 10^n = 10^{m+n}$; no matter how small the n is, as a positive quantity, 10^n is greater than 1. For n any positive fraction, $\frac{p}{q}$, 10^n represents the qth root of the pth power of 10, wherein p and q are integers. Now 10^p will be an integer 10, or greater than 10, and the qth root of this integer will be a number greater than 1 as it will be a number which raised to the qth power equals 10, 100, 1000, or some greater number. It could not be less than 1, as every positive integral power of a number less than 1 is also less than 1.

4. Logarithms, characteristic and mantissa. — Any two-place number lies between 10 and 100; the logarithm will lie between 1 and 2. Any four-place number lies between 1000 and 10,000; the logarithm will lie between 3 and 4, since as the number increases the logarithm increases. The fraction .07 is greater than .01 and less than .1; the logarithm is then greater than -2 and less than -1.

The logarithm of any number between 1 and 10 is a fraction, expressed commonly as a decimal between 0 and 1.

The logarithm of any given number which is expressed in decimal notation can be expressed as an integer, positive or negative, called the *characteristic*, plus the positive decimal fraction, the *mantissa*, which is the logarithm of that number between 1 and 10, having the same succession of digits as the given number. Initial zeros are not included in the succession.

Let k represent any number between 1 and 10, written in our ordinary decimal notation; then $10^n \cdot k$, n any positive or negative integer, can represent any number written in decimal notation.

Exponential Form		Column of Numbers	Column of Numbers :	Column of Logarithms
. . .	. . .	. . .	. . .	. . .
10^{-12}	=	.000000000001	log .000000000001	$= -12$
. . .	. . .	. . .	. . .	. . .
10^{-5}	=	.00001	log .00001	$= -5$
10^{-4}	=	.0001	log .0001	$= -4$
10^{-3}	=	.001	log .001	$= -3$
10^{-2}	=	.01	log .01	$= -2$
10^{-1}	=	.1	log .1	$= -1$
10^{0}	=	1	log 1	$= 0$
10^{1}	=	10	log 10	$= 1$
10^{2}	=	100	log 100	$= 2$
10^{3}	=	1000	log 1000	$= 3$
10^{4}	=	10000	log 10000	$= 4$
10^{5}	=	100000	log 100000	$= 5$
. . .	. . .	. . .	. . .	. . .
10^{12}	=	1000000000000	log 1000000000000	$= 12$
. . .	. . .	. . .	. . .	. . .

These positive numbers, middle columns, are arranged vertically in order of magnitude; the exponents (left) or logarithms (right) also are arranged vertically in order, increasing from -12 (or from $-\infty$ indicated by dots above) to -5, to -4, to -3, $\cdots$ to 0, to 1, $\cdots$ to 12, $\cdots$ to ∞. As the number increases the logarithm increases. Placing any number, not an integral power of 10, in its proper place as to magnitude on such a diagram, the logarithm has for integral part the logarithm of the preceding number in the table. Thus, 75.64 falls between 10 and 100 and its logarithm will be 1^+; .07564 falls between .01 and .1 and its logarithm will be -2^+, meaning -2 plus some positive fraction.

$$\log 10^n k = \log 10^n + \log k = n \log 10 + \log k = n + \log k.$$

n is the *characteristic* ; $\log k$ is the *mantissa.*

Thus, $\log 3.16229 = .50000$, since 3.16229 is the approximate square root of 10.

$$\begin{aligned} \log 31622.9 &= \log (10)^4(3.16229) = \log 10^4 + \log 3.16229 \\ &= 4 + \log 3.16229 \\ &= 4.50000. \\ \log .000316229 &= \log (10)^{-4}(3.16229) = \log (10)^{-4} + \log 3.16229 \\ &= -4 + .50000 \\ &= \bar{4}.50000, \text{ only the 4 is negative} \\ &= 6.50000 - 10, \text{ since } -4 \text{ equals } 6 - 10 \\ &= 16.5000 - 20 \\ &= 26.50000 - 30, \\ &\quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}$$

The minus sign over the 4 indicates that only the characteristic is negative; the alternate forms for writing a negative characteristic are frequently found convenient to use, particularly in extracting roots and with the trigonometric functions.

RULE. — *The logarithm of any number greater than unity has as characteristic a positive integer* (0 *included*) *which is* 1 *less than the number of digits to the left of the decimal point.*

The logarithm of any decimal fraction has as characteristic the negative of a positive integer (0 *not included*) *which is* 1 *greater than the number of zeros between the decimal point and the first significant digit* (i.e. *digit other than* 0) *to the right of the decimal point.*

5. Computation of logarithms. — Logarithms are actually calculated by a series derived from formulas obtained in higher mathematics. However, a simple although laborious method of computing logarithms approximately may make the significance of the logarithm somewhat clearer.

$$2^{10} = 1024, \qquad 2^{20} = 1,048,576.$$

Evidently, $10^6 < 2^{20} < 10^7$, since 1,048,576 is greater than 1,000,000 and less than 10,000,000.

Extracting the twentieth root,

$$10^6 < 2^{20} < 10^7$$

gives

$$10^{\frac{6}{20}} < 2 < 10^{\frac{7}{20}},$$

or

$$10^{.30} < 2 < 10^{.35}.$$

Hence, 2 is greater than 10 with an exponent .30, and less than 10 with an exponent .35.

$$2^{40} = (1,048,576)^2 < 1,100,000,000,000$$
$$> 1,000,000,000,000;$$

$2^{80} < 1.21 \times 10^{24}$ (2^{40} by 2^{40}) and greater than 1×10^{24}.

Whence $2^{100} < 1.33 \times 10^{30}$ and greater than 1×10^{30}, whence $10^{30} < 2^{100} < 10^{31}$, whence

$$10^{.30} < 2 < 10^{.31}, \text{ or } \log_{10} 2 = .30^+.$$

Similarly, 3^{20} is 3,486,784,401, or greater than 10^9 and less than 10^{10}; hence, log 3 lies between .45 and .50, the computation of 3^{20} is easily made by using $3^4 = 81$ and $3^{20} = (81)^5$, multiplying each time by 81 instead of by 3; the partial product by the 1 does not need to be rewritten.

6. Tables of logarithms. — The exponents or logarithms to the base 10 of all numbers up to 100,000 have been computed by methods of higher mathematics; these logarithms are arranged in tabular form in the natural order of the corresponding numbers, so as to be convenient for computation purposes. Our tables give the mantissas of the logarithms of all numbers between 100 and 999; by the insertion of the proper characteristic the logarithms of all numbers having one, two, or three significant figures, *i.e.* disregarding initial and terminal zeros, are given by our tables.

EXERCISES ON TABLE OF LOGARITHMS, PP. 498–499

1. Find the logarithms of the following numbers, writing first the proper characteristic, following the rules given on page 45, before employing the table.

a. log 234 =
b. log 46900 =
c. log 2.91 =
d. log 8450 =
e. log 6.06 =
f. log 1000 =
g. log .543 =
h. log .00902 =
i. log .00032 =
j. log .999 =
k. log .001 =
l. log 3 =

2. Find numbers corresponding to the following logarithms:

a. log $= 0.0414$
b. log $= 2.3096$
c. log $= 4.8500$
d. log $= 6.6425$
e. log $= 3.0000$
f. log $= 1.3010$
g. log $= 9.7396 - 10$
h. log $= 6.9996 - 10$
i. log $= 8.9031 - 10$
j. log $= 5.8904 - 10$
k. log $= 9.0086 - 10$
l. log $= 7.3010 - 10$

3. Given $\log 2 = .3010$, write the equivalent statement in exponential form.

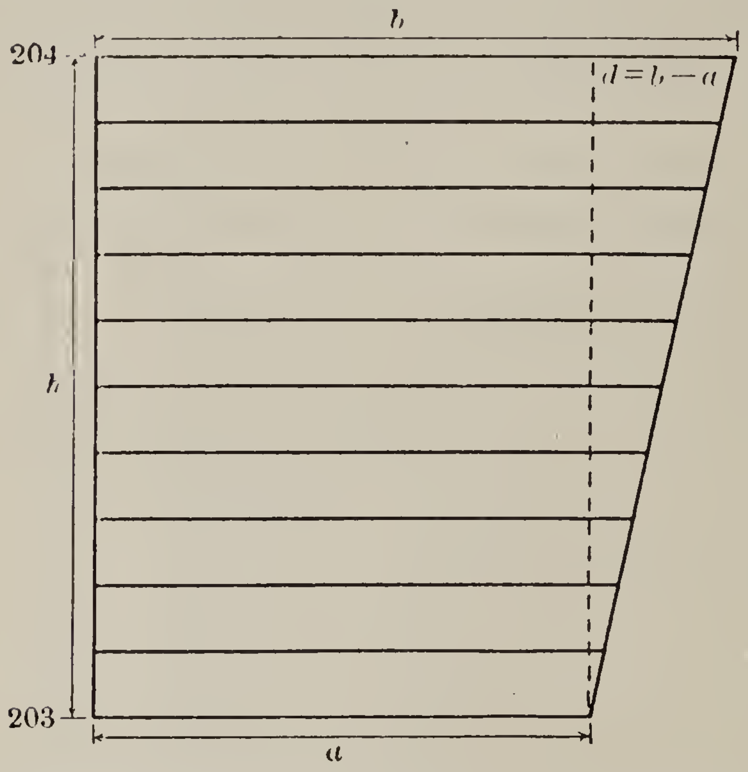
4. Given $10^{.699} = 5$, write the equivalent statement in terms of logarithms.

5. Given $10^{.699} = 5$, and $10^{x} = 2$, multiply and find the value of x.

7. Logarithms. Interpolation. — The process employed in extending the use of a table of logarithms of numbers with three significant figures, so as to give logarithms of numbers having four significant figures, is called *interpolation.* The method applies to increase in a similar manner the scope of any table of logarithms so as to give the logarithms of numbers having at least one more significant figure beyond those of the numbers in the tables.

The numerical process employed in interpolation may be illustrated graphically.

Given a trapezoid with bases a and b, and altitude h, we wish to divide the altitude into 10 equal parts and to find the lengths of the dividing lines parallel to the bases. Evidently each of these lines differs from the preceding line by one-tenth of the difference, $b - a = d$, between the bases. The upper line b may represent the logarithm of some number, *e.g.* of 204; the lower line may represent the logarithm of a number smaller by one unit, *e.g.* of 203; $d = b - a$ represents then the difference of the logarithms. We assume then that the nine intervening lines represent approximately the logarithms of 203.1, 203.2, . . . up to 203.9.



Graphical Interpolation between 75 and 96 (2.3075 and 2.3096), nine values interpolated

N	0	1	2	3	4	5	6	7	8	9
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404

This section of a logarithm table gives the logarithms of all numbers with three significant figures, having the succession of digits beginning 20 or 21. Note that the differences begin at 22 in the first line and drop to 19 in the second.

$$\begin{aligned} \log 203 &= 2.3075 \\ \left.\begin{array}{r} \log 203.1 = \\ \log 203.2 = \\ \text{to} \\ \log 203.9 = \end{array}\right\} &\longrightarrow \\ \log 204 &= 2.3096 \end{aligned}$$

These logarithms lie between 2.3075 and 2.3096; they increase steadily; we assume that they increase by uniform amounts, which we see in the table is roughly true for numbers from 200 to 219. The uniform increase of these logarithms of 203.1 to 203.9 is $\frac{1}{10}$ of the difference between the logarithms of 203 and 204, *i.e.* 2.1 of the units in the last place of our logarithms. It would not be correct to increase the number of places in our logarithms as our process is only an approximation not correct to further places even as our logarithms are only approximations to four decimal places. The logarithm of two is to five places .30103, to ten places .3010299957, to twenty places .30102999566398119521.

$$\begin{array}{rlrl}
\log 203.0 &= 2.3075 & \log .203 &= \bar{1}.3075 \\
203.1 &= 2.3077 & .2031 &= \bar{1}.3077 \\
203.2 &= 2.3079 & .2032 &= \bar{1}.3079 \\
203.3 &= 2.3081 & .2033 &= \bar{1}.3081 \\
203.4 &= 2.3083 & .2034 &= \bar{1}.3083 \\
203.5 &= 2.3086 & .2035 &= \bar{1}.3086 \\
203.6 &= 2.3088 & .2036 &= \bar{1}.3088 \\
203.7 &= 2.3090 & .2037 &= \bar{1}.3090 \\
203.8 &= 2.3092 & .2038 &= \bar{1}.3092 \\
203.9 &= 2.3094 & .2039 &= \bar{1}.3094 \\
204.0 &= 2.3096 & .2040 &= \bar{1}.3096
\end{array}$$

A four-place table of numbers, with logarithms to five places, gives simply:

	0	1	2	3	4	5	6	7	8	9
203	30750	30771	30792	30814	30835	30856	30878	30899	30920	30942
204	30963	30984	31006	31027	31048	31069	31091	31112	31133	31154

The appropriate characteristic must be inserted by the computer. A careful examination will show that our interpolation gives an incorrect four-place result for 203.4, with an error of half a unit. This type of error is inevitable, using four-place tables. In general, such errors tend to equalize each other; where absolute accuracy to four places is necessary, five- or even six-place tables must be used. Even with four-place tables it is evident that with a difference, called the tabular difference, of 21 to 24 the normal difference will be 2 units and 1 to 4 extra units will have to be distributed in the addition.

In the illustration above it may be noted that log 203.4 as 2.30835 does not inform us, strictly, as to whether log 203.4 to four places is 2.3083 or 2.3084; the latter is the case here since if log 203.4 is found to further places than five the fifth decimal place is actually a 5. Whenever the five-place logarithm is a terminal 5 which has been obtained by increasing a 4 to 5 the four-place logarithm would not be increased by one unit; thus, log 2007 to 5 places is 3.30255, to 6 places log 2007 is 3.302547, and hence to four places log 2007 is 3.3025. In some tables of logarithms a terminal 5 due to an increase, as in log 2007 = 3.30255, is marked with a superimposed negative sign or other mark.

In physical problems, measurement to three places permits of computation to three places only; the fourth place by interpolation assures accuracy, in general, of the third place.

The reverse process is to find the number, given the logarithm; thus, to find the number whose logarithm is 2.3088, we see that

the table gives log 203 = 2.3075
and log 204 = $\underline{2.3096}$
The tabular difference is 21
The given logarithm is 2.3088
The difference between it and the smaller logarithm nearest to it is 13

	Diff. 21
1	2.1
2	4.2
3	6.3
4	8.4
5	10.5
6	12.6
7	14.7
8	16.8
9	18.9

The table of tenths of the tabular difference, which is frequently given in tables of logarithms, shows that 13 is nearest to .6 × 21. The number is 203.6. Had the given logarithm been 2.3087, we would find as the number again 203.6, since the actual difference is 12, which also is nearer to .6 of 21 than to .5 of 21. Note that in the table, and in the difference table, the logarithm or part of the logarithm lies always to the right; the number, or part of the number, lies to the left or above.

8. Historical note. — Fundamentally the notion of logarithms is intimately connected, as we have shown, with the notion of exponents. The one-to-one correspondence between exponents and a series of successive powers of a given number was noted many years before logarithms were developed, in many arithmetical textbooks of the fifteenth and sixteenth centuries. A German mathematician, Stifel (1486–1567), published a work entitled *Arithmetica Integra*, 1544, in which these "exponentes," as he termed them, were extended to the left. Stifel spoke of the great possible use of such series for computation in which addition would replace multiplication, and subtraction replace division; but he developed the idea no further.

Exponents	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
Numbers	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32	64	128	256	512	1024

Figure from Stifel's Arithmetica Integra, 1544

In 1614 John Napier, Baron of Merchiston, a Scotchman, revolutionized computation processes by the composition of logarithmic tables, based on the idea of the comparison of two series essentially of the kind indicated above. The adoption of 10 as the base of a logarithmic series was due to a friend of Napier, Henry Briggs, who published in 1617 the decimal logarithms of the first thousand numbers.

In recent years the widespread adoption of computing machines which carry multiplications and divisions to fifteen and twenty places is somewhat replacing logarithms in the offices of great insurance companies and, to some extent, in observatories.

Logarithms. Illustrative Problems. — I. Find by four-place logarithms:

a. 203×137; *b.* $\frac{2.03}{137}$; *c.* $\frac{137}{2.03}$; *d.* $(203)^3$.

a. log 203 = 2.3075
log 137 = 2.1367
log product = 4.4442
product = 27810

By interpolation since 2, the difference found, is in tenths nearest to .1 of 16, the tabular difference.

b. $\log 2.03 = 0.3075$
$\log 137 = \underline{2.1367}$
$\log \text{quotient} = \bar{2}.1708$

or

$\log 2.03 = 10.3075 - 10$
$\log 137 = \underline{\ \ 2.1367\qquad\quad}$
$\log \text{quotient} = \ \ 8.1708 - 10$
$\text{quotient} = .01482$

The 2 is obtained by interpolation since 5, the difference found, is nearest to .2 of 29, the tabular difference.

c. $\log 137 = 2.1367$
$\log 2.03 = \underline{\ .3075}$
$\log \text{quotient} = 1.8292$

Quotient is 67.48, since the difference 5 is in tenths nearest to .8 of 6, the tabular difference.

d. $\log 203 = 2.3075$
$\underline{\qquad\qquad 3}$
$\log (203)^3 = 6.9225$
$(203)^3 = 8{,}366{,}000$

If these numbers represent physical constants, obtained by measurement, our computations would be slightly more accurate than the measurements warrant. Even in prices, in general, these results would be sufficiently accurate for purposes of business. Thus 137 pounds of cheese at 20.3 ¢ per pound would be found to be worth \$ 27.81; 137 tons of mine-run coal at \$ 2.03 per ton is worth \$ 278.10, with a possible error of 10 ¢; 1370 tons of coal at \$ 2.03 would be \$ 2781 with a possible error of 50¢, which is negligible in business of this magnitude.

II. Find by logarithms, four-place, to four places:

a. 203.8×137.5; *b.* $\frac{20.38}{13750}$; *c.* $(203.8)^3$; *d.* $(.02038)^{\frac{3}{5}}$.

a. $\log 203.8 = 2.3092$
$+ \log 137.5 = \underline{2.1383}$
$\log \text{product} = 4.4470$
$\text{product} = 27.790$

c. $\log 203.8 = 2.3092$
$\underline{\qquad\qquad 3}$
$\log (203.8)^3 = 6.9276$
$(203.8)^3 = 8{,}464{,}000$

b. $\log 20.38 = 1.3092$
$- \log 13750 = \underline{4.1383}$
$\log \text{quotient} = \bar{3}.1709$
$\text{or } 7.1709 - 10$
$\text{quotient} = .001482$

By interpolation the given difference of 6(.1709 – .1703) is nearest in tenths to .2 of the tabular difference, 29(.1732 – .1703).

d. $\log (.02038)^{\frac{3}{5}} = \frac{3}{5} \log .02038$

$$\begin{array}{rrcr} & \log .02038 = \bar{2}.3092 & & 8.3092 - 10 \\ & \underline{3} & \text{or} & \underline{3} \\ 5 & \bar{6}.9276 & & 24.9276 - 30 \end{array}$$

The division by 5 causes trouble because of the negative characteristic. To avoid the difficulty, write $\bar{6}.9276$ as 44.8276 – 50. Dividing this by 5 gives 8.9855 – 10 or $\bar{2}.9855$.

$$(.02038)^{\frac{3}{5}} = .09672.$$

EXERCISES

1. Using a watch with a second hand, time yourself on looking up the logarithms of the following 20 numbers; as a preliminary exercise look up the logarithms of these numbers without using the fourth significant figures, not requiring interpolating.

log 314.6 =
log 813.2 =
log 5.462 =
log .003468 =
log .3085 =
log 769.9 =
log 67870 =
log 368.60 =
log 53.85 =
log 19.26 =
log 14.32 =
log 1876000 =
log 81930 =
log .08764 =
log 3.250 =
log .7263 =
log 200.4 =
log 399.8 =
log 210.4 =
log .03899 =

Twenty minutes should be the maximum time on this list; practice until you can do all 20 with interpolations within 15 minutes or even 10 minutes; all 20 characteristics should be written before you begin to use the tables.

2. Find by four-place tables the logarithms of the following numbers, interpolating: 326, .08342, 10,050, .008766, 5499, 3.482×10^6, 37.04, 290.40, .9647, 38.55, .06948, 3001, 2.777×10^{-6}, 784.4, 6,934,000, 5.341, 70.98, .1237, 8462, 3740.

Time yourself; the 20 should not take more than 12 minutes.

3. Find the numbers corresponding to the following logarithms — no interpolation is necessary. Take the time of looking up the numbers and writing these in a prepared form as answers; the time should not exceed 8 minutes.

log = 3.8414
log = 0.9996
log = $\bar{2}.6866$
log = $\bar{1}.7482$
log = 6.3010
log = 8.0086 – 10
log = 4.2856
log = 1.8837
log = 9.3201 – 10
log = 2.7789
log = 1.0492
log = 5.9063
log = 2.9557
log = 3.1732
log = $\bar{6}.2672$
log = $\bar{5}.7832$
log = 1.1761
log = 9.5024 – 10
log = 2.4786
log = $\bar{3}.1673$

The time includes only looking up the logarithm and writing it down; it should not include copying the problem. A piece of blank paper may be placed alongside of each column and the logarithms written upon the paper; by folding the paper lengthwise the two columns may be placed upon the same sheet. Practice with timing.

4. Find the numbers corresponding to the following logarithms, interpolating wherever necessary; time yourself, in looking up numbers, first writing the given logs in column form, and not counting that time. The 20 should not take more than 15 minutes; 10 minutes is slightly better than average time.

log = 3.5861
log = $\bar{5}.6427$
log = $\bar{1}.4436$
log = 4.7320
log = 6.9428
log = $\bar{2}.4415$
log = 6.4893 – 10
log = 5.8662
log = 1.5729
log = 2.9990
log = 8.5418 – 10
log = 2.0923
log = 2.9844
log = 3.3080
log = 1.218
log = 7.8419 – 10
log = 0.4630
log = $\bar{1}.7848$
log = 0.9618
log = 1.7276

5. By chaining, the sides of a rectangular field are found to measure 413.2 feet and 618.4 feet. Find the area in square feet and in acres. What effect upon the computed area does an error .1 of a foot in measurement make? Consider this fact in making the computation, not assuming that these lengths are more accurately given than to .1 of one foot.

6. Use Hero's formula, $A = \sqrt{s(s-a)(s-b)(s-c)}$, wherein a, b, c are the sides and $\frac{(a+b+c)}{2} = s$, to compute the area of a triangle whose sides are 413.2 feet, 618.4 feet, and 753.2 feet. Discuss errors as before. Do all the preliminary computation of s, $s-a$, $s-b$, $s-c$ before looking up any logarithms.

7. Time yourself on finding the 20 numbers corresponding to the following logarithms:

2.1120
1.6150
$8.9312 - 10$
4.8990
$\bar{1}.2480$
5.4151
3.5674
1.1174
$\bar{1}.9462$
0.5132
3.3808
$7.8973 - 10$
$\bar{2}.7455$
0.0420
1.2222
$\bar{1}.3660$
4.3925
$\bar{6}.4105$
2.9213
$\bar{2}.0621$

8. Perform the following computations, using logarithms:

a. 54.04×376.2; *b.* $\frac{5.404}{3762}$; *c.* $(54.04)^2(3.762)$;
d. $\sqrt[3]{54.04}$; *e.* $\sqrt{54.04 \times 376.2}$.

9. Find the volume in gallons of a cylindrical can, 18 inches in diameter and 30 inches high. (1 gallon = 231 cubic inches.)

10. Show that the capacity of a cylindrical can in gallons can be written as $\frac{1}{3}$ of 1 % of d^2h + 2 % of the $\frac{1}{3}$ of 1 % of d^2h, or as $1.02 \times 0.00\frac{1}{3} \times d^2h$, given d and h in inches.

11. Find the volume in cubic feet of a silo 16 feet in diameter and 32 feet high. Compute for 15.9, 15.95, 16.05, and 16.1 feet in diameter. If the measurements are correctly given within .1 of one foot, how accurately can the volume be given, with a 16-foot diameter?

12. Find approximate value of 2^{50}, 2^{-50}, $2^{\frac{1}{10}}$, and $2^{-\frac{1}{10}}$ by logarithms.

13. Extract the cube roots of 2, 3, 4, 5, 6, 7, and 8 by logarithms. Multiply the value of $2^{\frac{1}{3}}$ by the value of $3^{\frac{1}{3}}$, and compare with your value of $6^{\frac{1}{3}}$.

14. Given a pendulum of length l centimeters and time of oscillation t seconds, you have the following formula connecting l, and t:

$$t = \pi\sqrt{\frac{l}{980}}.$$

Given $t = 1$, compute l; given $l = 60$, compute t; given that $t = 1$ but that instead of 980 you have 981, compute l.

15–30. Solve the problems at the end of Chapter II using logarithms.

9. Compound interest. — When interest is added to the principal at the end of stated intervals forming a new principal which is to continue to draw interest, the total increase in the original principal which accumulates by this process continued for two or more intervals is called the compound interest on the original principal.

Let P represent the principal, i the interest rate per interval, and n the number of intervals, and S the amount at the end of n intervals. Given interest compounded at rate i per annum for n years.

At the end of 1 year you have $P + iP = P(1 + i)$.

At the end of 2 years you have

$$P(1 + i) + iP(1 + i) = P(1 + i)^2.$$

At the end of 3 years you have

$$P(1 + i)^2 + iP(1 + i)^2 = P(1 + i)^3.$$

. . . .

At the end of n years you have $S = P(1 + i)^n$.

Or you may say that since the interest for 1 year increases the principal P to $(1+i)P$, then in 1 further year this new principal $P(1+i)$ will be increased in the same ratio, giving $P(1+i)\times(1+i)$ or $P(1+i)^2$, and for each further year the factor $(1+i)$ is introduced. Hence the amount at the end of n years is $P(1+i)^n$.

If interest is compounded at the end of every three months, or every six months, you substitute for i, $\frac{i}{4}$ or $\frac{i}{2}$, and for n, $4\,n$ or $2\,n$, since the number of intervals of three months in n years is $4\,n$ and of six months is $2\,n$.

The formula $S=P\left(1+\frac{j}{m}\right)^{mn}$ is used for an interest rate given as j per annum, but compounded m times per annum, at rate $\frac{j}{m}$ for each interval.

Problems in compound interest lend themselves to solution by logarithms.

Given,

$$S=P(1+i)^n.$$

$$\log S=\log P+\log(1+i)^n.$$

$$\log S=\log P+n\log(1+i).$$

$$\log P=\log S-n\log(1+i).$$

$$n=\frac{\log S-\log P}{\log(1+i)}.$$

$$\log(1+i)=\frac{\log S-\log P}{n}.$$

Note that it is better not to use these as formulas, memorizing them, but rather to go back to the fundamental relation, $S=P(1+i)^n$. Note also that the formula holds for other than integral values of n; thus at 6 % per annum the interest on the amount P for six months or eight months is defined as $P(1+.06)^{\frac{1}{2}}-P$ or $P(1+.06)^{\frac{2}{3}}-P$, respectively. Hence for $n+\frac{1}{2}$ years the amount would be $P(1+i)^n(1+i)^{\frac{1}{2}}=P(1+i)^{n+\frac{1}{2}}$.

PROBLEMS

1. Find the amount of \$ 1000 at interest 4 % annually, compounded for 20 years. Find the amount when compounded semi-annually at a nominal rate of 4 % per annum, *i.e.* 2 % semi-annually.

2. In how many years will money double itself at 4 %, 5 %, 6 % interest, compounded annually?

3. Given that at the end of 20 years \$ 1000 amounts to \$ 1480, what is its approximate rate of interest?

4. Given that at 5 % interest, compounded annually, \$ 1000 amounts to \$ 1480, what is the approximate number of years?

5. Find the compound amount of \$ 1400 at 5 % interest, compounded semi-annually, for 10 years, 11 years, 12 years, up to 20 years.

6. If \$ 100 is left to accumulate at 3 % interest, compounded annually, what will it amount to in 100 years? Solve by logarithms. What amount put at 3 % interest will amount in 100 years to \$ 1,000,000? What is the present equivalent of \$ 1000 to be paid at the end of 100 years, money worth 3 %, compounded annually?

7. Solve problem 6 for 4 %, 5 %, and 6 % interest, compounded annually. For 6 % per year, compounded semi-annually, for 50 years.

8. Benjamin Franklin, who died in 1790, left 1000 pounds to "the town of Boston" and the same to the city of Philadelphia. His will directed that this amount should be loaned at interest to young artisans, and thus accumulated for 100 years until the principal should have increased to 130,000 pounds. He directed further that at that time the major portion of this amount should be expended for some public improvement and the residue left to accumulate, similarly, for another hundred years. What rate of interest did Franklin assume that his money would earn? In Boston the amount,

$ 5000 approximately, accumulated to about $ 400,000. Find the average rate of interest earned annually. Assuming that $ 5000 was kept aside in 1891, as directed, what will this amount to in 1991, compounded at 4 % annually?

9. Find the amount at the end of 200 years of $ 5000, interest at 4 %, 5 %, and 6 %, compounded annually.

10. If a business doubles its capital, out of earnings, in 12 years, what rate of interest on capital invested does this represent per year? If in 20 years the capital is doubled, find the rate of interest earned.

11. The United States has increased in population from 7.2 million in 1810 to 101.1 million in 1910; find the approximate rate of increase per year, and for each ten-year period. Compare with the figures on page 65.

12. The city of New York increased in population from 120,000 to 4,769,000 in the interval from 1810 to 1910. Compute the average annual rate of increase, using the formula, $120{,}000(1+i)^{100} = 4{,}769{,}000$. Compute the average ten-year increase and compare with the actual statistics on page 66.

CHAPTER IV

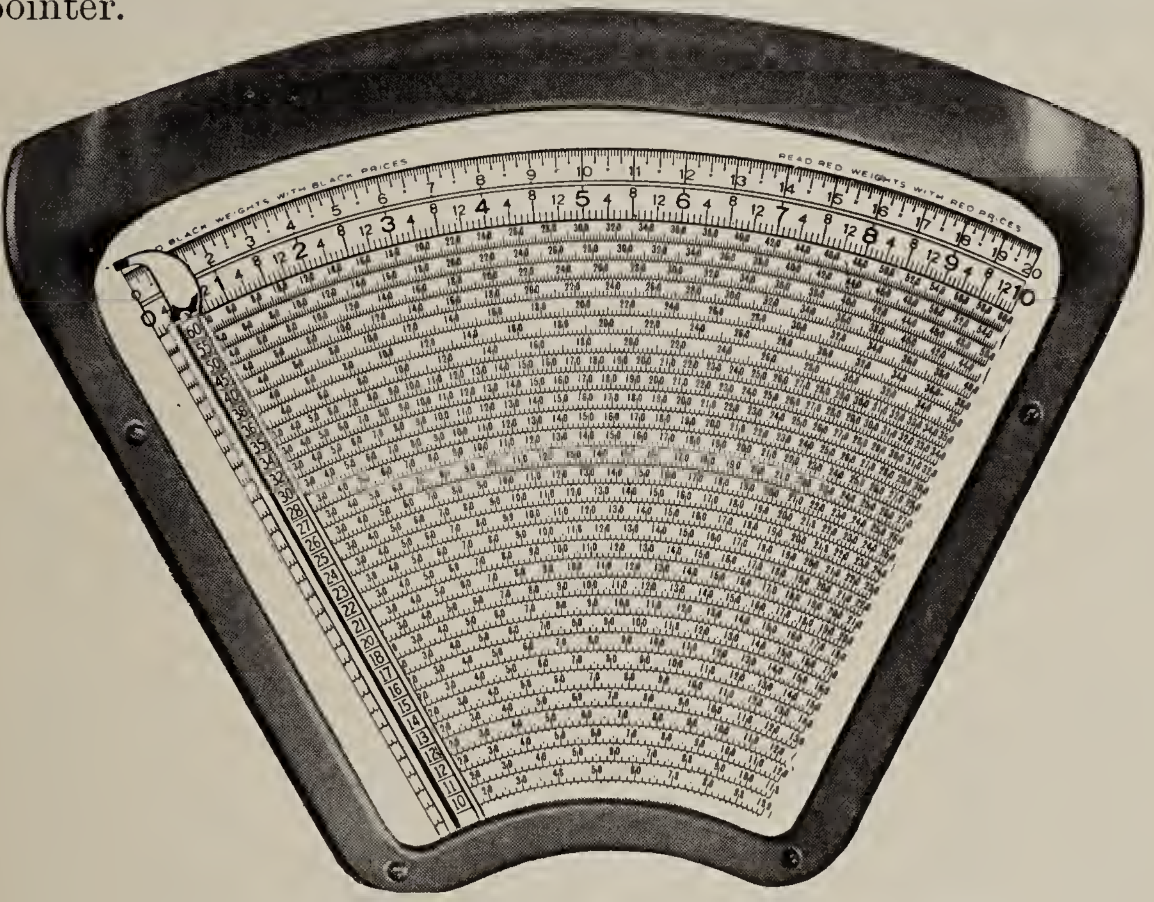
GRAPHICAL REPRESENTATION OF FUNCTIONS

1. Functional relationships. Expressions of the form $3x + 5$, $ax + b$ are called linear or first degree functions of the variable x; in elementary algebra such expressions have been combined according to the fundamental operations and subject to the laws given in a preceding chapter. Further, some attention is given in elementary algebra to expressions of the form $ax^2 + bx + c$, the general quadratic function of x, and expressions involving higher powers of x. The expression $ax^n + bx^{n-1} + \cdots$ is called an algebraic function of x of the nth degree when n is a positive integer and the coefficients $a, b, \cdots$ are constants. This represents of course a number for any value of x.

$F(x)$, $G(x)$, $\phi(x)$, $\mathrm{A}(x)$, $\cdots$ are methods of representing functional relationships; $F(x)$, (read F of x or F function of x), means that this expression assumes various values as x varies, these values being determined by some law. In the equation, $y = 3x + 5$, y is explicitly given as a function of x; y is here a linear function of x. In the equation, $y = x^2 + 4x - 5$, y is an *explicit* function of x; as x varies, so does y. In $x^2 + y^2 = 25$, as x takes on different values so does y, but one must solve for the corresponding values of y. Here y is called an *implicit* function of x.

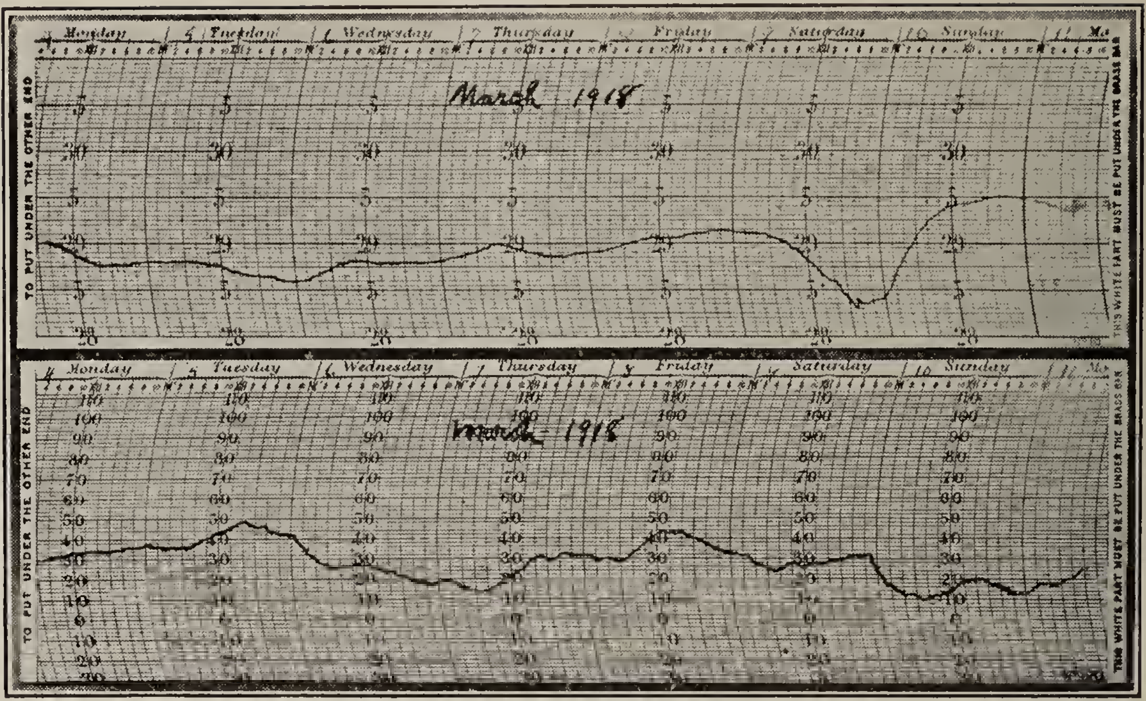
When two variable quantities are so related that the variation of one of these depends upon the variation of the other, either is said to be a function of the other. Thus the production of wheat in the United States from 1900 to 1915 is a variable quantity depending upon the year of production. The height of a given tree is a function of its age; to each number expressing in any convenient unit of time the age of the tree corresponds a given number expressing the height of the tree. Similarly the weight of a tree is a function of the age of the tree. This type of relationship cannot be expressed algebraically. It may be exhibited by the two series of numbers, or it may be expressed graphically.

2. Graphical representation of statistics. — Since two variable quantities are to be represented, two sets of numbers must be indicated; this could be done by placing the two sets upon two lines straight or curved, drawn parallel to each other. This is the form used upon grocers' scales wherein the variables of weight and corresponding price are placed upon concentric circular arcs; corresponding numbers are cut by the pointer.



Series of corresponding numbers graphically represented

It is commonly more convenient to place the two scales for representing the two variable quantities upon two lines perpendicular to each other. Upon the following figure the temperature and barometric pressure are indicated by the diagram for the week, March 4–11, 1918, at Ann Arbor, Michigan.



Temperature and barometric chart by moving pointer
The sharp break in barometer curve corresponds to a violent rainstorm.

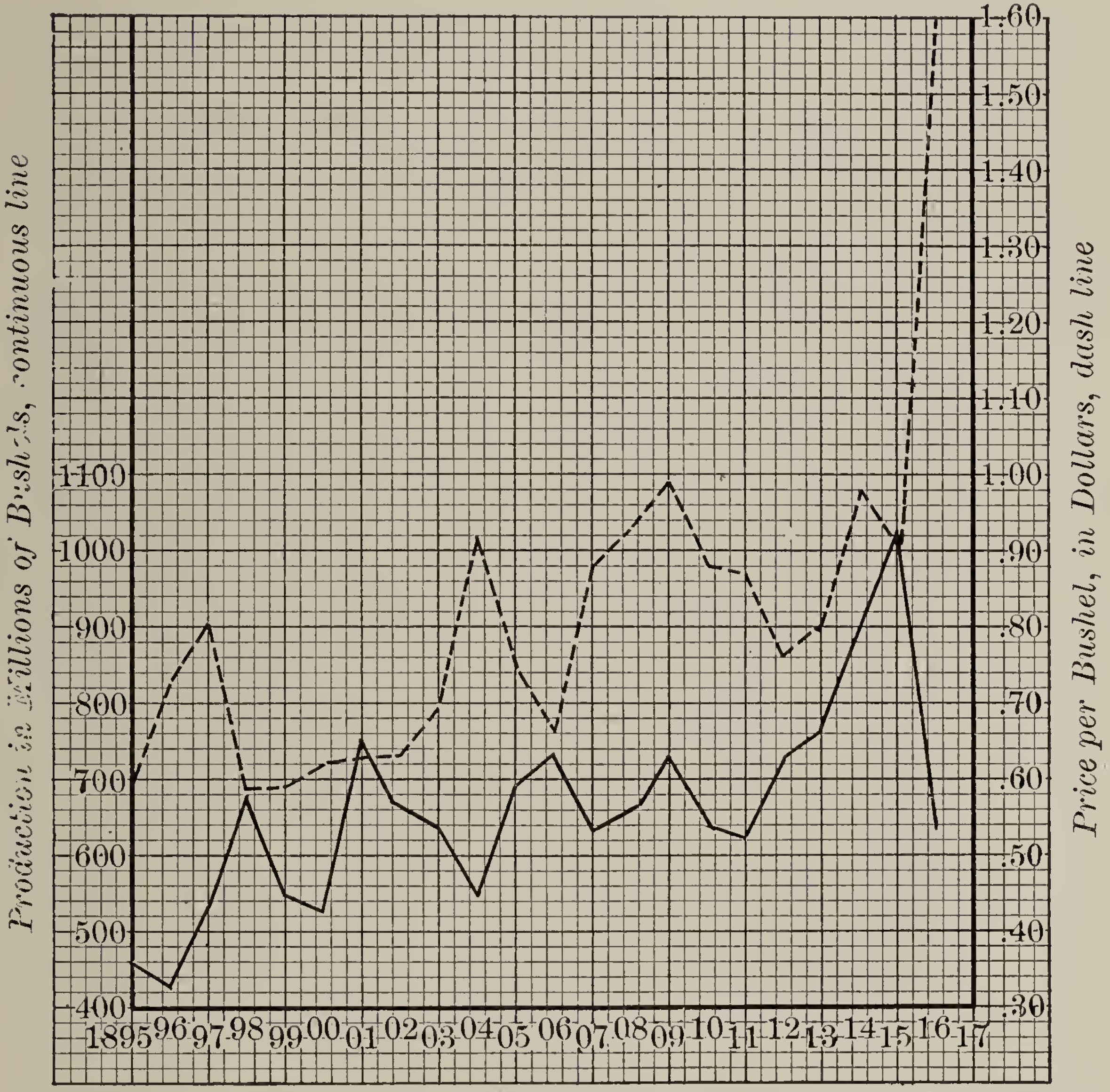
The horizontal displacement of any point on either graph, located by the vertical rulings, indicates the time of the observation ; the corresponding temperature or pressure is indicated by the vertical displacement.

ILLUSTRATIVE EXERCISES

1. Production and price of wheat in the U. S. from 1895 to 1916 are given in statistical form and graphically.

Year	Production	Exports	Price	Year	Production	Exports	Price
	Millions of Bushels		Cents		Millions of Bushels		Cents
1895	467	126	50.9	1907	634	163	87.4
1896	428	145	72.6	1908	665	114	92.8
1897	530	217	80.8	1909	737	87	98.6
1898	675	222	58.2	1910	635	69	88.3
1899	547	186	58.4	1911	621	80	87.4
1900	522	216	61.9	1912	730	143	76.0
1901	748	235	62.4	1913	763	146	79.9
1902	670	203	63.0	1914	891	332	98.6
1903	638	121	69.5	1915	1,026	243	91.9
1904	552	44	92.4	1916	640		160.3
1905	693	98	74.8	1917			
1906	735	148	66.7				

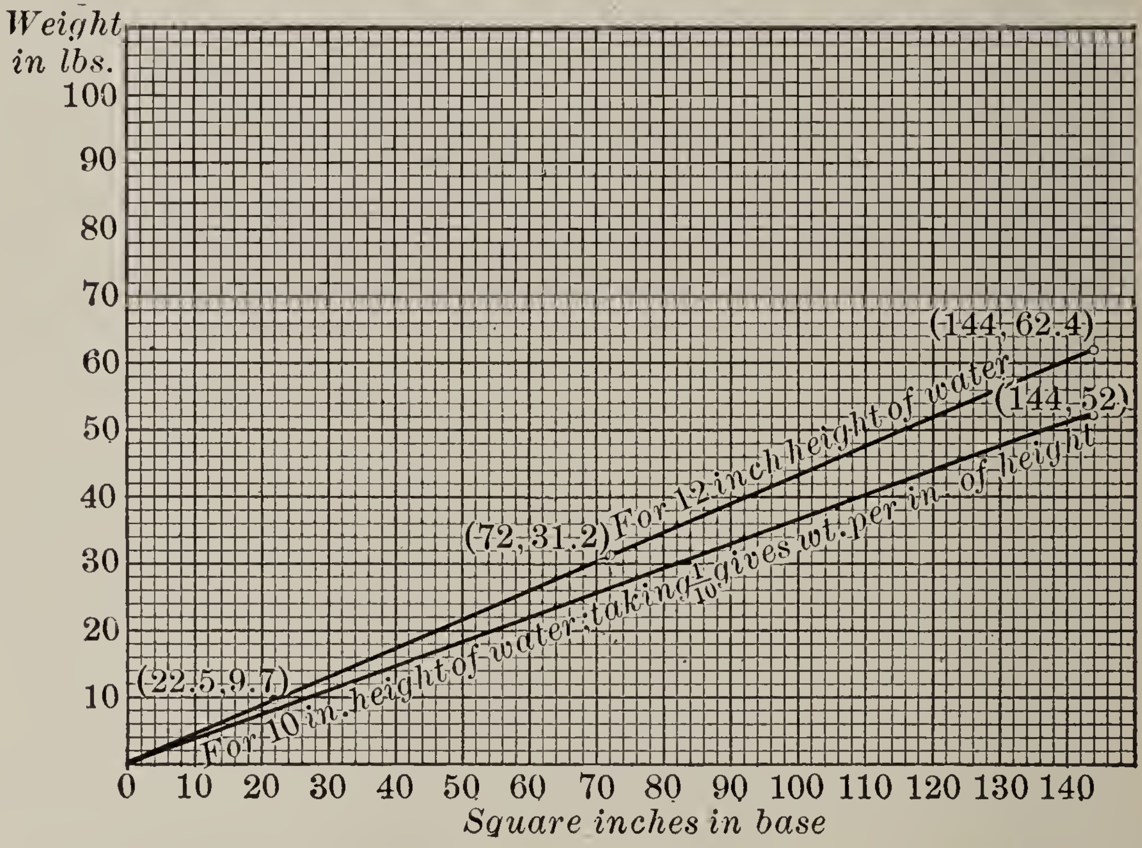
Statistics from the Yearbook of the U.S. Department of Agriculture



Graphical representation of wheat production (continuous line) **and price** (broken line) **in the United States, 1895–1917**

Note that the graphical form of these statistics brings out several points of interest. In the first place the maximum price paid for wheat in the interval is immediately found, and so also the minimum price of 51¢ (50.9¢) in 1895. Further, the diagram shows very pointedly that a large production under normal circumstances is accompanied by a fall in price, and an immediate diminution of production. In 1917, under war conditions, both production and price increased greatly.

2. The weight of water per cubic foot, or 60 pints, is 62.4 pounds. For cylindrical vessels filled to a height of 12 inches the weight for an area 144 square inches in the base would be 62.4 pounds; for 72 square inches in the base the weight of water would be 31.2 pounds; for 0 square inches the weight is 0 pounds. On coördinate paper represent square inches of base on the horizontal line, taking 1 major division to represent 10 square inches, and represent weight on the vertical line.



Weight of water in cylindrical vessels with varying base when filled to a height of 12 inches or 10 inches

Note that the weight of 12 inches of water in a vessel with a base containing 50 square inches is 21.5 pounds, and conversely if a cylindrical vessel contains 21.5 pounds in 12 inches of height, the base contains 50 square inches, and similarly, of course, for other values.

PROBLEMS

1. Plot the temperature, as vertical lengths, and the time, by hours, as horizontal lengths, for 24 hours.

2. Plot the contents in pints of cylindrical vessels 12 inches in height, with varying bases; take that with base 144 square inches, the capacity is 60 pints; with 72 square inches

in the base, 30 pints; with 0 base, 0 pints. The straight line joining these points can be used to give the base in square inches of any cylindrical vessel whose capacity for a height of 12 inches is known. What would be the base of a vessel that contains 10 quarts when filled to the height of 12 inches?

3. Plot cubic inches against pints, taking 1728 cubic inches as 60 pints.

INCREASE IN VOLUME WITH TEMPERATURE INCREASE

As liquids are heated the volume changes, generally increasing; thus water increases in volume when heated except between 0 and + 4° C. Given 1000 cu. cm. of water at 4° C. and 1000 cu. cm. of mercury at 0° C., the volume at other temperatures is given by the following table:

TEMPERATURE	VOLUME OF WATER	VOLUME OF MERCURY
0°	1000.13	1000.00
1°	1000.06	1000.2
4°	1000.00	1000.9
8°	1000.13	1001.4
10°	1000.27	1001.8
15°	1000.87	1002.7
20°	1001.77	1003.6
25°	1002.94	1004.5
30°	1004.35	1005.4
35°	1005.98	1006.3
40°	1007.82	1007.2

4. Plot the increase above 1000 cu. cm., or decrease, in cu. cm. in volume of the water, using 1 half-inch for 5° on horizontal axis and 1 half-inch for 1 cu. cm. on the vertical axis. Note that by adding 1000 to the given readings, actual volumes can be read.

5. Plot the same curve for the increase in volume of the mercury. It is evident that the increases in volume of the mercury are approximately proportional to the increases in temperature.

Statistics on Weight and Height

From an investigation of the statistics giving characteristics of a group of over 200,000 men and 130,000 women, the following facts are obtained on average height. The facts are given for groups of 1000.

Height	Frequency or No. in Group	
	Men	Women
4′ 9″	0	1
4′ 10′	0	4
4′ 11″	0	10
5′ 0″	2	40
5′ 1″	2	55
5′ 2″	5	107
5′ 3″	12	135
5′ 4″	30	184
5′ 5″	55	167
5′ 6″	99	134
5′ 7″	127	83
5′ 8″	169	48
5′ 9″	145	18
5′ 10″	147	8
5′ 11″	104	3
6′ 0″	66	1
6′ 1″	22	0
6′ 2″	11	0
6′ 3″	3	0
6′ 4″	1	0

Weight (to Nearest Integer in 5 or 0) of Men; Ages 35–39; Height 5′ 10″	Frequency or Number in Group
125	4
130	14
135	33
140	60
145	78
150	114
155	95
160	106
165	90
170	87
175	72
180	59
185	48
190	37
195	25
200	32
205	12
210	14
215	4
220	8
225	3
230	1

In any such group the number of individuals having any given characteristic is called the frequency corresponding to the given characteristic.

6. Plot the frequency curve of heights of men and women, taking $\frac{1}{2}$ inch as corresponding to 1 inch of height on the horizontal axis and taking $\frac{1}{2}$ inch vertical for 30 individuals. This curve represents very nearly what is termed a normal symmetrical distribution.

7. Plot the frequency curve for weights of men between 35 and 39.

Agricultural Statistics

In the *Yearbook* of the Department of Agriculture statistics of production and prices of standard crops and farm products are given, covering a period frequently of 50 years. Use this *Yearbook* to obtain the data for the following curves:

8. Plot the curve showing the production of corn in the United States from 1866 to the present time. Use 200,000,000 bushels as a vertical unit, taking $\frac{1}{4}$ inch as the unit; take one year as $\frac{1}{10}$ of an inch.

9. Plot prices on the diagram of 8, using a right-hand scale.

10. Plot similarly statistics for the amount and price of sugar produced in the United States.

11. Plot the average price in the United States of eggs by months for the current year; plot butter prices similarly.

Population Statistics

The population statistics of the United States by 10-year intervals as given by the Statistical Atlas of the U. S. Bureau of Census are:

Date	U. S. (Millions)	New York (Thousands)	Texas (Thousands)
1790	3.9	340	
1800	5.3	589	
1810	7.2	959	
1820	9.6	1,373	
1830	12.9	1,919	
1840	17.1	2,429	
1850	23.2	3,097	213
1860	31.4	3,881	604
1870	38.6	4,383	819
1880	50.2	5,083	1,592
1890	63.0	6,003	2,236
1900	77.3	7,263	3,049
1910	101.1	9,114	3,897
1920	117.0	10,942	4,734

12. Plot the curve of population of the United States.

13. Plot the population curve for Michigan, and estimate the population for the 5-year periods.

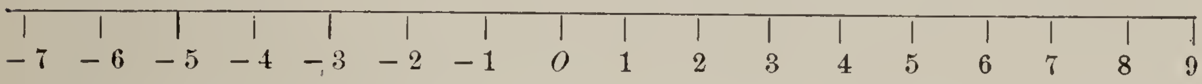
Date	Michigan (Thousands)	Date	New York City (Thousands)	Date	Michigan (Thousands)	New York City (Thousands)
1837	175	1790	49	1864	804	
		1800	79	1870	1,184	1,478
		1810	120	1874	1,334	
		1820	152	1880	1,637	1,912
		1830	242	1884	1,854	
				1890	2,094	2,507
1840	212		391	1894	2,242	
				1900	2,421	3,437
1845	303			1904	2,530	
				1910	2,810	4,769
1850	398		696	1920	3,205	5,621
1854	507					
1860	749		1,175			

14. Plot the population curve of New York City. What has been the average rate of increase for 10-year intervals and for yearly intervals, approximately, since 1810? Note that this requires the solution of the equations $120(1+i)^{10}=4769$, and $120(1+i)^{100}=4769$; solve by taking the logarithm of both sides.

15. Discuss the increase of the population of the United States from 1820 to 1920 as in problem 14 the population of New York City is discussed.

3. Graphical representation of algebraic functions. — To represent a point on a given line only one number is necessary with a point of reference and some unit of length. To every number corresponds one point and only one and conversely to every point corresponds one number and only one number.

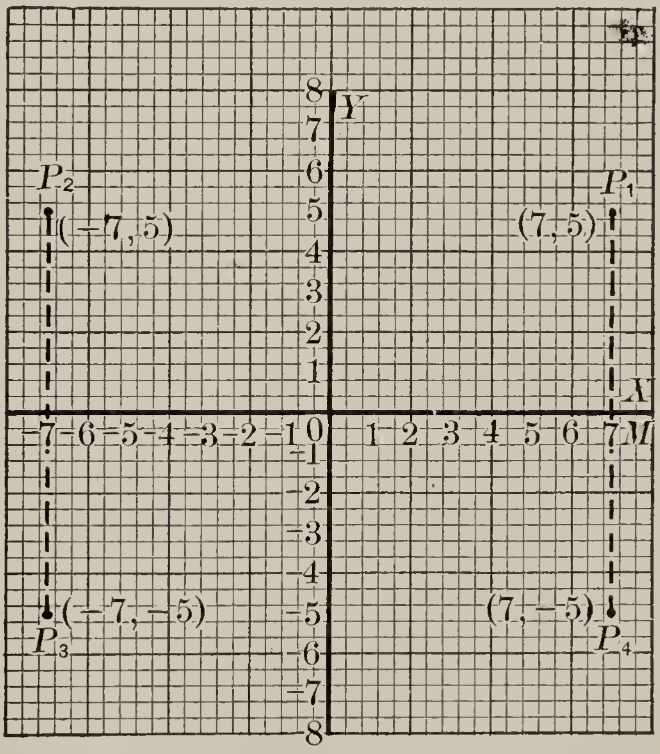
The distance cut off from O, the origin, by any point on this line may be called the abscissa of the point; a moving point upon this line may be designated by the variable x, which is then thought of as assuming different values, corresponding to the different positions of the point upon the line.



If another scalar line, OY, be taken intersecting OX at 90°, the two lines may be conveniently used to represent the position of any point in the plane of the two lines. The two lines of reference are called commonly the x-axis and the y-axis respectively.

The position of a point on the earth's surface is given by a pair of numbers representing in degrees longitude and latitude; the + and – of our numbers are replaced by E. and W. in longitude, and by N. and S. in latitude. If we agree to give longitude first, then + and – could, in both terms, replace the letters, and position on the earth's surface of any point can be given by a pair of numbers. The system of representing points in a plane is not essentially different.

Given any point in the plane as P, a perpendicular is dropped to the horizontal line. The distance cut off on this horizontal line is called the abscissa or x-coördinate of P; the distance cut off on the vertical line OY by a perpendicular from P to OY is called the ordinate or y-coördinate of the point P and it is evidently equal to the ⊥ PM dropped



Location of points in a plane

to the axis OX. The two numbers together, abscissa given first, serve to locate the point; thus a point P_1 7 units to the right of OY and 5 units above OX is located on our diagram. To this point corresponds the pair of numbers (7, 5) (read "seven, five") and to the pair of numbers (7, 5) corresponds point P_1. The point P_4 symmetrical to P_1 with respect to OX, is $(7, -5)$ the negative ordinate indicating that the point is below the x-axis. P_2 $(-7, 5)$ and P_3 $(-7, -5)$ are located upon the diagram.

A moving (or a variable) point P in the plane is designated by (x, y), which is read "x, y" (not "x AND y"), and the coördinates, abscissa and ordinate, of P are a different pair of numbers for each position of the point, *i.e.* x and y are variable.

Every point represents a pair of numbers, and consequently a series of points will represent a series of pairs of numbers. In the statistical diagrams the pairs of numbers are numbers functionally related. In an algebraic function, $y = 3x + 5$, we have involved a relationship corresponding to a mass of statistical information, and the pairs of numbers can be represented upon a diagram just as before. Corresponding numbers, a pair of numbers, are obtained by giving a value to x and computing the value of y. The points are seen to lie upon a straight line, which we shall see includes all points and only those points whose coördinates, abscissa and ordinate, when substituted for x and y, respectively, satisfy our given equation. This line is called the graph of the function, $3x+5$, or the locus of the equation, $y = 3x + 5$; the operation of locating the points and connecting them is termed plotting the graph.

To represent on cross-section paper any equation in two variables x and y, t and s, u and v, or by whatever letters designated, two intersecting scales as axes of reference OX and OY, OT and OS, or OU and OV are taken, and pairs of values which satisfy the functional relationship are plotted as above.

4. Historical note. — The invention, or more properly the discovery, of analytical geometry was made in the early part of the seventeenth century. The first work directly on the subject was published by René Descartes in 1637, *La Géométrie*, a work small in compass but great in its effect upon the development of mathematics and science. Almost simultaneously another Frenchman, Pierre Fermat, also discovered the methods independently of Descartes.

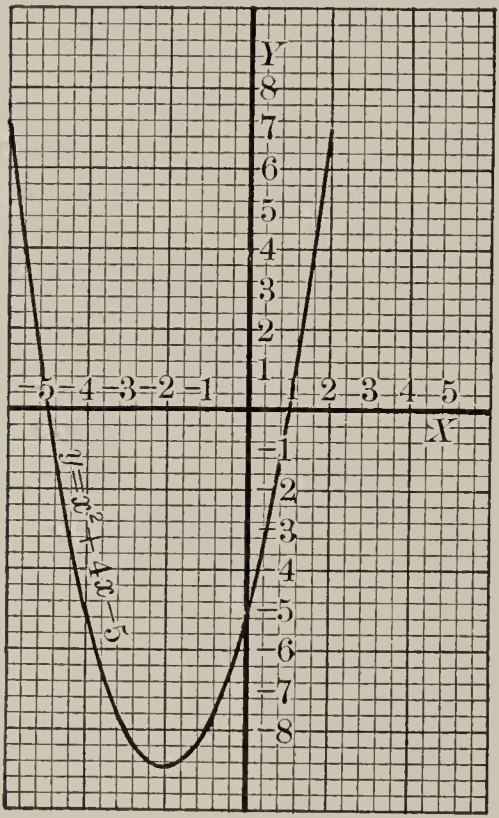
The idea of coördinates, called Cartesian after Cartesius (Latin form of Descartes) was not new; in fact, as we have noted, this idea is found in the latitude and longitude of Hipparchus (200 B.C.). The idea of coördinates for drawing similar figures was known even to the early Egyptians, and this idea was used for surveying purposes by Heron of Alexandria (c. 100 B.C.). The idea of fundamental properties of any curve as related to its axis or axes or to tangent lines and diameters was also not new. The new point was to combine these ideas, referring several curves and straight lines to axes geometrically independent of the curves, using letters to represent constant and variable distances associated with the curves and lines involved; the graphical representation of negative quantities is a vital part of the analytical geometry. These developments were made both by Descartes and by Fermat.

Modern mathematics begins with this analytical geometry and with the calculus which was developed within a century after Descartes by Newton and probably independently by Leibniz.

5. Industrial applications. — At the present time the graphical representation of statistics is playing an increasingly important rôle in many industrial enterprises. Curves derived from observations, empirical curves, are expressed in graphical form for convenience of reference and, frequently, for interpolation between observed values. The normal distribution curve is employed not only by statisticians but also in the production departments in many factories in the classification of their products.

ILLUSTRATIVE EXAMPLES

To plot a function of x, *give x values, find the corresponding values of y, or conversely, and plot the points.* Connect by a smooth curve passing through all the points in succession moving continuously from left to right.



Graph of $y = x^2 + 4x - 5$

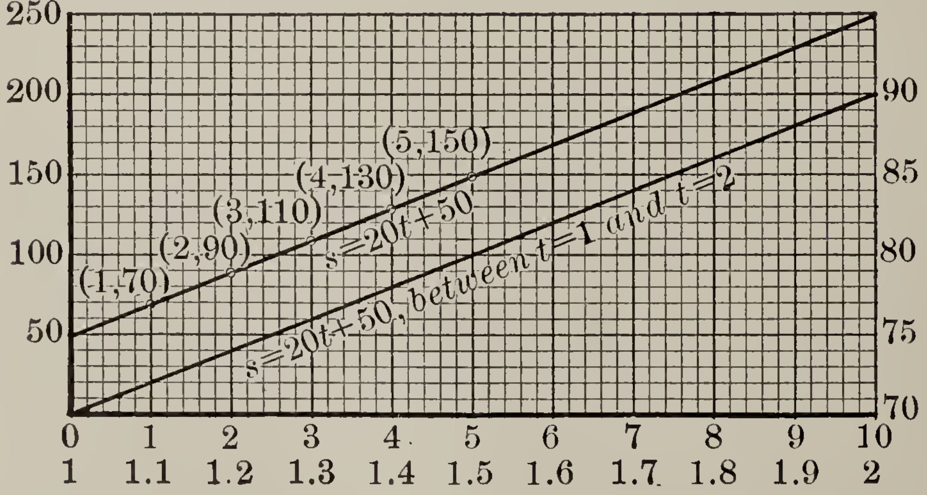
Points on the Curve	
x	y
−6	+7
−5	0
−4	−5
−3	−8
−2	−9
−1	−8
0	−5
1	0
2	7
3	16

1. Plot the graph of the function x^2+4x-5, *i.e.* plot the locus of the equation, $y=x^2+4x-5$.

Give to x the values from 0 to 3 and from 0 to −6; beyond these values in either direction the values of y evidently become very large. The curve is evidently symmetrical with respect to a line parallel to the axis and 2 units to the left.

The points where this curve crosses the x-axis represent solutions of the equation $x^2 + 4x - 5 = 0$.

2. Plot $s = 20t + 50$.



Points		Points	
0	50	1	70
1	70	1.1	72
2	90	1.2	74
3	110	1.3	76
4	130	1.4	78
5	150	1.5	80
6	170	1.6	82
7	190	1.7	84
8	210	1.8	86
9	230	1.9	88
10	250	2	90

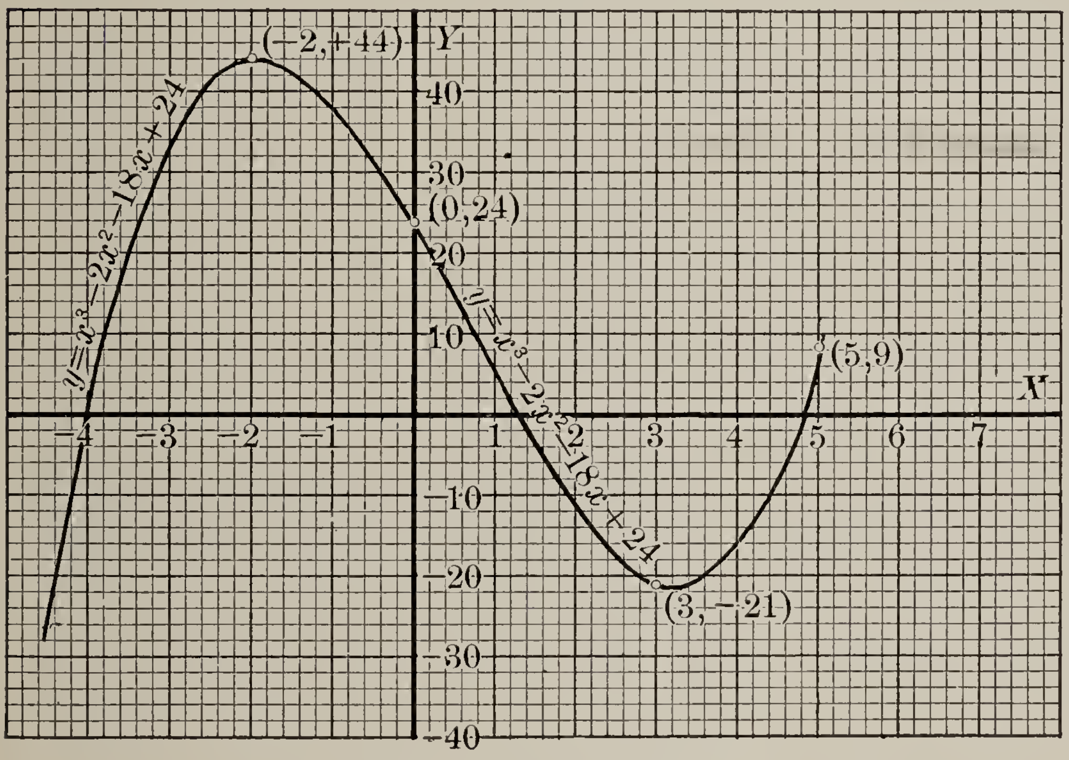
Upper graph, $s = 20t + 50$ **from** $t = 0$ **to** $t = 10$**, upper and left-hand scales**
Lower graph, $s = 20t + 50$ **from** $t = 1$ **to** $t = 2$**, lower and right-hand scales**

The lower and right-hand scales would be used if you were interested in the behavior of the function in the interval from $t = 1$ to $t = 2$. By the tenfold enlargement you can read values to the third significant figure.

This may represent the motion of a body which starting at a point 50 feet from the given point of reference moves away from that point in a straight line at the rate of 20 feet per second. The units might be miles and hours, so that the speed would be given as 20 miles per hour; this may represent then the motion of a train.

3. Plot $y = x^3 - 2x^2 - 18x + 24$.

The values of y are so large that the figure occupies too much space vertically. To obviate this difficulty one square on the axis of y is taken to represent ten units of y and one square on the x-axis is taken to represent one unit of x. This serves to compress or telescope the curve, but the essential peculiarities are preserved. In particular the points at which the curve crosses the x-axis, the values of x which make $x^3 - 2x^2 - 18x + 24 = 0$, remain unchanged. These values, the roots of $x^3 - 2x^2 - 18x + 24 = 0$, are seen to be -4, 1.3, approximately, and 4.8 approximately.



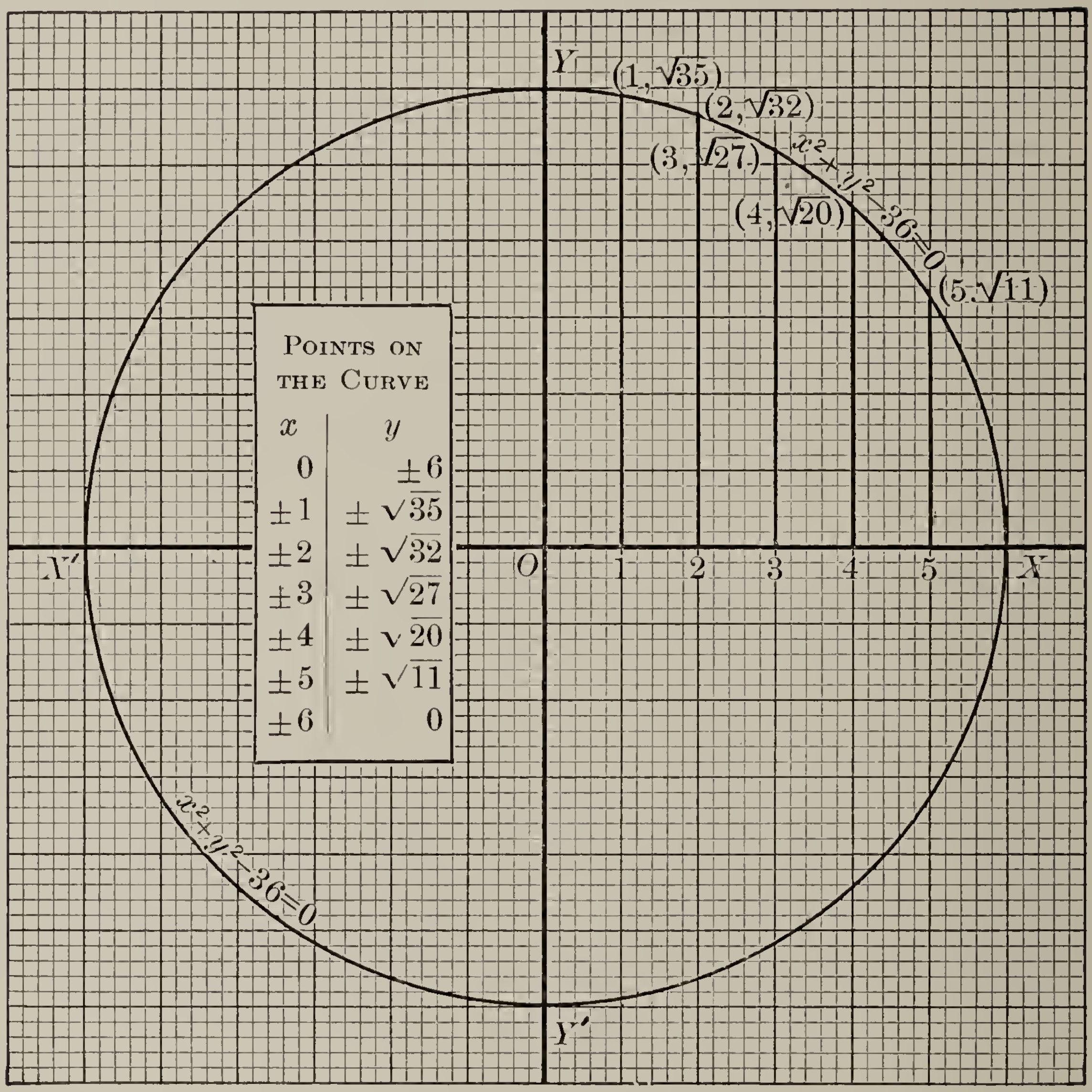
Graph of $y = x^3 - 2x^2 - 18x + 24$

Points on the Curve

x	y
-5	-61
-4	0
-3	33
-2	44
-1	39
0	24
1	5
2	-12
3	-21
4	-16
5	9
6	48

In general an algebraic equation of this type is not likely to have a rational root, such as the -4 above.

4. Plot $x^2 + y^2 - 36 = 0$.



POINTS ON THE CURVE

x	y
0	± 6
± 1	$\pm \sqrt{35}$
± 2	$\pm \sqrt{32}$
± 3	$\pm \sqrt{27}$
± 4	$\pm \sqrt{20}$
± 5	$\pm \sqrt{11}$
± 6	0

Graph of $x^2 + y^2 - 36 = 0$

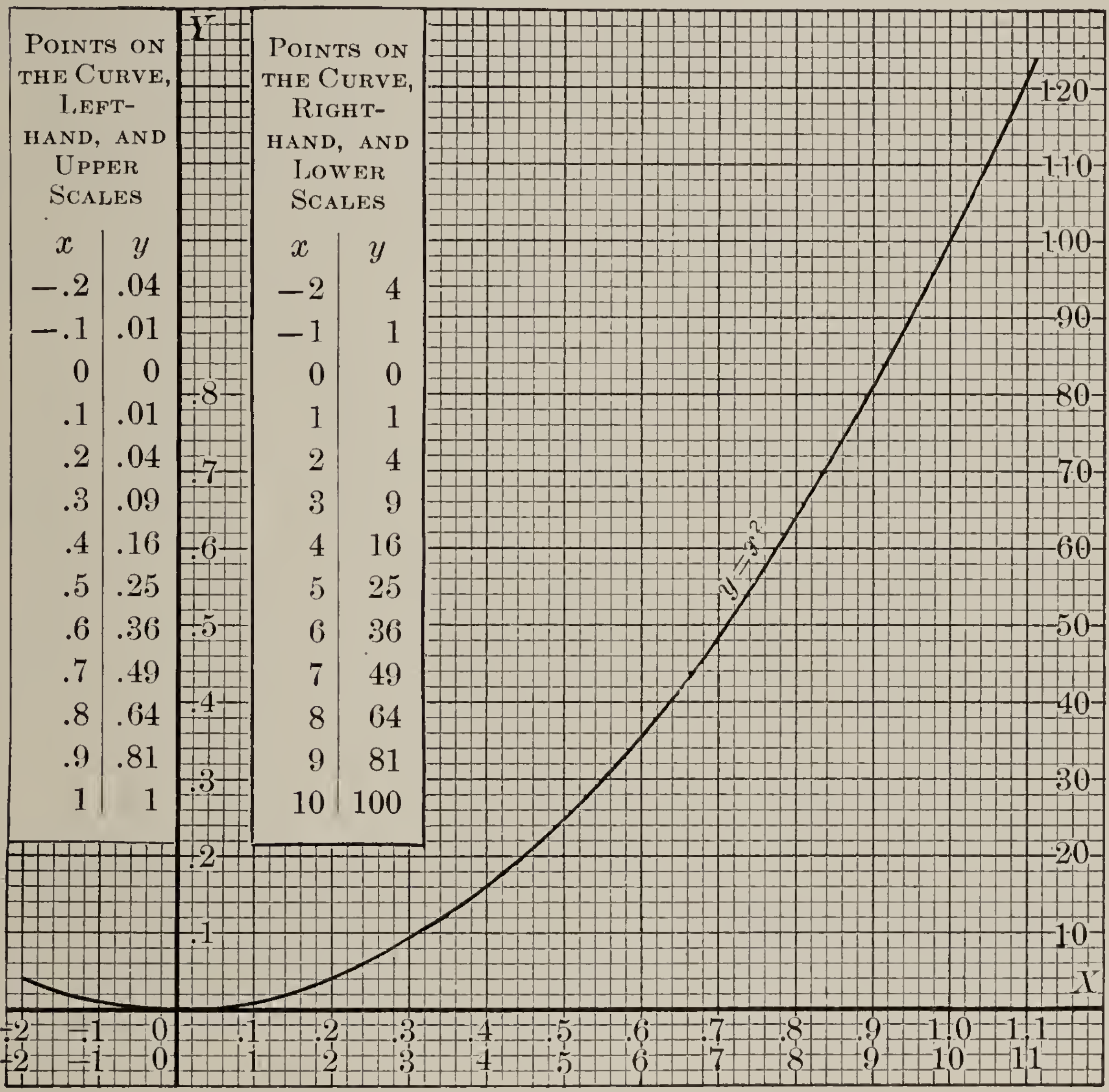
In drawing the graph of this function of x (implicit), it is important to note that there are two values of y corresponding to each value of x, and that these two values are symmetrically distributed with respect to the x-axis. Similarly this curve is symmetrical with respect to the x-axis, since any value of y gives two corresponding values of x, numerically equal but opposite in algebraic sign. The points when located are connected by a smooth curve which is here a circle.

To this diagram reference has been made in problem 7, page 17. As a circle of radius 6 the ordinates at $x = 1, 2, 3, 4$, and 5, respectively, give graphically the square roots of 35, 32, 27, 20, and 11.

The more complete discussion of equations of this type is given in Chapter XIV.

5. Plot $y = x^2$ for $x = 0, .1, .2, \cdots, 1.0, 1.1, 1.2, 1.3$.

Note that precisely the same curve is obtained if units 0, 1, 2, 3, ⋯ 15 are taken instead of tenths on the x-axis and tens in the place of tenths on the vertical axis, as indicated on the lower and right-hand scales.



Points on the Curve, Left-hand, and Upper Scales

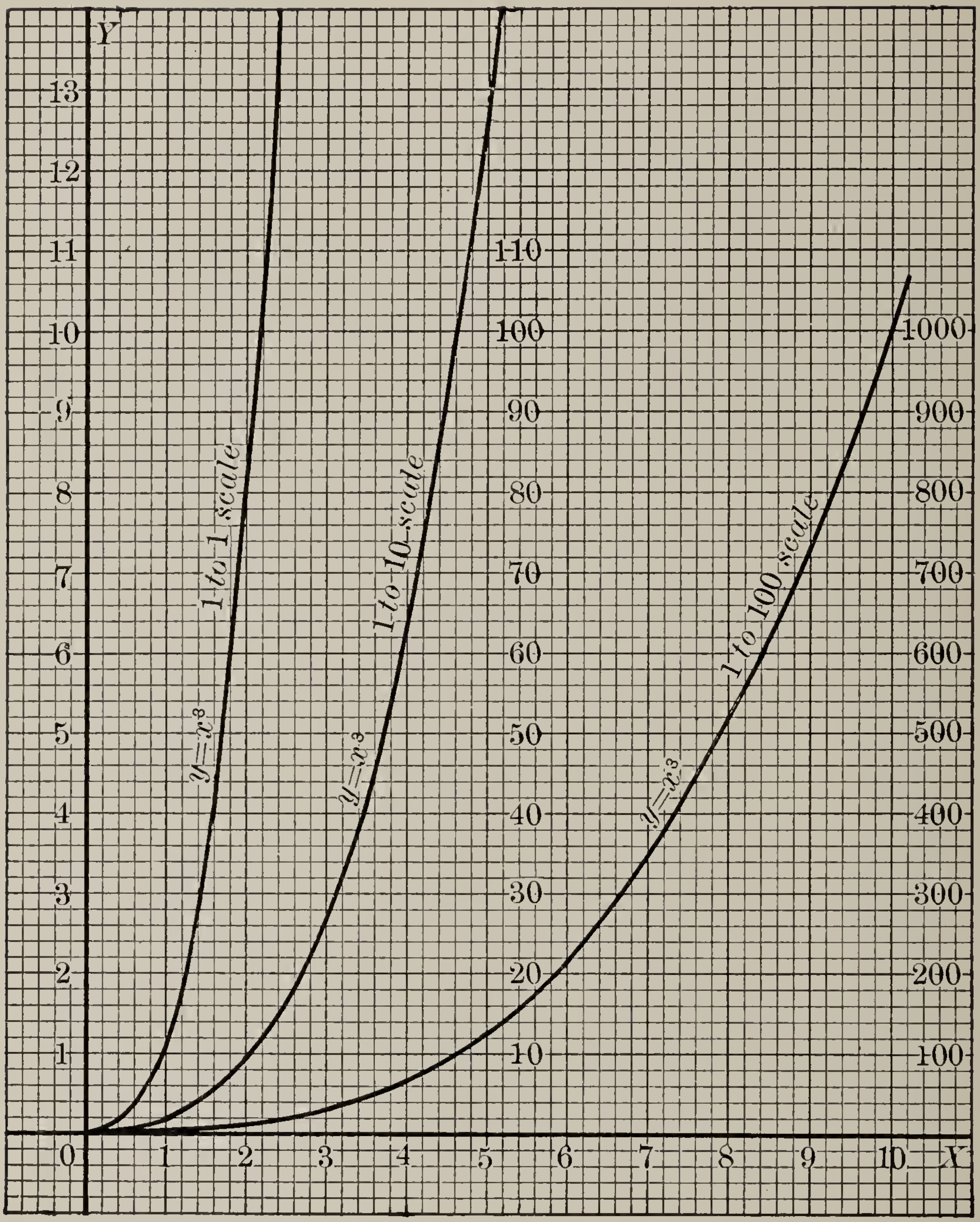
x	y
−.2	.04
−.1	.01
0	0
.1	.01
.2	.04
.3	.09
.4	.16
.5	.25
.6	.36
.7	.49
.8	.64
.9	.81
1	1

Points on the Curve, Right-hand, and Lower Scales

x	y
−2	4
−1	1
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100

Graph of $y = x^2$

This line if drawn somewhat carefully can be used to read squares of numbers of two places to two or three places. Thus $(.85)^2 = .72$; $(.96)^2 = .92$; $(.73)^2 = .54$. Square roots can also be read from this curve by noting the horizontal length, corresponding to any given vertical length. The square root of 20 is read as 4.45, of 30 as 5.50, of 40 as 6.35, of 50 as 7.05, of 60 as 7.75, of 70 as 8.38, of 80 as 8.95, of 90 as 9.5; the square root of 630 must read as 6.3 down between 2 and 3 on the horizontal scale, evidently about 25.



$y = x^3$

x	y
0	1
.2	.008
.4	.064
.6	.216
.8	.512
.9	.729
1	1.
1.2	1.728
1.4	2.744
1.6	4.096
1.8	5.832
2.	8
3	27
4	64
5	125
6	216
7	343
8	512
9	729
10	1000

Graph of $y = x^3$, using three different scales

The portion of the cubical parabola,

$$y = x^3,$$

given by positive values of x from 0 to 10.

The cubes of numbers from 0 to 10 can be read quite accurately to two significant figures, with an approach to the third.

Thus $(8.4)^3$ is read on the 1 to 100 scale curve as 595, instead of 593; $(4.4)^3$ is read on the 1 to 10 scale curve as 85.0 instead of 85.2; $(1.8)^3$ is read on the 1 to 1 scale curve as 6.00 instead of 5.83.

PROBLEMS

1. Plot $y = x^3$, from $x = 1$ to $x = 5$, using one half inch on the vertical axis to represent 10.

2. Plot $y = x^3$, from $x = 1$ to $x = 10$, using one half inch on the vertical axis to represent 100. See graph, above.

3. Plot $y = x^3$, from $x = 0$ to $x = 1$ by tenths, using 10 half-inches for 1 unit on each axis.

4. Read from the above curves the following cubes, to 2 significant figures:

$$3.2^3,\ 4.7^3,\ .82^3,\ 1.5^3,\ .64^3,\ .58^3,\ 7.1^3,\ 9.2^3.$$

5. Plot the graph of $s = 200\,t - 16\,t^2$, from $t = 1$ to $t = 12$. This represents the height at time t of a ball thrown upward with a velocity of 200 feet per second.

6. Plot the graph of $s = 1000 - 200\,t - 16\,t^2$, from $t = 1$ to $t = 4$; this represents the height above the earth of a ball thrown down from the top of the Eiffel tower with a velocity of 200 feet per second.

7. Plot the graph of $x^2 + y^2 = 100$ using $\frac{1}{4}$ inch as 1 unit on each axis. Find from the graph ten pairs of numbers whose squares summed equal 100.

8. The area of a circle is given by the formula $A = \pi r^2$; plot the graph of the function A from $r = 0$ to $r = 10$ inches; use 3.14 for π.

9. The capacity in gallons of a cylindrical can of height 10 inches having a diameter of d inches is given, within $\frac{1}{10}$ of 1 %, by the formula:

$$C = \tfrac{1}{30} \cdot d^2 + \tfrac{1}{1500} \cdot d^2.$$

Plot the graph for $d = 1$ to 20 and interpret as gallons per inch of height. How could you find from the graph the capacity of a can having a diameter of 8 inches and a height of 9 inches? Check by cubic inches, using 231 cubic inches to the gallon.

CHAPTER V

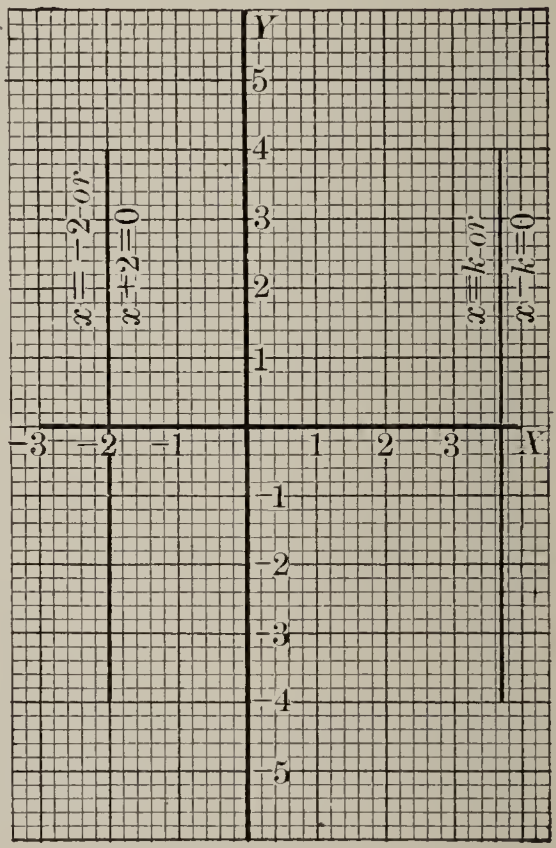
THE LINEAR AND QUADRATIC FUNCTIONS OF ONE VARIABLE

1. Theorem. — *Any equation of the first degree in two variables (x and y) has for its graph a straight line.*

Proof. — The general equation of the first degree may be written $Ax + By + C = 0$. This equation can always be put in one of the four forms:

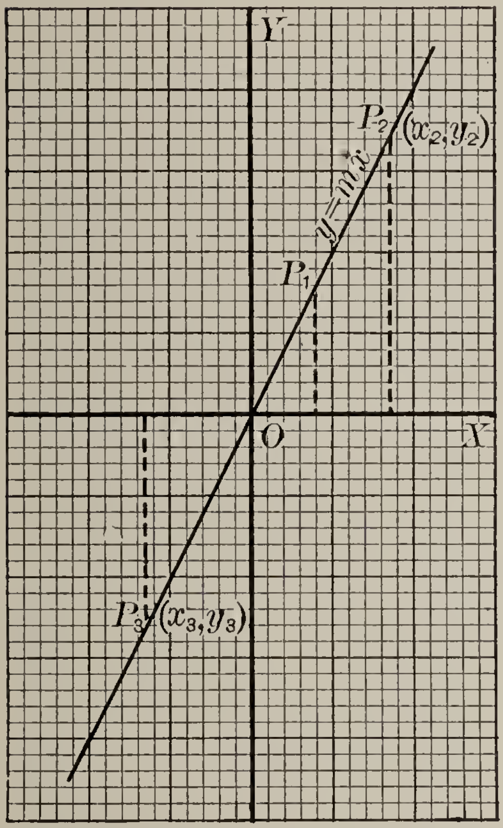
$$x = k, \quad \text{if } B = 0, \text{ or if } B = 0 \text{ and } C = 0;$$
$$y = k, \quad \text{if } A = 0, \text{ or if } A = 0 \text{ and } C = 0;$$
$$y = mx, \text{ if } C = 0,$$
$$\text{or } y = mx + k.$$

$x = k$ represents a straight line parallel to the y-axis, at a distance k units from it; on such a line the abscissa of any point is constant. The coördinates of any point on the line satisfy the equation, and any point whose coördinates satisfy the equation lies upon the line. Thus, $x = -2$ or $x + 2 = 0$ represents a straight line, parallel to and 2 units to the left of the y-axis. Similarly, $y = k$ represents a straight line parallel to the x-axis and at a distance of k units from it.



Graph of $x + 2 = 0$
and Graph of $x - k = 0$

$y = mx$. Assume different points which satisfy this relation; the origin lies upon the locus; (x_1, y_1) and (x_2, y_2) satisfy, if $y_1 = mx_1$, $y_2 = mx_2$.



Graph of $y = mx$

Any two points (x_1, y_1), (x_2, y_2) which satisfy this equation can be shown to lie upon the straight line connecting either one with the origin, which evidently satisfies the equation.

Consider first m to be positive; the point (x_1, y_1) may be taken in the first quadrant.

Since $y_1 = mx_1$, and $y_2 = mx_2$,

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = m.$$

The right triangle containing x_1 and y_1 as the sides is similar to the right triangle with x_2 and y_2 as sides, since $\frac{y_1}{x_1} = \frac{y_2}{x_2}$. Hence the corresponding angles at O are equal and the points (x_1, y_1) and (x_2, y_2) lie upon a straight line through the origin.

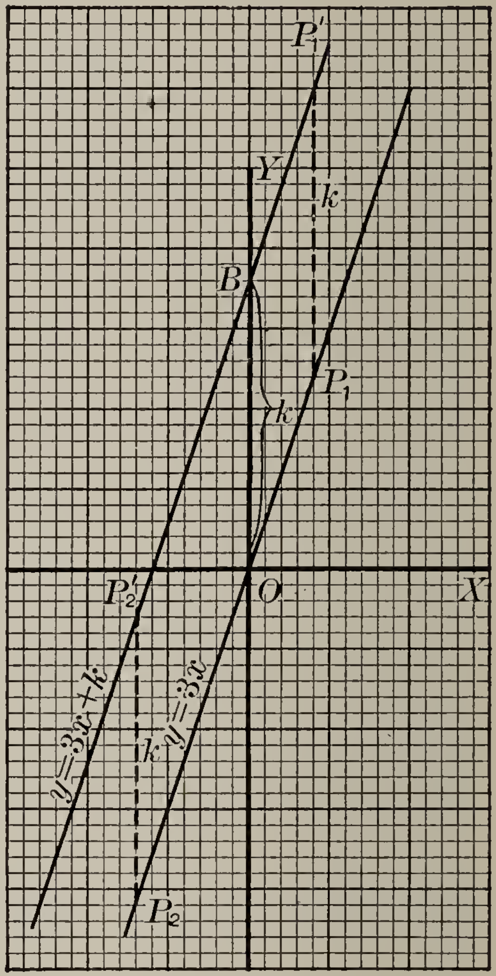
NOTE. — If y_2 and x_2 are negative, $\frac{y_2}{x_2}$ is in truth to be replaced by $\frac{-y_2}{-x_2}$, since only positive quantities are involved in plane geometry.

Conversely any point (x, y) which lies upon the given line drawn satisfies the given equation. For, by similar triangles, $\frac{y}{x} = \frac{y_1}{x_1} = m$, whence $y = mx$.

When m is negative, the argument is slightly changed, since any point (x_1, y_1) which satisfies the equation must have coördinates opposite in sign; then $\frac{y_1}{-x_1}$ or $-\frac{y_1}{x_1}$ equals $-m$.

The value m represents the rate of change of y compared with the rate of change of x of a point (x, y) moving on the line.

In the equation $y = mx + k$ for any values of x, the corresponding values of y are greater by k than the corresponding ordinates of $y = mx$. Construct any three such ordinates of $y = mx + k$. Since these extensions are parallel and equal in length, parallelograms are formed; the inclined sides of these parallelograms are parallel to the line $y = mx$ and consequently to each other. Since any two of these parallel lines have a point in common, they coincide and form one straight line (by plane geometry). The value m is called the slope of the line $y = mx + k$, and evidently varies as the angle which the line makes with the x-axis varies. The angle which a line makes with the x-axis is termed the *slope angle* of the line.



Conclusion. — Since every equation of the first degree,

$$Ax + By + C = 0,$$

can be put into one of four forms, mentioned, every equation of the first degree represents a straight line.

Conversely, every straight line is represented by an equation of the first degree; if the line is parallel to one of the axes, the form of the equation is evidently $x = k$ or $y = k$; if the line passes through the origin, the form is $y = mx$; and every point on the line can be shown to satisfy this equation, for let (x_1, y_1) be any fixed point on the line and (x, y) any point whatever on the line, then

$$\frac{y_1}{x_1} = \frac{y}{x}, \text{ by similar triangles, whence}$$

$$\frac{y}{x} = m, \text{ a fixed value, and } y = mx.$$

Any other line will be parallel to a line through the origin, and its points will satisfy an equation of the form

$$y = mx + k.$$

On the line $y = mx + k$, if at any point the value of x is increased by one unit, the value of y, the function of x, is increased by m units; on this line everywhere y increases m times as fast as x; the ratio of the increase of y to the unit increase of x, m, gives on the straight line the rate of change of the function y as compared with the rate of change of x.

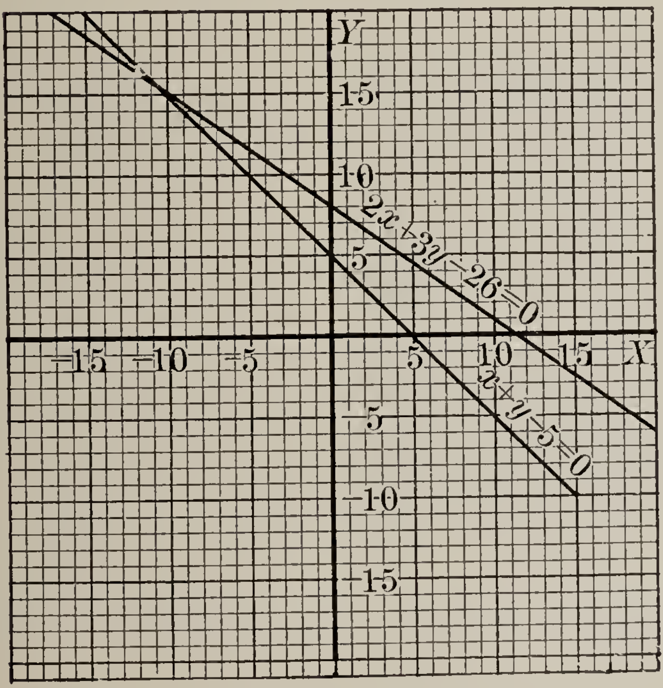
2. Intersection of graphs. — Plot

$$2x + 3y - 26 = 0,$$
$$x + y - 5 = 0.$$

Every point on the first line is such that its coördinates (x, y) when substituted in $2x + 3y - 26 = 0$, satisfy the equation; there are an infinite number of such points, *e.g.* $(0, \frac{26}{3})$, $(1, 8)$, $(2, \frac{22}{3})$, $(13, 0)$, $(\frac{23}{2}, 1)$, $(-1, \frac{28}{3})$, $\cdots$. By substituting 0 or 1 or 2 or -1, -2, $\cdots$, for x and solving for y, or conversely, points are obtained whose coördinates satisfy the given equation; similarly every point on the second line is such that its coördinates satisfy the equation,

$$x + y - 5 = 0;$$

the point of intersection satisfies both equations, and its coördinates can be obtained by solving the two equations as simultaneous. The argument is



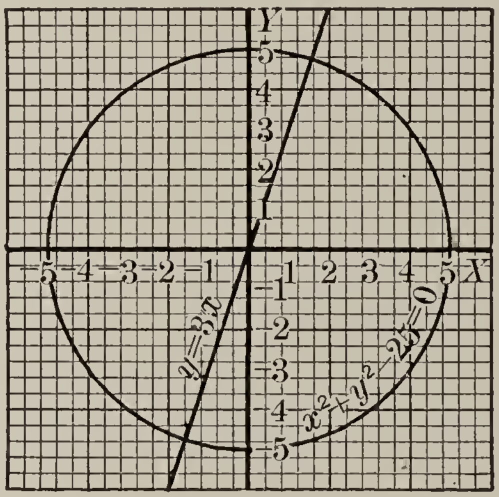
Graphs $\begin{cases} 2x + 3y - 26 = 0, \\ x + y - 5 = 0 \end{cases}$

entirely similar for the points of intersection of any two loci, representing algebraic equations; the points of intersection satisfy both equations, and give a graphical method of approximating the solutions of the equations regarded as simultaneous.

To review this demonstration, answer the questions below, and read the discussion.

What is true concerning the coördinates of every point on the first line? on the second line? What is true concerning the point of intersection so far as the two given equations are concerned? The drawing shows that (– 11, 16) satisfies both the equations, and substitution shows that this is precisely correct. In general the graphical solution is only approximate, the degree of accuracy depending upon the accuracy of the drawing and the scale used.

The point of intersection of two straight lines represents graphically the solution obtained by solving the two equations as simultaneous.



Graphs $\begin{cases} y = 3\,x, \\ x^2 + y^2 - 25 = 0 \end{cases}$

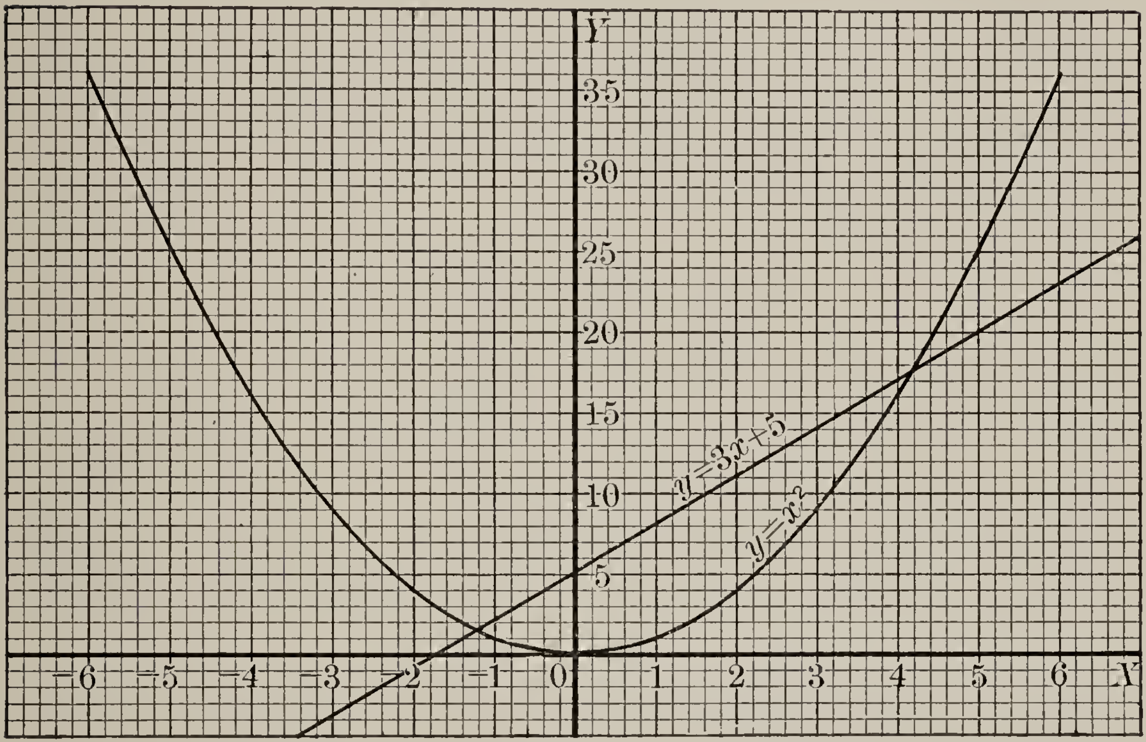
Intersections of $y = 3\,x$, and $x^2 + y^2 = 25$.

The graphical presentation shows very plainly that the solution is, approximately,

$$x = 1.6$$

and $$y = 4.8.$$

Graphical Solution of Simultaneous Equations



Graphs $\begin{cases} y = x^2, \\ y = 3x + 5 \end{cases}$

Intersections of $y = x^2$, $y = 3x + 5$.

The graph shows that there are two solutions; in the one,

$$x = -1.2, \quad y = +1.4,$$

and in the other, $x = 4.2, \quad y = 17.8.$

These are approximate values.

Plot carefully the graphs of the preceding problems, checking on the work presented by the graphs.

Plot carefully these two lines and verify the statements made:

$$\left.\begin{array}{l} 2x + 3y - 26 = 0, \\ 2x + 3y - 8 = 0. \end{array}\right\} \begin{array}{l}\text{Graphically, parallel;} \\ \text{algebraically, no solution.}\end{array}$$

The point of intersection of two graphs represents graphically the solution of the two equations regarded as simultaneous.

PROBLEMS

1. Solve

$$y - 3x - 5 = 0,$$
$$3x + 2x - 7 = 0,$$

both graphically and algebraically.

2. Plot the graphs of

$$y - 3x - 5 = 0,$$
$$3y + 2x - 7 = 0,$$
$$x + y - 2 = 0.$$

Do these three lines appear to meet in one point on your diagram? Have these three equations a common solution? Substitute the solution of the first pair (obtained in problem 1) in the third equation. Later it will be shown that a point whose coördinates when substituted in a first-degree expression give a small numerical value is near the straight line represented by the equation formed by putting that expression equal to zero.

3. Plot 15 points whose coördinates satisfy the equation

$$2y + 3x - 11 = 0.$$

4. Plot the lines $x = 3$ and $y = 4$; what point is represented by these equations? Note that the Cartesian system (x, y) of representing points implies each point as the intersection of two lines.

5. Solve

$$2y + 3x - 5 = 0,$$
$$3y - 4x - 8 = 0,$$

both graphically and algebraically.

3. Intercepts. — Any given line or curve cuts off on the coördinate axes distances that are called the intercepts of the line or curve. The x-intercept is obtained analytically by substituting $y = 0$ and solving, *i.e.* by *solving as simultaneous* the equations of the x-axis and the given line; the y-intercept is obtained by substituting $x = 0$.

The x-intercept of $2x + 3y - 26 = 0$ is 13, obtaining by substituting $y = 0$ in $2x + 3y - 26 = 0$; the y-intercept is $+\frac{26}{3}$; of $x^2 + y^2 = 25$, the x-intercepts are ± 5, the y-intercepts are also ± 5.

Note that the problem of finding the intercepts of a given graph is a special case of the problem to find the intersections of two given curves; the x-intercept designates the intersection of the given curve with the x-axis, $y = 0$, and similarly the y-intercept refers to the intersection with $x = 0$.

Rule. — To find the x-intercept, put $y = 0$, and solve; similarly for the y-intercept.

4. Pencil of lines. — The straight lines which pass through a common point constitute what is termed a *pencil* of lines. If the common point is determined as the intersection of two given lines, we may write the equation of the pencil of lines in terms of the two expressions which put equal to zero represent the given lines.

The pencil of lines through the intersection of

$$y - 3x - 5 = 0 \qquad (l_1)$$

$$3y + 2x + 7 = 0 \qquad (l_2)$$

is given by the linear equation, k being assumed constant,

$$(3) \qquad y - 3x - 5 + k(3y + 2x + 7) = 0. \qquad (l_3)$$

Evidently any point on the first line, l_1, makes $y - 3x - 5 = 0$, and any point on the second line, l_2, makes $3y + 2x + 7 = 0$; the point of intersection substituted in our equation (3) gives $0 + k \cdot 0$ or 0, hence the point of intersection of l_1 and l_2 satisfies equation (3) for all values of k.

By giving k successive values l_3 can be made to pass through any point of the plane. Thus to pass through (1, 5) substitute (1, 5) in l_3 and solve for k, giving

$$5 - 3 - 5 + k(15 + 2 + 7) = 0,$$

or $24k = 3$, $k = \frac{1}{8}$. The line $y - 3x - 5 + \frac{1}{8}(3y + 2x + 7) = 0$, or $8y - 24x - 40 + 3y + 2x + 7 = 0$ reduces to $11y - 22x - 33 = 0$, or $y - 2x - 3 = 0$ when simplified.

In solving as simultaneous the two equations $y - 3x - 5 = 0$, and $3y + 2x + 7 = 0$, the particular lines parallel to the axes of reference and passing through the point of intersection of l_1 and l_2 are sought. Thus, after multiplying the upper expression by -3 and adding, you get the line l_3 with $k = -\frac{1}{3}$.

$$y - 3x - 5 - \tfrac{1}{3}(3y + 2x + 7) = 0$$

gives $-11x - 22 = 0$, or $x = -2$.

To eliminate x we multiply the upper expression by 2 and the lower by 3 and add; this gives, $11y + 11 = 0$, or $y = -1$. The same line given by $11y + 11 = 0$ is obtained from line l_3 with $k = \frac{3}{2}$; *i.e.*:

$$y - 3x - 5 + \tfrac{3}{2}(3y + 2x + 7) = 0$$

gives $11y + 11 = 0$, or $y + 1 = 0$.

The point of intersection of the two lines, $(-2, -1)$, is given as the intersection of $x = -2$ and $y = -1$.

PROBLEMS

1. Write the equation of the family of lines through the intersection of the two lines:

$$y - 3x - 5 = 0,$$
$$3y + 2x - 7 = 0.$$

Determine k so that this line shall pass throught the point $(0, 0)$; through $(1, 5)$.

2. Find the x-intercept and the y-intercept of the line,

$$3y + 2x - 7 = 0.$$

Draw the graph of this line.

3. Write the equation of the family of lines passing through the point of intersection of the lines,

$$y - 4 = 0,$$

and $$x - 3 = 0.$$

Draw the graph of $y - 4 + k(x - 3) = 0$, for $k = 0$, for $k=1$, for $k = -1$, for $k = 3$.

4. Plot the three lines below, and obtain their three points of intersection graphically and algebraically:

$$\begin{aligned} y - x - 5 &= 0, \\ 3y + 2x - 10 &= 0, \\ y - 3x + 15 &= 0. \end{aligned}$$

Does the drawing on coördinate paper give an indication of the area of the triangle formed?

5. Write the equation, $3y + 2x - 10 = 0$, in the form, $y = mx + k$; what is the value of m?

6. The weight of a cylindrical vessel of water when filled to a height of 10 inches is 6.8 pounds, when filled to a height of 6 inches it is 4.4 pounds; plot the two points (6, 4.4) and (10, 6.8). The straight line joining these two points gives the weight of the vessel when filled to any height from 0 to 10. The equation may be written $w = k \cdot h + c$, where w and h are the variable weight and height, k and c are constants. This equation is the simple statement of the fact that the weight of the water and the container for any height h is the weight of the vessel (c) plus h, the height, times the weight of the water which fills the container to a height of one inch. Note the significance of the intercepts.

7. The equation, $w = \frac{62.4}{1728}v$, may be used to express the relation between the volume in cubic inches and the weight in pounds of a given mass of water. Plot this carefully and find approximately the weights of 100 cubic inches, 500 cubic inches, and 700 cubic inches of water. Find the volume of 15 pounds of water; the volume of 25 pounds; of 30 pounds.

8. The volume of mercury at any temperature between 0 and 40° C. is given by the equation $V = k(1 + at)$, wherein $a = .00018$; for $k = 1000$ cu. cm. this becomes $V = 1000 + .18t$.

Plot this equation taking the horizontal axis as at 1000. This is equivalent to plotting the increase in volume, $I = .18\,t$. Plot for 0 to 40° C. and find the increase in volume when 1000 cu. cm. of mercury at 0° C. are heated to 27° C.

9. Find the equation of the straight line through $(-3, 5)$ and through the intersection of $3x - y - 7 = 0$ and

$$5x + 12y - 17 = 0.$$

10. Plot degrees Fahrenheit as abscissas and degrees Centigrade as ordinates, connecting (32° F., 0° C.) to (212° F., 100° C.), by a straight line. Find the equation of this straight line. Find the Centigrade reading corresponding to 0° Fahrenheit, to 100° F. Discuss the meaning of the slope of the line.

11. Find the intercepts of the line $9y - 5x = -160$. Compare with your result in the preceding problem.

12. Plot the graph of $s = 16\,t^2$, for values of t from $t = 0$ to $t = 5$, using one inch for 1 second on the horizontal axis, and 1 inch for 100 feet on the vertical axis. Find value of s when $t = 4.3$ from the graph. Check by computation.

13. Plot carefully $x^2 + y^2 = 64$, and $y = 3x - 5$. From the graph get the approximate solution.

14. Show graphically how to change a system of marks from a scale of 100 to a scale of 75; from 75 to 100.

15. Sound travels at the rate of 1089 feet per second in air at 32° F. (or 0° C.); at the rate of 1130 feet per second in air at 70° F. The formula, $v = 1054 + \frac{13\,t}{12}$ gives very closely the velocity in feet per second at temperature t° Fahrenheit. Plot the graph of the function, plotting the excess above 1000 feet as ordinates and temperature Fahrenheit up to 80° F. as abscissas. At what temperature is the velocity 1100 feet per second? How would you adapt these figures to the Centigrade scale for temperature beginning 0° C.? $v = 1089 + 2\,t$ is the resulting equation.

16. The velocity after t seconds of a bullet shot straight upwards at 800 feet per second is given by the equation $v = 800 - 32\,t$. Plot the graph, taking 100 feet as $\frac{1}{2}$ inch on the vertical axis, and 5 seconds as $\frac{1}{2}$ inch on the horizontal axis; negative values of v mean that the bullet is descending.

17. Plot $v = 600 + 32\,t$, and interpret as downward velocity of an object thrown downwards from a height.

18. Time yourself on plotting the following 10 lines; five may be plotted with respect to one set of axes:

a. $3\,x + 4\,y - 12 = 0$.
b. $3\,y = 2\,x - 5$.
c. $x - y - 8 = 0$.
d. $7\,x + 3\,y - 18 = 0$.
e. $2\,y + x + 10 = 0$.
f. $5\,x + 12\,y - 10 = 0$.
g. $\frac{x}{3} - \frac{y}{5} = 1$.
h. $y = \frac{2}{3}\,x - \frac{3}{4}$.
i. $x = \frac{3}{5}\,y - \frac{7}{3}$.
j. $3\,x - 7\,y = 0$.

19. Time yourself (*a*) on finding the slope of each of the ten lines in problem 18; (*b*) on finding the x-intercept of each line; (*c*) on finding the ordinate of the point whose abscissa is 2; (*d*) on finding to one decimal place the ordinate of the point on each line whose abscissa is 2.4; (*e*) on putting these lines in slope form.

20. Plot, using values correct to 1 decimal place, the following lines:

a. $3.1\,x + 4.5\,y - 12 = 0$.
b. $3.2\,y = 2.6\,x - 5.7$.
c. $.9\,x - 4.8\,y - 8.3 = 0$.

5. The quadratic equation in one variable. — Any equation of the form $ax + b = 0$ is called a linear, or first-degree equation, in the variable x; the solution is given by $x = -\frac{b}{a}$; the graph of the function, $y = ax + b$, is a straight line of slope a, with the y intercept equal to b, and with the x intercept representing the solution of the equation, $ax + b = 0$.

Any equation of the form $ax^2 + bx + c = 0$ is called a quadratic equation in x; a, b, and c are to be regarded as constants. The graphical solution of one equation of this type has been presented (page 70) and the algebraic solution is given in elementary algebra, but will be given here a rapid review.

Algebraical solution of $2x^2 + 8x + 7 = 0$ and of the general equation, $ax^2 + bx + c = 0$;

$$\begin{array}{ll}
2x^2 + 8x + 7 = 0, & ax^2 + bx + c = 0, \\[1ex]
x^2 + 4x = -\tfrac{7}{2}, & x^2 + \dfrac{bx}{a} = -\dfrac{c}{a}, \\[2ex]
x^2 + 4x + 4 = \tfrac{1}{2}, & x^2 + \dfrac{b}{a}x + \left(\dfrac{b}{2a}\right)^2 = \dfrac{b^2}{4a^2} - \dfrac{c}{a}. \\[2ex]
(x+2)^2 = .5. & \left(x + \dfrac{b}{2a}\right)^2 = \dfrac{b^2 - 4ac}{4a^2}, \\[2ex]
x + 2 = \pm .71 \text{ (or .707 to 3 places)}, & x + \dfrac{b}{2a} = \dfrac{\pm\sqrt{b^2 - 4ac}}{2a}, \\[2ex]
x = -1.29 \text{ or } -2.71, & x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\end{array}$$

The necessary third term to complete the square is obtained by comparison with $(x \pm k)^2 = x^2 \pm 2kx + k^2$.

Graphically the equation $y = 2x^2 + 8x + 7$ represents a curve which intersects the x-axis, $y = 0$, in the two points whose abscissas satisfy the equation, $2x^2+8x+7=0$. $y=2x^2+8x+8$ represents a curve which is tangent to the x-axis, corresponding to the fact that the roots of the equation, $2x^2 + 8x + 8 = 0$, are equal to each other. The equation $y = 2x^2 + 8x + 11$ represents a curve which does not cut the x-axis, corresponding to the fact that the quadratic $2x^2 + 8x + 11 = 0$ has for solutions, $x = \dfrac{-4 \pm \sqrt{-6}}{2}$, values corresponding to no points on the x-axis, *i. e.*, to imaginary values of x. Plot the graphs indicated.

The quadratic equation is solved algebraically by reducing the problem to the solution of two first-degree equations:

$$x + \frac{b}{2a} = \frac{+\sqrt{b^2 - 4ac}}{2a},$$

$$\text{and } x + \frac{b}{2a} = \frac{-\sqrt{b^2 - 4ac}}{2a}.$$

The quantity $b^2 - 4ac$ which appears under the radical sign is called the discriminant of the quadratic. The nature of the roots of the quadratic equation is determined by this discriminant, when a, b, c represent real quantities, *i.e.*, a, b, and c having values which can be represented by points upon a scalar line. When

$b^2 - 4ac > 0$, *i.e.* positive, the two roots are real and unequal, when
$b^2 - 4ac = 0$, the roots are real and equal, and when
$b^2 - 4ac < 0$, *i.e.* negative, the roots are imaginary.

Further, the condition that the roots of the quadratic should be equal given by $b^2 - 4ac = 0$, may be obtained by inspection, or by actually setting the two roots equal to each other and simplifying; $ax^2 + bx + c$ may then be written $a\left(x + \frac{b}{2a}\right)^2$. Graphically these conditions correspond to the fact that the curve $y = ax^2 + bx + c$ cuts the x-axis in two points, is tangent to the x-axis, or does not intersect it at all, according as $b^2 - 4ac$ is greater than, equal to, or less than 0.

Frequently the two roots of the quadratic $ax^2 + bx + c = 0$ are designated by x_1 and x_2.

$$\text{Thus} \qquad x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

$$\text{and} \qquad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The sum and the product of the roots, $x_1 + x_2$ and x_1x_2, are given, respectively, by $-\frac{b}{a}$ and $+\frac{c}{a}$. The expressions $x_1 + x_2$ and x_1x_2 are representative symmetric functions of the roots of a quadratic function of one variable, being expressions which remain unchanged when x_1 and x_2 are interchanged.

6. Historical note. — The solution of linear equations was known four thousand years ago to ancient Egyptians. The equation $x + \frac{x}{7} = 19$, was proposed and solved in the work of an Egyptian writer named Ahmes; the problem reads, with "ahau" representing "heap" or "unknown," "*ahau* and its seventh, it makes 19." In other ancient Egyptian documents problems leading to pure quadratics are found. The Greeks were able to give as early as 450 B.C. a geometrical solution of any quadratic having positive roots; the numerical application appears in Greece somewhat later. In India numerical quadratics were solved in the fifth and sixth centuries A.D. The first systematic treatise combining clearly analytical statement with geometrical illustration is given by an Arab, Mohammed ibn Musa al-Khowarizmi, about 825 A.D. His work continued in use for centuries. The complete quadratic with general, literal coefficients, did not come, of course, until after the introduction of literal coefficients by Viète late in the sixteenth century.

7. Graphical solution of the general quadratic equation. — The general quadratic equation $ax^2 + bx + c = 0$ can be solved graphically by means of one fixed curved line, $y = x^2$, and a variable straight line. The intersection of

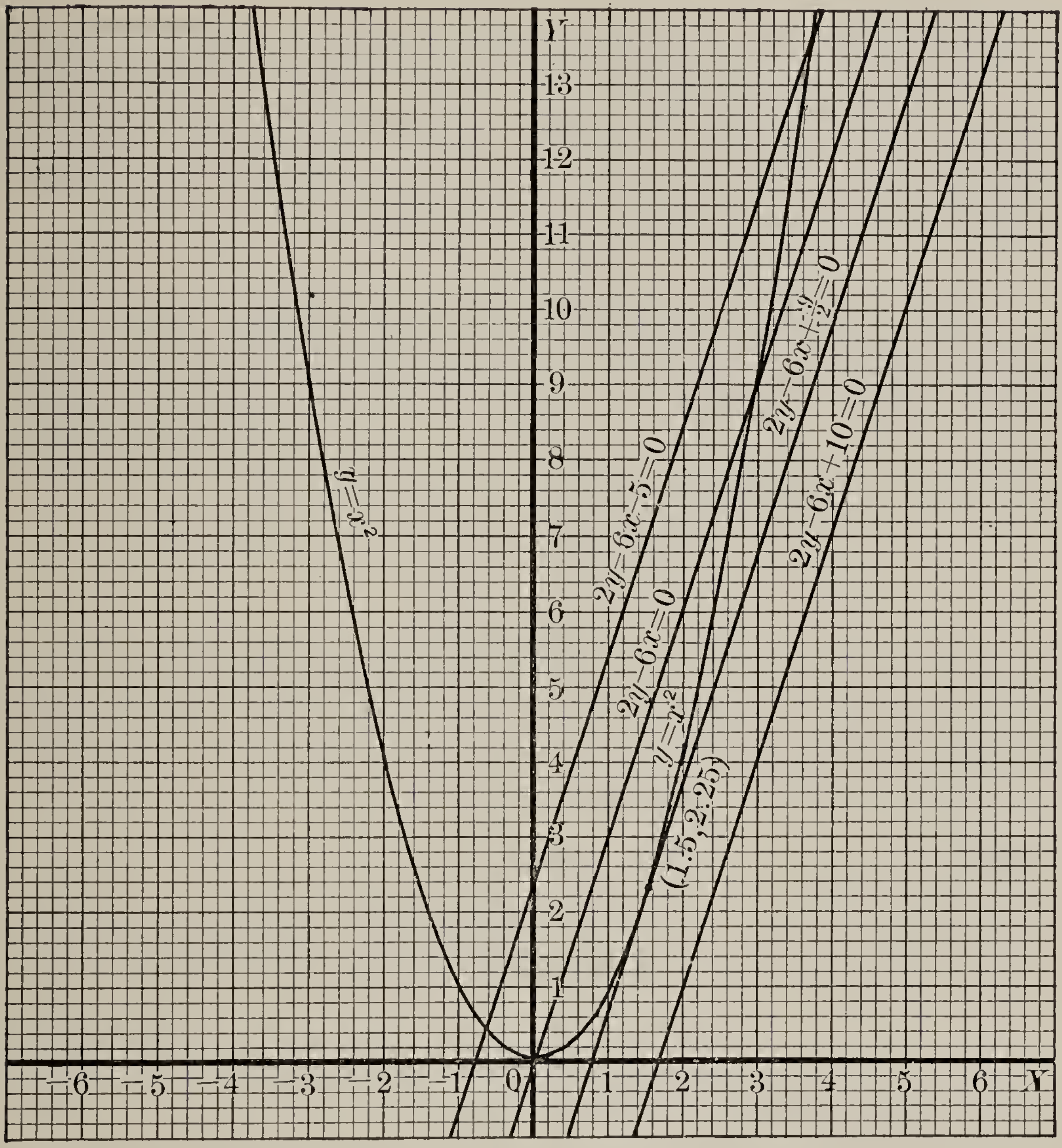
$$y = x^2$$

and

$$ay + bx + c = 0$$

gives the solution of the equation $ax^2 + bx + c = 0$, for the solution is obtained algebraically by substituting for y its value x^2 in $ay + bx + c = 0$, giving $ax^2 + bx + c = 0$.

The graphical solution of the quadratics, $2x^2 - 6x - 5 = 0$, $2x^2 - 6x = 0$, $2x^2 - 6x + \frac{9}{2} = 0$, and $2x^2 - 6x + 10 = 0$, is presented upon the diagram; the student is urged to solve these equations algebraically and to trace the correspond-



Graphical solution of quadratics

$2x^2-6x-5=0$; $2x^2-6x=0$; $2x^2-6x+\frac{9}{2}=0$; $2x^2-6x+10=0$

$$y = x^2,$$
$$2y-6x-5=0.$$

Two real solutions.

$$y = x^2,$$
$$2y-6x+\tfrac{9}{2}=0.$$

Two coincident solutions.

$$y = x^2,$$
$$2y-6x=0.$$

Two real solutions.

$$y = x^2,$$
$$4y-6x+10=0.$$

Two imaginary solutions

ence between the algebraic and graphical solutions. Two sets of real and different roots are indicated by two of these straight lines on our diagram; one set of equal roots is indicated; one pair of imaginary roots is indicated by the line which does not meet the curve.

8. The quadratic function. — The expression, $ax^2 + bx + c$, assumes different values when different values are assigned to the variable x. The variation in value of the function, $ax^2 + bx + c$, for given values of a, b, and c, is most easily given by drawing the graph of

$$y = ax^2 + bx + c.$$

The maximum or minimum value of the function, $ax^2 + bx + c$, for any real value of x, may be found by solving, $ax^2+bx+c=y$, and determining whether there is any greatest or least value which you have for real values of x.

Thus,
$$y = 2x^2 - 6x - 5,$$
$$2x^2 - 6x - (5 + y) = 0,$$
$$x = \frac{6 \pm \sqrt{36 + 8(5 + y)}}{4} = \frac{6 \pm \sqrt{76 + 8y}}{4}.$$

If y is less than $-\frac{76}{8}$, the values of x become imaginary; consequently $-\frac{76}{8}$ is the minimum value which y can have.

If $ax^2 + bx + c = 0$ has real roots, and a is positive, the function $ax^2 + bx + c$ is negative for values of x between the two roots and positive for all other values; if the equation has imaginary roots, a being positive, $ax^2 + bx + c$ is positive for all real values of x.

$a(x - x_1)(x - x_2)$ is positive for values of x greater than both x_1 and x_2, negative for values of x between x_1 and x_2, and zero for $x = x_1$ or $x = x_2$; x_1 and x_2 are supposed to be real, and a positive.

$$ax^2 + bx + c \equiv a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$$

is positive, a being positive, when $4ac > b^2$.

9. Summary. $ax^2 + bx + c = 0.$

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

$$x_1 + x_2 = -\frac{b}{a}, \quad x_1 x_2 = \frac{c}{a}.$$

Given a, b, and c, real numbers, the condition:

$b^2 - 4ac > 0$, gives **real** and **unequal** roots.

If a, b, and c are rational, and $b^2 - 4ac$ a perfect square, then the roots are also rational.

$b^2 - 4ac = 0$, roots are **real** and **equal**.

$b^2 - 4ac < 0$, roots are **imaginary**.

PROBLEMS

1. Solve by completing the square:

a. $x^2 - 4x - 5 = 0$; *b.* $2x^2 + 3x - 5 = 0$; *c.* $3x^2 - 4x - 9 = 0.$

2. Solve by formula:

a. $3x^2 - 4x - 9 = 0$; *b.* $7x^2 - 3x - 10 = 0$; *c.* $3y^2 + 2y - 5 = 0.$

3. Time yourself in solving the following 10 quadratics, writing the roots in simplest form but not approximating the square root.

a. $2x^2 + 3x - 5 = 0.$
b. $3x^2 - 2x + 7 = 0.$
c. $5y^2 + 12x + 3 = 0.$
d. $y^2 - 3y - 7 = 0.$
e. $2v^2 - 10v - 35 = 0.$
f. $9x^2 = 20x + 10.$
g. $7t^2 = 24t - 5.$
h. $t^2 + 4t = 1.$
i. $9v^2 = 16 - 24v.$
j. $8u + 5 = 3u^2.$

4. Time yourself in finding to one decimal place the roots in the above 10 equations.

5. Find the nature of the roots, without completely solving, in the following equations:

a. $x^2 + 3x - 5 = 0.$
b. $x^2 + 3x - 8 = 0.$
c. $x^2 + 3x - 40 = 0.$
d. $x^2 - 3x + 40 = 0.$
e. $x^2 - 3x + \frac{9}{4} = 0.$
f. $4x^2 - 12x + 9 = 0.$

6. The velocity of a freely falling body is given by the formula, $v = 32t$, when falling from rest; or $v = 32t + k$, where k represents the velocity at the instant when $t = 0$, or k is the velocity at the instant when you begin to measure the time. Plot for values of t from 0 to 10.

7. A bullet shot straight up into the air at a velocity of 1000 feet per second has its height above the earth given by the equation $h = 1000t - 16t^2$. Plot this equation for values of t increasing by intervals of 5 seconds from $t = 0$ to $t = 70$. If the bullet is shot at an angle in such a way that the vertical velocity when leaving the gun is 1000 feet per second, the given equation continues to hold for the height of the bullet above the earth. The resistance of the air (considerable at the velocity mentioned) is neglected in these equations.

8. Plot the graph of $y = 3x - 7$; give to x the integral values from -2 to $+5$ and find the values corresponding of y.

9. A freely falling body falls from rest in t seconds a distance s, given by $s = 16t^2$; plot points given by corresponding values, using horizontal axis as t-axis, and vertical axis for distance. Take values of t from 0 to 10, and as s will vary from 0 to 1600 take 1 cm. to represent 100 on the s-axis.

10. For what values of x are the following expressions:

a. $(x - 3)(x - 5)$; *b.* $(x + 1)(x - 4)$; *c.* $2x^2 + 3x - 5$;
d. $4x^2 - 12x + 9$; *e.* $4x^2 - 8x + 9$; *f.* $3x^2 + 2x - 7$.

11. Given $h = 800t - 16t^2$, find t when $h = 100$, 1000, 10,000, 12,000 respectively. This equation represents the height to which a bullet would rise when shot vertically upwards at a velocity of 800 feet per second, neglecting air-resistance. Interpret your results. This bullet has a velocity at time t, $v = 800 - 32t$; find the velocity at the various heights mentioned.

12. Solve $16t^2 - 800t + h = 0$ for t, regarding h as a constant. See preceding problem and find maximum value h can have.

13. In solving $16t^2 - 800t + h = 0$, two roots are obtained; find the sum of these roots and the product. Interpret the sum, *i.e.* give the physical meaning.

14. Discuss the changes in value of $3x^2 + 2x - 5$ as x changes from -10 to $-\frac{5}{3}$, to 0, to 1, to 2, to 10, to "positive infinity."

15. Determine the nature of the roots:

a. $4t^2 - 16t - 160 = 0.$ *c.* $3v^2 + 16v + 20 = 0.$

b. $7t^2 + 16t - 160 = 0.$ *d.* $3v^2 + 16v + 25 = 0.$

16. Plot the graphs of the functions in 15.

17. Solve the equations of 15 graphically, using the intersection with $y = t^2$ or $y = v^2$ (one half-inch may be taken for 10 units on the vertical axis).

18. Find the sum and the product of the roots in the problems of 14 and 15.

19. Solve $\begin{cases} x^2 + y^2 = 36, \\ y = 3x + 5, \end{cases}$ by substitution.

20. Solve $\begin{cases} s = 16t^2, \\ s = 3t - 5, \end{cases}$ by substitution. Draw graphs.

21. Solve $.1t^2 - 50t - 30 = 0$, to 2 places of decimals.

22. Solve $t^2 - 50t - .0001 = 0$, to 2 places of decimals.

23. Find the maximum or minimum values of the following quadratic functions:

a. $x^2 - 4x + 4.$ *c.* $x^2 - 4x + 6.$

b. $2x^2 + 3x + 5.$ *d.* $x^2 + 4.$

24. For what values of x is $x^2 - 6x - 16$ positive?

10. Equations reducible to quadratics. — The solution of

$$ax^2 + bx + c = 0$$

is a value of the variable x, which when it is substituted in $ax^2 + bx + c$, makes the expression 0; similarly this solution gives a value of the variable v, or t, or p, or t^2, or $t^{\frac{1}{2}}$, or $3t^2 - 1$, or $7t^2 + 2t - 3$, which makes the expression of the same form in that variable zero; viz., a value which makes $av^2 + bv + c$ equal zero when the value is put for v, or $a(t^2)^2 + bt^2 + c$, equal zero when the value is put for t^2, or $a(3\,t^2 - 1)^2 + b(3\,t^2 - 1) + c$ equal zero when the value is put for $3\,t^2 - 1$. Equations which can be put in the form $ax^2 + bx + c = 0$ are called equations in quadratic form, the term being applied, in general, to expressions which are not quadratics directly in the principal variable. Thus, in any expression involving x, x^2, x^3, x^4, x^5, or x^6, the value of the expression depends primarily upon the principal variable, x; an expression like $3\,x^4 - 2\,x^2 - 7$, involving the variable x^2, its square, and constants as coefficients, is said to be in quadratic form, and it is a quadratic in the variable x^2, but a quartic in x.

9. Illustrative exercises.

1. Solve $3\,t^4 - 5\,t^2 - 7 = 0$.

As a quadratic in t^2, the formula for the solution of a quadratic gives

$$t^2 = \frac{5 \pm \sqrt{25 + 84}}{6} = \frac{5 \pm \sqrt{109}}{6}, \text{ whence}$$

$$t = \pm\sqrt{\frac{5 \pm \sqrt{109}}{6}}.$$

There are four values represented here, of which two are imaginary.

2. Solve $x^3 = 1$, or $x^3 - 1 = 0$, and $x^3 - 8 = 0$; these illustrate a type of equation reducible to a quadratic by factoring.

$$x^3 - 1 \equiv (x - 1)(x^2 + x + 1) = 0.$$

$$x - 1 = 0, \qquad x = 1$$

$$x^2 + x + 1 = 0,\ x = \frac{-1 \pm \sqrt{1 - 4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}.$$

These values $\frac{-1 + \sqrt{-3}}{2}$, $\frac{-1 - \sqrt{-3}}{2}$ and 1 are called the cube roots of unity; note that $\sqrt{-3}$ is defined as a quantity whose square is -3; the systematic discussion of such numbers is deferred until a later chapter. Squaring either of the two imaginary cube roots of unity gives the other; these roots may then be designated as 1, w, w^2. The cube roots of 8 are 2, $2\,w$, and $2\,w^2$; of 7 are $7^{\frac{1}{3}}$, $7^{\frac{1}{3}}\,w$, and $7^{\frac{1}{3}}\,w^2$, wherein $7^{\frac{1}{3}}$ denotes the real cube root of 7.

PROBLEMS

1. Solve for t^3, and then for t. $t^6 - 7\,t^3 - 8 = 0$.

2. Solve and check by substitution:

$$x^{-4} + 3\,x^{-2} - 5 = 0.$$

3. $2\,x^{\frac{3}{2}} - 7\,x^{\frac{3}{4}} - 5 = 0$.

4. $\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) - 1 = 0$.

Note that this expression when cleared of fractions gives

$$x^4 + x^3 + x^2 + x + 1 = 0,$$

a factor of $x^5 - 1 = 0$; the imaginary roots of x which are obtained by solving are the other four fifth-roots of unity.

5. $(3\,x^2 - 5)^2 + 2(3\,x^2 - 5) - 7 = 0$.

6. $v + v^{\frac{1}{2}} = 10$.

7. $\dfrac{t}{t^2 - 1} + \dfrac{t^2 - 1}{t} = 2$.

8. Find the value of $x^{\frac{1}{2}}$, and of x in

$$x + 3\,x^{\frac{1}{2}} - 7 = 0.$$

9. Find the value of $x^{\frac{1}{2}}$, in

$$x - 3\,x^{\frac{1}{2}} - 7 = 0,$$

and compare with the preceding. The real test of a value found as a root is obtained by substituting the value in the given expressions. Squaring may introduce a new root; thus squaring $x = 2$, gives $x^2 = 4$, or is equivalent to multiplying $x - 2 = 0$, member by member by $x + 2$.

10. $3\,x + \sqrt{x + 5} = 7$.   11. $3\,x - \sqrt{x + 5} = 7$.

10. Limiting values of *a*, *b*, *c*. — As c approaches more and more nearly to zero as compared with a and b, it is evident that some value of x also near to zero will satisfy the equation $ax^2 + bx + c = 0$; this value will be of the same sign as c if b is negative, and opposite in sign to c if b is positive. This may be obtained by taking the approximate value of $\sqrt{b^2 - 4\,ac}$ as $b - \frac{4\,ac}{2\,b}$ (see page 24), giving $-\frac{c}{b}$ and $-\frac{b}{a} + \frac{c}{b}$ as the approximate values of the two roots when c (or a) is small in comparison with b.

Thus,
$$3\,x^2 - 2\,x - .000001 = 0,$$
$$x = \frac{2 \pm \sqrt{4 + .000012}}{6}$$
$$= \frac{2 \pm 2.000003}{6} = \frac{4.000003}{6} \text{ or } -\frac{.000003}{6}$$
$$= \tfrac{2}{3} \text{ or } -.0000005.$$

Similarly in $1000\,x^2 - 3000\,x - 1 = 0$, one solution will be small, approximately $\frac{-1}{3000}$. When $c = 0$, the roots of $ax^2 + bx + c = 0$ are the roots of $ax^2 + bx = 0$, giving $x(ax + b) = 0$; whence $x = 0$ and $x = -\frac{b}{a}$. When both b and c approach zero, both roots of the quadratic $ax^2 + bx + c = 0$ approach zero.

When a approaches zero in comparison with b and c, one root of the quadratic becomes very large and the other approaches $-\frac{c}{b}$. Thus in the quadratic
$$x^2 - 1000\,x - 3000 = 0,$$
$$x = \frac{1000 \pm \sqrt{1000000 + 12000}}{2} = \frac{1000 \pm 1006}{2} = 1003 \text{ or } -3.$$
$\left(\frac{1000 \pm 1005.982}{2}\right.$ are the more exact values, giving 1002.991 or $\left.-2.991.\right)$

As a approaches nearer and nearer to zero one root becomes larger and larger without limit. Thus if above we had
$$.001\,x^2 - 1000\,x - 3000 = 0$$
$$x = \frac{1000 \pm \sqrt{1000000 + 12}}{.002} = \frac{1000 \pm 1000.006}{.002}$$
$$= 1000001.5 \text{ or } -3 \text{ (more exactly 2.999991 as before).}$$

Both roots become large if both a and b become small as compared with c.

PROBLEMS

Find first approximate values, and verify by solving the quadratic:

1. $3x^2 - 7x - .0001 = 0$.

2. $5x^2 - 7x - .1 = 0$.

3. $5x^2 - .007x - .001 = 0$.

4. $4000 = 3000t - 16t^2$; one root is the number of seconds for a bullet to rise 4000 feet, initial velocity 3000 feet per second, air resistance neglected; what does the other root represent?

5. $.01x^2 - 300x - 500 = 0$.

6. $.003t^2 + 2t - 42 = 0$; this gives a more exact equation for the temperature at which the velocity of sound in air becomes 40 feet greater than it is at 0° C.

7. $1000000x^2 - 3000000x - 5 = 0$.

REVIEW PROBLEMS

1. Plot the graph of $y = 3x - 5$.

2. Plot the graphs of the following functions:

   a. $y = x^2 - 4x + 5$.
   b. $y = x^2 - 4x + 4$.
   c. $y = x^2 - 4x$.
   d. $y = x^2 - 4x - 2$.

3. For what values of x is y equal to 0 in the four functions of the preceding question? The graphical solution is desired.

4. Plot 15 points from $x = -.5$ to $x = +8$ and join by a smooth curve representing

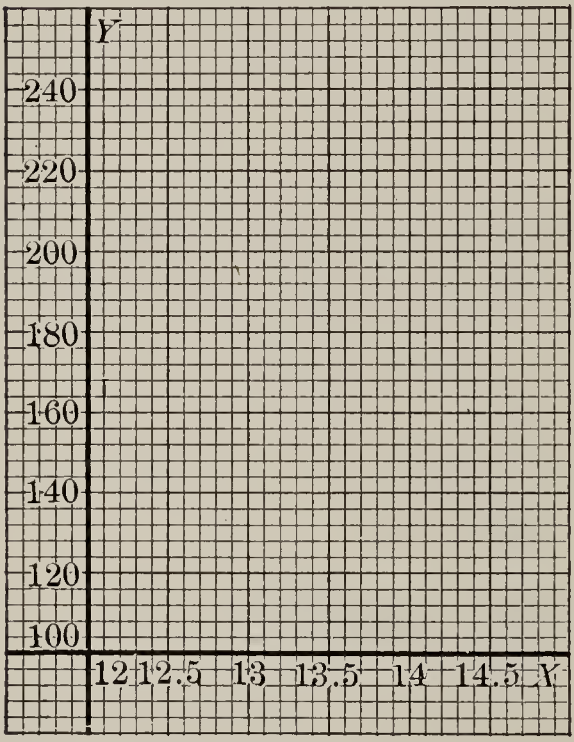
$$y = 2x^3 + 6x^2 - 10x - 8;$$

for what values of x is y equal to zero?

5. $s = 20t + 50 - 16t^2$. This equation represents the motion of a body thrown from a height of 50 feet straight up into the air with a velocity of 20 feet per second. Plot the graph and locate the position of the body at the end of 1 second; at the end of 5 seconds.

6. Plot the graph of $s = 800t - 16t^2$, for values of t from 0 to 50; note that it is desirable to get the values of s first for intermediate values and to choose the y-scale accordingly. This equation represents approximately the height after t seconds of a bullet shot straight into the air with a velocity of 800 feet per second.

x-axis and y-axis off the paper



Shifted lines of reference

7. Plot the graph of $V = \frac{\pi d^2}{4}$ between $d = 12$ and $d = 20$, taking the scales so as to enable you to read volumes as correctly as possible within these limits. Plot only values above 100 on the y-scale, and to the right of 12 on the x-scale. This gives the volume in cubic units per unit of height of cylindrical containers which have radii varying from 12 to 20 units. Apply this to cans and to silos.

8. Plot the graph of $t^2 = \frac{\pi^2 l}{980}$; this gives the time of beat of a pendulum l centimeters long where gravity is 980 cm. per sec. per sec.

9. Plot the graph of $y = x^{\frac{3}{2}}$, for values of x from 0 to 8.

10. Plot the graphs of the following linear functions:

a. $y = 3x - 5$.
b. $y = 3x$.
c. $v = 10 + 8t$.
d. $s = 6 - 3t$.
e. $s = 100 - 40t$.
f. $y = -2x + 10$.

CHAPTER VI

STRAIGHT LINE AND TWO-POINT FORMULAS

1. Slope-intercept formula: $y = mx + k$.

The equation $y = mx + k$, into which form the equation of any straight line can be put, is called the slope-intercept form of the equation of a line; m represents the slope of the line and k is the intercept on the y-axis. The equation of a line parallel to the y-axis, $x = k$, cannot be placed precisely in this form, as the y-intercept is infinite.

2. Point-slope formula: $y - y_1 = m(x - x_1)$.

As it is frequently desired to find the equation of a line of given slope and passing through a given point, a separate equation in terms of the slope and coördinates of the given point is desirable. Let the equation of the line be conceived as in the form, $y = mx + k$; since (x_1, y_1) is on the line, $y_1 = mx_1 + k$; subtracting gives $y - y_1 = m(x - x_1)$, the equation of the straight line in terms of m, the given slope, and (x_1, y_1) the coördinates of the given point.

3. Two-point formula: $\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$.

The equation of the straight line through $(x_1, y_1)(x_2, y_2)$ is also easily derived from the slope-intercept form.

As before $$y_1 = mx_1 + k,$$
$$y_2 = mx_2 + k,$$
whence $$y_2 - y_1 = m(x_2 - x_1),$$
and $$m = \frac{y_2 - y_1}{x_2 - x_1},$$ giving m, the slope of the line, in terms of x_1, y_1, x_2, and y_2.

Hence, $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ is the equation of the line in a form involving only the given constants.

The expression, $m = \frac{y_2 - y_1}{x_2 - x_1}$, represents the slope of a line joining (x_1, y_1) to (x_2, y_2). Similarly $\frac{y - y_1}{x - x_1}$ represents the slope of the line joining any point (x, y) to (x_1, y_1). The preceding equation of the line in the form $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$ is an equality of two slopes.

The formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ should be memorized.

This formula gives the rate of increase of y in the interval from (x_1, y_1) to (x_2, y_2) as compared with the increase of x in the same interval; it compares the change in y in the interval with the change in x in the same interval.

If $x_1 = x_2$, the line joining (x_1, y_1) to (x_2, y_2) should be given directly as $x = x_1$, parallel to the x-axis, and similarly if $y_1 = y_2$.

PROBLEMS

1. Find the equation of the line of slope 3 and y-intercept 5; with $m=3$, $k=-5$; $m=-3$, $k=8$; $m=0$, $k=4$; $m=5$, $k=0$.

2. Put the following equations into slope-intercept form:

a. $3y - 2x + 5 = 0.$

b. $3x + 2y - 7 = 0.$

c. $x + 2y = 0.$

d. $y - 3x - 7 = 0.$

e. $y + 5 = 0.$

f. $x + 3 = 0.$

3. Write the equation of the straight line through $(-2, 5)$ and $(1, 4)$; through $(3, -5)$ and $(2, 1)$. Find intercepts on both axes and the slope in each case.

4. Write the equation of the straight line through $(1, 5)$ having the slope 3. Find the x and y intercepts.

5. Find the equation of the straight line through $(a, 0)$ and $(0, b)$, *i.e.* the line having intercepts a and b, respectively, and put this equation into the form $\frac{x}{a}+\frac{y}{b}=1$. This is called the intercept form of the equation of a straight line.

6. Given $9\,C=5\,F-160$, the formula connecting centigrade and Fahrenheit readings of temperature, find the slope and the x and y intercepts. Find the slope of the line joining (32, 0) to (212, 100). What is the rate of change of C in the interval as compared with the change in F? What physical meaning have the intercepts?

7. Given that 1000 cu. cm. of mercury at 0° C. increases to 1007.2 cu. cm. at 40° C., find the rate of change of volume per degree of temperature, and finally per cu. cm. Note that it is not necessarily true that this rate found for an interval of 40° C. should be the uniform rate everywhere in the interval. Write the equation representing the volume in terms of temperature, assuming that the relation is linear, *i.e.* that the increase in volume is proportional to the temperature. Mercury expands differently at different temperatures, but the variation is slight in the interval from 0° to 40°, not varying by more than $\frac{2}{3}$ of 1% from .00018 cu. cm. per degree for 1 cu. cm.

8. Join (0, 0) to (100, 39.37) and interpret for converting centimeters to inches and inches to centimeters; what is the meaning of the slope? Find the value in inches of 18 cm., 39 cm., 47 cm. Note that 100 cm. = 39.37 inches.

9. 59.8 pints of water weigh approximately 62.4 lb. Draw the graph connecting (0, 0) to (59.8, 62.4) which will give the approximate weight of any given number of pints of water. How could you read the weight of quarts or gallons? Use $\frac{1}{2}$ inch for 10 units on both scales, in plotting.

10. Find the equations of the straight lines joining the following pairs of points, timing yourself:

a. $(3, 5)$ to $(-2, 7)$.
b. $(3, 5)$ to $(2, -7)$.
c. $(0, 8)$ to $(7, 0)$.
d. $(0, 8)$ to $(7, -6)$.
e. $(0, 8)$ to $(0, 5)$.
f. $(1, -3)$ to $(-1, -5)$.
g. $(1, -3)$ to $(1, 6)$.
h. $(-1, -3)$ to $(-3, -5)$.

i. $(-1, -3)$ to $(3, 3)$.
j. $(8, -3)$ to $(-3, 2)$.
k. $(-3, 5)$ to $(7, 0)$.
l. $(-3, 5)$ to $(-7, -2)$.
m. $(-3, 5)$ to $(7, -2)$.
n. $(2, 2)$ to $(-2, -2)$.
o. $(100, 60)$ to $(0, 0)$.

11. Find the equations of the following lines:

a. of slope 3, y-intercept 5.
b. of slope 3, x-intercept 5.
c. of slope -2, y-intercept 0.
d. of slope -1, through $(2, 2)$.
e. of slope $+4$, through $(2, 2)$.
f. having intercepts of 7 and 5 on the x- and y-axes respectively.

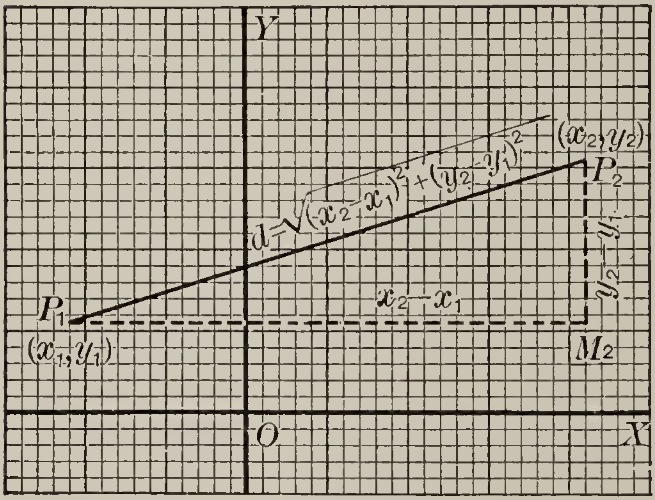
4. Distance between two points: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

$$d^2 = P_1P_2^2 = P_1M_2^2 + M_2P_2^2.$$

Since $\qquad P_1M_2 = x_2 - x_1$, and $M_2P_2 = y_2 - y_1$,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Whatever the positions of P_1 and P_2, parallels to the x- and y-axes through P_1 and P_2 form a rectangle (a straight line if $x_1 = x_2$ or $y_1 = y_2$) whose sides are in absolute value $|x_2 - x_1|$ and $|y_2 - y_1|$; the bars indicate that only the numerical value is considered. As a positive distance P_1M_2, if $x_1 > x_2$, would have to be written $x_1 - x_2$ or $-(x_2 - x_1)$. But since the numerical value of the expression $(x_2 - x_1)^2$ is the same as the value of $(x_1 - x_2)^2$ we may use in every case $(x_2 - x_1)$ for P_1M_2 in the above expression for d wherein only the square of P_1M_2 enters.



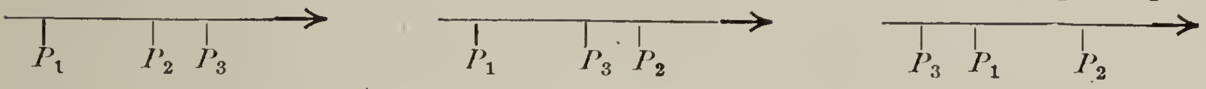
Distance between two points
$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

$$d = P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The distance from any point (x_1, y_1) to any point (x_2, y_2) is given by this formula; this distance is taken in general as a positive quantity.

This formula may be used to derive the equation of the straight line joining $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ for any point $P(x, y)$ on the line is such that $PP_1 + P_1P_2 = PP_2$; and for no point not on the line is this relation true.

5. Point of division formula: $x_3 = \frac{k_1x_2 + k_2x_1}{k_1 + k_2}$; $y_3 = \frac{k_1y_2 + k_2y_1}{k_1 + k_2}$.



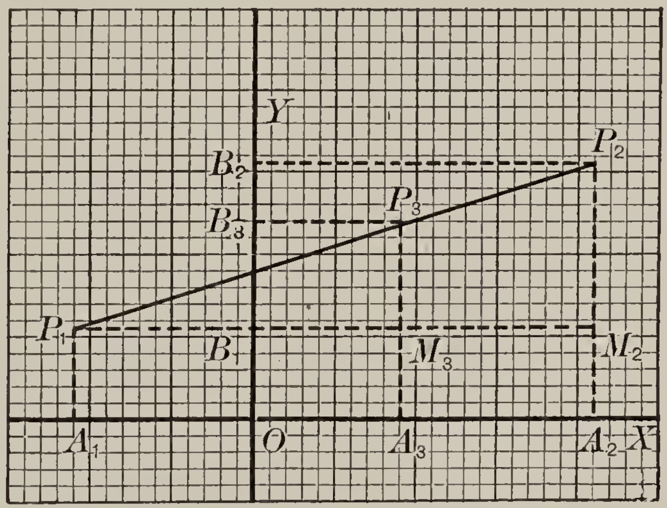
$$P_1P_2 + P_2P_3 = P_1P_3.$$

For any three points P_1, P_2, and P_3 on a directed line we have $P_1P_2 + P_2P_3 = P_1P_3$; if P lies between P_1 and P_3, all three segments have the same algebraic sign but otherwise positive and negative segments are involved.

$OP_1 + P_1P_2 = OP_2$ is then, similarly, the fundamental relation true for any three points on a directed line, whence

$$P_1P_2 = OP_2 - OP_1 = x_2 - x_1.$$

In words the distance on the x-axis (or any other line parallel to the x-axis) from any point whose abscissa is x_1 to any point whose abscissa is x_2, is given by $x_2 - x_1$. Similarly with respect to points on the y-axis, or two points on a line parallel to the y-axis, the distance from the point whose ordinate is y_1 to the point whose ordinate is y_2 is $y_2 - y_1$.



Point of division formula

$$\frac{P_1P_3}{P_3P_2} = \frac{k_1}{k_2} \cdot \quad \frac{A_1A_3}{A_3A_2} = \frac{B_1B_3}{B_3B_2} = \frac{k_1}{k_2}.$$

To find the coördinates of the point P_3 which divides the line joining P_1P_2 into two segments which bear to each other the ratio $\frac{k_1}{k_2}$, note that

$\frac{P_1P_3}{P_3P_2}=\frac{k_1}{k_2}$. By drawing lines through P_1, P_2, and P_3 parallel to the axes, similar triangles are formed, or the proposition of plane geometry that a series of parallels cut off on transversals proportional parts may be directly used.

$$\frac{P_1M_3}{M_3M_2}=\frac{A_1A_3}{A_3A_2}=\frac{x_3-x_1}{x_2-x_3}=\frac{k_1}{k_2}.$$

Whence

$$x_3=\frac{k_1x_2+k_2x_1}{k_1+k_2}=\frac{x_1+\frac{k_1}{k_2}x_2}{1+\frac{k_1}{k_2}}=\frac{x_1+rx_2}{1+r}, \qquad \text{wherein } r=\frac{k_1}{k_2}.$$

Similarly, $\frac{B_1B_3}{B_3B_2}=\frac{k_1}{k_2}$; whence $\frac{y_3-y_1}{y_2-y_3}=\frac{k_1}{k_2}$.

$$y_3=\frac{k_1y_2+k_2y_1}{k_1+k_2}=\frac{y_1+\frac{k_1}{k_2}y_2}{1+\frac{k_1}{k_2}}=\frac{y_1+ry_2}{1+r}, \qquad \text{wherein } r=\frac{k_1}{k_2}.$$

If $P_3(x_3, y_3)$ divides the line P_1P_2 externally in the ratio $\frac{k_1}{k_2}$, or r, the segments must be regarded as of opposite signs and consequently, the ratio $\frac{k_1}{k_2}$, or r, is negative. Either k_1 or k_2 can be regarded as negative; shifting the sign from k_2 to k_1 is equivalent to changing the sign of the numerator and denominator in the value of x_3 and y_3, no change is necessary in our above derivation of the values of x_3 and y_3.

By eliminating k_1 and k_2 between the two equations,

$$x=\frac{k_1x_2+k_2x_1}{k_1+k_2},$$

$$y=\frac{k_1y_2+k_2y_1}{k_1+k_2},$$

the equation of the straight line joining P_1 and P_2 is obtained.

Mid-Point:

Place $k_1 = k_2$, or place $r = 1$,

$$x = \frac{x_1 + x_2}{2},$$

$$y = \frac{y_1 + y_2}{2}.$$

This mid-point formula is of such frequent use that it should be separately memorized; the truth of it is obvious from the figure.

PROBLEMS

1. Find the mid-point, the points of trisection, and the point dividing the segment externally in the ratio, $-2:5$, in each of the following line segments:

a. $(-3, 4)$ to $(6, 7)$. *b.* $(3, -2)$ to $(-5, 4)$. *c.* $(0, 0)$ to $(9, 12)$.

Locate the points in each of these line segments both graphically and analytically; find the length of each segment.

2. Find the length and slope of the line joining $(8, -3)$ to $(-4, 2)$.

3. Plot the graph of $5y + 2x - 5 = 0$.

4. Plot $p = 5l + 50$.

5. Find the equation of the straight line joining $A(-2, 5)$ to $B(3, 7)$. Find slope of this line. Find length of AB. Find the point of trisection nearest A. Find a point on BA extended that divides the segment BA externally in the ratio $1:2$.

6. Given that the velocity of sound at 0° C. is 1090 feet per second, and at 30° C. is 1150 feet per second, find the velocity at 20° C., assuming that the relation is linear; the point dividing the line joining (0, 1090) and (30, 1150) in the ratio 2 : 1 will give the velocity as the ordinate. At what temperature will the velocity be 1100 ft. per second? What are the velocity and temperature at the middle point of the range given?

7. The resistance of wire increases uniformly with the temperature, $r = r_0(1 + at)$, the rate of increase depending upon the material of the wire; r_0 is the resistance at 0° C. and a is a constant. If a given piece of wire has a resistance of 200 ohms at 10° C. and of 208.4 at 30° C., find the resistance at the middle point [of (10, 200) and (30, 208.4)]. Find the equation for r in terms of t. Find the value of r when $t = 0$; interpret; find the value of t when $r = 0$. The theory is that at a temperature of absolute zero ($-273°$ C. or thereabouts) the resistance would be zero. *Ans.* $r = 195.8 + .42\,t$.

8. The resistance of copper wire of fixed diameter varies with the length. If the resistance of 1450 feet of a given spool is 184 ohms, and the resistance of 0 feet is 0 ohms, find the equation for r in terms of l. Plot (0, 0) and (1450, 184). What would be the resistance of 5280 feet of this wire?

9. Between $(-1, 5)$ and $(8, 37)$ insert 9 points dividing the line into ten equal parts, using the formulas

$$x = \frac{k_1x_2 + k_2x_1}{k_1 + k_2} \text{ and } y = \frac{k_1y_2 + k_2y_1}{k_1 + k_2},$$

rearranged as follows:

$$x = \frac{k_2x_1 + k_1x_1 + k_1x_2 - k_1x_1}{k_1 + k_2} = x_1 + \frac{k_1}{k_1 + k_2}(x_2 - x_1),$$

and similarly

$$y = y_1 + \frac{k_1}{k_1 + k_2}(y_2 - y_1).$$

Note that $k_1 + k_2$ is constant, 10, and k_1 changes for the nine points from 1 to 9. Use this method in problems 10, 11, and 14–17 below.

10. Between (21, .3584) and (22, .3746) insert 5 values dividing the interval into 6 equal parts.

11. Between (10, .3611) and (20, .3638) insert 9 values dividing the line joining these points into ten equal parts.

12. Find the point P_3 dividing the line joining $P_1(-1, 5)$ to $P_2(8, 37)$ externally in the ratio 1 to 7; externally in the

ratio $\frac{1}{11}$; externally in the ratio $\frac{1}{2}$. Note that either k_1 or k_2 must be made negative, or r taken as negative.

13. Find the point dividing (21, .3584) to (22, .3746) externally in the ratio $\frac{1}{7}$, $\frac{2}{8}$, $\frac{3}{9}$.

14. $\log 1 = 0$; $\log 2 = .301$; find 9 values dividing (1, 0) and (2, .301) into ten equal parts. Compare with the logarithms of 1.1, 1.2, 1.3, 1.4, $\cdots$ 1.9. See problem 9 above.

15. $\log 200 = 2.3010$; $\log 210 = 2.3222$; insert 9 values between (200, 2.3010) and (210, 2.3222), comparing with the logarithms of 201, 202, $\cdots$ 209.

16. $\log 200 = 2.3010$ and $\log 201 = 2.3032$; insert 9 values between (200, 2.3010) and (201, 2.3032) and interpret.

17. Given $y = 32\,x - 17$; find the corresponding values of y when $x = 10$ and $x = 20$. Find the points dividing this line in the ratio $\frac{1}{9}$, $\frac{1}{4}$, $\frac{3}{7}$, $\frac{2}{3}$, $\frac{1}{1}$. What points of division are obtained?

18. Given $P_1(-1, 5)$ and $P_2(8, 37)$; on the line joining these two points find the point whose abscissa is 3, without finding the equation of the line. Find the point whose abscissa is 7. Find the point whose ordinate is 16. Find the point whose ordinate is 0. Find the point whose abscissa is $+14$.

19. Eliminate r between the two equations $x = \dfrac{-1 + 8\,r}{1 + r}$ and $y = \dfrac{5 + 37\,r}{1 + r}$. These two equations constitute what are known as parametric forms of the equation of the straight line joining $(-1, 5)$ to (8, 37).

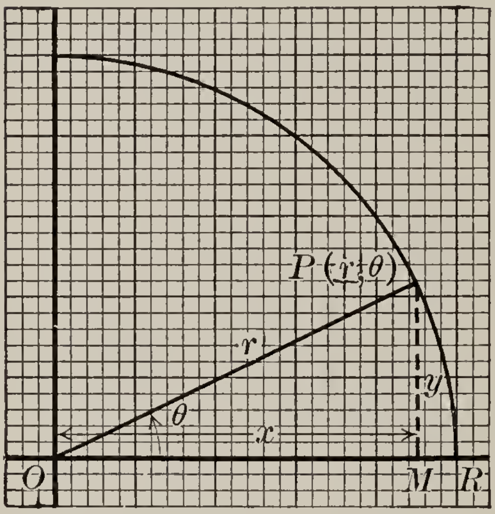
20. Write the equations of the line joining $(3, -2)$ to (15, 28) in parametric form. Find 6 points on this line.

21. Prove analytically that: *a.* The medians of any triangle meet in a point, trisecting each median. *b.* The diagonals of any parallelogram bisect each other.

CHAPTER VII

TRIGONOMETRIC FUNCTIONS

1. Angles and angular measurement. — The angle made by any line *OP* with the horizontal line *OX* is regarded as generated by a moving line, an arm or ray, starting from the position *OX* and turning about the point *O* as a pivot, moving always in the same plane. This moving ray if rotated in the sense contrary to that in which the hands of a clock move, counter-clockwise, is regarded as generating a positive angle; clockwise rotation generates a negative angle. A natural unit of angular magnitude is the complete rotation which brings the moving arm back to its original position. This measure is used in giving the speed of rotation, *e.g.* the angular speed of rotating shafts and wheels is measured in revolutions per minute or per second. The angle generated when the moving ray is in the same straight line with its original position, but extending in the opposite direction, is called a straight angle; half of this angle is the right angle, which was probably the earliest measure of angles used. Thus our terms acute and obtuse relate to the right angle. If the angle is conceived as given by the relative position of two lines non-directed, it is evident that only angles less than a straight angle would be discussed.



Angle generated by rotation

In the ancient development of geometry the right angle, so necessary in building, was fundamental; in Greece up to about 150 B.C. the right angle was used as the unit of measure. The artificial division of the complete rotation into 360 equal angular units called degrees is due to the Babylonians, who made this subdivision as early as 1000 B.C. The Babylonians used 60 as a unit of higher order much as we use ten, and it is probable that they divided another natural angular unit, one sixth of a perigon, given by the easy construction of a regular hexagon, into 60 equal parts called degrees; each degree was divided by them into 60 minutes (*partes minutiae primae*, in Latin, whence "minutes") and the minute into 60 seconds (*partes minutiae secundae*).

Another natural system of measuring angles is of fundamental importance in mathematical work. This is the circular system, in which the unit angle, called a radian, is the angle measured at the center of a circle by an arc whose length is the radius. The radius can be laid off on the circle 2π, $6\frac{2}{7}$ or 6.2832, times, and since equal angles at the center are intercepted by equal arcs on the circumference, this angle can be placed 2π times around the center, or approximately $6\frac{2}{7}$ times in a complete revolution. Just as 1° is used for 1 degree, so 1^r is used for 1 radian, and similarly for other numerical values; when no angle sign is used radians are understood.

$$2\pi^r = 360^\circ. \qquad \frac{\pi^r}{3} = 60^\circ.$$

$$\pi^r = 180^\circ. \qquad \frac{\pi^r}{6} = 30^\circ.$$

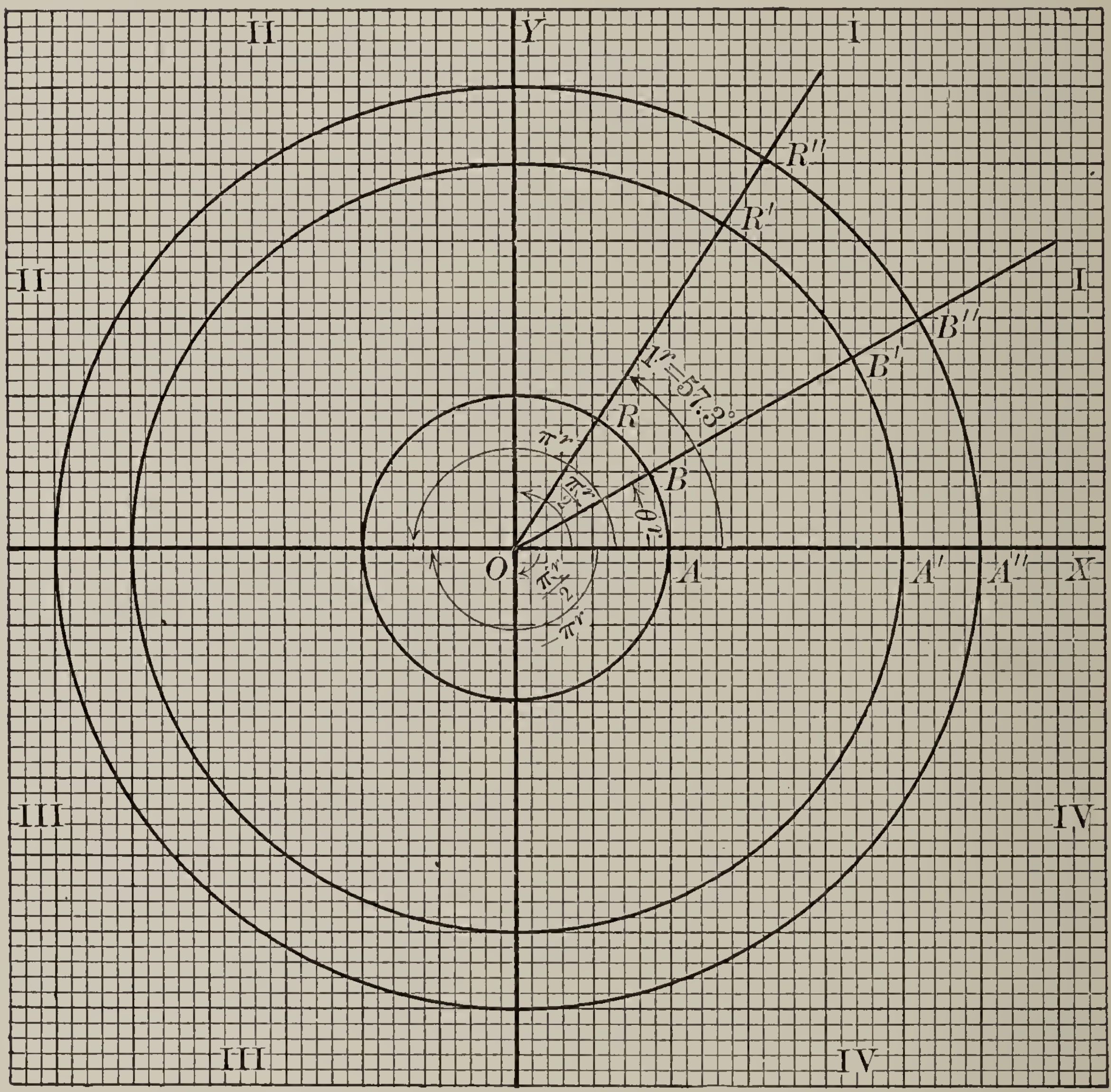
$$\frac{\pi^r}{2} = 90^\circ. \qquad \frac{\pi^r}{4} = 45^\circ.$$

The student should accustom himself to expressing angles in radians, particularly the angles of 30°, 45°, 60°, and 90°, and those which depend directly upon them.

Thus $150^\circ = \frac{5\pi^r}{6}$; $135^\circ = \frac{3\pi^r}{4}$; or, with many writers, simply $\frac{3\pi}{4}$ designates 135° in radians.

A natural system of measurement of angular magnitude

The arc AR equals the radius OA; arc $A'R' = OA'$; arc $A''R'' = OA''$.



Radian system of measuring angles

In these circles it is true that $\dfrac{AB}{OB} = \dfrac{A'B'}{OB'} = \dfrac{A''B''}{OB''}$. It is evident that if the circumference of one of these circles is divided into any number of equal parts, then lines from O to these points of division will divide, when extended, the other circles into the same number of equal parts. The angle at the center is measured, we may say, by the intercepted arc; or $\theta = \dfrac{a^r}{r}$, wherein a stands for the length of the arc and r for the radius. The length of the arc is given by the formula, $a = r\theta$, when θ is measured in radians; the area A of the corresponding sector of angle θ is $A = \frac{1}{2} r^2\theta$. Since $3.141593^r = 180°$, $1^r = \dfrac{180°}{3.141593} = 57.29578°$, $1° = .0174533$ radian.

2. Quadrants. —

The two lines OX and OY divide the plane into four quadrants, numbered as indicated on the preceding diagram, I to IV; we will commonly designate a quadrant by its numeral. In trigonometric work we conceive angles as placed with the vertex at O, one arm falling upon the OX axis to the right, and the other arm falling in one of the four quadrants, or upon one of the axes. We think of the angle, in effect, as generated by an arm rotating about O from the initial position OX. Under this assumption it is evident that the terminal arm of an angle may fall in any quadrant either by a positive rotation or by a corresponding negative rotation, the difference between the two angles being 360°. Rotations of greater than one revolution reproduce in order the positions on the diagram produced by rotations of less than one revolution, *e.g.* angles of 30°, $-330°$, $+390°$, $+750°$, $-690°$, and in general terms, $n \times 360° + 30°$ where n is any integer, are represented by the same figure. In radians we may say that α^r and $(2n\pi + \alpha^r)$ are represented by the same diagram for all integral values of n.

PROBLEMS

1. Using $3\frac{1}{7}$ for π, compute the value of 1^r in degrees. What is the percentage error?

2. Using $3\frac{1}{7}$ for π, compute the value of 1° in radians. Percentage error?

3. Give the value in degrees of $\frac{1}{2}$ revolution; π^r; $\dfrac{\pi^r}{6}$; $\dfrac{5\pi^r}{6}$; 1 straight angle; $\frac{2}{3}$ of one right angle; 3^r; $\dfrac{5}{2}\pi^r$; $\dfrac{\pi^r}{3}$; $\dfrac{\pi^r}{4}$; $\dfrac{3\pi^r}{4}$; $\dfrac{5\pi^r}{4}$.

4. Give the value in radians of 1 revolution; 180°; 45°; 135°; 60°; 120°; 225°; 3 right angles; 390°; 765°.

5. What is the percentage error in using 57.3° as the value of 1 radian?

6. What error in seconds is introduced by using 57.3° for 1 radian in finding the value of 3 radians? $3^r = 171.9°$; an error of 1 % would be approximately 1.7°.

7. A bicycle rider pedals at the rate of 20 miles per hour; how many revolutions does the rear wheel, diameter 28 inches, make per minute? The rear sprocket wheel, diameter 4 inches, makes the same number of revolutions as the rear wheel; how many revolutions does the front sprocket wheel, diameter 10 inches, make? Changing gear shifts the chain to a smaller rear sprocket; what speed will be attained at the same rate of pedaling by shifting to a 3-inch rear sprocket?

8. Place the following angles in their proper quadrants: 150°, 240°, 760°, $-840°$, $\frac{10\pi^r}{3}$, $\frac{5\pi^r}{4}$, $-\frac{10}{3}\pi^r$. Give the corresponding positive angles less than $2\pi^r$.

9. In the circle of radius 10 what is the length of the arc of an angle at the center of 60°? What is the difference between an arc of 60° and an angle of 60°? What is the length of the arc of 30°, 45°, $\frac{\pi^r}{6}$, $\frac{5\pi^r}{6}$?

10. What is the angle at the center in radians and degrees, in a circle of radius 100, subtended by an arc of length 100? 50? 30? 100π? Find the areas of the corresponding sectors of the circle.

11. In the artillery service angles are measured in "mils"; a "mil" is defined as $\frac{1}{6400}$ of a complete revolution. Compute the value in radians of one mil.

12. On the "mariner's compass" the complete revolution is divided into 32 parts, called "points" of the compass; compare the "points," with degrees, "mils," and radians.

13. Compute the value of the "mil" in minutes and give approximate formulas for converting "mils" into minutes and conversely.

14. At what rate per second in degrees, and in radians, do the hands of a clock turn?

15. A grindstone of diameter 18 inches is turning 246 times per minute. Compute the linear velocity of a point on the rim.

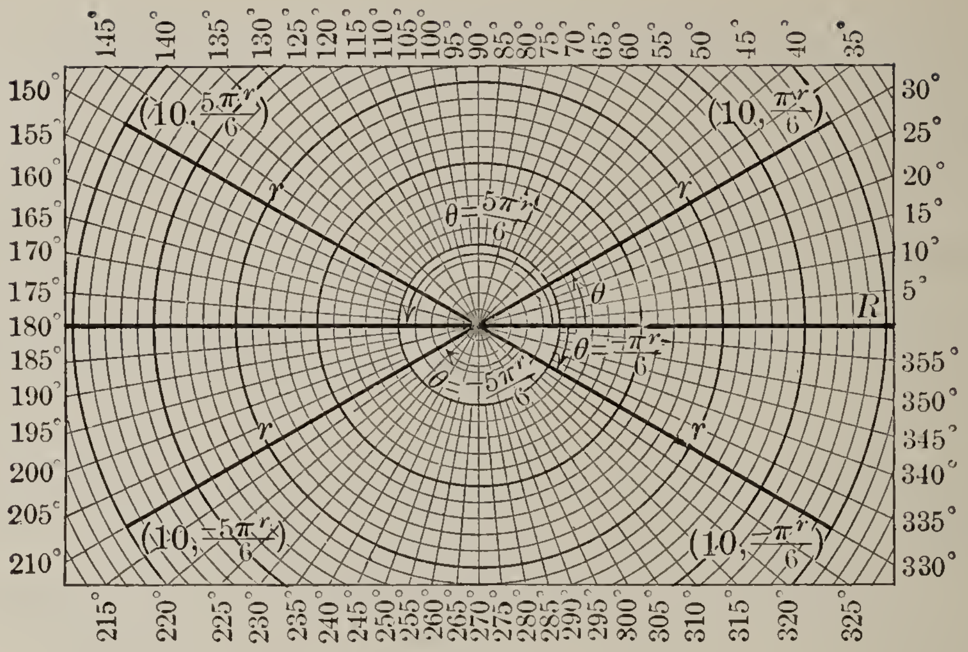
16. In grinding certain tools the linear velocity of the grinding surface should not exceed 6000 feet per second. Find the maximum number of revolutions per second of a 10-inch (diameter) emery wheel and of a 5-inch wheel.

17. Find the angular velocity in revolutions and in radians of an Ohio grindstone, 2 feet in diameter, which should have a circumferential speed of 2500 feet per minute.

18. The path of the earth is approximately a circle with radius 93,000,000 miles; find the distance traveled in 1 day. What percentage of error would be introduced by using 365 instead of $365\frac{1}{4}$ days? Show that the fact that we give r as 93,000,000 implies that the position of the point on the earth would not affect our computation.

3. Polar coördinates and angular variables. — Any point P in the plane may be located by giving its distance from a fixed point O, called the pole, and the angle which a line from the pole to the point P makes with a fixed line OR, called the polar axis. In general terms the polar coördinates of any point, of a variable point, are designated by r and θ, *radius vector* and *vectorial angle.* (See p. 116.)

r will be assumed to be a positive quantity, and θ may be assumed as the angle generated by the rotation of the vector OP from an initial position on OR. A negative angle is generated with the polar axis by a line which turns from the polar axis, about O, in the clockwise direction. Thus the $\angle ROP$ is taken as $+30°$; this same figure may also be con-



Polar coördinate paper

$$\text{Location of}\left\{\begin{array}{l}(10,\ 30°),\quad (10,\ 150°),\quad (10,\ -150°),\ \text{and}\quad (10,\ -30°),\\ \left(10,\ \frac{\pi^r}{6}\right),\ \left(10,\ \frac{5\,\pi^r}{6}\right),\ \left(10,\ \frac{-5\,\pi^r}{6}\right),\ \text{and}\ \left(10,\ \frac{-\pi^r}{6}\right).\end{array}\right.$$

ceived as representing – 330°. Angles which differ by multiples of 360°, generated by lines rotating from an initial position upon the polar axis, are represented by the same diagram; two such angles are commonly called "congruent" angles. Each rotation of 360° brings a line back to its starting place.

PROBLEMS

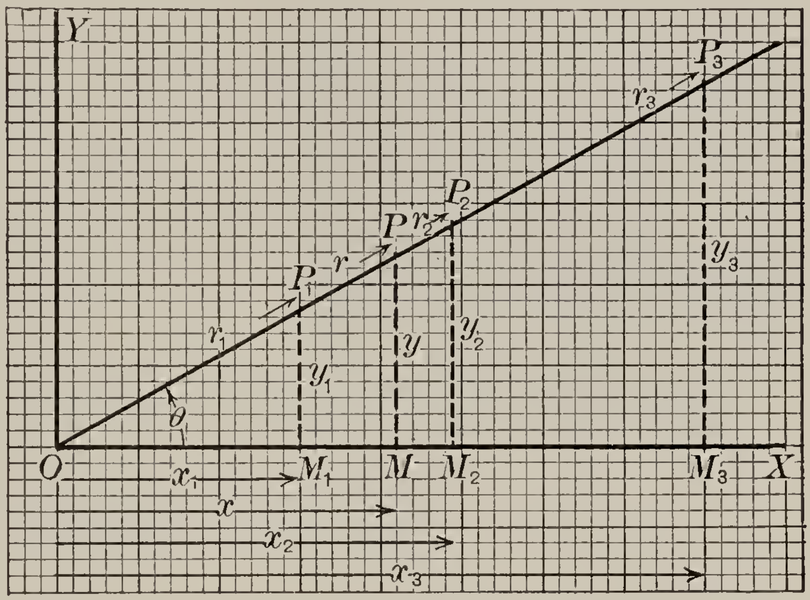
1. Locate the points (3, 30°), (6, 90°), (4, 45°), (8, 135°), (3, 270°), (6, – 90°), (5, 180°), and (2, 390°).

2. Locate the points $\left(5,\ \frac{\pi^r}{6}\right)$, $\left(4,\ \frac{\pi^r}{2}\right)$, (6, 0), $\left(3,\ -\frac{\pi^r}{4}\right)$, $(\sqrt{2},\ \pi^r)$, and $(3,\ 3\,\pi^r)$.

3. What is common to all points on OR?

4. What curve is represented by $r = 10$?

5. What curve is represented by $\theta = 30°$ or $\theta = \frac{\pi^r}{6}$?



$\theta = \angle XOP$; $\sin\theta = \frac{y_1}{r_1} = \frac{y_2}{r_2} = \frac{y_3}{r_3} = \frac{y}{r}$; $\cos\theta = \frac{x}{r}$.

4. Trigonometric functions — sine and cosine. — Assume an x-axis to coincide with the polar axis, and a y-axis to be drawn perpendicular to the polar axis at the pole. When θ is any fixed angle, the coördinates (x, y) in rectangular coördinates and (r, θ) in polar coördinates, of points upon the ray making the angle θ with OX, are connected by the following relations :

$$\frac{y_1}{r_1} = \frac{y_2}{r_2} = \frac{y_3}{r_3} = \frac{y}{r}, \text{ for points upon the ray,}$$

$$\text{and } \frac{x_1}{r_1} = \frac{x_2}{r_2} = \frac{x_3}{r_3} = \frac{x}{r},$$

$$\text{and } \frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{y_3}{x_3} = \frac{y}{x}.$$

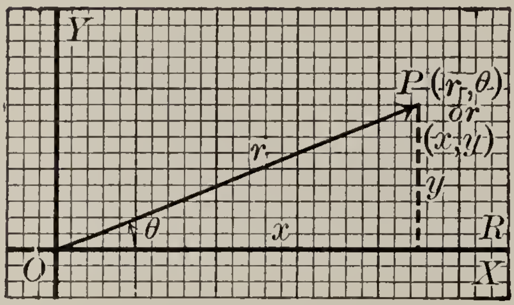
$$x^2 + y^2 = r^2, \text{ for any point } (x, y) \text{ in the plane.}$$

We may say that $\frac{y}{r}$ is a constant for any given angle θ; this constant changes as θ changes. It is evidently a function of θ. Since r remains positive, this function is positive for all angles θ represented in the upper quadrants; negative for angles in quadrants III and IV. This constant is $\frac{1}{2}$ for $\theta = 30°$ $\left(\text{or } \frac{\pi^r}{6}\right)$, $\frac{\sqrt{2}}{2}$ or .707 for $\theta = 45°$, $\frac{1}{2}\sqrt{3}$ or .866 for $\theta = 60°$, 1 for $\theta = 90°$, .866 for $\theta = 120°$, .707 for $\theta = 135°$, $\frac{1}{2}$ for $\theta = 150°$, and 0 for $\theta = 180°$, all by elementary geometry. When θ is an angle which lies in quadrant III or IV, *i.e.* values of θ between $+180°$

and $+360°$, this function of θ becomes negative. This function of θ is called the sine of θ, or $\sin\theta$.

$$\sin\theta = \frac{y}{r}.$$

Similarly the ratio $\frac{x}{r}$ is a constant, whose value depends entirelv upon the position of the moving ray; this function of θ we define as cosine θ.



$\sin\theta = \frac{y}{r}.$

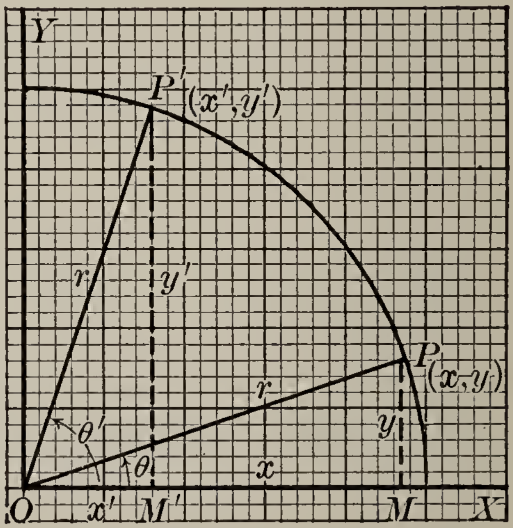
$\cos\theta = \frac{x}{r}.$

Polar coördinates, r and θ
Rectangular coördinates, x and y

$$\cos\theta = \frac{x}{r}; \qquad \sin\theta = \frac{y}{r}.$$

The consideration of the changes in value of these functions of θ, $\sin\theta$, and $\cos\theta$, as θ changes, is facilitated by thinking of the moving ray as fixed in length.

For positive values of θ less than 90°, θ in I or, in symbolic language, $0 < \theta < 90°$, it is evident that the complementary angle to any angle θ gives a triangle similar to the triangle involving θ. In this second triangle the ordinate and abscissa correspond respectively to the abscissa and ordinate in the original triangle, whence $\frac{y'}{r} = \frac{x}{r}$. Now $\frac{y'}{r} = \sin(90° - \theta)$, and $\frac{x}{r} = \cos\theta$; hence $\cos\theta = \sin(90° - \theta)$, or, in words, the cosine of any angle $\theta\,(0 < \theta < 90°)$ is the sine of the complement of θ. This explains the name, cosine θ, which is simply the "complement's sine."



Complementary angles,
$\theta + \theta' = 90°$

Further, $\frac{x'}{r} = \frac{y}{r}$, whence $\cos(90° - \theta) = \sin\theta$.

Either one of the triangles may be regarded as the original, the complementary angle will be found in the other; the demonstration, as given, applies in either case. The above figure serves, then, to demonstrate the two formulas, $\sin(90° - \theta) = \cos\theta$ and $\cos(90° - \theta) = \sin\theta$, for any positive acute angle θ. Later these formulas will be shown to hold for all angles θ, without restriction as to magnitude or sign.

The formula $\cos(90° - \theta) = \sin\theta$ may be derived from $\sin(90° - \theta) = \cos\theta$ by substituting for θ the value $90° - \theta'$, and finally replacing θ' by θ. Since θ may vary from 0 to 90°, $90° - \theta$ varies between the same limits.

$$\sin(90° - \theta) = \cos\theta,$$
$$\cos(90° - \theta) = \sin\theta.$$

5. Historical note. — The function $\sin\theta$ is Hindu in its origin, dating back probably to the fourth century A.D. The Hindus called the sine "ardha-jiva," meaning half-chord. In the eighth century A.D. the Arabs becoming familiar with Hindu astronomy and trigonometry, as used in astronomical work, transliterated the word "jiva" or "jiba" into "geib"; the word in Arabic means curve and in the twelfth century European translators into Latin of Arabic works of science translated this word as "sinus." Into English the word comes by transliteration again, the sound and not the sense being preserved.

Plane trigonometry is possible using the chords instead of the half-chords; this system was developed by the Greeks, but it leads to much more complicated formulas and methods.

6. Tangent and the reciprocal functions. — The quotient $\frac{\sin\theta}{\cos\theta}$ varies as θ varies; this is then a function of θ. This function is called the tangent. By definition,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \left(= \frac{y}{x}\right).$$

The reciprocals of sin θ, cos θ, and tan θ are also functions of θ; to these the names cosecant θ, secant θ, and cotangent θ have been given. The six fundamental definitions follow:

$$\sin\theta = \frac{y}{r}, \qquad \text{cosecant } \theta, \text{ or } \csc\theta = \frac{1}{\sin\theta},$$

$$\cos\theta = \frac{x}{r}, \qquad \text{secant } \theta, \text{ or } \sec\theta = \frac{1}{\cos\theta},$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}, \qquad \text{cotangent } \theta, \text{ or } \cot\theta = \frac{1}{\tan\theta}.$$

PROBLEMS

1. Given $\sin\theta = .29$, find $\cos\theta$ using the formula $\sin^2\theta + \cos^2\theta = 1$. The negative value has a meaning.

2. In what quadrants is sin θ positive? in what quadrants is cos θ positive?

3. Given $\sin\theta = .29$, in what quadrants may θ lie?

4. In what quadrants is tan θ positive?

5. As a rotating arm of length 10, moving about O from OX, turns through 90°, discuss the changes in value of the y of the end of the moving arm; consider r as 10 and discuss the change in value of sin θ as the angle generated increases from 0° to 90° to 180°. What change in sin θ as θ increases beyond 180°?

6. Discuss similarly the changes in values of cos θ as θ varies from 0° to 90°; from 90° to 180°.

7. Discuss the possible values of tan θ. Take $x = 1, \frac{1}{2}, .1, .01, .001$ and compute y in a circle of radius 10. Discuss the values of tan θ. When $x = .000001$, $y = 9.99999999999995$ what is the approximate value of tan θ?

8. Given $\tan\theta = 3$, find sec θ from the formula

$$\sec^2\theta = 1 + \tan^2\theta.$$

Compute both the positive and the negative values of cos θ.

9. Express in terms of the sine of the complementary angle: cos 48°, cos 84°, cos 56°, cos 48° 10′, cos 90°.

10. Express in terms of the cosine of the complementary angle sin 48°, sin 84°, sin 56°, sin 48° 10′, sin 90°.

11. Complete the following table:

$\cos 45^\circ = .7071 = \sin 45^\circ$
$\cos 46^\circ = .6947 = \sin 44^\circ$
$\cos 47^\circ = .6820 = \sin$
$\cos 48^\circ = .6691 = \sin$
$\cos 49^\circ = .6561 = \sin$
$\cos 50^\circ = .6428 = \sin$

Reverse the table, beginning $\sin 40^\circ =$
$\sin 41^\circ =$

12. Complete the following table:

$\sin 35^\circ = .5736 = \cos$
$\sin 35^\circ\, 10' = .5760 = \cos$
$\sin 35^\circ\, 20' = .5783 = \cos$
$\sin 35^\circ\, 30' = .5807 = \cos$
$\sin 35^\circ\, 40' = .5831 = \cos$
$\sin 35^\circ\, 50' = .5854 = \cos$
$\sin 36^\circ = .5878 = \cos$

Notice that the sines of 35° + some minutes are cosines of angles 54° + some minutes; the cosines of 35° + minutes are sines of the complements, 54° + minutes. In our tables you have written at the left of the table 35° and 54° at the right; sin at the top and cos at the bottom.

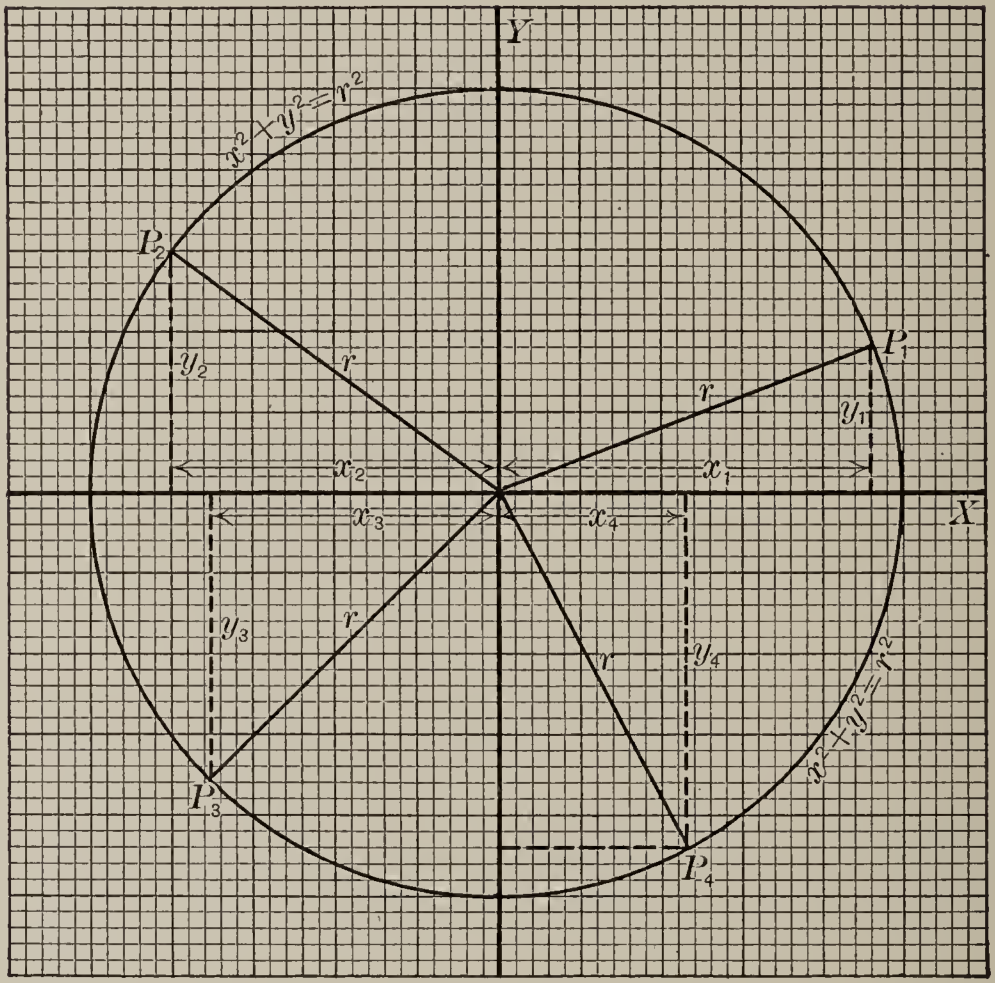
35°

′	sin	cos	
0	.5736	.8192	60
10	.5760	.8175	50
20	.5783	.8158	40
30	.5807	.8141	30
40	.5831	.8124	20
50	.5854	.8107	10
60	.5878	.8090	0
′	cos	sin	′

54°

7. Fundamental formulas. — Since $x^2 + y^2 = r^2$, for any point on this circle of radius r,

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1.$$



Note that although x or y or both may be negative, the relation continues to hold since $(-x)^2 = x^2$, and $(-y)^2 = y^2$. Whence, by substitution, $\cos^2\theta + \sin^2\theta = 1$, for all values of θ.

By division by $\cos^2\theta$,

$$1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta},$$

or $1 + \tan^2\theta = \sec^2\theta$, for all values of θ.

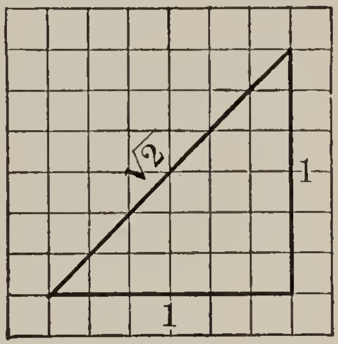
Similarly, $1 + \cot^2\theta = \csc^2\theta$.

$$\boldsymbol{\sin^2\theta + \cos^2\theta = 1.}$$

$$\boldsymbol{1 + \tan^2\theta = \sec^2\theta.}$$

These formulas are of fundamental importance. They should be memorized.

8. Functions of 0°, 30°, 45°, 60°, and related angles. — By plane geometry the values of these functions can be precisely determined for the angles which can be geometrically constructed with ruler and compass. The most important of these angles are 30°, 45°, 60°, and 72°; the values are evident for 0 and 90° (as limits).



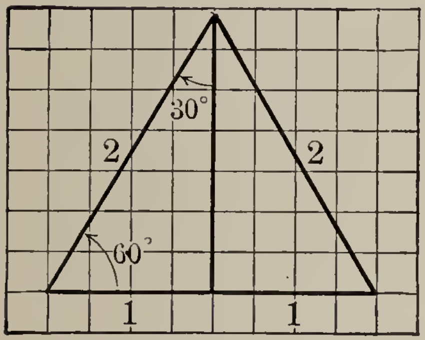
Functions of 45°

One half a unit square.

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = .707.$$

$$\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = .707.$$

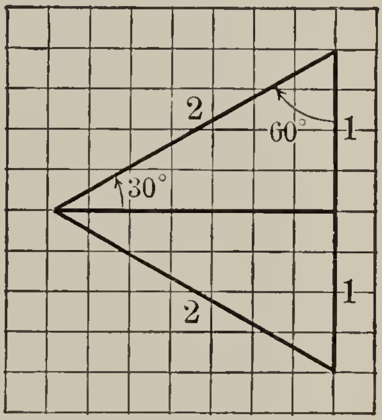
$$\tan 45° = 1 = \cot 45°.$$



Functions of 60°

Equilateral triangle.

$$\sin 60° = \frac{\sqrt{3}}{2}$$
$$= \cos 30°.$$
$$\cos 60° = \tfrac{1}{2}$$
$$= \sin 30°.$$
$$\tan 60° = \sqrt{3}$$
$$= \cot 30°.$$
$$\tan 30° = \frac{1}{\sqrt{3}}$$
$$= \cot 60°.$$



Functions of 30°

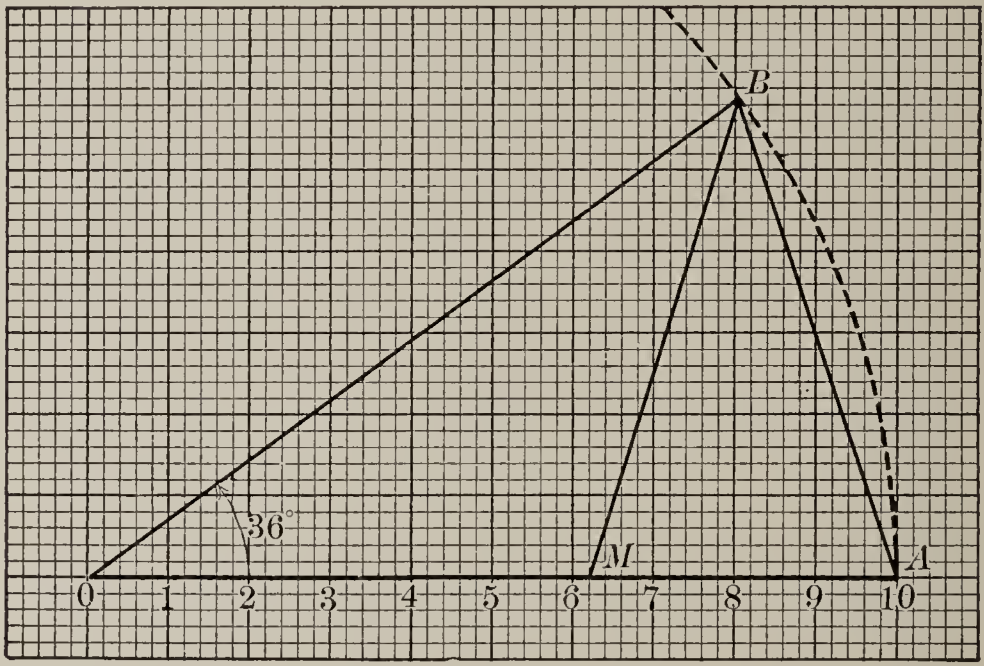
Equilateral triangle.

$$\frac{\sqrt{3}}{2} = .866. \qquad \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = .577.$$

These diagrams should be memorized as half of a unit square for 45°, and half of an equilateral triangle placed vertically for the functions of 60° and directly related angles, and the same placed horizontally for the functions of 30° and related angles (– 30°, 150°, 210°).

$$\sin 0° = 0. \qquad \sin 90° = 1.$$
$$\cos 0° = 1. \qquad \cos 90° = 0.$$
$$\tan 0° = 0. \qquad \tan 90° = \infty.$$

The meaning of the expression $\tan 90° = \infty$ (infinity) is that as the angle θ approaches nearer to 90° the tangent becomes larger than any quantity we may assign, however large; strictly at 90° the tangent function has no meaning, as a division by zero is involved. The expression $\tan 90° = \infty$ is not, then, an equality, like $\tan 60° = \sqrt{3}$.



Construction of the regular decagon

OM divides *OA* in " extreme and mean " ratio.
Algebraical method by solving, $x^2 = 10\,(10 - x)$.

The method of constructing a decagon combined with the solution of a quadratic equation enables us to find the sine of 18°. The radius of the circle is divided in extreme and mean ratio to obtain the side of the inscribed decagon: $10(10-x)=x^2$, in a circle of radius 10. Whence,

$$x^2 + 10\,x - 100 = 0,$$

$$x = -\,5 \pm \sqrt{125} = -\,5 \pm 11.1803 = -\,16.180 \text{ or } +\,6.1803,$$

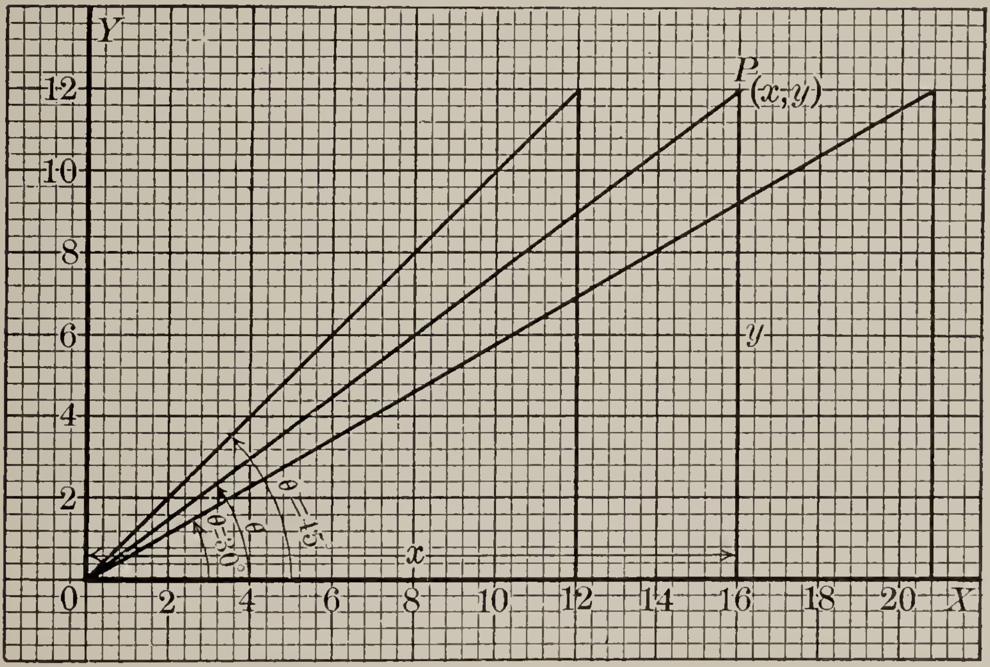
of which we take the positive value. One half of this value is the value of y in the triangle of reference for 18° when $r = 10$. Hence the sine of 18° is

$$\frac{3.090}{10} = .3090.$$

9. The Greek method, using chords. — By the methods of plane geometry, using chords instead of half-chords, the sine of half an angle and the sine of the sum and the difference of the two given angles can be computed. One theorem involved, in addition to the Pythagorean theorem, is not given in many geometries. It is called Ptolemy's theorem, as it is fundamental in the method of computing chords developed by Ptolemy, a Greek writer of the second century A.D., whose text-book on astronomy, the Almagest, continued in active use for fifteen hundred years. The theorem is that in an inscribed quadrilateral the product of the diagonals is equal to the sum of the products of the opposite sides.

From the chord of 60° one can compute the chord of 30°; thus the sine of 15° is obtained. From 36° and 30° the sine of 3° can be obtained by using half the chord of the difference of two given arcs; from this the sine of $1\frac{1}{2}^\circ$, $\frac{3}{4}^\circ$, $\frac{3}{8}^\circ$, $\frac{3}{16}^\circ$, $\frac{3}{32}^\circ$, $\frac{3}{64}^\circ$, $\frac{3}{128}^\circ$, $\frac{3}{256}^\circ$, $\cdots$ can be computed. The sine of 1° cannot be obtained by this process, nor can the sine of $\frac{1}{2}^\circ$; these are found by other methods, giving approximations as accurate as desired for any practical purposes.

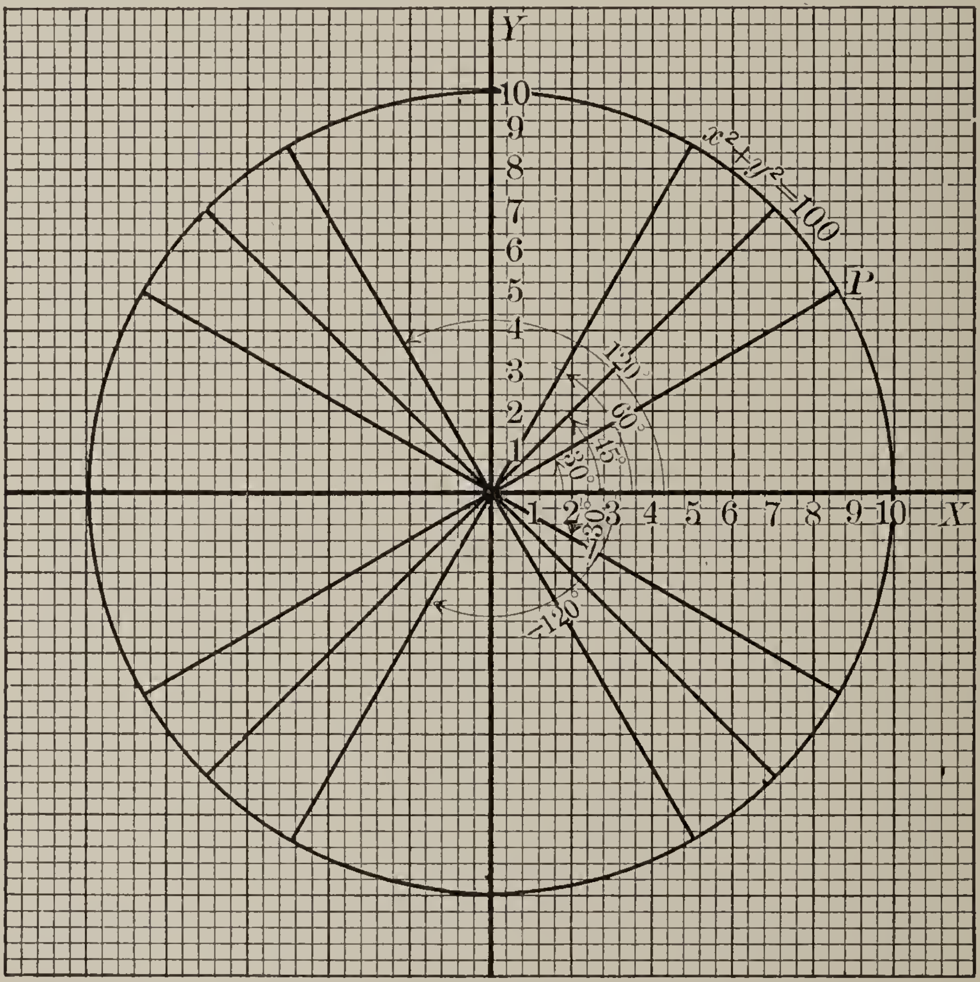
10. Origin of the tangent and cotangent functions. — In the study of astronomy the angle of inclination to the horizon of the sun and of other heavenly bodies is important. The ratio of the length of the shadow to the length of the vertical object



Arabic shadow function

The shadow varies as the cotangent of the angle of inclination of the sun.

casting the shadow gives the cotangent of the angle of inclination of the sun. This function of the angle appeared before the tangent function in the works of the Arabic astronomer, Al-Battani, of the tenth century A.D., and it was called the shadow and later, right shadow or second shadow. The tangent function, being the ratio of the length of the shadow cast on a vertical wall to the length of a stick placed horizontally out from the wall, was called later the first shadow. The Arabs took the length of the stick as 12.



Variation of sine and cosine as θ varies.

11. Variation. — As θ varies the trigonometric functions also vary; it is desirable to fix in mind the changes of the three principal functions, viz. $\sin\theta$, $\cos\theta$, and $\tan\theta$, as θ changes by rotation of the moving arm.

Taking $r = 10$, it is an easy matter to follow on the graph the changes in the x and y of the end of the moving ray. As the moving ray starts from OX, an angle of 0°, the y or ordinate is zero. So we have that the sine, $\frac{y}{r}$, begins at zero for 0°; as θ increases the y increases, reaching a maximum of 10 when θ is 90° and the maximum value then of $\sin\theta$ is $\frac{10}{10}$ or 1. As θ increases beyond 90°, the ordinate begins to decrease, arriving finally at 0 when the moving arm is on OX'. For angles greater than 180° up to 270° the ordinate decreases, finally reaching a minimum or lowest value of -10; the corresponding minimum of $\sin\theta$ is -1; from 270° on to 360°, completing a revolution, the sine *increases* from -1 up to 0.

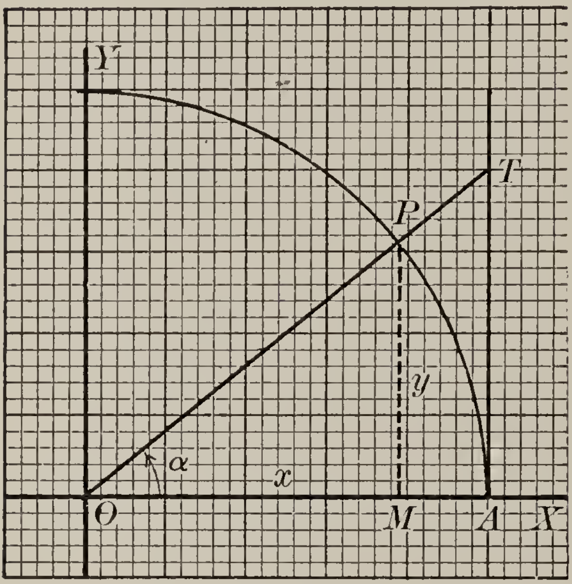
For angles greater than 360°, or for negative angles, the moving ray would move through no new positions; for any such angle the trigonometric functions are equal to the functions of the corresponding positive angle having the same position.

The limits $+1$ and -1 of the sine function and cosine function are evident, of course, in the figure. In any position of the moving ray x and y are the sides of a right triangle of which r is the hypotenuse, except that on the axes x or y equals r; hence the quotients $\frac{y}{r}$ and $\frac{x}{r}$ are either numerically less than 1 or at most equal to 1.

Note particularly on the diagram the sines of 30°, 45°, and 60°, as $\frac{5}{10}$, approximately $\frac{7.1}{10}$, and $\frac{8.7}{10}$; the values, .500, 0.707, and 0.866 may well be memorized. On the diagram it is a simple matter to read the sines of 10°, 20°, 30°, 40°, 50°, 60°, 70°, 80°, and 90° as the corresponding ordinates divided by 10, correct to two decimal places. The cosines of these angles are read as the corresponding abscissas divided by 10.

The tangent as $\frac{y}{x}$ is not in a form to give the numerical value without computation; however, by drawing the tangent line to the circle at A and producing r to cut the tangent line at

T, you have $\frac{AT}{OA}=\frac{y}{x}$; whence $\frac{AT}{10}=\tan\alpha$, and so the value of the tangent of the angle can be read as the ordinate at A divided by 10.



The tangent read as a length
$AT = 10 \tan\alpha$.

$$\tan 30° = \frac{1}{\sqrt{3}} = \frac{5.8}{10} = .58;$$

$$\tan 45° = \frac{10}{10} = 1;$$

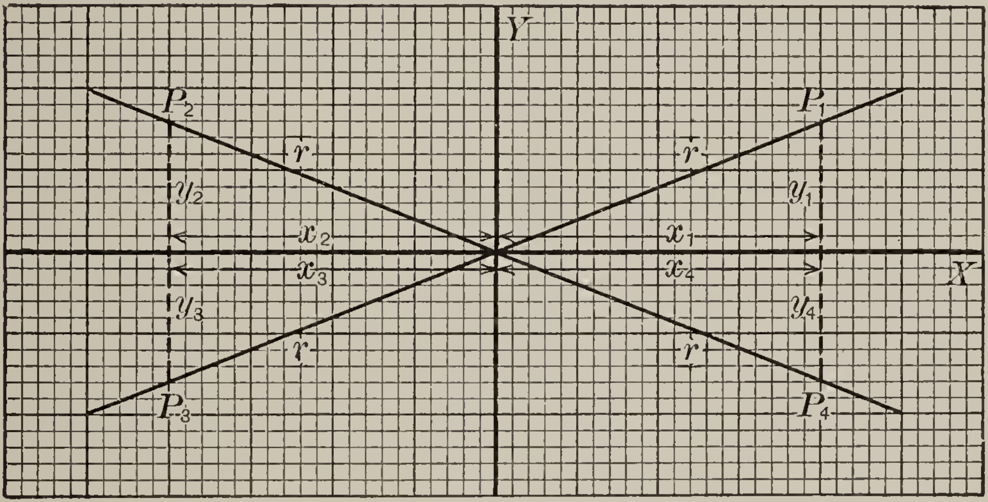
$$\tan 60° = \frac{\sqrt{3}}{1} = \frac{17.3}{10} = 1.73.$$

When the angle increases beyond 90° the position of the terminal arm fixes the sign of each function; the sine is positive when the arm is in the upper quadrants, I and II, and negative in the lower, and the cosine positive to the right, I and IV, and negative in II and III. The tangent is positive in I and III, and negative in II and IV; when positive the corresponding vertical lengths are cut off above A on the tangent at A, and when negative, in II and IV, the corresponding vertical lengths are cut off below A on the tangent.

If the radius is taken as unity, the ordinate, abscissa, and tangent length represent numerically and in algebraic sign the sine, cosine, and tangent values of the corresponding angle. However it is usually more convenient to take a radius of 10, 25, 50, or 100 and to interpret the trigonometric functions as ratios, as indeed they are.

12. Related angles. — From our definitions it is evident that $\sin\theta$ has the same value for two angles, symmetrically placed with reference to the y-axis, θ and $180° - \theta$; $\cos\theta$ has the same value for two angles symmetrically placed with respect to the x-axis, θ and $-\theta$, or θ and $360° - \theta$; $\tan\theta$ has the same value for two angles which differ by 180°, θ and $180° + \theta$. All functions are the same for angles which differ by 360°, or by any integral (positive or negative) multiple of 360°, for the terminal arms of such angles will coincide when the angles are placed in position to determine the trigonometric functions.

The trigonometric functions of $360° - \theta$, $180° - \theta$, $180° + \theta$, $90° + \theta$, $90° - \theta$, and $-\theta$, in terms of the functions of θ are of particular importance in later work. In the figure the vectors OP_1, OP_2, OP_3, and OP_4 are the terminal arms of related angles in quadrants I, II, III, and IV. The vector OP_1 determines, we may say, a positive acute angle XOP_1, and, further, any angles which differ from the positive acute angle by any integral multiple of 360°; OP_2 represents the terminal arm of $180° - \theta$, OP_3 of $180° + \theta$, and OP_4 of $360° - \theta$, or of $-\theta$.



θ, $-\theta$, $180° - \theta$, $180° + \theta$. Related angles

Read the corresponding functions on the diagram.

If θ is the angle represented in quadrant I, $180° - \theta$ is the angle here represented in II; and conversely, if θ is in II, $180° - \theta$ is in I; if θ is the angle in III, $180° - \theta$ is the angle in IV, and conversely. Evidently if θ in I is 30°, $180° - \theta$ is 150°, represented in II, and if $\theta = 150°$, $180° - \theta = 30°$; further, if θ is $-330°$ in I, differing from 30° by $-360°$, $180° - \theta$ will be $180° - (-330°)$ or 510° which is in II, $360° + 150°$, differing from $180° - 30°$ by 360°. The ordinates in I and II are equal and of the same sign, and similarly the ordinates in III and IV are algebraically equal; r is the same in all. Hence for *all* angles θ,

$$\sin (180^\circ - \theta) = \sin \theta.$$

The abscissas in I and II are numerically equal but opposite in sign, similarly in III and IV; the vectors are the same and positive.

Hence $$\cos (180^\circ - \theta) = - \cos \theta.$$

By definition, $\tan (180^\circ - \theta) = \frac{\sin (180^\circ - \theta)}{\cos (180^\circ - \theta)}$, for all angles.

By substitution,

$$\tan (180^\circ - \theta) = \frac{\sin (180^\circ - \theta)}{\cos (180^\circ - \theta)} = \frac{\sin \theta}{- \cos \theta} = - \tan \theta;$$

this formula also holds for all angles θ, since every formula involved has been shown to hold for all angles θ.

If θ is an angle in I, $180^\circ + \theta$ is in III; the corresponding positions are represented for any such angles by our figure. If θ is represented by the vector in II, $180^\circ + \theta$ is represented by the vector in IV. The ordinate in III equals numerically the ordinate in I, but is opposite in sign; similarly the abscissas of I and III; similarly the ordinates and abscissas, respectively, of II and IV are equal in value and opposite in sign.

Hence $$\sin (180^\circ + \theta) = - \sin \theta,$$

and $$\cos (180^\circ + \theta) = - \cos \theta.$$

These equalities hold for all angles θ.

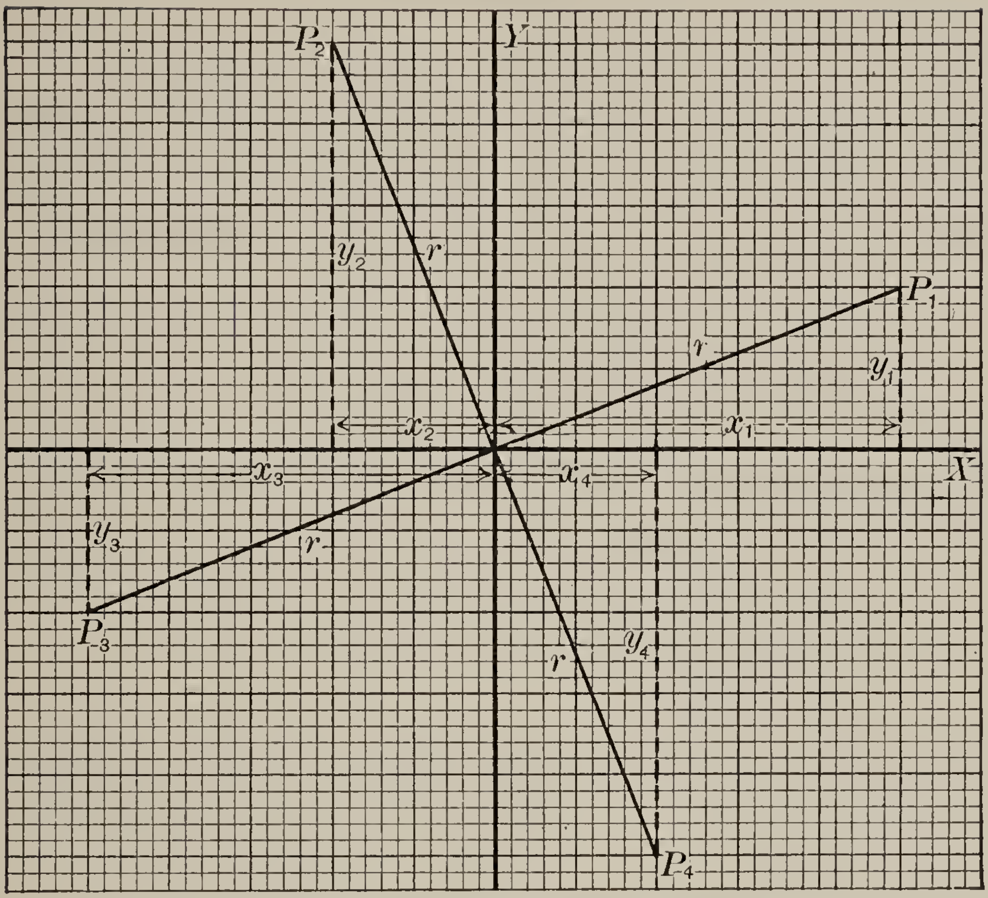
By definition, $\tan (180^\circ + \theta) = \frac{\sin(180^\circ + \theta)}{\cos(180^\circ + \theta)} = \frac{-\sin \theta}{-\cos \theta} = \tan \theta$, which holds for all angles θ.

In precisely the same way

$$\sin (- \theta) = - \sin \theta,$$

$$\cos (- \theta) = \cos \theta,$$

and $$\tan (- \theta) = - \tan \theta, \text{ for all values of } \theta.$$



θ, $90° + \theta$. Related angles which differ by 90°

Our second figure can be used to show that

$$\sin(90° + \theta) = \cos\theta,$$
$$\cos(90° + \theta) = -\sin\theta,$$

for all values of θ. If θ is the angle represented in I, $90° + \theta$ is the angle here represented in II; if θ is in II, $90° + \theta$ is represented in III; if θ is in any quadrant, $90° + \theta$ is in the quadrant following in the counter-clockwise sense. Now the ordinate in any quadrant here equals numerically and algebraically the preceding abscissa; thus $y_2 = x_1$; $y_3 = x_2$; $y_4 = x_3$; $y_1 = x_4$.

For any angle θ, **$\sin(90° + \theta) = \cos\theta$.**

Similarly $$\cos(90° + \theta) = -\sin\theta,$$

$$\tan(90° + \theta) = \frac{\sin(90° + \theta)}{\cos(90° + \theta)} = \frac{\cos\theta}{-\sin\theta}$$
$$= -\cot\theta.$$

The following relations have now been established for all angles θ:

$$\sin(180° - \theta) = \sin\theta, \qquad \sin(180° + \theta) = -\sin\theta,$$
$$\cos(180° - \theta) = -\cos\theta, \qquad \cos(180° + \theta) = -\cos\theta,$$
$$\tan(180° - \theta) = -\tan\theta, \qquad \tan(180° + \theta) = \tan\theta,$$

$$\sin(-\theta) = -\sin\theta,$$
$$\cos(-\theta) = \cos\theta,$$
$$\tan(-\theta) = -\tan\theta.$$

The formulas,

$$\sin(90° - \theta) = \cos\theta,$$
$$\cos(90° - \theta) = \sin\theta,$$

and

$$\tan(90° - \theta) = \cot\theta,$$

have been established for acute angles. However, the preceding formulas for $90° + \theta$ which we have established for all values of θ, positive and negative, can be used to prove that these formulas for $90° - \theta$ hold for all values of θ.

Thus $\sin(90° - \theta)$ can be considered as $\sin(90° + (-\theta))$, and as the formulas for $90° + \theta$ hold for all values of θ, we have:

$$\sin(90° + (-\theta)) = \cos(-\theta) = \cos\theta,$$

and

$$\cos(90° + (-\theta)) = -\sin(-\theta) = -(-\sin\theta) = +\sin\theta.$$

Hence

$$\sin(90° - \theta) = \cos\theta,$$

and

$$\cos(90° - \theta) = \sin\theta, \text{ for all values of } \theta.$$

Further

$$\tan(90° - \theta) = \frac{\sin(90° - \theta)}{\cos(90° - \theta)} = \frac{\cos\theta}{\sin\theta} = \cot\theta,$$

again for all values of θ.

The student will do well to remember the diagrams and to connect the formulas with these. It is necessary to recollect only the representation for an acute angle θ; it is more desirable to connect the formulas with the diagrams than merely to memorize the formulas.

PROBLEMS

1. Find the sine, cosine, and tangent of 210°. The triangle is the same as that used for the functions of 30°, but it is placed in III.

2. $\sin 150° =$; $\cos 150° =$; $\tan 150° =$

3. $\sin 315° =$; $\cos -45° =$; $\tan -45° =$

4. $\sin 225° =$; $\sin 495° =$; $\tan 750° =$

5. Express the following in terms of functions of positive angles less than 45°:

a. $\sin 170° =$ *b.* $\sin 130° =$

c. $\cos 170° =$ *d.* $\cos 130° =$

e. $\sin 220° =$ *f.* $\tan -40° =$

6. Express in terms of functions of x:

a. $\sin (x - 90°) =$

HINT. — Use first $\sin (-\theta) = -\sin \theta$.

b. $\sin (270° + x) =$

HINT. — Express first as

$(180° + \theta)$; *i.e.* $\sin (180° + \overline{90° + x}) = -\sin (90° + x) = \cdots$.

c. $\cos (x - 270°) =$ *d.* $\tan (360° - x) =$

7. Draw one quadrant of a circle of radius 10 half-inches; construct the angles of 30°, 45°, and 60° and read their values. Bisect the angle of 30° and so obtain the values of the functions of 15°. Make a table of values of sines, cosines, and tangents, advancing by 15°. Note that the chord of 30° may readily be computed; one half of this chord divided by the radius gives the sine of 15°. Find the sine of $7\frac{1}{2}°$ similarly. From the table of sines of acute angles from 0 to 90° by 15° intervals, give the sines of the related obtuse angles up to 180°.

8. Given $\tan \theta = 3$, find $\sec \theta$ and $\cos \theta$; what is the significance of the double sign in the answer?

9. Express in terms of functions of positive angles less than 45° :

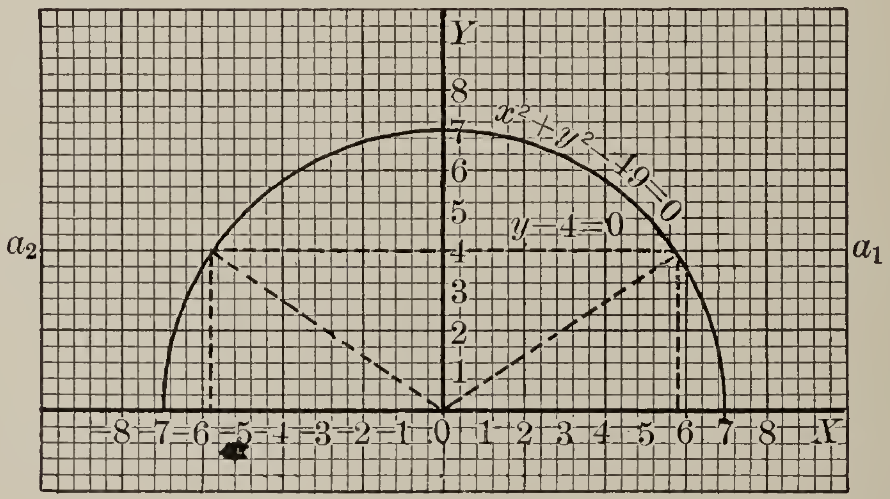
a. sin 100°.
b. cos 100°.
c. tan 100°.
d. sin 200°.
e. tan 200°.
f. sin 300°.
g. cos (− 60°).
h. cos (− 160°).
i. tan (− 420°).
j. sin 750° 50′.
k. cos 1030° 40′.
l. tan 218° 10′.

13. Angles constructed from given functions. — Given

$$\sin\theta = \tfrac{4}{7},$$

construct both values of θ (in I and II). The problem is in geometrical language to construct a right triangle with the hypotenuse and one side given, since $\sin\theta = \frac{y}{r}$.

Take r as 7 and y as 4; since $\sin\theta = \frac{y}{r}$, 7 is to be one side of our angle; with 7 as a radius and O as center describe a semicircle above the x-axis; $y = 4$ is a line parallel to the axis of x, cutting the circle, $x^2 + y^2 = 49$, in two points. Find the intersections a_1 and a_2; geometrically the angle is found. Using the Pythagorean theorem $x^2 + 4^2 = 7^2$, and $x^2 = 33$, $x = \pm 5.75$. The positive value of x is to the right and the negative to the



Graphical solution of $\sin\theta = \frac{4}{7}$

left; to the first corresponds the acute angle θ, and to the other θ with the terminal arm in II.

In I, $\sin\theta = \frac{4}{7} = .57.$

$$\cos\theta = \frac{5.75}{7} = .82.$$

$$\tan\theta = \frac{4}{5.75} = .70.$$

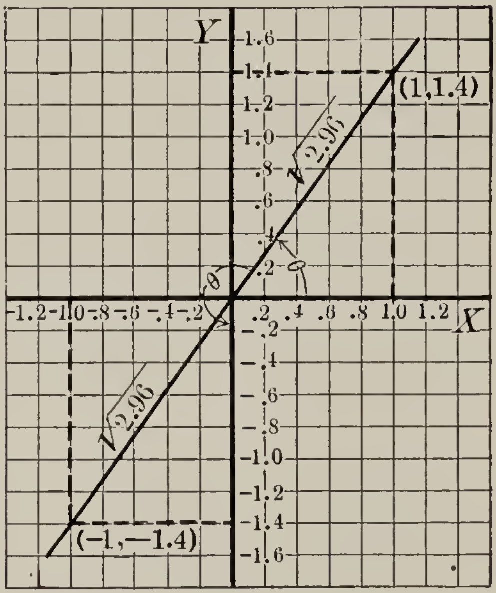
In II, $\sin\theta = \frac{4}{7}.$

$$\cos\theta = -\frac{5.75}{7} = -.82.$$

$$\tan\theta = -\frac{4}{5.75}$$
$$= -.70.$$

This problem should also be solved using the formula

$$\sin^2\theta + \cos^2\theta = 1.$$



Graphical solution of tan θ = 1.4

Given $\tan\theta = 1.4$, find the other functions of θ and discuss the two solutions.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}.$$

θ can be in I or III.

Take y as 1.4, x as 1 (or y as 14, $x = 10$, or other values as convenient); evidently x as -1 and y as -1.4 gives θ in III.

Since $x^2 + y^2 = r^2$, $r^2 = 1 + 1.96 = 2.96.$

$$r = 1.72^+.$$

In I, $\sin\theta = \frac{1.4}{1.72}$ or $\frac{1.4}{\sqrt{2.96}} = \frac{1.4}{2.96}\sqrt{2.96} = .81$ (or .814 to three places).

$$\cos\theta = \frac{1}{1.72} = .581^+.$$

In III, $\sin\theta = -.814,$

$$\cos\theta = -.581.$$

This problem should be solved also by using the formula

$$1 + \tan^2\theta = \sec^2\theta.$$

EXERCISES

1. Use 10 of the larger units on a sheet of cross-section paper and find by construction the sines of the angles 0°, 10°, 20°, 30°, ⋯ 180°. Compare with the table. Note that the values for 10° and 170°, 20° and 160°, 30° and 150° ⋯ correspond. Use the formula $\cos\theta = \sin(90° - \theta)$ to find the cosines of 0°, 10°, 20°, ⋯ 90°. Note that in the second quadrant the cosines become negative.

2. Construct an angle of 60°, using 10 as side of the equilateral triangle used. Find cos 60°, sin 60°, tan 60° to 2 places.

3. Find the sine of 150°, 210°, 330°.

Use half the equilateral triangle, placed horizontally with vertex of 30° angle placed at the origin.

4. Find the sine and cosine of 120°, 135°, 225°, $-30°$.

5. Find the tangent of 120°, 135°, 225°, $-30°$, from the data of the preceding problem; find tan 120°, tan 135°, tan 225°, $\tan(-30°)$ from the geometrical figure.

6. By construction of a square, side 10 units, find approximate values of the functions of 45°. Find the values using the Pythagorean theorem.

7. Construct an angle of 30°, and find values of the functions.

8. Construct angles of 15° and $7\frac{1}{2}°$, and find the values of the functions.

9. Given $\sin\theta = \frac{5}{12}$, find $\cos\theta$ and $\tan\theta$; indicate both solutions.

10. Given $\cos\theta = .432$, compute $\sin\theta$ and $\tan\theta$ to 3 places, θ in I.

11. Given $\tan\theta = 4.32$, compute $\sin\theta$ and $\cos\theta$ for θ in III.

12. Given $\tan\theta = \frac{7}{4}$, construct θ geometrically.

13. Construct θ geometrically, given $\sin\theta = \frac{7}{12}$.

14. Given $\sin\theta = -\frac{5}{12}$, find $\cos\theta$, θ in IV.

15. Given $\sin\theta = .43$, find $\cos\theta$, θ in III.

16. Given $\tan \theta = -.43$, construct θ in II, and find values of $\sin \theta$ and $\cos \theta$ from the figure.

14. The inverse functions. — If we are given the sine s of an angle and desire to speak of the angle we can say "the angle whose sine is s" and we can abbreviate this expression in writing to arc sin s or to $\sin^{-1} s$. Similarly the angle whose cosine is m is written arc cos m, or $\cos^{-1} m$. Note that in $\sin^{-1} s$, $\cos^{-1} m$, and $\tan^{-1} k$, the -1 is not at all a negative exponent; these expressions for angles are read anti-sine s, anti-cosine m, and anti-tangent k, or sometimes, inverse sine s, etc., respectively.

In what follows we shall use mainly the symbols arc sin, arc cos, arc tan, arc csc, arc sec, and arc cot, although the other symbols are also in common use. Whether the "arc" or "-1" symbols are used the student is strongly advised to read "arc tan t" or "$\tan^{-1} t$" always as "the *angle* whose tangent is t," and similarly expressions like arc cos x, arc sin $\frac{1}{2}$, and $\sin^{-1} k$.

A given angle has only one sine, but a given number is the sine of many different angles. A similar remark applies to the other five functions. To illustrate: sin 30° is 0.5 and no other value. But arc sin 0.5, the angle whose sine is 0.5, may be 30° or $-330°$ or 390° or 750° or 150° or 510° or 870° or any angle differing from 30° or 150° by an integral multiple of 360°. The sine of any one of these various angles is 0.5;

$$\sin (k\,360° + 30°) = .5 \text{ and } \sin (k\,360° + 150°) = .5,$$

where k is any integer.

PROBLEMS

1. Given arc $\cos \frac{1}{3} = \theta$, construct θ both in the first and in the fourth quadrant. Note that the problem is precisely the same as though the requirement were to construct θ when given that $\cos \theta = \frac{1}{3}$; or to construct arc $\cos \frac{1}{3}$.

2. Between what values must k lie to have any solution for $\theta = \text{arc} \cos k$?

3. Given that the angle, arc sin $\frac{1}{3}$, is obtuse, construct the angle.

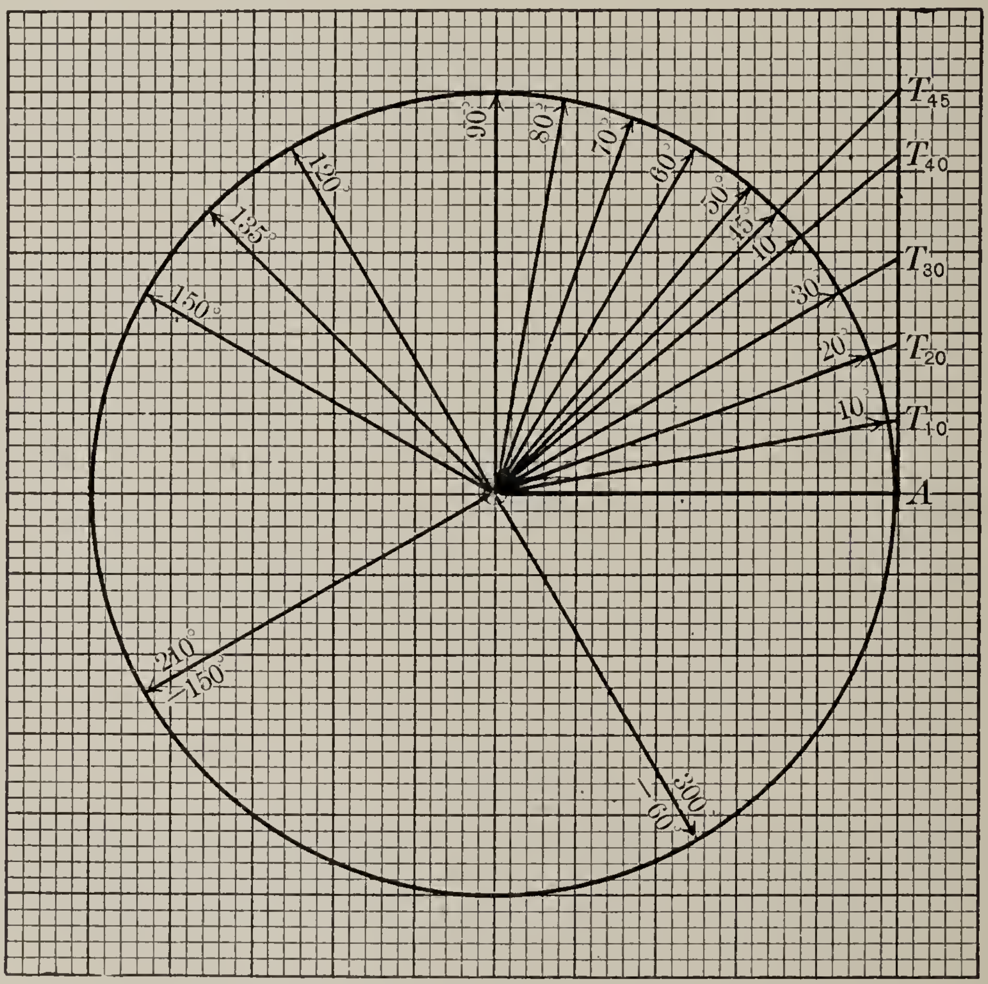
4. Construct the following angles of the first quadrant: arc sin $\frac{3}{7}$, arc tan $(+2)$, arc cos $\frac{7}{12}$, arc sin .43. Give the approximate value of the other two principal functions in each case.

5. Give five solutions of each of the following:
arc cos $\frac{1}{2} =$; arc tan $1 =$; arc sin $0 =$; arc cos $1 =$.

6. If arc sin $.438 = 26°$, what is arc cos .438?

7. What is the value of arc sin $-.438$? Give four answers. Give the general formula representing angles θ which satisfy $\theta = \text{arc sin}\,(-.438)$. What is arc cos $-.438$?

8. On the following diagram, regarding the circle as having a radius of 100, read the numerical value to two decimal places, of the sine, cosine, and tangent of each angle represented. Each minor division represents 4 units.



CHAPTER VIII

TABLES AND APPLICATIONS

1. Tables. — The tables of the trigonometric functions are computed by processes dependent upon formulas derived in the higher mathematics. We have shown the graphical method of finding sine, cosine, and tangent, which serves also to bring out the fact that the sines of angles from 0 to 45° are at the same time cosines of the complementary angles; similarly since $\tan(90° - x) = \cot x$, it follows that the tangents of angles from 0 to 45° are cotangents of the complementary angles, from 90° down to 45°. Since tables are given of both sine and cosine it is necessary to give values of both functions only up to 45°, and similarly with tangent and cotangent. Thus $\sin 26° 10'$ is found in the table of sines which reads down with 26° at the left, and below 10′ as given at the top; if the $\cos 63° 50'$ is sought we look for 63° at the right of the table of sines with the minutes to be read below; and we find that the cosine table is the same as the table of sines, but reading up; this brings us to precisely the same place in the tables as $\sin 26° 10'$, the complementary angle; similarly $\sin 63° 50'$ is sought in the row marked 63 at the left of the table and leads to the value which read as a cosine represents $\cos 26° 10'$.

For angles greater than 90° the formulas which we have given for related angles are applied. Probably the simplest formulas to apply to obtain the functions of obtuse angles are the formulas,

$$\sin(90° + x) = \cos x,$$

and

$$\cos(90° + x) = -\sin x.$$

Thus
$$\sin 128^\circ\ 35' = \cos 38^\circ\ 35';$$
$$\cos 128^\circ\ 35' = -\sin 38^\circ\ 35'.$$

It is well to note that subtracting 90° from angles greater than 100° and less than 200° simply increases the tens' digit of the angular measure by one, dropping the hundreds' digit.

The formulas for $180^\circ + \alpha$, and for $360^\circ - \alpha$ or $-\alpha$, are used for angles in III or IV.

Since computation is largely effected by means of logarithms, it becomes desirable to have separate tables of the logarithms of the trigonometric functions. The sines and cosines of all angles are numerically less than 1 and so are tangents of angles less than 45°; hence the logarithms of these numbers will have negative characteristics. In the logarithms of the trigonometric functions, -10 is to be annexed to the logarithm as given in the table for sines, cosines, and tangents up to 45°. Thus $\log \sin 30^\circ$ is $9.6990 - 10$; $\log \sin 56^\circ\ 10' = 9.9194 - 10$; $\log \tan 34^\circ\ 10' = 9.8317 - 10$; but $\log \tan 56^\circ\ 10' = .1737$.

2. Interpolation. — The insertion, by interpolation, of the natural and logarithmic functions of angles lying between those expressly given in the tables follows precisely the same lines as in the corresponding problem in the logarithms of numbers. Our tables give these functions for angles increasing by multiples of 10 minutes; interpolation enables us to compute the functions of angles to minutes; in using tables giving the functions to minutes interpolation enables us to compute to tenths of a minute. Note that the assumption is always that if angles are read to minutes you compute only to minutes; the tables used should correspond to the precision of measurement of the given data. Four-place tables are, in general, sufficiently accurate for measurements which are made to four places in numbers, and to minutes in angular measurement.

Illustrative problems. 1. Find by interpolation (*a*) $\sin 36^\circ\ 15'$, (*b*) $\log \sin 36^\circ\ 16'$, (*c*) $\log \cos 36^\circ\ 18'$, and (*d*) $\log \tan 36^\circ\ 14'$.

TABULAR VALUES — *Compare with your tables*

angle	sin	log sin	cos	log cos	log tan	
36° 10′	.5901	9.7710	.8073	9.9070	9.8639	53° 50′
36° 20′	.5925	9.7727	.8056	9.9061	9.8666	53° 40′
	cos	log cos	sin	log sin	log cot	angle

The values to four decimal places of the functions of angles between 36° 10′ and 36° 20′ evidently lie between the values which are here given. Thus sin 36° 10′ is .5901 and sin 36° 20′ is .5925, an increase of 24 units of the fourth place; this 24 is called the tabular difference. To the ten equal steps of increase from 36° 10′ to 36° 20′, by minutes, correspond ten increases approximately equal to each other, in the sines of these angles, making a total increase in ten steps of 24 units of the fourth place. The tenths of 24 are respectively,

.1	.2	.3	.4	.5	.6	.7	.8	.9
2.4	4.8	7.2	9.6	12	14.4	16.8	19.2	21.6

In adding, as our logarithms are given only to four places, we add rejecting tenths, and retaining in the last place the nearest unit. Thus for tenths of 24 we use always 2, 5, 7, 10, 12, 14, 17, 19, and 22. The interpolation does not always give the correct result to four places, although in the values of the sine the error is always less than 1 unit of the fourth place. In the above values of sin 36° 11′ to 36° 19′ as given by the addition 2, 5, 7, 10, 12, 14, 17, 19, and 22 units of the fourth place to .5901 the first value .5903 should be, to 4 places, .5904; the error is less here than $\frac{1}{50}$ of 1 % of the value taken.

1. *a.* $\sin 36° 15' = .5901 + \frac{5}{10}$ of $.0024 = .5913$.

Method: Tabular difference is 24; to .5701 add .5 of 24 units of the fourth place.

b. $\log \sin 36° 16' = 9.7710 - 10 + \frac{6}{10}(.0017) = 9.7720 — 10.$

Tabular difference is 17; 10.2 is replaced by 10, and this is added in the third and fourth decimal places to 9.7710.

c. $\log \cos 36° 18' = 9.9070 - 10 - \frac{8}{10}(.0009) = 9.9063 - 10.$

Tabular difference is 9; cosine and log cosine are decreasing functions; 7 units of the fourth place must be subtracted.

d. $\log \tan 36° 14' = 9.8639 - 10 + \frac{4}{10}(.0027) = 9.8650 - 10.$

Tabular difference is 27; 10.8 is replaced by 11.

2. Find to minutes, by interpolating, the angle when given (*a*) $\sin \alpha = .5919$, (*b*) $\log \sin \alpha = 9.7717$, and (*c*) $\log \cot \alpha = 9.8650$; find α in each case.

3. (*a*) $\sin \alpha = .5919$; tabular difference is 24; given difference .5901 to .5919 is 18 units of the fourth place. Among the tenths of 24 find the nearest to 18; 16.8 and 19.2, respectively .7 and .8 of 24, are equally near and the even number of tenths is commonly taken, in such cases, by computers.

$$\sin \alpha = .5919; \qquad \alpha = 36° 18'.$$

(*b*) $\log \sin \alpha = 9.7717$; $\alpha = 36° 10' + \frac{7}{17}$ of $10'$ (to minutes).

$$\alpha = 36° 14'.$$

Tabular difference is 17; 7 is nearest to .4 of 17.

(*c*) $\log \cot \alpha = 9.8650$; $\alpha = 53° 40' + \frac{16}{27}$ of $10'$.

Tabular difference is 27, a decrease; given decrease is 16; among the tenths of 27 the nearest to 16 is 6; hence $\alpha = 53° 46'$. Had $\log \cot \alpha$ been given as $9.8651 - 10$ or $9.8649 - 10$, the angle α would again be given as $53° 46'$.

PROBLEMS

1. Find the 20 natural trigonometric functions following, without interpolation; time yourself; limit 6 minutes.

a. $\sin 36° 10'$.
b. $\tan 63° 20'$.
c. $\cos 34° 10'$.
d. $\cot 80° 00'$.
e. $\sin 59° 30'$.
f. $\cos 48° 50'$.
g. $\tan 70° 30'$.
h. $\sin 28° 50'$.
i. $\tan 16° 20'$.
j. $\cos 8° 40'$.
k. $\sin 157° 10'$.
l. $\cos 214° 10'$.

m. cot 141° 00′.
n. tan 329° 30′.
o. cos 136° 50′.
p. cos − 28° 10′.
q. tan − 64° 20′.
r. sin 384° 00′.
s. cot 756° 00′.
t. sin 242° 40′.

2. Find the logarithms of the above 20 trigonometric functions, timing yourself. Limit 7 minutes.

3. Find the following 20 logarithms, interpolating; time yourself. Limit 12 minutes.

a. log sin 36° 14′.
b. log tan 63° 29′.
c. log cos 34° 14′.
d. log cot 80° 06′.
e. log sin 59° 32′.
f. log cos 48° 57′.
g. log tan 70° 33′.
h. log sin 28° 51′.
i. log tan 16° 22′.
j. log cos 8° 48′.
k. log sin 152° 15′.
l. log cos 214° 26′.
m. log cot 141° 05′.
n. log tan 329° 33′.
o. log cos 136° 57′.
p. log cos − 28° 11′.
q. log tan − 64° 26′
r. log sin 384° 03′.
s. log cot 756° 08′.
t. log sin 242° 44′.

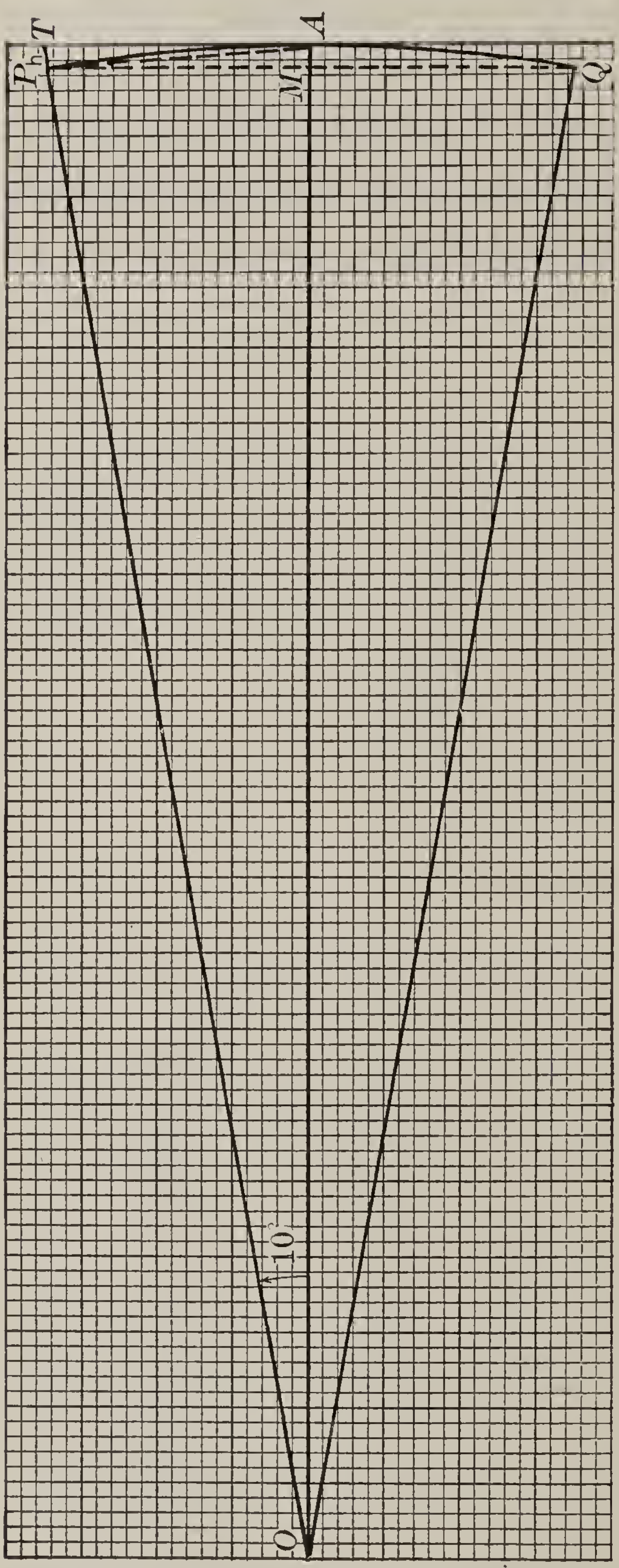
4. Find the angles less than 90° corresponding to the following 20 logarithms; no interpolation; time 6 minutes.

a. $\log \sin \alpha = 9.6878 - 10$
b. $\log \cos \alpha = 9.9954 - 10$
c. $\log \tan \alpha = 9.4898 - 10$
d. $\log \cot \alpha = .5102$
e. $\log \cos \alpha = 9.8241 - 10$
f. $\log \tan \alpha = 9.7873 - 10$
g. $\log \sin \alpha = 9.3179 - 10$
h. $\log \tan \alpha = .2155$
i. $\log \cos \alpha = 8.9816 - 10$
j. $\log \cot \alpha = 9.9341 - 10$
k. $\log \sin \alpha = 9.9499 - 10$
l. $\log \cos \alpha = 9.8081 - 10$
m. $\log \cot \alpha = .8904$
n. $\log \tan \alpha = 8.9420 - 10$
o. $\log \cos \alpha = 9.9640 - 10$
p. $\log \cos \alpha = 9.9757 - 10$
q. $\log \tan \alpha = .5720$
r. $\log \sin \alpha = 8.9403 - 10$
s. $\log \cot \alpha = .0152$
t. $\log \sin \alpha = 9.9977 - 10$

5. Give in each case another angle which would satisfy the above relationship, in problem 4; *e.g.* if $\log \sin \alpha = 9.6990 - 10$, $\alpha = 30°$ or $150°$.

6. Find the following 20 angles; interpolate; time yourself.

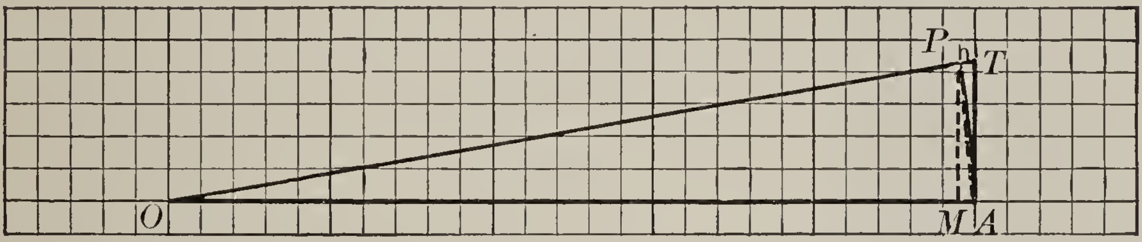
a. $\log \sin \alpha = 9.6881 - 10$
b. $\log \cos \alpha = 9.9955 - 10$
c. $\log \tan \alpha = 9.4861 - 10$
d. $\log \cot \alpha = .5104$
e. $\log \cos \alpha = 9.8228 - 10$
f. $\log \tan \alpha = 9.7879 - 10$
g. $\log \sin \alpha = 9.3200 - 10$
h. $\log \tan \alpha = .2144$
i. $\log \cos \alpha = 8.9912 - 10$
j. $\log \cot \alpha = 9.9358 - 10$
k. $\log \sin \alpha = 9.9502 - 10$
l. $\log \cos \alpha = 9.8092 - 10$
m. $\log \cot \alpha = .8955$
n. $\log \tan \alpha = 8.9492 - 10$
o. $\log \cos \alpha = 9.9645 - 10$
p. $\log \cos \alpha = 9.9753 - 10$
q. $\log \tan \alpha = .5699$
r. $\log \sin \alpha = 8.9404 - 10$
s. $\log \cot \alpha = .0137$
t. $\log \sin \alpha = 9.9978 - 10$



Angle 10° in a circle of radius 5 inches
$PM = .868$ in.; arc $PA = .873$ in.;
$AT = .882$ in.

3. Angles near 0° and 90°. — For angles near zero, from 0° to 2°, the cosines vary only from 1.0000 to .9994; the cosine function to 4 places cannot then be used for determination of the angle to minutes. Similarly, of course, the sines of angles from 88° to 90° vary between the same limits. For ordinary purposes it will suffice to avoid the use of the cosine in the interval from 0° to 2° or 3° or 4°; the method of avoidance is explained below.

In computing graphically the values of $\sin\theta$ and $\tan\theta$ even with a radius of 10 cm., or of 5 inches, the difference between $\tan\theta$ and $\sin\theta$ becomes too small to read accurately when θ is less than $\frac{\pi^r}{24}$ (*i.e.* $7\frac{1}{2}°$; $.131^r$). For 10° which is .1745 radian, $\sin .1745^r$ is .1736 and $\tan\theta$ is .1763; for 5° or $.0873^r$, $\sin\theta = .0872$ and $\tan\theta$ is $.0875^r$; for 1° or $.01745^r$, $\sin\theta = .01745$ and $\tan\theta$ is .01746 or 5 places are necessary to exhibit any difference between θ, $\sin\theta$, and $\tan\theta$.



$\sin\theta < \theta < \tan\theta$

for small acute angles θ, θ is measured in radians

Evidently, triangular area $OAP <$ sector $OAP <$ area OAT, but the area of the triangle

$$\begin{aligned} OAP &= \tfrac{1}{2}\, OA \times MP \\ &= \tfrac{1}{2}\, r \times r\sin\theta \\ &= \tfrac{1}{2}\, r^2 \sin\theta. \end{aligned}$$

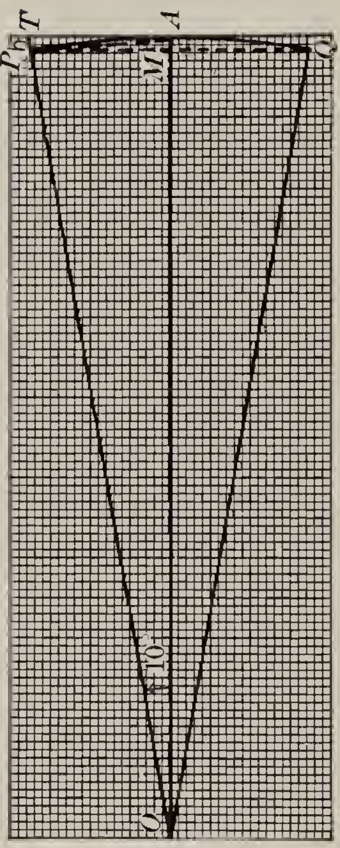
The area of the sector $OAP = \frac{1}{2}r^2\theta$, since θ is measured in radians, and the area $OAT = \frac{1}{2}r^2\tan\theta$.

Whence, by substituting,

$$\tfrac{1}{2}r^2\sin\theta < \tfrac{1}{2}r^2\theta < \tfrac{1}{2}r^2\tan\theta.$$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Whence, as θ diminishes, $\frac{\theta}{\sin\theta}$, lying between 1 and a number approaching 1, can be made as near to 1 as we please. By methods of plane geometry, using 30°, 15°, $7\frac{1}{2}°$, $3\frac{3}{4}°$, together with 72°, 60°, 12°, 6°, and 3° it can be established that $\cos \frac{3}{4}°$ differs from 1 by less than $\frac{1}{100}$ of 1 %; $\cos \frac{3}{4}° = .99991$; for any angle θ, less than $\frac{3}{4}°$, θ will exceed $\sin\theta$ by less than $\frac{1}{100}$ of 1 % and $\tan\theta$ will exceed θ by less than $\frac{1}{100}$ of 1 %. Similarly the discrepancy between $\sin\theta$ and $\tan\theta$ for θ, any angle less than $3\frac{3}{4}°$, is less than $\frac{1}{4}$ of 1 %, and for any angle up to 8° the difference is less than 1 % of either value.



Arc *PAQ* on the earth's surface

TA, horizon distance.

PT, height of observer.

On the earth's surface ordinary distances are regarded as straight lines. However for many purposes the deviation from a straight line is of importance; thus particularly with projectiles of long range, the deviation is of vital importance. In the figure given if *PA* represents an arc on the earth's surface, *PT* may be regarded as the altitude of a balloon, aëroplane, or top of a mountain, and *TA* gives the distance of the horizon. $\angle\, TOA$ is equal to the dip of the horizon. *AM* is the drop in the distance twice *PA*, *i.e.* from *T* an observer would note, on the ocean, the complete disappearance of a ship of height *AM* when the ship is at *Q*. By algebraic process, $AT = \sqrt{2\,rh + h^2}$; when h is measured in feet, r in miles, and *TA* in miles, this gives for values of h less than 15 miles, $AT = \sqrt{\frac{3\,h}{2}}$, correct to $\frac{1}{5}$ of 1 %. Check using 3960 miles as r.

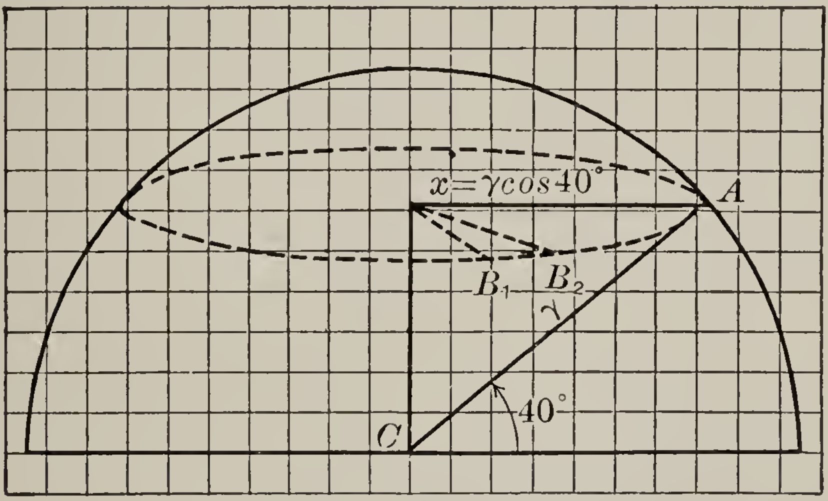
PROBLEMS

1. Given that an observer is at a height of 1000 feet, compute the distance to the horizon, $r = 3960$ miles. What is the dip of the horizon? Note that the tangent of the dip-angle is the horizon distance divided by the radius.

2. Find the angle subtended at the center of the earth by an arc of length 1 mile, 10 miles, 20 miles.

3. What is 1° of latitude in miles?

4. Degrees of longitude vary in length from degrees on a great circle of the earth at the equator to 0 at the poles. Find the radius of the small circles on which degrees of longitude are measured, for 40° north latitude. Where else on the earth's surface would degrees of longitude be the same? From 35° to 45° N. latitude discuss the percentage variation in degrees of longitude, as compared with degrees of longitude at 40° N. latitude.



Circle of 40° N. latitude

5. How far below the arc of 1 mile on the earth does the corresponding chord fall at the lowest point? Find the same distance in inches for arcs of 2 miles, 8 miles, 10 miles, 16 miles, 20 miles.

6. What part of the height of a mountain, measured on the altitude, is not visible from a point 20 miles distant?

7. From what distance can the top of a mountain 10,000 feet high be seen?

8. What distance from shore is a ship whose masts, 55 feet high, are just disappearing from view?

9. Using the figure in the text, find an approximation for TA in miles when h is small and measured in feet.

$$AT=\sqrt{\frac{2\,h\cdot r}{5280}+\left(\frac{h}{5280}\right)^2}=\sqrt{\frac{2\cdot h\cdot 60}{80}+\left(\frac{h}{5280}\right)^2}$$

$$=\sqrt{\frac{6}{4}h+\left(\frac{h}{5280}\right)^2}=\sqrt{1.5\,h}\,;$$

for values of h less than 5×5280, the $\left(\frac{h}{5280}\right)^2$ can be neglected.

10. Find the "dip" of the horizon and the distance from the balloon for $h=100$, 500, and 1000 feet.

11. Find the distance from the point below the balloon on the earth's surface to the points on the horizon viewed by the observer in the balloon.

12. According to the approximate formula of Huyghens the length of a circular arc, a, is connected with the chord, c, of the arc and the chord, h, of half the arc, by the formula $a=\frac{8\,h-c}{3}$. Compute the actual length.

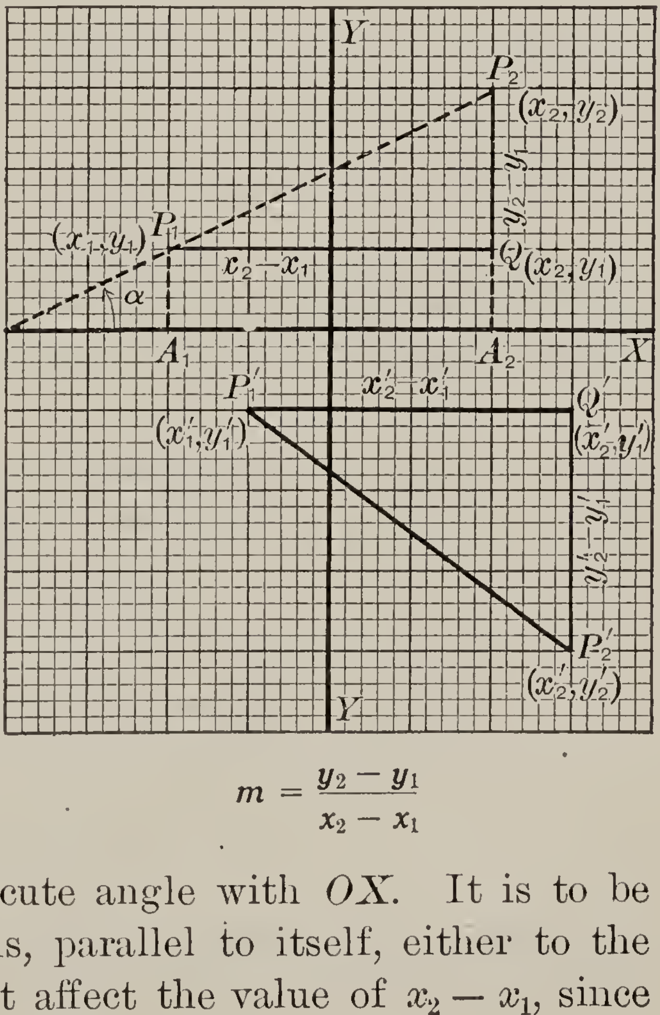
CHAPTER IX

APPLICATIONS OF TRIGONOMETRIC FUNCTIONS

1. Parallel and perpendicular lines. — The slope of the line joining (x_1, y_1) to (x_2, y_2),

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{QP_2}{P_1Q},$$

evidently represents the tangent of the angle which the line joining these two points makes with the positive ray of the x-axis, *i.e.* the angle from the x-axis to this line. We have taken $P_2(x_2, y_2)$ to the right of $P_1(x_1, y_1)$, but obviously interchanging $P_2(x_2, y_2)$ and $P_1(x_1, y_1)$ simply changes sign of both numerator and denominator of the fraction representing the slope m; Q is in each figure the point (x_2, y_1), and P_1Q and QP_2 have like signs if P_1P_2 or P_2P_1 makes a positive acute angle with the positive ray of the x-axis; and P_1Q and QP_2 have unlike signs in the contrary case when P_1P_2 or P_2P_1 makes a negative acute angle with OX. It is to be noted that shifting the y-axis, parallel to itself, either to the right or to the left does not affect the value of $x_2 - x_1$, since



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

whatever the position of O, $A_1A_2 = P_1Q = OA_2 - OA_1 = x_2 - x_1$; similarly no change is made in the value of the slope by shifting the x-axis parallel to itself, up or down.

Given $y = m_1x + k$, any straight line, m_1 represents the tangent of the angle which this line makes with the positive ray of the x-axis. Any parallel line has the same slope; $m_2 = m_1$ for two parallel lines. Any perpendicular line has the slope angle

$$\alpha_2 = 90° + \alpha_1;\ \tan\alpha_2 = \tan(90° + \alpha_1) = -\cot\alpha_1 = \frac{-1}{\tan\alpha_1},$$

whence $m_2 = -\dfrac{1}{m_1}$. Of two parallel lines the slopes are equal, and of two perpendicular lines the slope of the one is the negative reciprocal of the slope of the other, *i.e.*

$$m_2 = -\frac{1}{m_1} \text{ or, by solving, } m_1 = -\frac{1}{m_2}.$$

Given $y = mx + b$, any family of parallel lines of slope m. $y = -\dfrac{1}{m}x + k$ represents the family of perpendicular lines.

Illustrative problem. — Given $3x + 4y - 7 = 0$, find the slope, the parallel line through the origin, the family of perpendicular lines, and the perpendicular line through $(-1, 5)$.

$$4y = -3x + 7.$$

$$y = -\tfrac{3}{4}x + \tfrac{7}{4},\ m = \tan\theta = -\tfrac{3}{4},\ \theta = -36° 52'.$$

$y = -\tfrac{3}{4}x$ is the parallel line through the origin.

Derive this both from $y = mx + b$ and $y - y_1 = m(x - x_1)$.

The perpendicular line has the slope, $m_2 = -\dfrac{1}{m_1} = +\dfrac{4}{3}$.

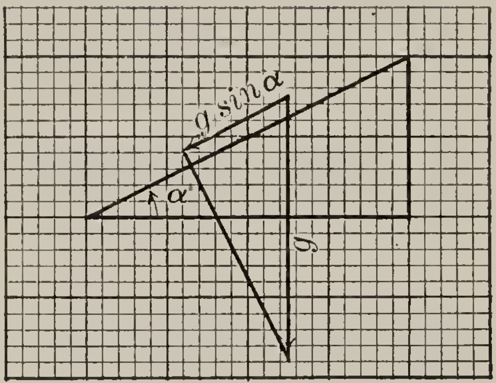
$y = \tfrac{4}{3}x + k$ is the family of perpendicular lines.

$y - 5 = \tfrac{4}{3}(x + 1)$ is the perpendicular line through $(-1, 5)$.

EXERCISES

1. Write the equations of the sides of the triangles used in finding the functions of 30°, 45°, and 60°.

2. Gravity imparts to a falling body a vertical velocity of $32\,t$ feet per second, with t seconds as time during which the body has fallen; on a smooth inclined plane gravity imparts a velocity of $32\,t \cdot \sin \alpha$ where α is the angle of inclination of the plane. Find the velocity imparted at the end of 1 second to a body sliding (without friction, assumed) on an inclined plane of slope 10°, 20°, 30°, 40°, 50°, ⋯ to 90°.



Acceleration down a plane, $g \sin \alpha$

3. In a freely falling body $s = 16\,t^2$; while on a plane $s = 16\,t^2 \cdot \sin \alpha$; find s for $t = 10$, $\alpha = 30°$, 45°, and 60°.

4. To pull the body up the plane requires a force of $W \sin \alpha + kW \cdot \cos \alpha$, where k is a constant dependent upon the friction. Find the force to pull a weight of 1000 lb. up an incline of 30°, $k = \frac{1}{3}$.

5. Find the slope of the line joining $(-3, 7)$ to $(5, 9)$; find the middle point of this line; find the equation of the perpendicular bisector of the segment.

6. Write the equation of the line through $(-3, 5)$ making an angle $\tan^{-1} \frac{5}{12}$ $(m = \frac{5}{12})$ with the x-axis, and write the equation of the perpendicular from $(1, 8)$ to this line.

7. Find the foot of the perpendicular line found in problem 6 and then find the distance between $(1, 8)$ and the original line, using the distance formula.

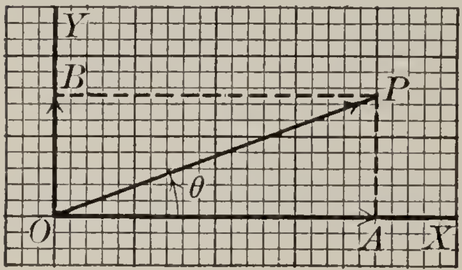
8. Find the slope angles in degrees and minutes of the following lines:

(a) $5\,y - 12\,x - 7 = 0$,
(b) $12\,y + 5\,x - 3 = 0$,
(c) $x - y - 5 = 0$,
(d) $3\,x - y - 8 = 0$.

9. Find lines through $(1, 5)$ parallel and perpendicular to each of the lines in the preceding exercise.

10. Find the perpendicular bisectors of the sides of the triangle formed by the three lines given by the equations, $5y-12x-7=0$, $12y+5x-3=0$, and $x+y-5=0$. Find the area of this triangle graphically and analytically.

2. Projections of vectors. — OP has been designated by r, for radius vector of the point P. The line OP has magnitude, given by r, and direction, given by the angle θ. We may use this system of representation to represent velocities, forces, and other physical quantities. As a velocity this vector may be resolved into two component velocities, represented by OA and OB. OA represents the velocity in the x direction, $x = r \cos\theta$; OB represents the y velocity, $r \sin\theta$, the vertical component of the velocity of a body moving with velocity represented by OP. The projection of any vector upon a directed line is defined as the directed distance between the perpendiculars dropt from the extremities of the given vector upon the line; it is given by $v \cos\alpha$ wherein v represents the vector and α is the angle between the positive rays of the two lines. Since $\cos(-\alpha) = \cos\alpha$, we do not need to distinguish between the two lines, *i.e.* the angle can be taken as obtained by rotation from the given line to the given vector, or vice versa.

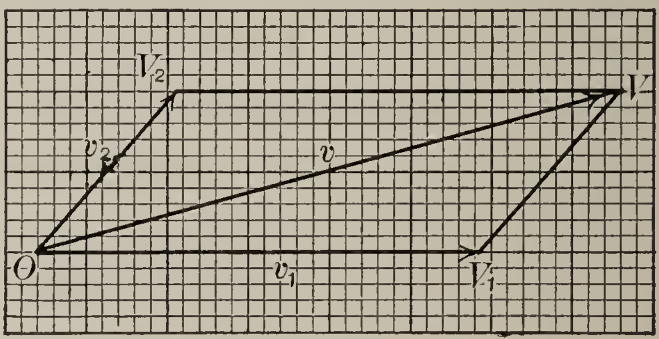


Components of a vector

OA represents the x component of OP.

OB represents the y component of OP.

OP is the resultant of OA and OB.



Vector parallelogram

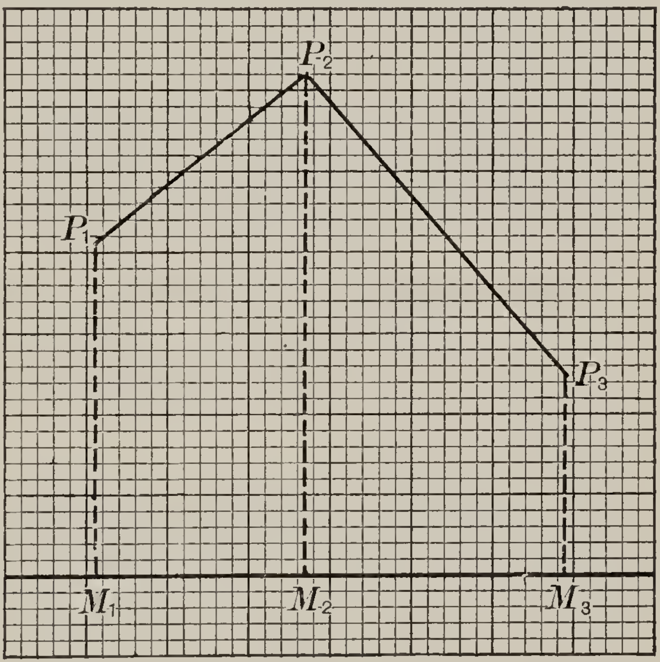
It is a fundamental assumption that any two vector quantities which may be represented acting together at the same point may be replaced by a single vector which is the diagonal of the parallelogram formed by the two given vectors. The process is called vector addition. This assumes that in space, for example, an imparted velocity S. E. of 50 miles per hour increased by a velocity N. E. of 30 miles per hour produces the same displacement whether the two forces which produce the velocities act together for one hour, or whether both act in succession each for an hour.

The projection of a broken directed line upon a given directed line is the same as the projection of the straight line joining the ends of the broken line.

This follows from the fact that on a directed line

$$M_1M_2 + M_2M_3 = M_1M_3,$$

whatever the relative positions of M_1, M_2, and M_3. The directed length M_1M_2 is the projection of P_1P_2, M_2M_3 is the projection of P_2P_3, M_1M_3 is the projection of P_1P_3. The physical interpretation is simply that the total component in the x direction (or any other) imparted by two (or more) vectors is the algebraic sum of the two (or more) x components of these vectors, taken separately.



Projection of a broken line on a directed line

When the velocity is given as v, v_x and v_y are commonly used to designate the x and y components of the velocity; evidently, also

$$v^2 = v_x^{\ 2} + v_y^{\ 2},$$
$$v_x = v \cos \theta,$$
$$v_y = v \sin \theta.$$

PROBLEMS

1. A bullet, muzzle velocity of 3000 feet per second, leaves the gun elevated at an angle of 10°. The position, neglecting air resistance, is determined at the end of t seconds by the two equations:

$$y = 3000\,t \sin 10° - 16\,t^2,$$
$$x = 3000\,t \cos 10°.$$

Find t when $y = 0$; when $y = 5$; explain the two values in each case. Find x for both values of t which make $y = 0$.

2. The velocity is a vector resolved into components $v_x = v \cos \alpha$ and $v_y = v \sin \alpha$. Find v_x and v_y when $\alpha = 10°$, 20°, 30°, 45°, 60°.

3. A ship sails S. E. for 2 hours at 8 miles per hour and E. N. E. ($22\frac{1}{2}°$ off East) for 2 hours at 6 miles per hour. Find the x and y of the resultant position.

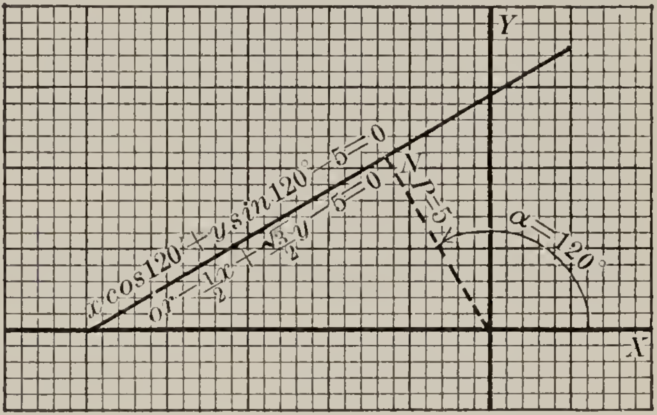
4. The propeller imparts to a steamer a velocity of 8 miles per hour S. E. ($-45°$) and the wind imparts a velocity of E. N. E. ($+22\frac{1}{2}°$) of 6 miles per hour. Find the position at the end of 1 hour.

5. A boy runs east on the deck of a steamer at the rate of 20 feet per second; the steamer moves south at the rate of 15 miles per hour. Find the actual direction in which the boy is moving and his total velocity.

6. Find the velocity in miles per hour of a point on the earth's surface due to the rotation of the earth on its axis; find the velocity per second due to the revolution about the sun; compare, and note that the resultant can never be greater than the sum nor less than the difference of the two. Take values only to 3 significant figures; 3960 mi. $= r$; 93,000,000 miles as distance from sun.

7. The United States rifle, model 1917, has a muzzle velocity of 2700 feet per second. Find the horizontal velocity of the bullet when the angle of elevation is 1°, 10°, 20°, 30°, and 45° respectively.

3. Normal form of a linear equation. — The slope-intercept, point-slope, and two-point formulas correspond to the fact that a straight line is determined when one point on the line [$(0, k)$ or (x_1, y_1) respectively] and the direction of the line are given, or when two points are given. A straight line may be determined in many other ways; one method which gives a further useful form of the equation of the straight line determines the line in terms of the length and direction of the perpendicular from the origin upon the line.



A line determined by the normal to it from the origin

Normal length, 5; $\alpha = 120°$.

Thus if a perpendicular from the origin upon a given line is 5 units long, and makes an angle of 120° with the x-axis (positive ray) geometrically we construct the line by constructing the ray of 120° and upon it taking a length of 5 units. At the extremity of this line of 5 units length a perpendicular is drawn which is the required line. The point N is readily found to be $(5 \cos 120°, 5 \sin 120°)$ and the slope is $\frac{-1}{\tan 120°}$, therefore the equation of the line to be found is

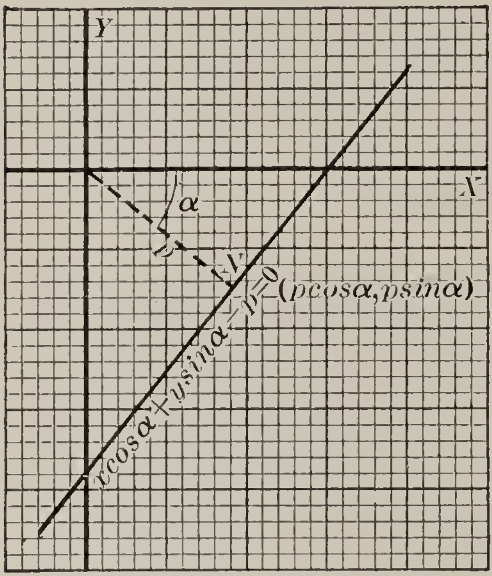
$$y - 5 \sin 120° = \frac{-1}{\tan 120°}(x - 5 \cos 120°).$$

$\tan 120° = \frac{\sin 120°}{\cos 120°}$; substituting, clearing of fractions, transposing

$$x \cos 120° + y \sin 120° - 5(\sin^2 120° + \cos^2 120°) = 0,$$

$$x \cos 120° + y \sin 120° - 5 = 0, \text{ since } \sin^2 \alpha + \cos^2 \alpha = 1,$$

$$-\frac{1}{2}x + \frac{\sqrt{3}}{2}y - 5 = 0.$$



$x \cos \alpha + y \sin \alpha - p = 0$
Normal form.

In general, given the normal ON to the line from the origin, of length p, and making angle α with OX, the extremity N is $(p \cos \alpha,\ p \sin \alpha)$; the slope is $-\dfrac{\cos \alpha}{\sin \alpha}$, and the equation becomes

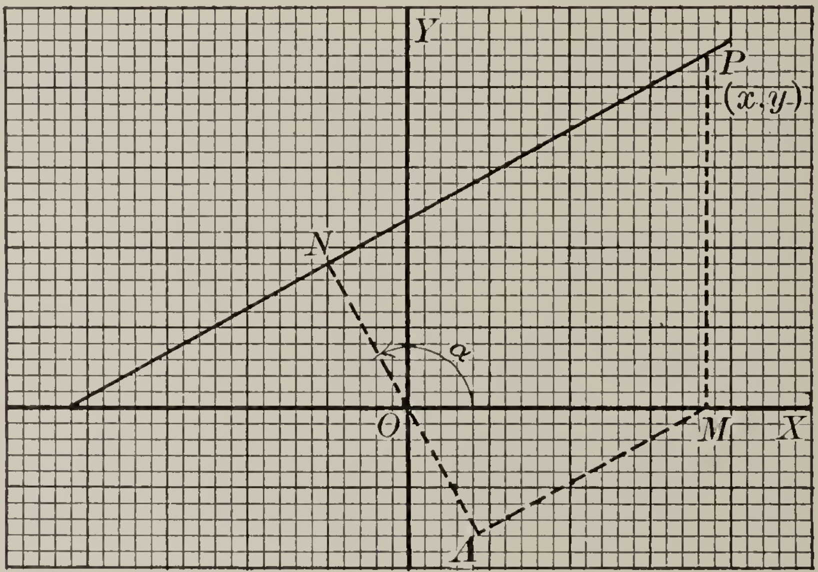
$$x \cos \alpha + y \sin \alpha - p = 0,$$

p is taken as a positive quantity just as r has been taken. Evidently if $p = 0$,

$$x \cos \alpha + y \sin \alpha = 0$$

represents a parallel line through the origin. Evidently also for parallel lines on opposite sides of the origin the angles α and α' differ by 180°; *i.e.* $\alpha' = 180° + \alpha$, whence

$$\sin \alpha' = - \sin \alpha.$$
$$\cos \alpha' = - \cos \alpha.$$



$$\left.\begin{array}{c}\textbf{The projection on } ON\\ \textbf{of } OM + MP\end{array}\right\}\ \textbf{equals}\ \left\{\begin{array}{c}\textbf{the projection on } ON\\ \textbf{of } OP\end{array}\right.$$

4. Normal form derived by projection. — We have shown that the projection of any broken line upon any given line is the same as the projection upon the given line of the vector joining the ends of the broken line. Let $P(x, y)$ be any point on the line whose equation is sought; drop PM the perpendicular from P to the x-axis; the projection of the broken line $OM + MP$ on the normal ON is equal to the projection of OP on ON. Now $OM = x$ makes the angle α, by hypothesis, with ON, and MP makes the angle $\alpha - 90°$; hence the projection of OM on ON is $x \cos \alpha$ (OA, negative in the figure since α is obtuse) and of MP on ON (AN in the figure) is $y \cos(\alpha - 90°)$; the projection of OP on ON is ON itself, or p; further $y \cos(\alpha - 90°) = y \cos(90° - \alpha) = y \sin \alpha$. Then, since projecting on the line ON,

projection of OM + projection of MP = projection of OP,

we have

$$\boldsymbol{x \cos \alpha + y \sin \alpha = p}, \text{ whence } \boldsymbol{x \cos \alpha + y \sin \alpha - p = 0}.$$

5. To put the equation of a straight line in normal form. —

Let the given equation be $3x - 4y + 7 = 0$, and let

$x \cos \alpha + y \sin \alpha - p = 0$ be the same equation in normal form.

If these two equations represent the same line, these lines must have the same slope and the same y (or x) intercept.

$$\frac{-\cos \alpha}{\sin \alpha} = \frac{3}{4}, \quad \frac{7}{4} = \frac{p}{\sin \alpha}.$$

$$\cos \alpha = \frac{-3}{4} \sin \alpha.$$

$$\cos^2 \alpha = \tfrac{9}{16} \sin^2 \alpha.$$

But $$\cos^2 \alpha = 1 - \sin^2 \alpha,$$

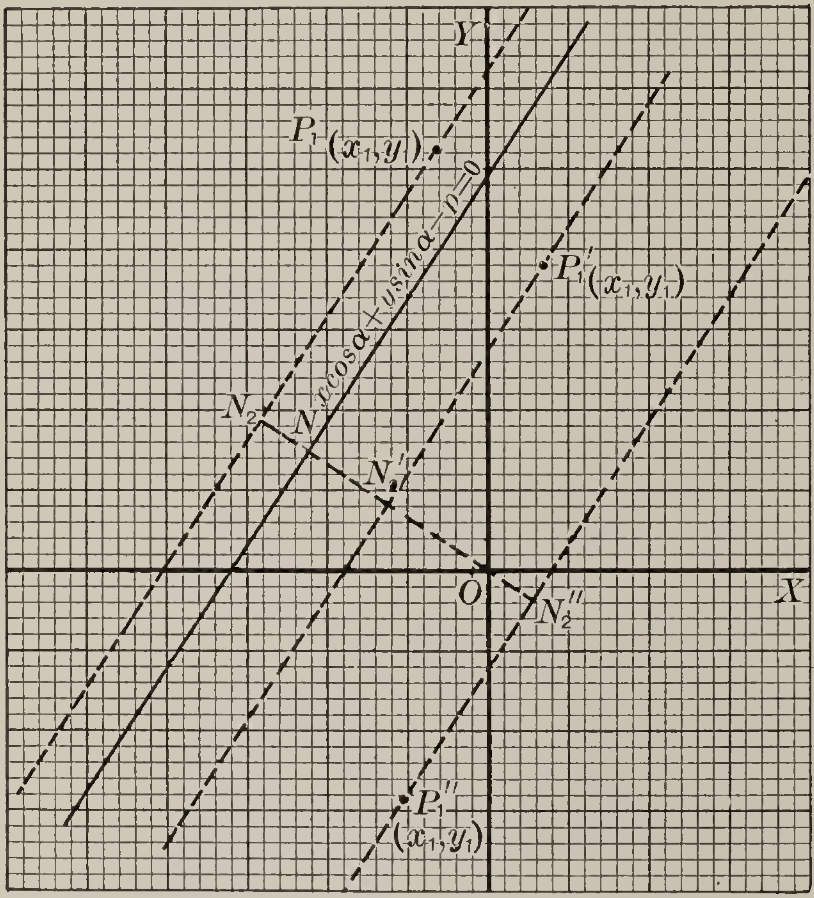
whence $1 - \sin^2 \alpha = \frac{9}{16} \sin^2 \alpha$; $\frac{25}{16} \sin^2 \alpha = 1$; $\sin \alpha = \pm \frac{4}{5}$.

$p = + \frac{7}{4} \sin \alpha$, whence since p is to be positive, $\sin \alpha$ must be taken as positive. Hence $\sin \alpha = + \frac{4}{5}$, $p = \frac{7}{5}$; $\cos \alpha = - \frac{3}{5}$; and thus the normal form is $- \frac{3}{5} x + \frac{4}{5} y - \frac{7}{5} = 0$. This equation is obtained by dividing each member of the original equation by -5.

In general to put $Ax + By + C = 0$ in the normal form, $x \cos \alpha + y \sin \alpha - p = 0$, one must multiply through by some quantity k, so that $kA = \cos \alpha$, $kB = \sin \alpha$, and $kC = -p$; $kC = -p$ shows that k must be chosen opposite in sign to C; squaring both members of the first two equations and adding gives $k^2(A^2 + B^2) = 1$, whence $k = \pm \frac{1}{\sqrt{A^2 + B^2}}$, of which the sign is taken as opposite to C.

Rule. — *To put an equation* $Ax + By + C = 0$ *in normal form divide through by* $= \pm\sqrt{A^2 + B^2}$, *with the sign taken opposite to that of the constant term.*

6. To find the perpendicular distance from a point to a line. — In solving this problem one considers the various forms of the straight line which may be employed. Evidently the normal form is most hopeful for use, since it involves the perpendicular distance of the given line from the origin. Through the point $P_1(x_1, y_1)$ draw a line parallel to the given line; evidently the difference between the normals to the two lines gives the distance. Three possibilities must be considered: 1. P_1, on the opposite side of the given line from the origin; 2. P_1,



Distance of a point from a line

on the same side of line as O, the origin, but such that the normal angle is the same, *i.e.* so that the parallel line through $P_1(x_1, y_1)$ falls on the same side of O as the given line, P'_1 on the figure; 3. P_1, on the same side as the origin, but the normal angle increased (or diminished) by 180°, designated by P''_1 on the figure.

Let $x \cos \alpha + y \sin \alpha - p = 0$ be the equation of the line.

1. $x \cos \alpha + y \sin \alpha - (x_1 \cos \alpha + y_1 \sin \alpha) = 0$ is the parallel line through $P_1(x_1, y_1)$, since this equation is evidently in normal form and the line passes through (x_1, y_1).

$$ON_2 = x_1 \cos \alpha + y_1 \sin \alpha.$$
$$d = ON_2 - ON = x_1 \cos \alpha + y_1 \sin \alpha - p.$$

The perpendicular distance is obtained then by writing the equation in normal form and substituting for (x_1, y_1) the coördinates of the given point. Evidently if $P_1(x, y)$ is on the line, this gives also the correct distance, which is then zero.

2. $x \cos \alpha + y \sin \alpha - (x_1 \cos \alpha + y_1 \sin \alpha)$ is the equation of the parallel line; again, $ON'_2 = x_1 \cos \alpha + y_1 \sin \alpha.$

$$d = ON - ON'_2: \text{ whence } \quad -d = ON'_2 - ON$$
$$= x_1 \cos \alpha + y_1 \sin \alpha - p.$$

The same rule holds, but the distance in this case is negative. Evidently the rule holds if ON'_2 is 0.

3. $\alpha' = 180 + \alpha$; $\cos \alpha' = - \cos \alpha$, $\sin \alpha' = - \sin \alpha$.

To write the equation of the parallel line in normal form, the coefficients of x and y must both be the negatives of the coefficients of x and y in the given equation.

$x(- \cos \alpha) + y(- \sin \alpha) - (- x_1 \cos \alpha - y_1 \sin \alpha) = 0$ is the equation of the parallel line in normal form.

$$ON''_2 = -(x_1 \cos \alpha + y_1 \sin \alpha).$$
$$d = ON''_2 + ON = - x_1 \cos \alpha - y_1 \sin \alpha + p,$$
or
$$- d = x_1 \cos \alpha + y_1 \sin \alpha - p.$$

Rule. — *To obtain the distance from a point to a line write the equation in normal form, substituting therein for x and y the coördinates of the given point. The resulting number gives the distance as positive if the point and the origin lie upon opposite sides of the given line, as negative if P_1 and O are upon the same side of the given line.*

$x \cos \alpha + y \sin \alpha - p$ represents the perpendicular distance then from $P(x, y)$ to the line $x \cos \alpha + y \sin \alpha - p = 0$. For all points on one side of this line the expression is positive, and on the other side, crossing the line to the origin side, the expression is negative.

A line which passes through the origin, $p = 0$, will be said to have its equation in normal form when $\sin \alpha$ is taken as positive, *i.e.* when the coefficient of y is made positive. Points on this line make $x \cos \alpha + y \sin \alpha = 0$; points above the line make the expression $x \cos \alpha + y \sin \alpha$ positive, and points below the line make it negative.

Thus $3x - 4y = 0$ is written $\frac{-3}{5}x + \frac{4}{5}y = 0$, or

$$\frac{-3x + 4y}{5} = 0,$$

to be in normal form. The perpendicular distance from any point to such a line will be positive for points above the line, and negative for points below the line.

7. Bisector of the angle between two lines. — Geometrically the bisector of an angle is the locus of the points equidistant from the two sides of the angle; analytically we express the condition that two distances should be equal to each other.

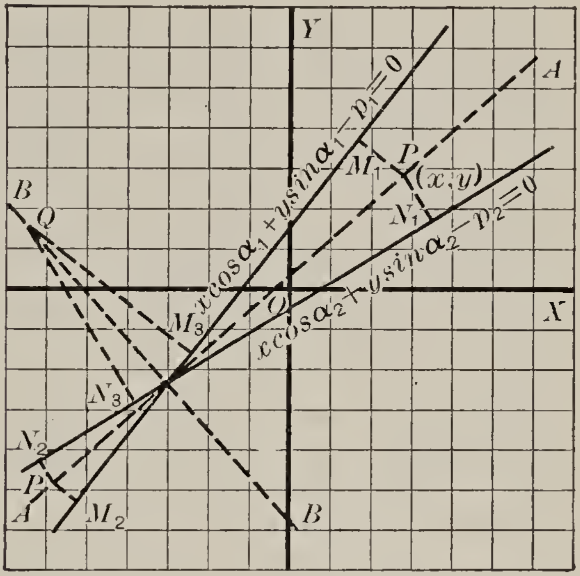
Let the equations be given in normal form, as

$$x \cos \alpha_1 + y \sin \alpha_1 - p_1 = 0 \text{ and } x \cos \alpha_2 + y \sin \alpha_2 - p_2 = 0.$$

Let $P(x, y)$ represent any point on either bisector of the given angle; analytically

$$x \cos \alpha_1 + y \sin \alpha_1 - p_1 = \pm (x \cos \alpha_2 + y \sin \alpha_2 - p_2).$$

Bisector A, in the opening which includes the origin, is obtained by taking the + sign since both perpendiculars are of the same sign for points on A. Bisector B is obtained by taking the negative sign since any point on B is on the same side as the origin with respect to one of the lines, and on the opposite side with respect to the other; hence if PM_1 comes out negative, PM_2 will be positive (by the formula) and the equality will be obtained by putting $PM_1 = - PM_2$.



Bisectors of the angles between two lines given normal form

Just as the two axes divide the plane into 4 quadrants in which the distances to these axes are + +, − +, − −, and + − respectively, so any two lines in the plane divide the plane into 4 sections in which the perpendicular distances, as given by our formula, to these lines are + +, + −, − −, and − + respectively. The + + and − − sections are separated by the + − and − + sections respectively, as it is evident that you pass from + + to + − by crossing the second line.

The bisector of the + + and − − opening is given by equating the left-hand members of the equations of the two lines in normal form; the bisector of the + −, − + opening is obtained by equating the one to the negative of the other left-hand member.

If one of the lines passes through the origin, or if both do, then the above-mentioned convention is necessary to establish the part of the plane in which the left-hand member of the equation of the line is positive. It is customary to make $\sin \alpha$ positive, which makes the portion of the plane above the line the positive side, *i.e.* the coördinates of any point above the line when substituted in the given equation give a positive value, and of any point below the line give a negative value.

PROBLEMS

1. If a line makes an angle of 30° with the x-axis what angle does the normal to the line make with the x-axis?

2. What is the slope of the line, $y = 2x + 5$? What is the slope angle? What is the slope of the normal to this line? What is the angle which this normal makes with the x-axis? Find from the tangent of the angle made by the normal with the x-axis the sine and cosine of the same angle. Write the equation in normal form and interpret the constants.

3. Given α and p, as below, slope angle of the normal and length of the normal from the origin to the line, find the equations of the lines, and draw the lines:

a. $\alpha = 30^\circ, \quad p = 5.$
b. $\alpha = -30^\circ, \quad p = 5.$
c. $\alpha = 150^\circ, \quad p = 5.$
d. $\alpha = 210^\circ, \quad p = 5.$
e. $\alpha = 137^\circ, \quad p = 5.$
f. $\alpha = 137^\circ, \quad p = 10.$
g. $\alpha = -63^\circ, \quad p = 10.$
h. $\alpha = 223^\circ\,15', p = 8.$

4. If α remains equal to 40° and p varies, what series of lines will be obtained? if p remains equal to 5, and α varies, what series of lines will be obtained?

5. Write the following equations in normal form:

a. $3x - 4y - 5 = 0.$
b. $5x + 12y + 7 = 0.$
c. $5x + 12y - 7 = 0.$
d. $3x - 5y - 4 = 0.$
e. $y = 2x - 14.$
f. $3y - 7x + 52 = 0.$
g. $\frac{x}{3} + \frac{y}{5} = 1.$

6. Find the distances of the points (1, 5), (2, 3), (0, 5), (0, − 5), (−2, − 3), and (−3, 7) from each of the lines in the preceding problem.

7. What is the distance of the point (x, y) from the line $3x-4y-5=0$? Under what circumstances does the formula give a negative value for this distance? What is the distance of any point (x, y) from $5x+12y+8=0$? What does equating these two expressions, *i.e.* the left-hand members of each normal form, give? Interpret on the diagram. What is obtained by setting one of these expressions equal to the negative of the other?

8. Find the bisectors of the angles between the following pairs of lines:

a. $3x+4y-5=0$ and $12x-5y-10=0$.
b. $y-2x-5=0$ and $2x+y+7=0$.
c. $y-2x-5=0$ and $3y+x-8=0$.
d. $y-2x=0$ and $3y+x-8=0$.
e. $y-2x=0$ and $3y+x=0$.

9. Find the distance of the points (1, − 3), (3, 0), (3, − 7), and (0, − 8) from each of the lines in the preceding problem.

10. Find the distance between the following pairs of parallel lines:

a. $y=2x-7$,
$y=2x+3$.

b. $4y-3x=5$,
$4y-3x-16=0$.

c. $4y-3x=0$,
$4y-3x-16=0$.

d. $x+2y-7=0$,
$2x+4y+17=0$.

e. $7.2x+8.3y-15=0$,
$7.2x+8.3y-8=0$.

11. In problem 8 show that each bisector obtained is one of the pencil of lines through the point of intersection of the given two lines.

12. Find the area of the triangle having as vertices the following points:

a. (3, 4), (0, 0), and (0, 8).
b. (3, 4), (0, 0), and (10, 2).
c. (1, 1), (4, 5), and (7, – 3).

13. Find the area of the triangle formed by the three lines:

$$\begin{aligned} &3x + 4y - 5 = 0, \\ &12x - 5y - 10 = 0, \\ \text{and}\quad &4x - 3y - 7 = 0. \end{aligned}$$

14. What is the distance of any point (x, y) from the point (0, 0)? What is the distance of any point (x, y) from the line $x - 5 = 0$? Equate these two expressions for distance and simplify. The resulting equation has for its graph all points which are equally distant from the point (0, 0) and the line $x - 5 = 0$.

15. Find the locus of all points which are equidistant from the point (0, 0) and the line $y - 8 = 0$. Let (x, y) represent any point satisfying the given condition.

16. Find the locus of all points at a distance 10 from the point (0, 0); from (1, – 3). Find the locus of all points at a distance 10 from the line $3x - 4y - 7 = 0$; at a distance – 10; explain graphically.

17. Find the locus of all points equally distant from $3x - 4y - 5 = 0$ and from (1, – 5).

18. In problem 12 find the equations of the three bisectors of the angles of the triangle formed; find the perpendiculars from the vertices to the opposite sides; find the perpendicular bisectors of the sides; show that in each instance you have three lines which have a point in common.

19. What points, when the coördinates are substituted for x and y, make the expression $4y - 3x - 5$ positive? What points make this expression zero? What points make this expression negative? Locate three points of each type, plot and discuss.

20. Substitute in the expression $x^2 + y^2 - 25$ for x and y the coördinates of the points (0, 3), (± 3, 2), (± 1, 4). Plot these points. Substitute (0, ± 5), (± 3, ± 4), (± 4, ± 3), and (± 5, 0). Plot. Substitute also (0, 8), (± 7, 0), (5, 3), and (± 4, 6). Note that the graph of $x^2 + y^2 - 25 = 0$ separates the plane into two parts; in the one part inside this curve are all points whose coördinates substituted for x and y, respectively, make the expression $x^2 + y^2 - 25$ negative, and in the part outside lie all points which make this expression positive.

21. Prove that the perpendicular bisectors of the sides of any triangle meet in a point; the vertices may be assumed as (x_1, y_1), (x_2, y_2) and (x_3, y_3) or as $(0, 0)$, $(x_1, 0)$, and (x_2, y_2).

22. Prove that the bisectors of the angles of any triangle meet in a point.

23. Given the three vertices of a parallelogram, how do you find the fourth vertex? Apply to (1, 5), (6, − 1), and (3, 2).

24. How do you find a line parallel to a given line at a given distance from it?

CHAPTER X

ARITHMETICAL SERIES AND ARITHMETICAL INTERPOLATION

1. Definition of an arithmetical series. — In the table of natural sines the values of the sines of 21° to 22° are given as follows,

sin 21°	sin 21° 10′	sin 21° 20′	sin 21° 30′	sin 21° 40′	sin 21° 50′	sin 22°
.3584	.3611	.3638	.3665	.3692	.3719	.3746

It is to be noted that each value differs from the preceding by .0027, and each angle differs from the preceding by 10′. Either of these sequences of numbers with a constant difference between each number and the preceding is termed an arithmetical series; the continuation of the lower series by the successive addition of .0027 gives indefinitely further values of the arithmetical series, but gives only 5 following sines. Of the series of numbers giving to four decimal places the values of the sines of angles which increase by intervals of 10′ it happens, for reasons which will be further discussed below, that twelve values beginning with the sine of 21° coincide with the first twelve terms of an arithmetical series; the sine series must not be confused with the arithmetical series, as it is only arithmetical in limited intervals and then only when approximate values are used. Thus if five place values of the sines of the angles above were given the series would no longer be arithmetical.

The type form of arithmetical series is

a	$a+d$	$a+2d$	$a+3d$	....	$a+9d$	....	$a+(n-2)d$	$a+(n-1)d$
1st term	2d	3d	4th	....	10th	....	$(n-1)$th term	nth term

each term is d greater than the preceding term of the series.

$l_n = a + (n - 1)d$; by l_n we designate the nth term of such a series. It is evident from the definition that the tenth term in such a series is $a + 9d$, since the common difference d appeared first in the second term and one further d was added in each subsequent term.

2. Last or *n*th term, and sum. — Strictly we should prove by a process called mathematical induction, that the formula,

$$l_n = a + (n - 1)d,$$

always represents the nth term. Evidently for $n = 1$ this does represent our first term; for $n = 2$ the expression does represent our second term; for $n = 3$ the expression $a + 2d$ does represent our third term; let us suppose that for n this does represent our nth term, then our $(n + 1)$th term, which is d greater, must be $l_n + d = a + (n - 1)d + d = a + nd$; now the formula gives $l_{n+1} = a + (\overline{n + 1} - 1)d = a + nd$; hence if this formula is correct for the nth term, the formula is correct for the next, the $(n + 1)$th term. However, we know that the formula is correct for the third term, hence it is, by our theorem just stated, true for the next, the fourth term; since it is true for the fourth it is, by the theorem, true for the fifth; so for every subsequent term.

Frequently the sum to n terms, s_n, of such a series is desired. To obtain a simple expression for s_n, we proceed as follows:

$$s_n = a + (a + d) + (a + 2d) + \cdots a + 9d + \cdots a + (n - 1)d$$

or l_n; reversing the series gives,

$$s_n = l_n + (l_n - d) + (l_n - 2d) + \cdots l_n - 9d + \cdots l_n - (n - 1)d;$$

adding,

$$2s_n = (a + l_n) + (a + l_n) + (a + l_n) + \cdots (a + l_n) + \cdots (a + l_n);$$

$$s_n = \frac{n}{2}(a + l_n).$$

Fundamental formulas:

$$\boldsymbol{l_n = a + (n - 1)d,}$$

$$\boldsymbol{s_n = \frac{n}{2}(a + l_n).}$$

3. Practical importance. — Arithmetical series are of great importance because of their occurrence in practical problems, and because they are fundamental in the applications of mathematics to statistical problems. In physical problems involving time, the time is commonly measured at the end of equal intervals, giving an arithmetical series for the time; in the tables of logarithms our numbers increase arithmetically, and so in the tables of trigonometric functions the angles increase arithmetically. Refinement of measurement is commonly made by subdividing the unit of measurement into smaller equal intervals, giving new arithmetical series.

PROBLEMS

1. Find the tenth and the twentieth terms of the series, 1, 3, 5, 7, ⋯; find the $(n+1)$th term.

2. Find the sum to 10 and to 20 terms of the series 1, 3, 5, ⋯ .

3. Show by mathematical induction that the sum of the first n odd numbers is n^2, by showing that if the sum is n^2 then the sum of the first $(n+1)$ odd numbers is $(n+1)^2$.

4. Solve $l = a + (n-1)d$, for d; solve for a; solve for n.

5. Given $l = 235$, $d = 7$, $n = 40$, find a.

6. Given $l = 235$, $n = 7$, $a = 5$, find d.

7. Given $d = 7$, $a = 5$, $n = 40$, find l.

8. One hundred men increase uniformly in height from 5.01 feet to 6 feet by .01 of a foot, find the total height; if their weights increase uniformly by half-pounds from 110 pounds, find the total weight of the group, and the average weight.

9. On an inclined plane, angle of 30°, a ball rolls approximately 8 feet in 1 second, 24 feet in the second second, 40 feet in the next, and in every second 16 feet more than in the preceding second. Find the distance a ball travels in 5 seconds; in 10 seconds. This formulation neglects the energy-loss due to rolling.

10. On a hill inclined at 30° a bob-sled moves approximately according to the law of the rolling ball in the preceding problem. Find the length of time to cover 1000 feet. Find the distance covered in the last second of the slide. Find the average velocity during the slide, and the average velocity during the last second. Reduce velocity to miles per hour.

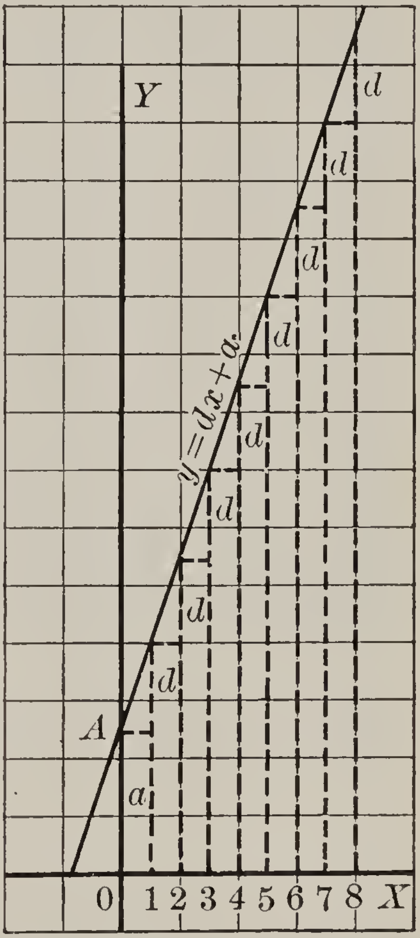
Note. — Average velocity is simply the space covered divided by the time required.

4. Graphical representation. — The arithmetical series is represented graphically by the straight line, and conversely any straight line represents an arithmetical series. For this reason the interpolation processes explained above under logarithms and under trigonometric functions are sometimes termed "straight-line interpolations"; the process is correct in those small intervals in which the curve representing the function is approximately a straight line.

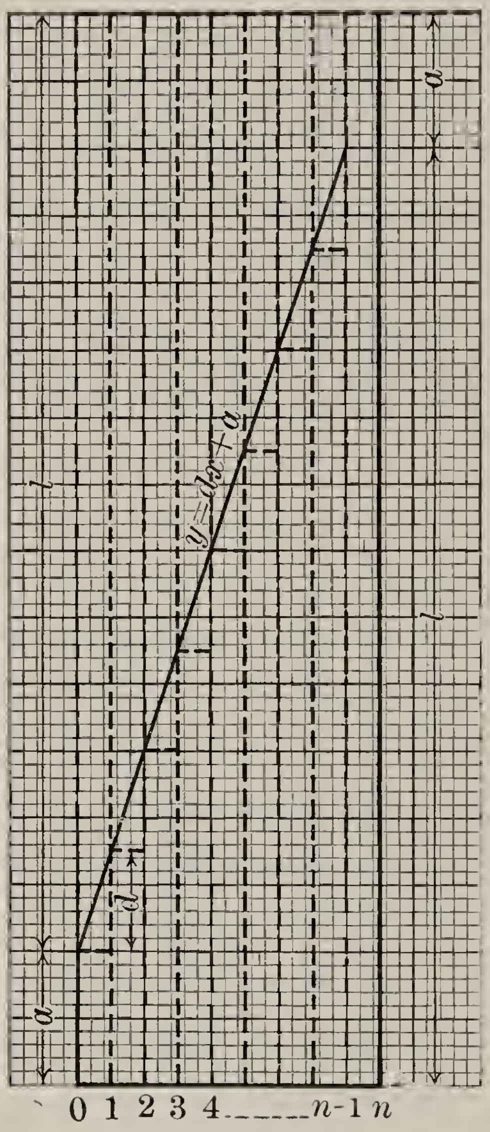
For the integral values of x, from $0, 1, 2, 3, \cdots$ up to $n-1$, the ordinates of the line

$$y = dx + a$$

represent graphically the terms of the type arithmetical series, $a, a+d, a+2d, \cdots$; for values of x from 0, $\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \cdots$ up to $\frac{n-1}{10}$, the ordinates represent the terms of an arithmetical series with first term a and the common difference $\frac{d}{10}$. To any series of equal increases or increments given to x there correspond a series of equal increments given to the ordinates;



The ordinates of $y = dx + a$ at $x = 0, 1, 2, 3, \cdots$ represent terms of an arithmetical series this depends upon the theorem of plane geometry that if a series of parallel lines cut off equal parts on one transversal they do on every transversal, and this theorem is equally true for any straight line in the plane.



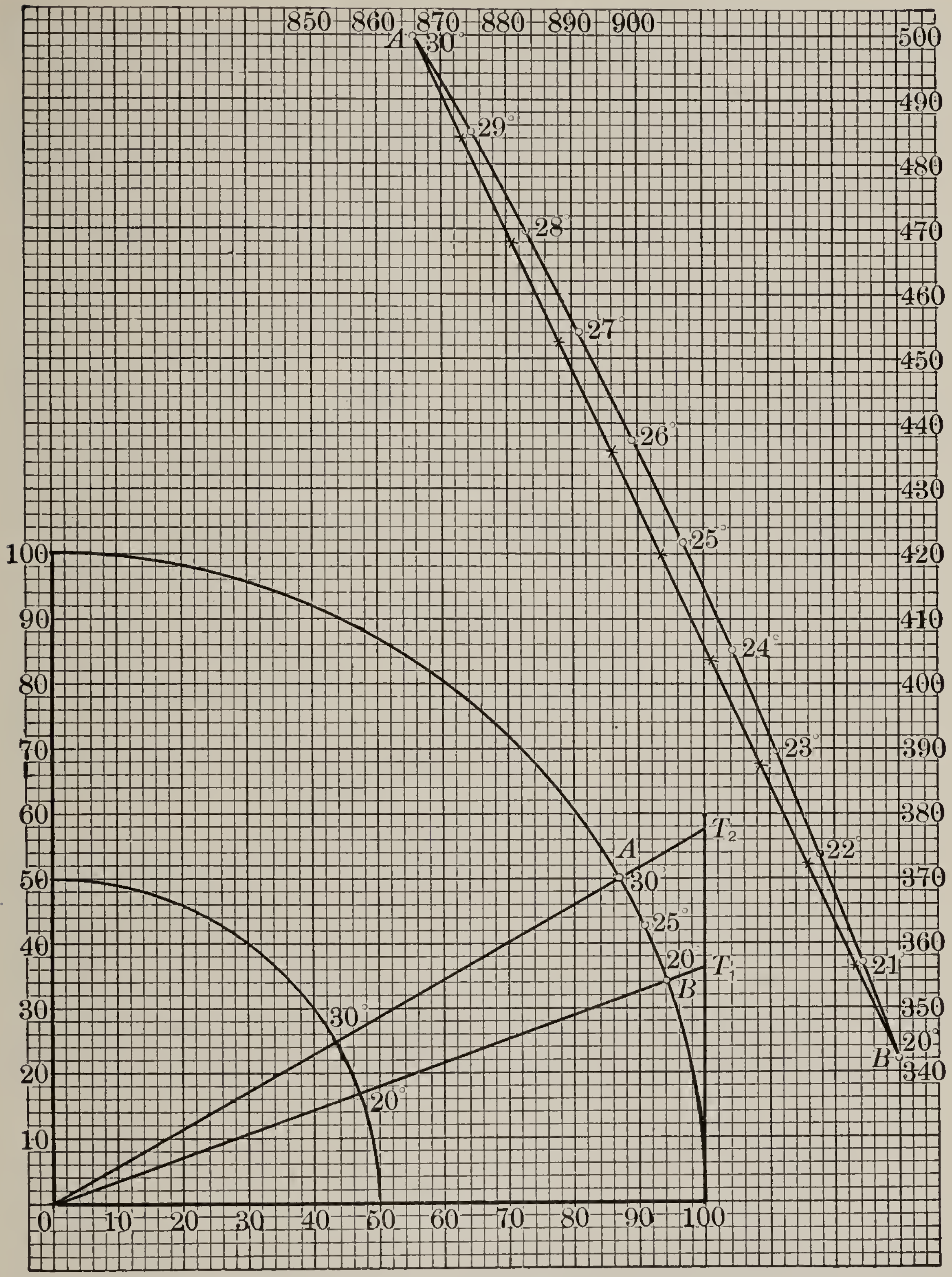
Graphical representation of the sum of an arithmetical series

$$s_n = n\frac{a+l}{2}$$

5. Interpolations in sines and other functions. — The value of the sines of the angles are given by the corresponding ordinates in a circle of radius unity, or the ordinates divided by 100 in a circle of radius 100, or the ordinates divided by 1000 in a circle of radius 1000.

On our diagram, with radius 100 the straight line joining the end of the ordinate corresponding to 20° to the end of the ordinate at 30° does not differ materially from the circular arc connecting these points. Were we to plot these angles in a circle of radius 1000 the points of intersection would appear as in the second part of the diagram, lettered *AB*, and constituting a tenfold linear enlargement of *AB*.

If the sines of the angles were given by intervals of ten degrees, interpolation by tenths would give the sines by degrees; the circle with radius 100 mm. (or 100 twentieths of an inch) permits the sine and cosine to be read to two places accurately, and this rather low degree of refinement corresponds to a table of sines given by intervals of ten degrees; interpolation would give substantially correct values to *two decimal places, e.g.* for the sines of 21°, 22°,



Arc of 20° to 30° in circles with radii 50, 100, and 1000 fortieths of an inch

The marks on the long chord indicate the points given by interpolating between sin 20° and sin 30°, and between cos 20° and cos 30°. Even on the arc with 25 inch radius nine interpolated points on the chord and corresponding points on the arc, between 21° and 22°, coincide.

··· 29° by interpolating between sin 20° = .34 and sin 30° = .50. The arc of 10° on this circle differs slightly but appreciably to the eye from the chord of 10°, but the interpolated points on the chord are not easily distinguished from the ten points on the curve.

Angles given by degrees permit interpolation by intervals of 6′ or by intervals of 10′, with substantially correct values to the third place if the values are given only to three places; values given to four places give by interpolation values substantially correct to the fourth place. On our circle with radius 1000 the sine and cosine can be read to three decimal places; interpolation between the values of sin 20° and sin 30° give points markedly different from the true points on the curve. These points are indicated by checks on the chord of 10°. Interpolating five points (for 10′, 20′, 30′, 40′, 50′) on the chord from the 20° point to the 21° point gives points not readily to be distinguished from the correct points on the arc. On this diagram it is not possible to distinguish the subdivisions for minutes on the arcs from the corresponding points on the chords of central angles of 10′.

With the proper changes, noting particularly that as θ increases cosine θ decreases, the argument given holds for interpolated values for cos θ.

Interpolation of the tangent values is similar, except in the neighborhood of 90° where the tangent changes very rapidly; in a separate table are given by minutes, the tangents of angles from 88° to 90°.

The graph of the function $y = \log_{10} x$, or $10^y = x$, is a continuous curve which for small arcs approximates a straight line. Similarly the graphs of the functions $y = \sin x$, and of $y = \sin x$, log cos x, log tan x and log cot x approximate straight lines within small intervals, and so are subject to our ordinary process of interpolation.

6. Arithmetical means. — If two numbers a and b are given, the arithmetical mean between the two is the number x which makes a, x, b three consecutive terms of an arithmetical series; to insert n arithmetical means it is necessary that a, the n means, and b form $n+2$ consecutive terms of an arithmetical series. Ordinary interpolation is the insertion between two tabular values of some particular one of 9 arithmetical means.

If a, x, b form an arithmetical series,

$$b - x = x - a,$$

$$\text{whence } x = \frac{a+b}{2}.$$

If $a, a+d, a+2d, a+3d, \cdots a+(n-1)d, a+nd, b$ form an arithmetical series, $b = a+(n+1)d$; whence $d = \dfrac{b-a}{n+1}$.

The sum of n terms of the series $a, a+d, a+2d, \cdots a+(n-1)d$, is

$$s_n = \frac{n}{2}(a + l_n);$$

the average value of these n numbers is

$$\frac{n(a+l_n)}{2 \cdot n} = \frac{a+l_n}{2},$$

termed the arithmetical mean of the n numbers; the sum of an arithmetical series is seen to be the "average value" multiplied by the number of terms. Similarly of any collection whatever of n quantities, the arithmetical mean is regarded as the total sum divided by the number of quantities. In statistical work the latter mean, total sum divided by the number of given quantities, is called the "weighted mean."

PROBLEMS

1. Between .3584 and .3746 insert 5 arithmetical means; if $.3584 = \sin 21°$ and $.3746 = \sin 22°$, what do these means represent? Between .3746 and .3584 insert 5 arithmetical means; interpret as cosines.

2. Given $\sin 21° = .3584$ and $\sin 21° 10' = .3611$, find 9 intermediate values; interpret.

3. Given sin $0° = 0$, sin $30° = .5000$, what value would arithmetical interpolation give for sin $21°$? What is the error?

4. Given sin $20° = .3420$ and sin $30° = .5000$; find to 4 places sin $21°$. How many terms in the arithmetical series which is implied?

5. What is the sum of the first ten integers?

6. If cards are marked 1 to 190, what is the total sum? What is the average value of the total group of numbers?

7. How many years of life have been lived by a group of 30 individuals, aged 21, 22, 23, . . . 50 years?

8. Falling from rest a body falls approximately 16 feet in the first second, and 48 in the second, and in each succeeding second 32 feet more than in the one which precedes. What distance will the body fall in 10 seconds? How long will it take such a body to fall 1000 feet?

9. If it takes a lead ball 8 seconds to fall to the earth from a balloon, what is the height of the balloon?

10. How long will it take a ball to reach the earth if dropped from the top of the Washington monument, 550 feet high?

11. Draw figures to show that between $x = 3$ and $x = 4$, the graph of $xy = 1$ approximates a straight line.

12. Write the equation of a straight line representing for integral values of x, from 0 to 10, the arithmetical series with $a = 10$, $d = -3$. Represent the series also by the series of rectangles, each of width 1. In summing the arithmetical series we reversed the series and added; show the geometrical equivalent on the figure, page 170, with the rectangles.

13. Given that the first term of an arithmetical progression is 8 and the last term 100, what equation must n and d satisfy? If $d = 2$, what is n? If $d = 3$, what is n? Interpret.

14. Show that in an arithmetical series of n terms (a, $a + d$, $\cdots$) the average value is $\frac{1}{2}$ the sum of the first and last term. Note that the average value of the terms is the total sum divided by the number of terms. This average value is termed the arithmetical mean of the n terms.

15. Find the average value of the following 10 heights, and the arithmetical mean:

40	6 feet	126	5 feet $9\frac{1}{2}$ inches
64	5 feet $11\frac{1}{2}$ inches	138	5 feet 9 inches
86	5 feet 11 inches	120	5 feet $8\frac{1}{2}$ inches
92	5 feet $10\frac{1}{2}$ inches	112	5 feet 8 inches
142	5 feet 10 inches	80	5 feet $7\frac{1}{2}$ inches

16. If there are 1000 men measured and they are grouped in height as above, find the average height of these men. Note that the easiest way to find this average is to take the variation above and below some one of the "middle" values, *e.g.* with reference to 5 feet 9 inches as origin, 6 feet is regarded as $+3$ inches and this group has a total of 120 inches excess above 5 feet 9 inches per individual; 5 feet $7\frac{1}{2}$ inches is regarded $-1\frac{1}{2}$ inches and the total group of 80 has a total deficiency of 120 inches, or -120 inches; the two neutralize each other. Whatever total remains is divided by 1000 and added, algebraically, to the 5 feet 9 inches.

17. Draw the graph of $l = a + (n - 1)d$; assuming a and d as constants.

18. *Historical problem.* — In the Egyptian manual mentioned above, occurs the following problem: If 100 loaves of bread are divided according to the terms of an arithmetical series among 5 people so that $\frac{1}{7}$ of what the first three receive equals what the last two receive, find the number received by each person. Solve the problem. The Egyptian reckoner assumes that the last person receives 1 loaf, and without any explanation, that the second receives $6\frac{1}{2}$ loaves, and so on in arithmetical progression; the sum he finds to be 60, and to arrive at the correct values all numbers are increased in the ratio of 100 to 60. Compare with your solution.

CHAPTER XI

GEOMETRICAL SERIES AND APPLICATIONS TO ANNUITIES

1. Geometrical series. — A series of terms in which each term is obtained from the preceding by multiplying by a fixed number is called a geometrical series; by definition the ratio of each term to the preceding is a constant. Designating this ratio by r, and the first term by a, the type series becomes:

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots\ \ldots\ \ldots\ ar^{n-1}.$$

$$l_n = ar^{n-1}.$$

The sum of such a series to n terms is obtained as follows:

$$\text{Let } s_n = a + ar + ar^2 + ar^3 + \ \ldots\ + ar^{n-1}.$$

$$rs_n = ar + ar^2 + ar^3 + \ \ldots\ + ar^{n-1} + ar^n.$$

$$s_n - rs_n = a - ar^n.$$

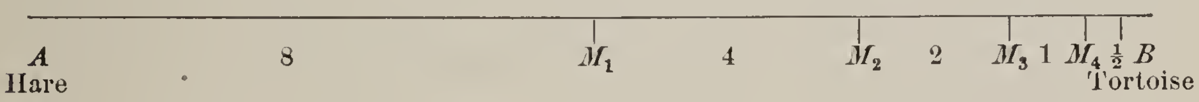
$$s_n(1-r) = a - ar^n.$$

$$s_n = \frac{a - ar^n}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r} = \frac{ar^n - a}{r-1}.$$

2. Sum to infinity, $r < 1$. — When r is numerically less than 1, the terms of a geometrical series become smaller and smaller without limit. The sum of n terms differs from the fixed quantity, $\frac{a}{1-r}$, by the quantity $\frac{ar^n}{1-r}$, which decreases as n increases; this difference between $\frac{a}{1-r}$ and $\frac{a}{1-r} - \frac{ar^n}{1-r}$ can be made smaller than any assigned quantity however small by taking n sufficiently large; the value $\frac{a}{1-r}$ is termed the "sum to infinity" of the series or, more strictly, the "limit" of the sum of the series as the number of terms increases indefinitely.

A simple and familiar illustration of a geometrical "sum to infinity" is found in the recurring decimals of elementary arithmetic. Thus .33333 ..., or $.\dot{3}$, is the series $\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\ldots$ with $r=\frac{1}{10}$, $a=\frac{3}{10}$; $.\dot{1}\dot{7}$ represents the series $\frac{17}{100}+\frac{17}{10000}+\frac{17}{1000000}+\ldots$ with $r=\frac{1}{100}$, $a=\frac{17}{100}$, $s=\frac{\frac{17}{100}}{\frac{99}{100}}=\frac{17}{99}$.

Another illustration of an infinite geometrical series is found in the total distance traversed in passing from one point to another by passing first through $\frac{1}{2}$ of the distance, then through half of the remaining distance, and successively in each subsequent movement through half of the distance which remains to the goal. One of the famous paradoxes of Zeno is to the effect that the hare cannot overtake the tortoise since the hare must first cover one half the distance intervening between their original positions, then one half of the remaining distance, then one half the remaining distance, and in each subsequent interval of time one half of the distance which remains; consequently the hare cannot overtake the tortoise as by this process there is always distance intervening, and in the meantime, further, the tortoise has advanced. If each of these intervals of space of the infinite series traversed required the same length of time, or any finite portion of time, the argument would be sound, but you have here two infinite series each with a finite sum. As each space interval becomes smaller, the time required to traverse that space becomes smaller. The total distance evidently is the sum of $8+4+2+1+\frac{1}{2}+\cdots$ which has the sum 16, both by the formula and by the figure; to cover the whole distance requires 1 hr. and this takes the hare up to B. The fallacy in Zeno's argument lies in restricting the discussion to limited intervals of space



and time preceding the instant and place at which the tortoise is overtaken.

3. Geometrical means. — If a, x, b form a geometrical series $\frac{b}{x} = \frac{x}{a}$, whence $x^2 = ab$, $x = \sqrt{ab}$.

If $a, ar, ar^2, ar^3, ar^4, \cdots ar^{n-1}, ar^n, b$ form a geometrical series, evidently

$$b = ar^n \cdot r = ar^{n+1},$$

whence $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$, and the series is

$$a,\ a\left(\frac{b}{a}\right)^{\frac{1}{n+1}},\ a\left(\frac{b}{a}\right)^{\frac{2}{n+1}},\ \cdots,\ a\left(\frac{b}{a}\right)^{\frac{n}{n+1}},\ a\left(\frac{b}{a}\right)^{\frac{n+1}{n+1}} \text{ or } b.$$

PROBLEMS

1. Sum to twenty terms the series 2, 4, 8, 16, ⋯.

In the following series give the sum to twenty terms, and where possible give the sum "to infinity."

2. $3, -6, +12, -24, +48, \cdots$.

3. $3, \frac{1}{2}, \frac{1}{12}, \frac{1}{72}, \cdots$.

4. $.1717171717 \cdots$, or $.\dot{1}\dot{7}$ (repeating decimal).

5. $.01717171717 \cdots$, or $.0\dot{1}\dot{7}$.

6. $3.16161616 \cdots$.

7. 5, 15, 45, 135, ⋯.

8. 4, 10, 28, 82, 244, ⋯ the nth term being $1 + 3^n$.

9. 3, 7, 11, 15, 19, ⋯.

10. $3, -7, -17, -27, \cdots$.

11. $v, v^2, v^3, v^4, \cdots$. Sum to n terms.

12. $1, r, r^2, r^3, r^4, \cdots$.

13. $(1 + i), (1 + i)^2, (1 + i)^3, (1 + i)^4, \cdots$. Sum to n terms.

14. The number of direct ancestors which an individual has is represented by the series, 2, 4, 8, 16, ⋯. Find the total number in the preceding 10 and in the preceding 20 generations.

15. According to Galton's law of heredity the parents contribute to the hereditary make-up of an individual $\frac{1}{2}$ of what is contributed by all the ancestors; the grandparents contribute $\frac{1}{4}$; the great-grandparents, of whom there are eight, contribute $\frac{1}{8}$; and so on; find the total contribution. Find the individual contribution of a single individual four generations back.

16. Insert three geometric means between 2 and 17; compute to one decimal place.

17. Insert one, two, and three geometric means between 1 and 2.

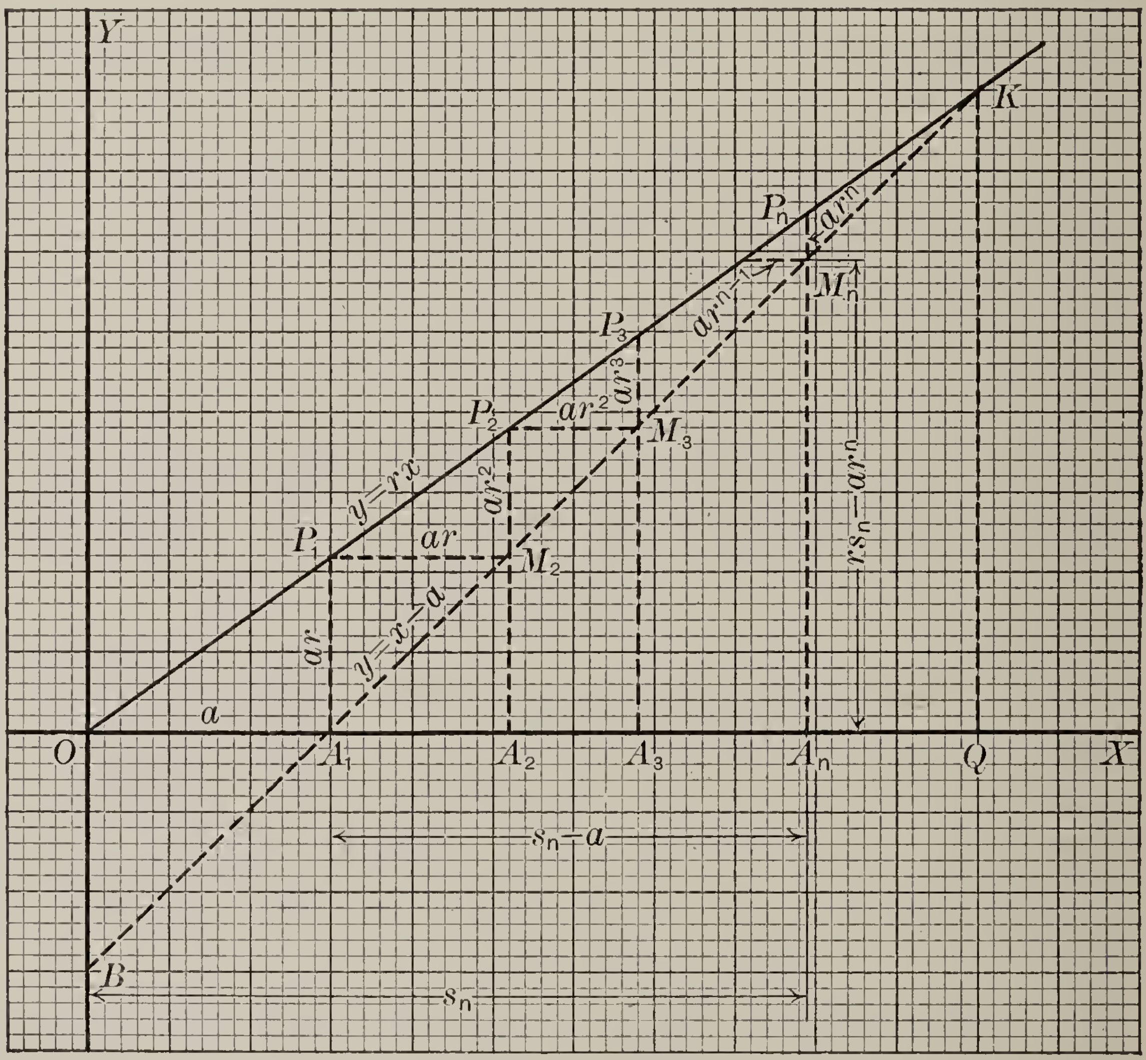
18. One of the so-called "three famous problems of antiquity" is to construct, using only ruler and compass, a cube which is double the volume of a given cube. This problem was very soon reduced to the insertion of two geometric means between a and $2\,a$; insert two geometric means between a and $2\,a$ and show that this gives the algebraic value of the side of a cube of double the volume; the geometrical solution has been demonstrated to be impossible if ruler and compass are the only instruments of construction.

19. In the population statistics on page 65 find the population of the United States in 1810 and 1910; between these two numbers insert 9 geometric means; find r; this represents the approximate decennial rate of increase in the population at 1810 which would give the final actual population in 1910. Compare with the actual census figures at the end of each ten years.

20. How would you find the regular annual rate of increase in the population of the United States, in other words the fixed annual percentage increase in population which would change the population from 7.2 millions in 1810 to 101.1 millions in 1910?

21. The plunger chamber of an air pump is approximately $\frac{1}{15}$ of the total air capacity; at each stroke $\frac{1}{15}$ of the air in the receiver is removed; after 10, 15, and 20 strokes find the proportionate amount of air remaining in the receiver; approximately how many strokes must be made to remove 99% of the original air?

4. Graphical representation — Geometrical series. — On the straight line $y = rx$ of slope angle α, with $\tan \alpha = r$, if $x = a$, ar, ar^2, ar^3, $\cdots$ the successive abscissas represent the terms of the geometrical series; by means of the 45° line through the origin these successive abscissas are readily constructed. However, a more favorable construction for a graphical treat-



Graphical summation of the geometrical series, $r < 1$

Note that the "sum to infinity," $r < 1$, is represented.

ment of the series regards each term of the series a, ar, ar^2, ar^3, $\cdots$ as an addition to the preceding abscissa; the successive additions to the ordinates, increments of the ordinates, will be ar^2, ar^3, ar^4, $\cdots$. By drawing the line,

$$y = x - a$$

at an angle of 45° with the x-axis, the successive abscissas are readily constructed; by drawing through the point $P_1(a, ar)$ a line parallel to the x-axis it intersects the 45° line drawn through $(a, 0)$ at a point M_2 such that $P_1M_2 = P_1A_1$, since $\angle P_1A_1M_2 = 45°$; a parallel to the y-axis through M_2 intersects the line $y = rx$ at a point P_2 such that $M_2P_2 = r \cdot P_1M_2 = ar^2$, and $A_2P_2 = ar + ar^2$. Similarly, if P_n represents the nth point found on our line, $y = rx$,

$$M_nP_n = ar^n,\ OA_n = s_n = a + ar + ar^2 + \cdots ar^{n-1},$$

$\frac{P_nA_n}{OA_n} = \tan \alpha = r$, whence $P_nA_n = rs_n$; $A_nM_n = sr_n - ar^n$; $A_1A_n = s_n - a$; but $A_1A_n = A_nM_n$, since the slope of the line $A_1M_2M_3$ is 1.

$$\therefore s_n - a = rs_n - ar^n,$$

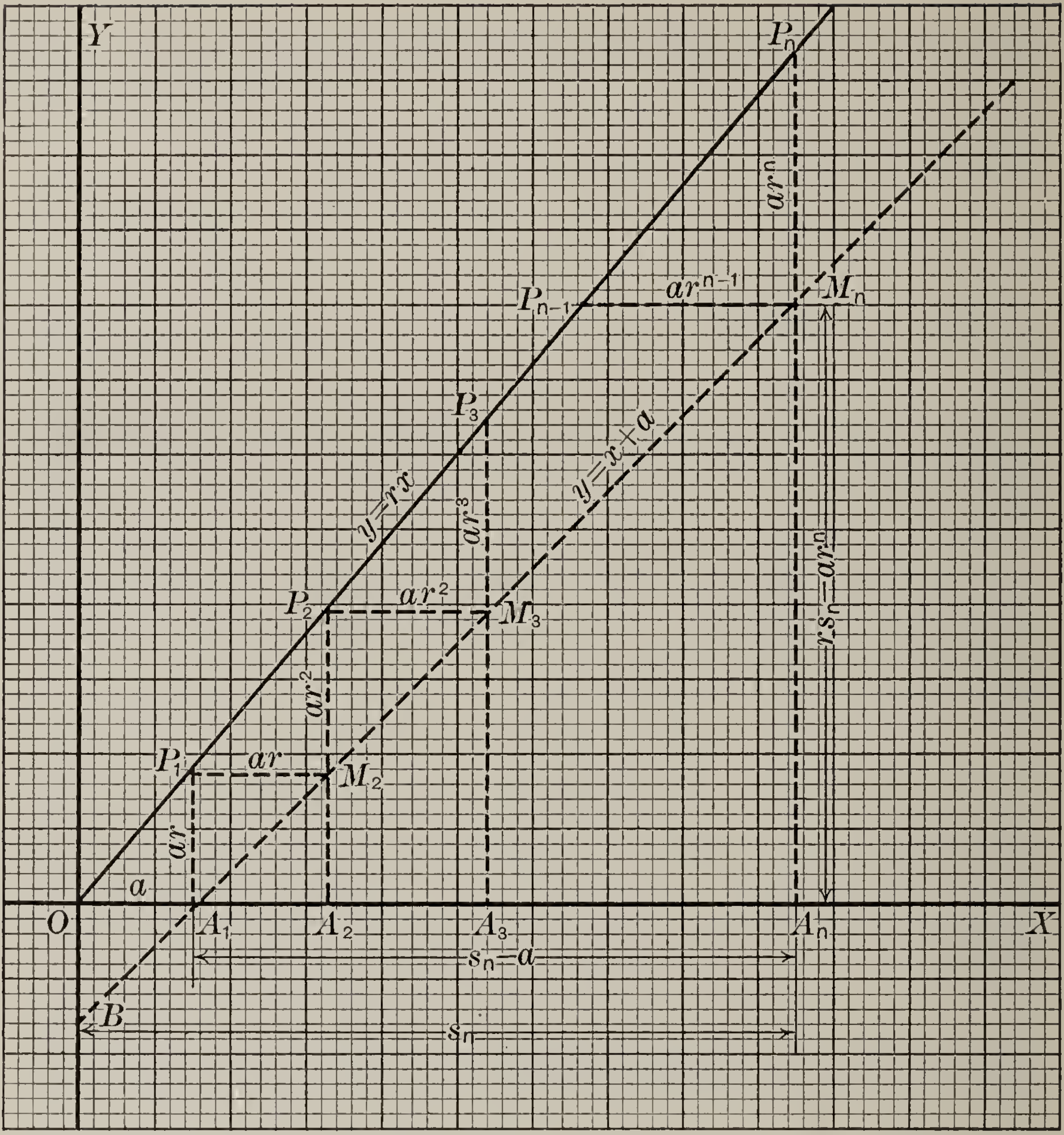
$$s_n(1 - r) = a - ar^n,$$

$$s_n = \frac{a - ar^n}{1 - r}.$$

If $r < 1$, as in our figure 1, $\alpha < 45°$, and the two lines intersect at K to the right of the origin; and the points of intersection M_2, M_3, $\cdots M_n \cdots$ will fall below K on the line BK. The abscissa of K represents the "sum to infinity" of our series, as it is evident that the series of triangles could be continued indefinitely in the opening OKA. Evidently also, solving $y = x - a$, $y = rx$,

$$rx = x - a,$$

$$x = \frac{a}{1 - r}.$$



Graphical summation of the geometrical series, $r > 1$

For $r > 1$, the figure is quite similar; the two lines, $y = rx$ and $y = x - a$, diverge; $\tan \alpha = r$, $\alpha > 45°$.

Evidently,
$$A_nM_n = A_1A_n,$$
$$A_nP_n = rs_n,$$
$$A_nM_n = rs_n - ar^n,$$
$$OA_n = s.$$
$$A_1A_n = s - a,$$
whence
$$s_n - a = rs_n - ar^n.$$
$$s_n(1 - r) = a - ar^n,$$
as before
$$s_n = \frac{a - ar^n}{1 - r} \text{ or } \frac{ar^n - a}{r - 1}.$$

5. Historical note. — Arithmetical and geometrical series are found in the oldest mathematical documents known, both in the remains of ancient Egypt and of ancient Babylon. The system of numbers used by the Babylonians as early as 2000–3000 B.C. was sexagesimal, increasing by powers of 60 in geometrical series. Further an early Babylonian clay tablet gives the portion of the moon's surface illuminated on each of fifteen successive nights from new moon to full moon by a geometric and an arithmetical series. The moon's surface is conceived as divided into 240 parts; on the first five nights 5, 10, 20, 40, and 80 parts, respectively, are illuminated and on the following ten nights, 96, 112, 128, 144, ⋯ and on in arithmetical progression to 240.

The Egyptian manual of mathematics of 1700 B.C. (or thereabouts), includes two rather complicated problems on arithmetical series, involving also the insertion of means, and one problem involving the summation of a geometric series.

The equivalent of a general formula for summation of a geometrical series was first established rigorously by the Greeks, and appears in Euclid's Elements, Book IX, prop. 36. The first summation "to infinity" of a decreasing geometrical progression was effected by the great Archimedes (287–212 B.C.) who employed the formula in finding the area of a segment of a parabola.

In a great part of the later development of mathematics such series have played a prominent rôle, in some measure because of their own intrinsic importance and in some measure as fundamental in the discussion of other types of series.

6. Annuity formulas. *a. Accumulated value of an annuity.*— The geometrical series plays a large rôle in the theory of investments and insurance. We have shown (p. 54) that at rate i per year, compounded annually, 1 will amount in n years to $(1+i)^n$; if 1 is invested at rate i at the end of each year, *i.e.* annually, for n years, these payments constitute an annuity of 1 for n intervals, at i per annum. The total accumulated value $s_{\overline{n|}}$, at the end of n years of such an annuity, is the sum of the geometrical series

$$(1+i)^{n-1},\ (1+i)^{n-2},\ (1+i)^{n-3},\ (1+i)^{n-4}, \cdots (1+i),\ 1,$$

since the first payment made at the end of the first year accumulates for $(n-1)$ years, the second for $(n-2)$ years, $\cdots$, and the nth payment of 1 is made at the end of the n years. The sum, called the amount of the annuity,

$$s_{\overline{n|}} = \frac{(1+i)^n - 1}{i},$$

represents the accumulated value of an annuity of 1 per interval for n years or intervals at a rate of i per year or interval.

b. The annuity which will accumulate to 1. — Very evidently K per annum will produce at the end of n years $Ks_n = K \cdot \frac{(1+i)^n - 1}{i}$; the annuity which in n years will amount to 1 is evidently the value of K which makes $Ks_n = 1$; this value is $\frac{1}{s_{\overline{n|}}} = \frac{i}{(1+i)^n - 1}$.

c. Present value of an annuity of 1. — The accumulated value of 1 to be paid at the end of n years at i per year is $(1+i)^n$; K will accumulate at the rate i in n years to $K(1+i)^n$; this means that K dollars (or units) in hand accumulates to $K(1+i)^n$ dollars, and that $K(1+i)^n$ dollars to be paid n years hence is worth K dollars now, money at rate i per year. Hence the present value of 1 to be paid n years hence is a value of K which makes $K(1+i)^n = 1$, or $K = \frac{1}{(1+i)^n} = v^n$, wherein $v = \frac{1}{1+i}$. The present value of an annuity of 1 per annum for n years when money is worth i per annum is the sum of the geometrical series,

$$v,\ v^2,\ v^3,\ v^4,\ v^5,\ v^6,\ v^7, \cdots v^n.$$

The first term v is the present value of the first payment of 1 which is to be made 1 year from date; the second term is the present value of the second payment of 1 to be made in 2 years; ⋯; v^n is the present value of the final payment of 1 to be made n years hence.

$$a_{\overline{n}|} = \frac{v - v^{n+1}}{1 - v} = \frac{1 - v^n}{\frac{1}{v} - 1} = \frac{1 - v^n}{i}.$$

d. The annuity which 1 *will purchase.* — The present value of K per annum for n years is $Ka_{\overline{n}|} = K\frac{1 - v^n}{i}$; the annuity which is worth 1 at the present time is evidently a value K which makes $Ka_{\overline{n}|} = 1$, whence $K = \frac{1}{a_{\overline{n}|}} = \frac{i}{1 - v^n}$. This is the annuity which 1 will purchase.

e. Summary of interest functions. —

These six functions,

$r^n = (1 + i)^n$, accumulation of 1,

$v^n = (1 + i)^{-n}$, discount value,

$s_{\overline{n}|} = \frac{(1 + i)^n - 1}{i}$, accumulated annuity value,

$a_{\overline{n}|} = \frac{1 - (1 + i)^{-n}}{i} = \frac{1 - v^n}{i}$, present value of annuity,

$\frac{1}{s_{\overline{n}|}} = \frac{i}{(1 + i)^n - 1}$, annuity to accumulate to 1,

and $\frac{1}{a_{\overline{n}|}} = \frac{i}{1 - (1 + i)^{-n}}$, the annuity which 1 will purchase,

are of fundamental importance in the valuation of bonds, and in all problems where stipulated payments are to be made at stipulated intervals, and also in the theory of interest.

If interest is to be compounded semiannually and payments are made semiannually, the interval can be considered as 6 months and the rate of interest as one half the stated rate; similarly the interval can be considered as 3 months and the rate of interest as one fourth the stated interest if interest is compounded quarterly and payments made quarterly. Other types of problems with payments falling between interest periods are beyond the scope of this work.

PROBLEMS

1. Compute the value of $s_{\overline{20}|}$ for 6 %, 5 %, 4 %, using logarithms to obtain $(1+.06)^{20}$, $(1+.05)^{20}$, and $(1+.04)^{20}$. What percentage of error is introduced using four-place tables? Discuss the effect in finding the accumulated value of an annuity of $100 per year for 20 years. Check by the tables given at the back of this book.

2. Compute $a_{\overline{20}|}$, and discuss as in 1.

3. Find by logarithms from your values of $s_{\overline{20}|}$ and $a_{\overline{20}|}$ the annuity which will accumulate to 1 in 20 years, and the annuity which 1 in hand will purchase.

4. If payments of $50 per year are made semiannually and interest is compounded semiannually, find the accumulated value of this annuity at the end of 20 years for a nominal interest rate of 6 %, 5 %, and 4 % respectively, per annum (3 %, $2\frac{1}{2}$ %, and 2 % per interval). Use the tables.

5. What annual payments continued for 10 years are equal to $1000 cash in hand?

6. What annual payments continued for 10 years are equivalent to $1000 to be paid at the end of 10 years? to $1000 to be paid at the end of 20 years?

7. Prove $s_{\overline{n}|}=(1+i)^n a_{\overline{n}|}$; and $a_{\overline{n}|}=v^n \cdot s_{\overline{n}|}$. Discuss.

8. Show algebraically that $\frac{1}{a_{\overline{n}|}}-\frac{1}{s_{\overline{n}|}}=i$.

NOTE. — The difference between 1 in hand and 1 to be paid in n years is simply the earning power of the 1 in hand for this period of n years; if money is worth 6 % per year, 1 in hand will earn every year for n years .06 in addition to preserving itself; 1 in n years is worth simply 1 then; $\frac{1}{a_{\overline{n}|}}$ is the annuity for n years which 1 in hand purchases, and $\frac{1}{s_{\overline{n}|}}$ is the annuity equivalent to 1 to be paid in n years; the difference is the annual earning of the 1 in hand, i.

9. Give the arithmetical series represented by the ordinates of $y = 3x + 7$, for integral values of x from 0 to 10. What is the sum of this series? Between 1 and 2 interpolate 9 values, at equal intervals, and state corresponding series; what corresponds to the tabular difference?

10. Discuss as in problem 9 the corresponding ordinates of $2y = -3x + 7$.

11. Sum to 20 terms the series 7, 5, 3, 1, -1, ⋯.

12. Write the 20th term of 7, 5, 3, 1, -1, ⋯.

13. Write the tenth term of 7, 5, $\frac{25}{7}$, $\frac{125}{49}$, ⋯.

14. Find the arithmetical and the geometrical means between 7 and 5.

15. *Historical problem.* — It is related that an Indian prince who wished to reward the inventor of the game of chess suggested to the inventor that he should name the reward he desired. The scholar replied that he would take 1 grain of wheat for the first square of the board, 2 for the second, 4 for the third, 8 for the fourth, 16 for the fifth, and so on in geometrical progression to cover the 64 squares. The prince agreed but found, on the computation, that the value exceeded that of his realm. Taking 10,000 grains as approximately a pint, make a rough calculation of the amount involved.

16. *Historical problem.* — In textbooks of the sixteenth century the following problem frequently appears. A blacksmith being asked his price for shoeing a horse replied that for the first nail he would charge one fourth of one cent (use this in place of farthing, or pfennig), $\frac{1}{2}$ cent for the second nail, 1 cent for the third, 2 for the fourth, and so on for the thirty-two nails. Compute the price.

7. Annuity applications. — Brief tables of the annuity functions are given at the back of this book; somewhat larger tables will be found in the Bulletin No. 136 of the U. S. Bureau of Agriculture, which includes also a more extensive treatment of the subject of bonds and annuities by Professor James W. Glover.

a. Common annuity. — To find the purchase price of an annuity of k dollars per interval for n intervals when the current rate is i per interval is obviously a direct application of the $a_{\overline{n}|}$ table, as the purchase price is simply the present value of the series of payments. If the first payment of the annuity is to be made r intervals hence, a deferred annuity for n intervals, the price may be considered as the difference between an annuity for $r-1$ intervals and an annuity for $n+r-1$ intervals. $a_{\overline{n+r-1}|}$ gives a payment every interval for $n+r-1$ intervals, the first made at one interval from the present time; $a_{\overline{r-1}|}$ gives a payment every interval for $(r-1)$ intervals, the first as before, and the last $(r-1)$ intervals from the present time; the difference is the value of the deferred annuity of n payments, first payment to be made at the end of r intervals.

Some large banks, trust companies, and insurance companies do this type of business. Frequently a purchaser desires an annuity to be paid annually terminating with the death of the purchaser; this involves then a life contingency, and the discussion and solution of this problem require new methods and new tables.

b. Farm loans. — To extinguish or amortize a debt by n annual payments of fixed amount is the type of problem which arises under the recent Farm Loan Act. Thus a farmer borrowing \$10,000 at the bank at 5% interest may desire to make such a payment as to extinguish the debt in 30 years, money being worth 5% annually. The problem may be solved by considering the annuity which \$10,000 will purchase at 5% interest for 30 years or \$10,000 × .06505, giving \$650.50. The problem may also be solved by considering the interest as paid each year, \$500, in addition to which an annual payment must be made to accumulate at 5 % to \$10,000 at the end of 30 years. This annual payment is found to be \$150.50. The value found, \$650.50, may be checked, as below, by using the $s_{\overline{n}|}$ table. Or another check is to find the value at the end of 30 years of the \$10,000 or \$10,000 $(1 + .05)^{30}$ and compare this with the value of \$650.50 $\times s_{\overline{30}|}$.

Suppose, on the other hand, that the borrower desires to pay approximately \$600 per year, applying the extra amount each year to the debt. The annuity which 1 will purchase, $\frac{1}{a_{\overline{n}|}}$, is the function involved. The question here may then be put, For what period of years at 5 % interest will \$10,000 purchase an annuity of \$600 per annum? We will consider only approximate solutions, taken from the tables.

The tables show that at 5 % interest \$10,000 will purchase an annuity of \$10,000 $\times \frac{1}{a_{\overline{36}|}} =$ \$10,000 $\times$ .06043 for 36 years, or \$604.30 annually for 36 years; \$10,000 will purchase an annuity of \$598.40 for 37 years. 36 years would be taken, and this period of 36 years is provided for, as an amortization term, by the government. By paying every year \$100 more than the interest, at the end of 36 years the accumulated value of this annuity, the excess \$100 over the interest, would be worth at 5 %: \$100 $\times s_{\overline{36}|} =$ \$100 $\times$ 95.8363 = \$9,583.63, leaving \$416.37 due at the end of 36 years. This amount with interest at 5 % should be the next and final payment.

c. *Sinking funds.* — If a city issues bonds to be redeemed 20 or 30 or 40 or n years hence, it is commonly desirable to provide for the repayment of the bonds by an annual (interval) payment allowed to accumulate at i per annum (per interval). Similarly in business a manufacturing concern using an expensive piece of machinery which has a probable lifetime of 20, 30, or 40 years must provide for the eventual replacement of this machine by an annual payment, out of earnings, into a sinking fund. In this type of problem the function involved is the annuity which will accumulate to 1 in n years. Thus to provide for replacement of a $ 10,000 piece of machinery in 30 years money at 5 % requires an annual payment of $\$10,000 \times \frac{1}{s_{\overline{30}|}}$ or $\$10,000 \times \frac{i}{(1+i)^{30}-1}$ which equals $ 150.50 per annum.

d. Bonds at premium and discount. — If a city issues bonds at 5 % when money is worth in the money market 4 %, the bonds will sell at higher than face value, since they pay on each $ 100 an annuity of $ 5.00 per year for the term of the bond, when investors are demanding only $ 4.00 per annum with security of capital. This higher price is, on the basis of money at 4 %, the present value of an annuity of $ 1.00 per annum for the term of the bond or $1 \times a_{\overline{n}|}$ at 4 %. The difference between the par value or face value of a bond and the price offered by investors is called the premium (or discount, when the price offered is less) on the bond.

If a city issues bonds at 5 % when investors are demanding 6 %, a bond for $ 100 will sell at a discount of $1 \times a_{\overline{n}|}$ at 6 %, since the investor receives from these bonds not $ 6.00 per annum but only $ 5.00 per annum. Evidently the longer the bond has to run the greater would be the discount.

In general terms the premium on a bond of face value C, paying a dividend rate g, bought to yield j per annum is

$$P = C(g - j)a_{\overline{n}|} \text{ at } j \text{ per annum.}$$

PROBLEMS

1. If a father sets aside annually $ 100 per year as a fund for his son when the latter becomes of age, to what will the fund amount at the end of 21 years, the money accumulating at 4 % interest?

2. If a farm mortgage of $10,000 draws 5 % interest and the farmer pays annually $ 600, what is the accumulated value at the end of 21 years of the excess payments of $ 100 per annum, accumulated at 5 % ?

3. What annual payment will accumulate at 5 % in 30 years to $ 10,000? What annual payment would have to be made on a $10,000 mortgage, to extinguish the debt in 30 years, money worth 6 % ?

4. Find in the tables the annuity for 30 years which $ 10,000 will purchase.

5. What semiannual payment will accumulate in 30 years (60 payments) to $ 10,000, interest being 5 % compounded semiannually?

6. Find the cost of an annuity of $ 10,000 per year to run for 10 years, 20 years, and 30 years, respectively, money being worth 4 %.

7. If a city issues $10,000 in bonds, what amount must be set aside annually to accumulate at 4 % interest to redeem the bonds at the end of 20 years?

8. Find the cost of an annuity of $500 per annum for 10 years, the first payment to be made 10 years hence, 20 years hence, and 30 years hence, respectively.

9. What premium can you afford to pay on a $ 10,000 bond drawing 5 % to run 20 years, if money is worth 4 % ? What discount should you receive if money is worth 6 % ?

10. What is the present value of $10,000 to be paid 20 years hence, money at 5 % ?

11. Which is the better offer for a piece of property, money being worth 5 %, a rental of $ 600 per year for 20 years, or a price of $ 10,000? A rental of $ 600 per year for 10 years, and $ 700 per year for the following 10 years, or $ 12,500? Assume no change in the price of the real estate in 20 years.

12. What sum at 4 % interest should a railroad set aside each year to replace engines worth $ 35,000, which have an estimated life of 25 years? to replace buildings worth $ 1,000,000 which have an estimated life of 100 years?

13. If a man invests $100 each year for 20 years, what annuity, for 20 years, can he purchase at the end of the first 20 years, money at 5 % interest?

14. If a man agrees to take $1000 a year for five years for a house originally offered at $5000, what is the discount when money is worth 5 % ?

15. At 5 % interest what annual payment for five years is equivalent to $5000 cash in hand ?

16. Answer questions 14 and 15, assuming that the first $1000 is to be paid immediately.

CHAPTER XII

BINOMIAL SERIES AND APPLICATIONS

1. Binomial series. — The expressions $(1+i)^4$, $(1+i)^{20}$, ... can be developed in powers of i by means of the binomial expansion,

$$(a+x)^n = a^n + \frac{n}{1}\cdot a^{n-1}x + \frac{n(n-1)}{1\cdot 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cdot a^{n-3}x^3$$
$$+ \cdots \frac{n(n-1)(n-2)\cdots \text{to } (r-1) \text{ factors}}{1\cdot 2\cdot 3\cdot 4\cdots(r-1)}\cdot a^{n-r+1}x^{r-1} + \cdots.$$

In particular, if $a=1$,

$$(1+x)^n = 1 + \frac{n}{1}\cdot x + \frac{n(n-1)}{1\cdot 2}\cdot x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cdot x^3$$
$$+ \cdots \frac{n(n-1)(n-2)(n-3) \text{ to } (r-1) \text{ factors}}{1\cdot 2\cdot 3\cdot 4\cdots(r-1)}\cdot x^{r-1} + \cdots.$$

This formula expresses a rule for the formation of successive terms of the expansion of $(a+x)^n$ or $(1+x)^n$; the first term contains a with the exponent n of the binomial; the second term has as coefficient the exponent, or index, of the binomial, x appears to the first power, and the exponent of a decreases by 1; the coefficient of the third term has two factors in numerator and denominator, in the numerator $n(n-1)$ and in the denominator $1\cdot 2$; a appears with exponent 1 less than in the preceding term, and x with exponent 1 greater; each following term can be obtained from the preceding by introducing one further factor in numerator and denominator and at the same time decreasing the power of a by one and increasing that of x by one; the further factor in the new numerator is one less than the last one introduced there, and the further factor in the new denominator is one greater than the last one of the preceding denominator; the coefficient of the term in x^{r-1} contains in the numerator $(r-1)$ integral factors from n down, and in the denominator $(r-1)$ integral factors from 1 up; x appears with exponent $r-1$ and a with the exponent which added to $r-1$ makes n, *i.e.* $n-r+1$.

Illustrations of the binomial expansion.

$$a.\quad (a+x)^3 = a^3 + \frac{3}{1}a^2x + \frac{3\cdot 2}{1\cdot 2}ax^2 + \frac{3\cdot 2\cdot 1}{1\cdot 2\cdot 3}x^3$$
$$= a^3 + 3\,a^2x + 3\,ax^2 + x^3.$$

$$b.\quad (a+x)^{15} = \overset{\text{1ST}}{a^{15}} + \overset{\text{2D}}{\frac{15}{1}a^{14}x} + \overset{\text{3D}}{\frac{15\cdot 14}{1\cdot 2}a^{13}x^2} + \cdots.$$
$$\overset{\text{10TH TERM}}{\frac{15\cdot 14\cdot 13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}a^6x^9} + \cdots.$$

Note that it is well in writing the tenth term to begin with x^9; then a enters to the sixth power as in every term of this expansion the exponents of a and x together make 15; the denominator contains 9 factors, 1, 2, 3, ⋯ 9; the numerator contains 9 factors, which should be counted as they are written; finally cancellation should be made, giving

$$5\cdot 7\cdot 11\cdot 13\,a^6x^9.$$

$$c.\quad (a-x)^{15} = a^{15} - \frac{15}{1}a^{14}x + \frac{15\cdot 14}{1\cdot 2}a^{13}x^2 - \frac{15\cdot 14\cdot 13}{1\cdot 2\cdot 3}a^{12}x^3$$
$$+ \cdots - \overset{\text{10TH TERM}}{5\cdot 7\cdot 11\cdot 13\,a^6x^9} + \cdots.$$

$$d.\quad (1+x)^{15} = 1 + 15\,x + \frac{15\cdot 14}{1\cdot 2}x^2 + \frac{15\cdot 14\cdot 13}{1\cdot 2\cdot 3}x^3 +$$
$$\frac{15\cdot 14\cdot 13\cdot 12}{1\cdot 2\cdot 3\cdot 4}x^4 + \cdots \overset{\text{10TH}}{5\cdot 7\cdot 11\cdot 13\,x^9}.$$

Note that the powers of a in each term can be dropped, as every power of one equals one.

e. Compute to 4 decimal places $(1 + .04)^{15}$.

$$(1+.04)^{15} = 1 + 15(.04) + \frac{15 \cdot 14}{1 \cdot 2}(.04)^2 + \frac{15 \cdot 14 \cdot 13}{1 \cdot 2 \cdot 3}(.04)^3 + \frac{15 \cdot 14 \cdot 13 \cdot 12}{1 \cdot 2 \cdot 3 \cdot 4}(.04)^4 + \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}(.04)^5 + \cdots.$$

.60 second	.0291	1.00000 first term
7	.12	.60000 second term
4.20	5).00349 fifth	.16800 third term
.04	.00070	.02912 fourth term
3).1680 third term	11	.00349 fifth term
.0560	.0077	.00031 sixth term
.04	.04	1.80092 *Ans.*
.002240	.00031 sixth	
.02240 10 times		
672 3 times		

.02912 fourth term, to be multiplied by .12

Note here the method of computation given at the left; each term is obtained from the preceding term; three new factors of which one is .04 enter into each succeeding term, two in the numerator and one in the denominator; these three factors after the second term (.60) are $\frac{14 \times .04}{2}$ or $7 \times .04$; then $\frac{13 \times .04}{3}$; then $\frac{12 \times .04}{4}$ or .12; then $\frac{11 \times .04}{5}$, which might well be treated as $.008 \times 11$; then the factor $\frac{10 \times .04}{6}$ would give the seventh term from the sixth, making about 2 in the fifth decimal place.

The expansion of

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots$$

may be written as follows:

$$(1+x)^n = 1 + nx + \frac{(n-1)x}{2}T_2 + \frac{(n-2)x}{3}T_3 + \frac{(n-3)x}{4}T_4 + \frac{(n-4)x}{5}T_5 + \cdots,$$

in which T_2, T_3, T_4, T_5, ⋯ designate the second, third, fourth, ⋯ terms respectively. This type of representation, in which each term is obtained from the preceding, is frequently of use in statistical work and in computation.

f. Write the sixth and sixteenth terms of $(1+x)^{17}$.

6TH TERM	16TH TERM
$\frac{17\cdot 16\cdot 15\cdot 14\cdot 13}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5$;	$\frac{17\cdot 16\cdots}{1\cdot 2\cdots}\cdot x^{15}$.

g. What decimal place is affected by the sixth term of $(1.06)^{17}$?

$$\frac{17\cdot 16\cdot 15\cdot 14\cdot 13}{1\cdot 2\cdot 3\cdot 4\cdot 5}(.06)^5;\quad (.06)^3 = .000216;\quad (.06)^4 = .00001296;$$

$(.06)^5 = .00000078^-$; multiply .00000078 by 4, this by 7, this by 13, and then by 17, rejecting any beyond 3 significant figures; this gives .00000312, .0000218, .000286, and finally .00476. Note that the computation of six terms of $(1+.06)^{17}$ involves not very much more numerical labor than this determination.

h. Write six terms of $(a-3x)^{12}$.

$$a^{12} - 12\,a^{11}(3x) + \frac{12\cdot 11}{1\cdot 2}a^{10}(3x)^2 - \frac{12\cdot 11\cdot 10}{1\cdot 2\cdot 3}a^9(3x)^3$$
$$+\frac{12\cdot 11\cdot 10\cdot 9}{1\cdot 2\cdot 3\cdot 4}a^8(3x)^4 - \frac{12\cdot 11\cdot 10\cdot 9\cdot 8}{1\cdot 2\cdot 3\cdot 4\cdot 5}a^7(3x)^5.$$

$$a^{12} - 36\,a^{11}x + 2\cdot 3^3\cdot 11\,a^{10}x^2 - 2^2\cdot 3^3\cdot 5\cdot 11\,a^9x^3 + 5\cdot 3^6\cdot 11\,a^8x^4$$
$$-\,2^3\cdot 3^7\cdot 11\,a^7x^5.$$

It is not necessary or desirable to perform the multiplication in such an expression as $2^2\cdot 3^3\cdot 5\cdot 11$; such terms, if desired numerically, are usually obtained progressively from preceding terms as in example (*e*) above.

PROBLEMS

1. Expand to 6 terms, $(a+x)^6$, $(a+x)^{14}$, $(a+2x)^{10}$, $(a-3x)^9$.

2. Write 5 terms in simplest form (prime factors) of $(1+x)^6$, $(1+x)^{14}$, $(1+2x)^{10}$, $(1-3x)^9$.

3. Compute to 4 decimal places $(1+.05)^6$, $(1+.05)^{14}$, $(1+.05)^{10}$, $(1-.05)^9$.

4. Compute to 2 decimal places the value at the end of 10 years of $100 placed at interest at 6% compounded annually; use $100 \times (1+.06)^{10}$. How could you use the result obtained to find the value at the end of 20 years?

5. From problem 3 give the amount at the end of 6, 14, and 10 years, respectively, of $256 at 5% interest, compounded annually.

6. Find the value at the end of 6, 14, and 10 years respectively of an annuity of 1 per annum, paid at the end of each year, interest at 5%. Use the formula $s_{\overline{n|}} = \frac{(1+i)^n - 1}{i}$ and the preceding results.

7. Compute the values in 3, 4, and 5 by logarithms and compare.

8. Given $2^{10} = 1024$, find to 1 decimal place $(2+.01)^{10}$. *Ans.* 1076.4.

Find also $(2.1)^{10}$ to one decimal place, and check by logs.

9. Find $(12.3)^3$ to five significant figures.

10. Find the amount at the end of 20 years of $100 placed at interest, 3%, compounded semiannually.

11. Write the 8th term of $(1-3x)^{17}$ and of $(1+x)^{29}$.

12. How many terms in $(1-3x)^{17}$? Write the middle terms.

13. Write in simplest form the coefficient of x^6 in $(1-2x)^{25}$.

14. Write the series for $(1+x)^5$, $(1+x)^6$, $(1+x)^7$, and $(1+x)^8$. What is the sum of the coefficients?

(NOTE. — Substitute 1 for x.)

15. What is the sum of the coefficients in $(1+x)^{19}$?

16. Write 7 terms of $(1+\sqrt{x})^{10}$ and of $(1+\sqrt[3]{x})^{12}$.

2. Proof of $(a+x)^n = a^n + \frac{n}{1}a^{n-1}x + \frac{n(n-1)}{1\cdot 2}\,a^{n-2}x^2 +$

$$\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\,a^{n-3}x^3 + \cdots \overset{r\text{th Term}\qquad\qquad \text{to a Total of } (r-1) \text{ Factors}}{\frac{n(n-1)(n-2)\cdots}{1\cdot 2\cdot 3\cdot 4\cdots(r-1)}\,a^{n-r+1}x^{r-1}} + \cdots.$$

The proof of this expansion for positive integral values of n is effected by the process called mathematical induction.

Evidently $(a+x)^2 = a^2 + 2\,ax + x^2$, follows the rule;

also $\qquad (a+x)^3 = a^3 + 3\,a^2x + 3\,ax^2 + x^3$, follows the rule.

By the rule,

$$(a+x)^4 = a^4 + \frac{4}{1}a^3x + \frac{4\cdot 3}{1\cdot 2}a^2x^2 + \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}ax^3 + \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}x^4$$
$$= a^4 + 4\,a^3x + 6\,a^2x^2 + 4\,ax^3 + x^4;$$

by actual multiplication we find the same series, showing that the rule as given holds for $n = 4$.

Assume

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1\cdot 2}\,a^{n-2}x^2 + \cdots$$
$$\frac{n(n-1)(n-2)\cdots(n-r+3)}{1\cdot 2\cdot 3\cdots(r-2)}\,a^{n-r+2}x^{r-2}$$
$$\overset{r\text{th Term}}{+\frac{n(n-1)(n-2)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots(r-1)}\,a^{n-r+1}x^{r-1}};$$

multiply by $\qquad a + x = a + x$

$$(a+x)^{n+1} = a^{n+1} + (n+1)a^{n-1}x + \left(n + \frac{n(n-1)}{1\cdot 2}\right)a^nx^2 + \cdots$$

$$\overset{\text{The New } r\text{th Term}}{\left(\frac{n(n-1)(n-2)\cdots(n-r+3)(n-r+2)}{1\cdot 2\cdot 3\cdots(r-2)(r-1)} + \frac{n(n-1)(n-2)\cdots(n-r+3)}{1\cdot 2\cdot 3\cdots(r-2)}\right)a^{n-r+2}x^{r-1}}$$

Note that these two coefficients of the new rth term have the first $(r-2)$ factors of numerator and denominator the same; multiply the denominator and numerator of the second term by $r-1$ and then add the numerators, taking out the common factors, giving

$$\left\{\frac{\overset{\text{NEW } r\text{TH TERM COMMON FACTORS}}{[n(n-1)\cdots(n-r+3)]}\,[\overset{\text{REMAINING FACTOR OF FIRST TERM}}{(n-r+2)} + \overset{\text{REMAINING FACTOR OF SECOND TERM}}{(r-1)}]}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(r-1)}\right\} a^{n-r+2}x^{r-1}.$$

The new rth term may be written, then,

$$\frac{(n+1)(n)(n-1)(n-2)\cdots(n-r+3)}{1\cdot 2\cdot 3\cdot 4\cdots(r-1)}\,a^{(n+1)-r+1}x^{r-1};$$

the rth term of $(a+x)^{n+1}$ is formed according to the rule with $(n+1)$ substituted throughout for n; the numerator contains $(r-1)$ factors beginning with $(n+1)$, and the denominator contains the same number of factors beginning with 1.

Hence if this expansion assumed for $(a+x)^n$ is correct for any value n, it is correct for a value one greater, $n+1$. The theorem is true, by trial, for $n=4$; hence it is true for $n=5$; since it is true for $n=5$ it is also true for $n=6$; $\cdots$ and so for every integral value of n.

3. Binomial series; any exponent. — The equation,

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^4 + \cdots$$

can be shown by methods of the higher mathematics to hold for all values of n when $-1 < x \leq 1$. This means that a series for $(1+x)^{\frac{1}{3}}$ can be obtained by substituting $n=\frac{1}{3}$ in the above formula. Similarly $(1+x)^{-3}$ and $(1+x)^{-7}$, $(1+x)^{-\frac{3}{2}}$ or $(1+x)^{\sqrt{2}}$ can be developed in powers of x, when $|x|<1$, by substituting for n the values -3, and -7, and the like in the above formula. These series are of frequent use in statistical work. Thus if money is worth 6 % *per annum* the interest for one half year is not 3 % of 1, since this rate continued for the full year would give at the end of one year in addition to the 6 % of 1, the interest on the 3 % of 1 for one half year; the interest on 1 for one half year is taken to be

$$j = (1 + .06)^{\frac{1}{2}} - 1,$$

and this rate of interest *per half year* accumulates at the end of two half years a principal of one to (1.06), or is equivalent to 6 % per year. Similarly the effective rate of 6 % per annum means that the interest for one fourth of a year will not be .015 times the principal, but rather $(1 + .06)^{\frac{1}{4}} - 1$, since this is the rate of interest for one quarter of a year which continued for four quarters will accumulate a principal of 1 to 1.06, since $[(1 + .06)^{\frac{1}{4}}]^4 = 1.06$.

In our illustrative problems we will assume that the terms that follow any given term in the expansion of an expression like $(1 + x)^{\frac{1}{2}}$ are together less than the last term given; the general proof of the convergence of these series is reserved for the calculus; however, it is evident here that each new factor, when x is less than 1, diminishes in value and finally the terms are in turn respectively less than the terms of a geometrical series with ratio x; thus, below in the expansion of $(1 + .06)^{\frac{1}{2}}$, beyond any given term the terms are respectively less, term by term, than the terms of a geometrical series with ratio .06; the sum of all terms beyond the fourth term is certainly less than $\frac{a}{.94}$, wherein a is the fourth term, since the corresponding geometrical series even to an infinite number of terms has only this sum, $\frac{a}{1-r} = \frac{a}{.94}$.

Note particularly that the expansion of $(1 + x)^n$ for values of n other than positive integers leads to a series which has no termination, *i.e.* to an infinite series; this series is valid and has meaning only when $|x| < 1$; similarly any infinite series given by $(a + x)^n$ has meaning only when $\left|\frac{x}{a}\right| < 1$. In our further discussion this limitation will be consistently assumed.

As an illustration of the possible absurdity from the point of view of finite series of the infinite series given by $(1 + x)^n$, let us take the fraction $\frac{1}{1 - x}$ which may be written $(1 - x)^{-1}$, and developed by the binomial theorem as,

$$1 + x + x^2 + x^3 + x^4 + x^5 + \cdots.$$

For values of x numerically less than unity this series is valid, but if you put x equal to 3, or -5, or other value greater than unity, you obtain an absurdity.

4. Illustrative problems. — *a.* Compute $(1 + .06)^{\frac{1}{2}}$ to 5 decimal places.

$$(1 + x)^{\frac{1}{2}} = 1 + \tfrac{1}{2} \cdot x + \frac{\frac{1}{2}(\frac{1}{2} - 1)}{1 \cdot 2} \cdot x^2 + \frac{\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)}{1 \cdot 2 \cdot 3} \cdot x^3 + \cdots$$

$$\begin{aligned}(1 + .06)^{\frac{1}{2}} &= 1 + \tfrac{1}{2}(.06) - \tfrac{1}{8}(.06)^2 + \tfrac{1}{16}(.06)^3 + \cdots \\ &= 1 + .03 - .00045 + .0000135 + \cdots \\ &= 1.02956.\end{aligned}$$

If the value of $(1.06)^{\frac{1}{2}}$ were desired to eight decimal places, progressive computation would be desirable, appearing as follows:

$$T_1 = 1$$

$$T_2 = \tfrac{1}{2}(.06)\,T_1 = .03$$

$$T_3 = -\frac{1}{2}\left(\frac{.06}{2}\right) T_2 = -.00045$$

$$T_4 = -\frac{3}{2} \cdot \frac{.06}{3}\, T_3 = +.0000135$$

$$\begin{array}{r} 4\,\underline{).0000135} \\ .000003375 \\ \underline{5} \\ .000016875 \\ \underline{.03} \\ .00000050625 \\ \underline{.042} \\ 1012 \\ \underline{2024} \\ .00000002125 \end{array}$$

$$T_5 = -\frac{5}{2} \cdot \frac{.06}{4}\, T_4 = -.000000506$$

$$T_6 = -\frac{7}{2} \cdot \frac{.06}{5}\, T_5 = +.000000021$$

$$T_7 = -\frac{9}{2} \cdot \frac{.06}{6}\, T_6 = -.000000001$$

$$\begin{array}{r} 1.030013521 \\ \underline{-.000450507} \\ 1.029563014 \end{array}$$

The letters T_1, T_2, T_3, T_4, $\cdots$ represent the first, second, third, fourth, and succeeding terms; each term is obtained from the preceding, introducing the new factors.

b. Write six terms of the series for $(1+x)^{\frac{1}{2}}$ and for $(1-x)^{\frac{1}{2}}$.

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\cdot\frac{-1}{2}}{2}x^2 + \frac{\frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}}{1\cdot 2\cdot 3}x^3 + \frac{\frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}\cdot\frac{-5}{2}}{1\cdot 2\cdot 3\cdot 4}x^4$$

$$+\frac{\frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}\cdot\frac{-5}{2}\cdot\frac{-7}{2}}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5 + \cdots$$

$$= 1 + \tfrac{1}{2}x - \tfrac{1}{4}x\cdot T_2 - \tfrac{1}{2}x\cdot T_3 - \tfrac{5}{8}x\cdot T_4 - \tfrac{7}{10}x\cdot T_5 - \cdots.$$

Alternate terms after the second are positive and negative.

$$(1-x)^{\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{-1}{2}\right)}{2}x^2 - \frac{\frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}}{1\cdot 2\cdot 3}x^3 + \frac{\frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}\cdot\frac{-5}{2}}{1\cdot 2\cdot 3\cdot 4}x^4$$

$$-\frac{\frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}\cdot\frac{-5}{2}\cdot\frac{-7}{2}}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5 + \cdots$$

$$= 1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4$$

$$= 1 - \tfrac{1}{2}x + \tfrac{1}{4}x\cdot T_2 + \tfrac{1}{2}x\cdot T_3 + \tfrac{5}{8}x\cdot T_4 + \tfrac{7}{10}x\cdot T_5 + \cdots.$$

Every term after the first is negative.

c. Compute $\sqrt{.98}$ to 5 places.

$$(1-.02)^{\frac{1}{2}} = 1 - \tfrac{1}{2}(.02) + \frac{\frac{1}{2}(-\frac{1}{2})}{1\cdot 2}(.02)^2 - \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1\cdot 2\cdot 3}(.02)^3 + \cdots$$

$$= 1 - .01 - .00005 - .0000005 - \cdots$$

$$= .9899495 \text{ or } .98995 \text{ to 5 places.}$$

d. Compute the cube root of (1012) to 6 significant figures.

$$(1012)^{\frac{1}{3}} = (1000)^{\frac{1}{3}}(1+.012)^{\frac{1}{3}} = 10(1+.012)^{\frac{1}{3}}.$$

$$(1+.012)^{\frac{1}{3}} = 1 + \tfrac{1}{3}(.012) + \frac{\frac{1}{3}(\frac{1}{3}-1)}{1\cdot 2}(.012)^2 + \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{1\cdot 2\cdot 3}(.012)^3 + \cdots$$

$$= 1 + .004 - .000016 + .0000001 - \cdots$$

$$= 1.003984.$$

$$(1012)^{\frac{1}{3}} = 10 \times 1.003984 = 10.03984.$$

e. Compute $(1.05)^{\frac{3}{2}}$ to 4 places; treat this as $1.05 \times (1.05)^{\frac{1}{2}}$; .05 may be taken as $\frac{1}{20}$. Check by logarithms.

f. Compute the square roots of 26 and 30 to 4 places.

$$\sqrt{26} = 26^{\frac{1}{2}} = (25+1)^{\frac{1}{2}} = 5(1+\tfrac{1}{25})^{\frac{1}{2}} = 5(1+.04)^{\frac{1}{2}}$$
$$= 5(1+.02-.0002+.000004)$$
$$= 5(1.019804) = 5.09902.$$
$$30^{\frac{1}{2}} = (25+5)^{\frac{1}{2}} = 5(1+.2)^{\frac{1}{2}} = 5(1+.1-.005+.0005-.0000625)$$
$$= 5(1.0954375) = 5.47718.$$

Check roughly, using logarithms.

g. Compute $(1.05)^{-7}$ to 5 significant figures.

$$(1.05)^{-7} = 1 - 7(.05) + \frac{-7(-8)}{1\cdot 2}(.05)^2 + \frac{-7(-8)(-9)}{1\cdot 2\cdot 3}(.05)^3$$
$$+ \frac{-7(-8)(-9)(-10)}{1\cdot 2\cdot 3\cdot 4}(.05)^4 + \frac{-7(-8)(-9)(-10)(-11)}{1\cdot 2\cdot 3\cdot 4\cdot 5}(.05)^5$$
$$= 1 - .35 + .07 - .0105 + .0013125 - .0001444 + .0000144 = .71068.$$

5. Historical note. — The binomial theorem as applied to $(a+b)^2$ and $(a+b)^3$ was well known to Euclid (320 B.C.) and other early Greek mathematicians. The great Arabic mathematician and poet Omar al Khayyam (died 1123 A.D.) extended the rule to other positive integers. In China there appeared in 1303 a work containing the binomial coefficients arranged in triangular form, the so-called Pascal (1623–1662) triangle of coefficients.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

PASCAL'S TRIANGLE OF COEFFICIENTS, PRINTED 1665.

1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	
1	3	6	10	15	21		
1	4	10	20	35			
1	5	15	35				
1	6	21					
1	7						
1							

CHINESE FORM OF TRIANGLE OF COEFFICIENTS.

1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

STIFEL'S (1486–1567) TRIANGLE OF COEFFICIENTS, 1544.

The general rule for any exponent $\frac{m}{n}$ was first discovered by Sir Isaac Newton, and made known in a letter of date Oct.

24, 1676, to a friend named Oldenburg. It is of interest to note that Newton wrote each coefficient in terms of the coefficient immediately preceding, following the lines indicated in our numerical problems above. The complete proof for the general case, any real or complex imaginary number, was finally effected by a brilliant Norwegian mathematician, Abel (1802–1829), in 1826.

PROBLEMS

1. Find the coefficients of the first five terms of $(1+x)^{\frac{1}{2}}$ and $(1-x)^{\frac{1}{2}}$ and use them in the next two problems.

2. Compute to 3 significant figures the square roots of 27, 28, and 29 as $5(1+.08)^{\frac{1}{2}}$, $5(1+.12)^{\frac{1}{2}}$, and $5(1+.16)^{\frac{1}{2}}$, respectively.

3. Compute the square roots of the first eleven integers, to 5 places, taking $\sqrt{2}$ as $\frac{1}{10}(196+4)^{\frac{1}{2}}=\frac{14}{10}(1+\frac{1}{49})^{\frac{1}{2}}$, $\sqrt{3}$ as $2(1-\frac{1}{4})^{\frac{1}{2}}$, $\sqrt{5}$ as $2(1+\frac{1}{4})^{\frac{1}{2}}$, $\sqrt{6}$ as $(4+2)^{\frac{1}{2}}=2(1+.5)^{\frac{1}{2}}$, $\sqrt{7}$ as $\frac{1}{2}(25+3)^{\frac{1}{2}}$, $\sqrt{8}$ as $2\sqrt{2}$, $\sqrt{11}$ as $\frac{1}{3}(100-1)^{\frac{1}{2}}$. Check the first four significant figures by logarithms.

4. Write 5 terms of $(1+x)^{\frac{1}{3}}$, and of $(1-x)^{\frac{1}{3}}$.

5. Use these to compute the values of $\sqrt[3]{7}$ and $\sqrt[3]{9}$, as $2(1-\frac{1}{8})^{\frac{1}{3}}$ and $2(1+\frac{1}{8})^{\frac{1}{3}}$.

6. From the cube root of 9 find the cube root of 3.

7. Find the cube root of 6.

8. Using the cube roots of 6 and 3, find the cube root of 2.

9. Compute $(1+.05)^{\frac{1}{4}}$ to 5 places.

10. Write 5 terms of $(2-3x)^{-\frac{1}{3}}$.

11. Write the 6th term of $(2-3x)^{-7}$.

12. Find the value to 4 significant figures of $\dfrac{1}{\sqrt{1-.02}}$ by expanding $(1-.02)^{-\frac{1}{2}}$.

13. Expand in powers of i to 4 terms $\frac{1}{1+i}$ and $\frac{1}{1-i}$. What is the approximate percentage error in using $(1-i)$ and $(1+i)$ as multipliers, respectively, instead of $\frac{1}{1+i}$ and $\frac{1}{1-i}$, when $i = .01$, $.05$, $.005$, and $.5$, respectively?

14. Find the fifth root of 35 correct to 2 decimal places. What is the shortest way? Compute the root to 5 places. What method can you use?

15. Write the term containing x^6 in the expansion of $(1-3x)^{-\frac{1}{2}}$.

16. Write the first five terms of $(10+.3)^8$; of $(10.3)^6$; of $(10.3)^{-6}$ and give the value to 4 significant figures.

17. Time yourself on writing and simplifying 10 terms of the following five expansions: $(1+x)^{\frac{1}{3}}$, $(1-x)^{\frac{2}{3}}$, $(1-2x)^{-\frac{1}{3}}$, $(1-\frac{3}{2}x)^{\frac{4}{5}}$, $(1-3x)^{-6}$.

18. Compute correctly to 4 significant figures, using the formulas for $\frac{1}{1+i}$ and $\frac{1}{1-i}$. Time yourself on the exercise.

$$\frac{18}{.98},\ \frac{27}{1.02},\ \frac{54}{1.03},\ \frac{62}{.99},\ \frac{68}{1.05}.$$

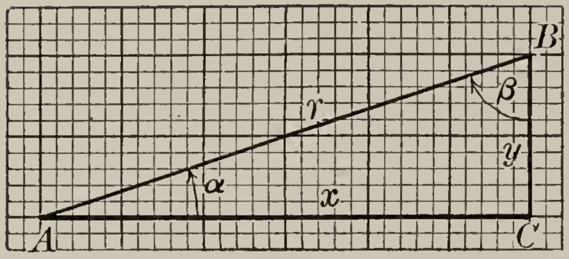
19. Compute the following to three significant figures, timing yourself:

$$(1.06)^{\frac{1}{3}},\ (1.06)^{\frac{1}{12}},\ (1.05)^{\frac{1}{2}},\ (1.06)^{\frac{2}{3}},\ (1.10)^{\frac{1}{5}},\ (1.06)^{-\frac{1}{3}}.$$

CHAPTER XIII

RIGHT TRIANGLES

1. Right triangles. — To apply our trigonometric work to the numerical solution of right triangles place the triangle under consideration in quadrant I in proper position to be able to read the trigonometric functions of one acute angle.



Fundamental formulas of the right triangle

$$x = r\cos\alpha = r\sin\beta$$
$$y = r\sin\alpha = r\cos\beta$$
$$\tan\alpha = \frac{y}{x} = \cot\beta$$
$$\cot\alpha = \frac{x}{y} = \tan\beta$$
$$x^2 + y^2 = r^2$$
$$\alpha + \beta = 90^\circ.$$

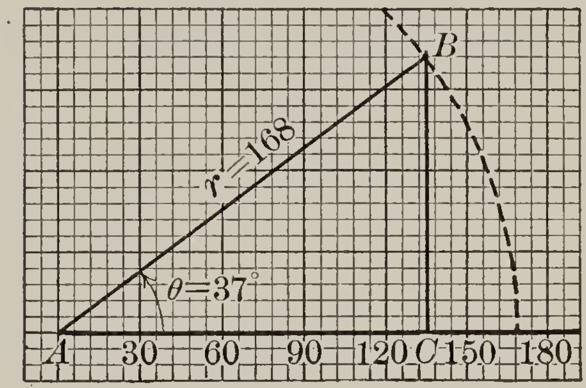
The equation $x = r\cos\alpha$ may be written $\cos\alpha = \frac{x}{r}$ or $r = \frac{x}{\cos\alpha}$, as occasion demands; similar transformations are to be effected upon the other equations given.

Given α with x, y, or r; or β with x, y, or r; or two of the lengths; these formulas enable us to solve the right triangle completely for the remaining three parts.

The student is advised to draw the figure to scale on coördinate paper, using a protractor to lay off correctly to degrees the angles given, before attempting to apply any formulas; then write the required equations directly from a consideration of the figure, and not by attempting to memorize the solutions for the different types of problems. The lengths as given graphically serve as a check upon the values obtained.

2. Right triangles. Type I. — Given hypotenuse and one angle, *i.e.* α and r, or β and r.

A guy wire 168 feet long reaches to the top of a tall chimney, making an angle of 37° with the ground. Find the height of the chimney and the distance from the supporting peg to the foot of the chimney.



Hypotenuse and one angle given

Solution. — First draw the figure, as indicated, using $\frac{1}{2}$ inch to represent 30 units.

$$y = 168 \sin 37^\circ$$
$$x = 168 \cos 37^\circ$$

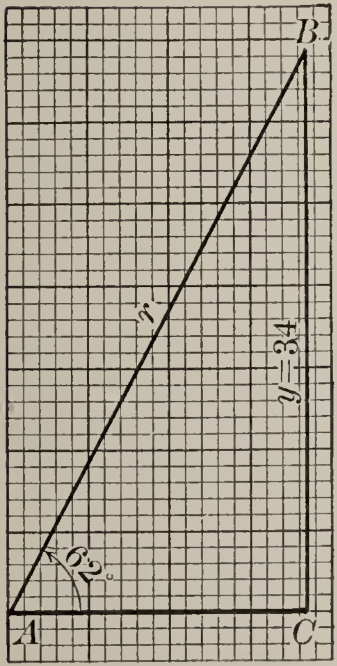
Check. $y = x \tan 37^\circ$

Using natural functions there are here three problems in multiplication. The logarithmic solution is as follows:

$$\begin{aligned} \log 168 &= 2.2253 \\ \log \sin 37^\circ &= \underline{9.7795 - 10} \\ \log y &= 2.0048 \\ y &= 101.1. \end{aligned} \qquad \begin{aligned} \log 168 &= 2.2253 \\ \log \cos 37^\circ &= \underline{9.9023 - 10} \\ \log x &= 2.1276 \\ x &= 134.2. \end{aligned}$$

Check.
$$\begin{aligned} \log x &= 2.1276 \\ \log \tan 37^\circ &= \underline{9.8771 - 10} \\ \log y &= 2.0047. \end{aligned}$$

Compare with preceding value of $\log y$; a disagreement in the fourth place is permissible. It would not affect the fourth significant figure of y in this case; nor would the measurements of height of chimney and length of guy wire be made with greater accuracy than to one tenth of a foot. $x^2 + y^2 = \overline{168}^2$ could be used as a check.



Angle and side given

3. Right triangles. Type II. — Given one leg and an angle.

If a telegraph pole is 34 feet high, and the supporting wire makes an angle of 62° with the ground, find the length of the wire and the distance from the foot of the supporting peg to the foot of the pole.

The corresponding formulas are as follows:

$$x = 34 \cot 62^\circ \qquad \textit{Check.} \quad x = r \cos 62^\circ$$

$$r = \frac{34}{\sin 62^\circ} \qquad \text{or} \quad 34^2 = r^2 - x^2 = (r-x)(r+x)$$

$$\begin{aligned} \log 34 &= 1.5315 \\ \log \cot 62^\circ &= \underline{9.7257 - 10} \\ \log x &= 1.2572 \\ x &= 18.08. \end{aligned} \qquad \begin{aligned} \log 34 &= 11.5315 - 10 \\ - \log \sin 62^\circ &= \underline{9.9459 - 10} \\ \log r &= 1.5856 \\ r &= 38.51. \end{aligned}$$

$$\textit{Check.} \quad \begin{aligned} \log r &= 1.5856 \\ + \log \cos 62^\circ &= \underline{9.6716 - 10} \\ \log x &= 1.2572, \textit{ checks.} \end{aligned}$$

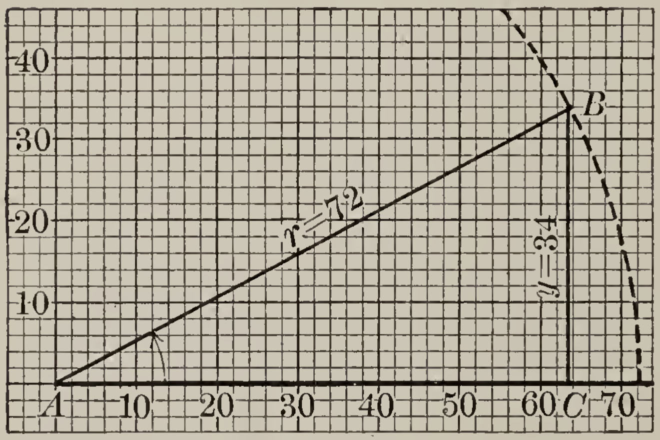
4. Right triangles. Type III. — Given a leg and the hypotenuse.

Let the side $a = 341$ and c, the hypotenuse, equal 725.

Evidently, $\sin \alpha = \frac{341}{725}$,

$$b = 725 \cos \alpha. \qquad \textit{Check.} \quad b = 341 \cot \alpha.$$

$$\begin{aligned} \log 341 &= 12.5328 - 10 \\ \log 725 &= \underline{2.8603} \\ \log \sin \alpha &= 9.6725 - 10 \\ \alpha &= 28^\circ\, 4'. \end{aligned} \qquad \begin{aligned} \log 725 &= 2.8603 \\ \log \cos \alpha &= \underline{9.9457 - 10} \\ \log b &= 2.8060 \\ b &= 639.7. \end{aligned}$$



Hypotenuse and one side given

Graphical solution gives a rough check to two significant figures.

$$\textit{Check.} \quad \begin{aligned} \log 341 &= 2.5328 \\ \log \cot \alpha &= \underline{.2731} \\ \log b &= 2.8059. \end{aligned}$$

Compare with above value $\log b = 2.8060$ as a check; the error of 1 here is inevitable with 4-place logarithms.

Note that on the small graph only two places are accurately representable.

5. Right triangles. Type IV. — Given the two legs.

Entirely similar except that the initial formula is for $\tan \alpha$ instead of $\sin \alpha$.

Note that commonly in lettering right triangles x and y or a and b are used for the legs, r or c for the hypotenuse, α and β for the angles at A and B, opposite a and b respectively.

In Type III if a and c are nearly equal we may avoid the use of the sine of the angle near to 90° by computing the other side using the formula

$$b^2 = c^2 - a^2 = (c - a)(c + a).$$

Thus if $a = 718,\ c = 725,\ b^2 = (725 - 718)(725 + 718)$

$$= 7(1443) = 10101.$$

$$b = 100.5,$$

either by inspection as in this case, or from a table of squares, or by logarithms.

PROBLEMS

Solve the following right triangles by logarithms:

1. Given $r = 240$, $\alpha = 30° \, 10'$.

2. Given $a = 368$, $\alpha = 30° \, 14'$.

3. Given $a = 368$, $r = 579$.

4. Given $a = 368$, $b = 275$.

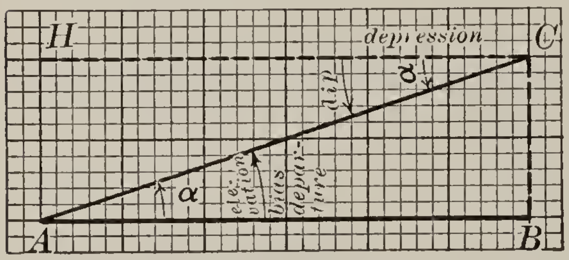
5. Solve for the missing parts the following ten problems, using logarithms; time yourself; the exercise should be completed within 30 minutes.

a. Given $r = 186$, $a = 84.3$.
b. Given $a = .394$, $b = .654$.
c. Given $a = 2.89$, $\beta = 68° \, 24'$.
d. Given $b = 706$, $\alpha = 70° \, 10'$.
e. Given $a = 930$, $\alpha = 24°$.
f. Given $b = 184$, $\alpha = 55° \, 15'$.
g. Given $r = .0936$, $b = .0418$.
h. Given $b = 3.24$, $\beta = 86° \, 14'$.
i. Given $b = 878$, $\alpha = 48° \, 19'$.
j. Given $r = 8.4 \times 10^6$, $\beta = 34° \, 16'$.

6. Area. — In computations of functions involving measured and computed values, measured values are taken, as far as possible, in preference to computed values. The computed value involves not only the inaccuracies or errors of measurement, but also the errors of computation, the inevitable errors of computation with approximate numbers as well as the avoidable errors. Among the following formulas for the area of a right triangle the student should select, in accordance with the principle mentioned, the formula to be used in each problem involving a right triangle.

$$A = \tfrac{1}{2}\,ab = \tfrac{1}{2}\,a^2 \cot \alpha = \tfrac{1}{2}\,b^2 \tan \alpha = \tfrac{1}{2}\,c^2 \sin \alpha \cos \alpha.$$

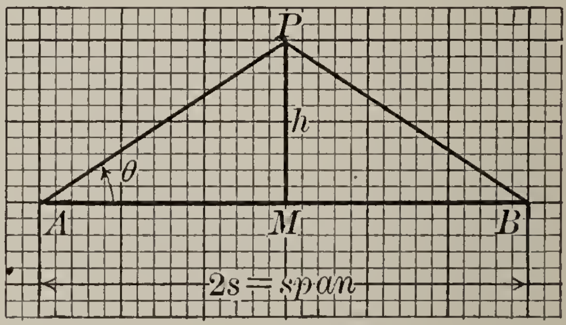
7. Applications. — In the application of the solution of right triangles to practical problems we find that the difficulty is frequently a matter of terminology rather than of principle. The student is urged to acquire some real familiarity with the industrial and scientific application of the principles explained.



Common terms relating to angles

Dip — depression — elevation — bias — departure.

The terms "elevation," "depression," "dip," "departure," and "bearing," all refer to angles. Thus in the figure ABC, if AC is in the direction of the sun or if C represents the top of a mountain viewed from A, then angle BAC is termed the angle of "elevation" of C as viewed from A; if the observer is at C, on a mountain or in an airship, HCA is the angle of "depression"; if A represents the horizon as viewed from C, then HCA is called the "dip" of the horizon. If CA represents a vertical section of a vein of coal, the angle HCA or BAC is called the "dip" of the vein; in navigation if AB represents east, then angle BAC represents the "departure" north of the line AC, whereas in surveying the angular deflection from north or south is given as the "bearing."



Pitch $= \dfrac{h}{2s}$

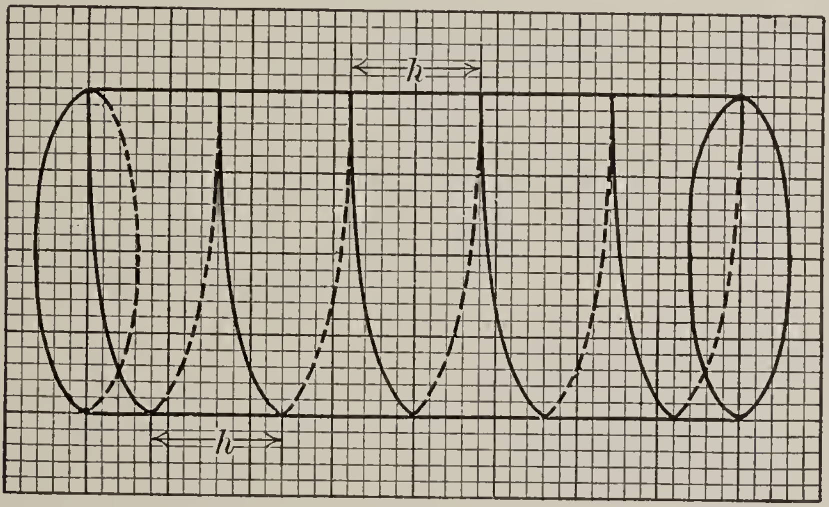
Frequently some function of an angle is given from which the angle must be determined. The pitch of a roof is given as the height divided by the span, whence the corresponding slope angle of the roof is θ, given by

$$\tan \theta = \frac{h}{s},$$

where h is the height and $2\ s$ is the total span.

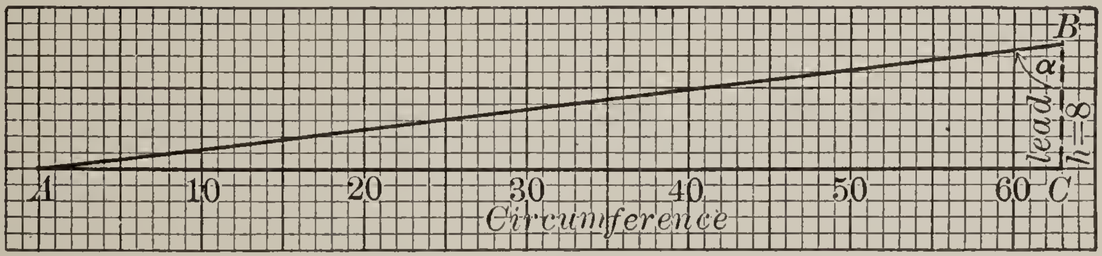
The slope of a railroad is commonly given as so many (h) feet of rise in 100 feet horizontally; this gives the slope angle θ from $\tan \theta = \frac{h}{100} = h\ \%$.

A spiral thread winds about a cylinder advancing a height h, called the "lead," in one complete turn; the circumference of the cylinder is the base and the "lead" is the altitude of a right triangle which may be regarded as wrapt about the cylinder to give the spiral. The angle α made with h by the spiral line is called the angle of the spiral; evidently $\tan \alpha = \frac{2 \pi r}{h}$.



Full size representation of a one-inch cylindrical screw

The "lead" is $\frac{8}{20}$ of an inch.



Circumference, *AC*, and length of one spiral, *AB*, of the above one-inch cylindrical screw

PROBLEMS

1. A standpipe subtends an angle of 4° at the eye of an observer; if its height is 280 feet above the level of the eye, find its distance from the observer.

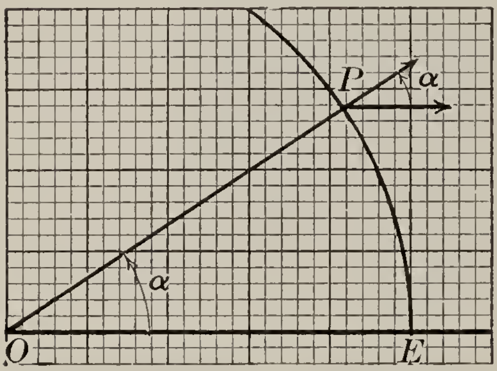
2. If the diameter of the standpipe in 1 subtends an angle of 15′ at the eye, what is the diameter in feet? Suppose that the angle at the eye lies between 10′ and 20′, what range of diameter would these values give?

3. The shadow of a flagstaff 60 feet high is 48 feet long. Find the angular elevation of the sun.

4. Using trigonometric functions, find the height of a building which at the same time casts a shadow 87 feet long.

5. Find the lengths of the circle of latitude and the circle of longitude through your home city.

6. When the sun is directly over the equator, the latitude of any place on the earth's surface from which the sun is visible is the angle between the zenith line (the vertical) and the line to the sun, when the sun is on the meridian. Find the length in feet of the shadow of a pole 100 feet high, at noon, latitude 40° N., and also for latitude 42° 18′ N.



Zenith distance representing latitude

Sun on celestial equator, direction *OE*. *OP* is zenith direction of *P*.

7. Compute the diameter of a circle circumscribed about an equilateral triangle of side 40.

8. Find in a circle of radius 486 cm. the chords of angles of 30°, 60°, 45°, 90°, 120°, 72°, 68°.

9. For any angle α, find the chord and the chord of half the angle in terms of the radius r. Apply the latter formula to obtain the results of problem 8.

10. A pendulum of length 34 inches swings between two points 10 inches apart; compute the arc of the swing. If this is a seconds pendulum, passing the vertical once every second, what is the velocity of the pendulum bob? Find the chord of this arc, and the difference between the chord and the arc.

11. A circular arch over a doorway is to be 4 feet wide and 20 inches high; compute the radius. Compute for heights of 10 to 24 inches by 2-inch intervals.

12. Given the radius 10 feet and the span 4 feet of a circular arch. Compute the height. Compute for spans of 2 feet to 20 feet, by 2-foot intervals.

13. Adapt the preceding results to a radius of 8 feet. How closely would interpolation give correct results? discuss by considering the problem graphically.

14. In a circle of radius 100 inches, compute to one decimal place the lengths of sides and the perimeters of regular inscribed polygons of 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12 sides. Time yourself on the exercise. State the general formulas involved.

15. Compute perimeters of regular circumscribed polygons of 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12 sides in a circle of radius 100. Time yourself on the numerical work; 30 minutes is ample time.

16. Compute the perimeter of a regular inscribed polygon of 96 sides, and of a regular circumscribed polygon of 96 sides, radius 100. How does the circumference compare with these two values? Archimedes computed these lengths by plane geometry methods and so found π to lie between $3\frac{1}{7}$ and $3\frac{10}{71}$. Check his result.

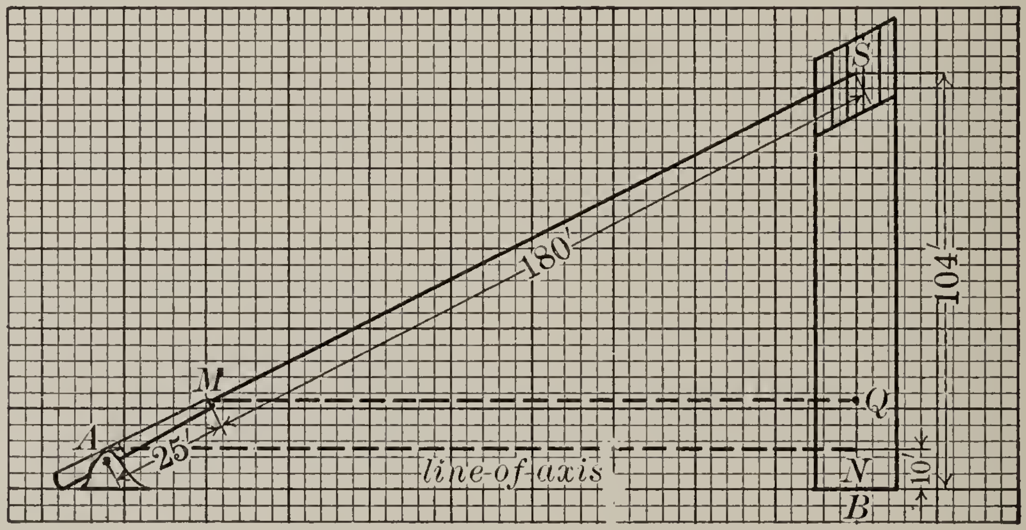
17. Frequently arches of bridges are circular segments; find the radius of the circular arch of the famous Rialto in Venice (see illustration, page 225). The width of the arch is 95 feet and the height is 25 feet. Draw the graph of the arch to scale.

18. One of the largest masonry bridges in the U. S. is the Rocky River bridge at Cleveland; the height of the circular arch is 80 feet and the span is 280 feet. Find the corresponding radius.

19. Find the angle of the spiral represented in the above diagram of the one-inch cylindrical screw.

20. Find the angle of a cylindrical screw of diameter $\frac{1}{4}$ inch which advances $\frac{1}{40}$ of an inch in one complete turn.

21. A twelve-inch gun has a muzzle velocity of approximately 2500 feet per second (f.s.). The velocity is tested by electrical means; screens are placed at a known distance apart and the projectile in passing through the screens breaks successively two electrical circuits which serve to give the time of flight of the projectile to thousandths of a second in passing through the known distance. The apparatus may also be used to determine the angle of elevation of the larger guns. In the figure TAT' represents the axis of trunnions of a twelve-inch gun; AM along the axis of the barrel is 25 feet; MS is 180 feet; the one screen is over the muzzle and the other screen is at a height of 94 feet above the axis of trunnions. Determine the angle of elevation of the gun and reduce to "mils." Find the horizontal distance MQ between



Determination of velocity of a projectile

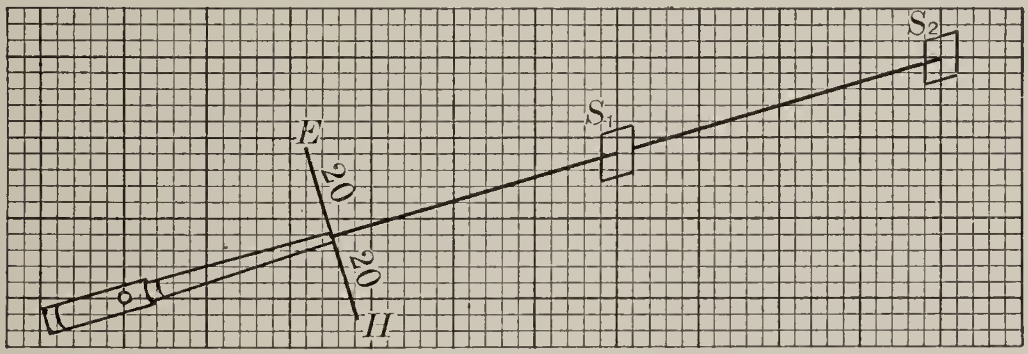
One screen is at the muzzle of the gun.

muzzle and screen and the vertical distance between muzzle and screen, QS; find the time of flight of the projectile, assuming 2500 f.s. as velocity; find "horizontal velocity," v_x, and "vertical velocity," v_y, by dividing MQ and QS respectively by this time of flight. The vertical velocity v_y divided by 32.2 gives approximately the time in seconds that the projectile will continue to rise; find this time; the position of the projectile after this interval of time is given approximately by the product of horizontal velocity, v_x, multiplied by the time, as horizontal distance from the gun, and by vertical component of velocity multiplied by the same value of t less 16.1 multiplied by t^2, as ordinate. The equations are

$$x = v_x t,$$
$$y = v_y t - 16.1t^2.$$

What error is possible in the angle measured if the height of S is given only within one foot? The aim is directed at a point two feet below the top of the screen, as, in general, there is a slight "jump" due apparently to the explosion. Estimate the jump in degrees and minutes, and in "mils" if the projectile hits the top of the screen.

22. When two screens are used with a large gun the distance between screens is sometimes measured by taking equal distances MH and ME at right angles to the line MS_1S_2 and measuring the angles MHS_1, MHS_2, MES_1, and MES_2. Note that the screens are 20 to 100 feet in the air on tall standards,



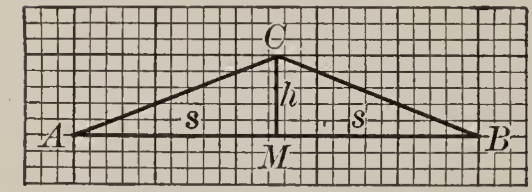
Distances of screens from muzzle, M, determined by right triangles

Two screens spaced 100 feet apart (horizontally).

making it inconvenient to measure the distance with a steel tape. Assuming that the distances ME and MH are taken as 20 feet and that the angles $MHS_1 = MES_1 = 70^\circ\, 10'$, and that angle $MHS_2 = MES_2 = 82^\circ\, 34'$, compute the distance MS_1, MS_2, and S_1S_2. If the screen S_1 is at an elevation of 34 feet and the screen S_2 is at an elevation of 83 feet, compute the angle of elevation of the gun, and the height of the muzzle above the plane of its axis if the muzzle is 25 feet long from the axis.

Note that the relative positions of the screens are usually determined by two observers in towers whose distance apart is fixed; these observers record positions of muzzle and each of the screens.

23. If a stick of length 12 units casts shadows of lengths 4, 6, 8, 10, 12, 15, 18, 30, and 40 units respectively, determine the angle of inclination of the sun. For angles of inclination of 10° to 20° by degrees, determine the corresponding shadow length to tenths of one unit. This type of table was the first appearance of the cotangent function as direct shadow; it appeared as early as 900 A.D. in the works of the great Arabic astronomer Al-Battani.



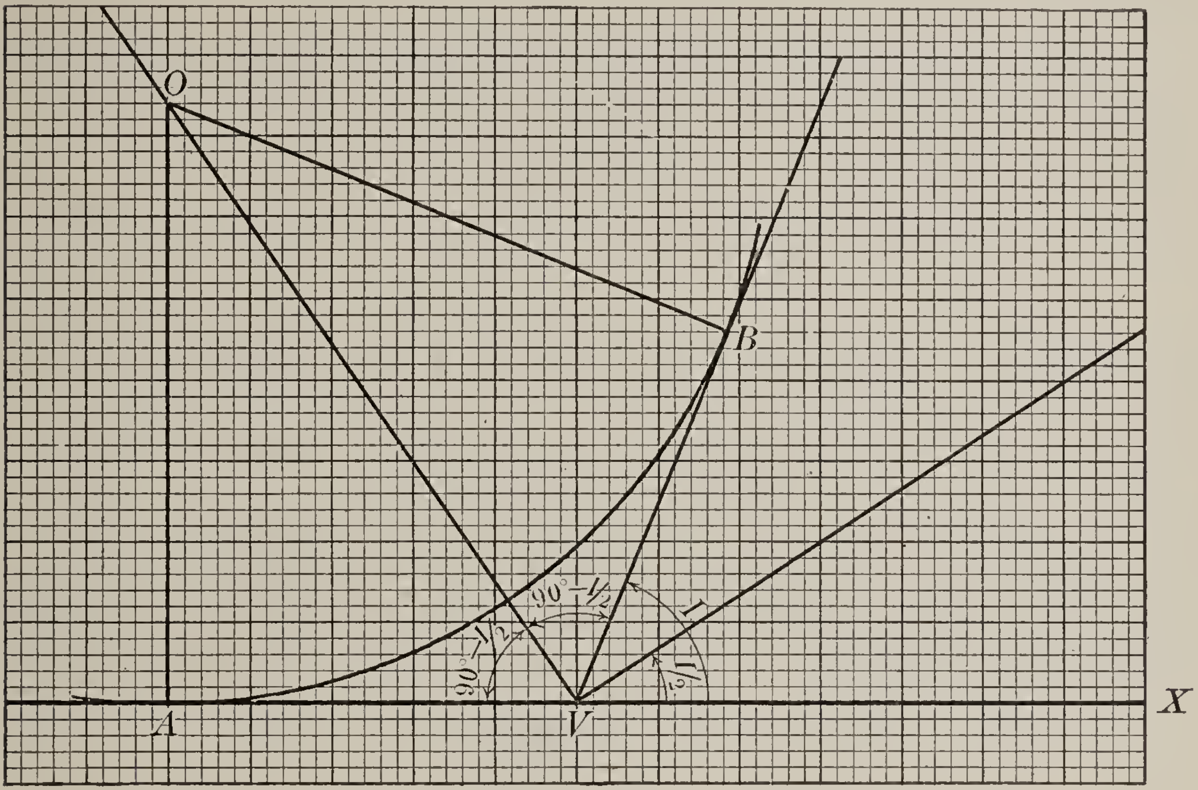
Pitch equals $\frac{h}{2s}$ when span is $2s$

24. The pitch of a roof is given by the vertical height h, from the point C to M, on the diagram, divided by the span, $2s$; thus $\frac{1}{2}$ pitch is a 45° slope. Find the slope angle of a roof of $\frac{2}{3}$ pitch, of $\frac{1}{3}$ pitch. If $2s$ is given as 48 feet, find the length of the rafters in each of the roofs mentioned.

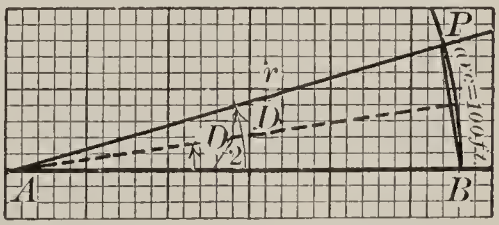
25. In a roof of span 62 feet find to the tenth of an inch the lengths of the rafters if the roof is inclined at 30°, 40°, 42°, 45°, 53°, and 60°. In each case determine the effect upon the length of the rafter of an error of one degree.

26. Find the pitch and the angle of inclination of the roof represented in the diagram above.

8. Railroad curves. — In so far as possible the track of a railroad is laid out in straight lines. Wherever the direction of the track is changed a curved line of track is introduced leading from the one straight track to a second; these straight portions of track must be tangent to the curve which joins them, and so they are commonly designated simply as *tangents*. Let AV and VB in the figure represent two such tangents, meeting at a point V, called the *vertex*; the exterior angle XVB is called the *deflection angle*, and is usually designated by I. A single circular arc, radius R, which joins two tangents is called a *simple curve* and is designated in American railroad practice by the number of degrees D at the center of the circle subtended by a chord whose length is 100 feet, the length of one chain used in surveying. On a simple curve the lengths



Simple curve at a turnout on a railroad track



Chord 100 feet; arc approximately 100 feet of two consecutive tangents, from intersection point to the circle, *i.e.* AV and VB, are equal; this length is called T. The angle D is commonly given only in degrees and half-degrees.

Relation between D and R. Let PB on the figure represent a chord of length 100 feet; drop the perpendicular from A, the center of the circular arc of radius R, bisecting PB. Evidently $\sin\frac{D}{2}=\frac{50}{R}$, whence $R=\frac{50}{\sin\frac{D}{2}}$. Now for any angle up to 4° the sine differs numerically from the angle expressed in radians by less than .1 of 1% of itself; hence you may replace $\sin\frac{D}{2}$ by $\frac{D}{2}\cdot\frac{\pi}{180}$, the value of $\frac{D}{2}$ in radians, with an error of less than .1 of 1% when D is any angle up to 8°. Note that the error is less than 5 feet in 5000; the circular measure of the angle is larger than the sine so that the error will be a deficiency.

$$R=\frac{18000}{\pi D}$$

gives the radius.

Relation between I, R, and T. On our figure in the right triangle OAV the angle AVO is $90°-\frac{I}{2}$, and the angle AOB is evidently equal to the deflection angle I.

$$\tan\left(90°-\frac{I}{2}\right)=\frac{OA}{AV}=\frac{R}{T},$$

whence $$R=T\cot\frac{I}{2},\quad\text{and}\quad T=R\tan\frac{I}{2}.$$

Evidently the radius R can be expressed in terms of the "degree" D of the curve, giving new formulas involving D, T, and I.

Elevation of outer rail. In turning a curve a railroad train tends to leave the track, due to the tendency of any moving body to continue its motion in a straight line. To keep the train on the track the flanges alone are not sufficient, but the outer edge must be elevated. The formula for ordinary speeds, giving number of inches of elevation, is $e = \frac{gv^2}{32\,R}$, wherein g is the gage of the track in feet, v the velocity of the train in feet per second, and R the radius of curvature in feet.

PROBLEMS

1. What radii have railroad curves of 8°, 7°, 6°, 5°, 4°, 3°, 2°, and 1°, respectively?

2. If a railroad curve is built with the radius of 2640 feet, compute D in degrees.

3. On a circular track of 100 miles' circumference what would be the number of degrees?

4. On English and continental railroads the curvature is usually given by the length of the radius; find the number of degrees, American D, corresponding to radii of 8000, 5000, 4000, 3000, 2000, 1000, 800, 600, and 400 feet, respectively. Do not compute beyond minutes. Find D for radii of 300 meters, 1000 meters.

5. Compute e, elevation of outer rail, for $g = 4$ feet 8.5 inches, standard gage on American railroads, when $v = 60$ miles per hour, and $R = 800$, 1000, 2000, 4000, and 5000 respectively. Compute for a one-degree and for a two-degree curve.

6. Given that two portions of straight track diverge at 22° 14′, and that the tangent distance is to be 300 feet, compute R; find R for T, the tangent distance, equal to 200, 250, and 350. Find the corresponding values of D. How could you determine the length of T, approximately 300, so that D will come out in degrees and half-degrees?

7. Compute R when $T = 400$, 500, and 600, respectively, the deflection angle being 60°; similarly when $I = 30°$.

8. Compute e for $g = 4$ feet 8.5 inches (4.71 feet), standard gage, $v = 60$ miles per hour, and curves of 1°, 2°, 5°, 6°, and 8°, respectively.

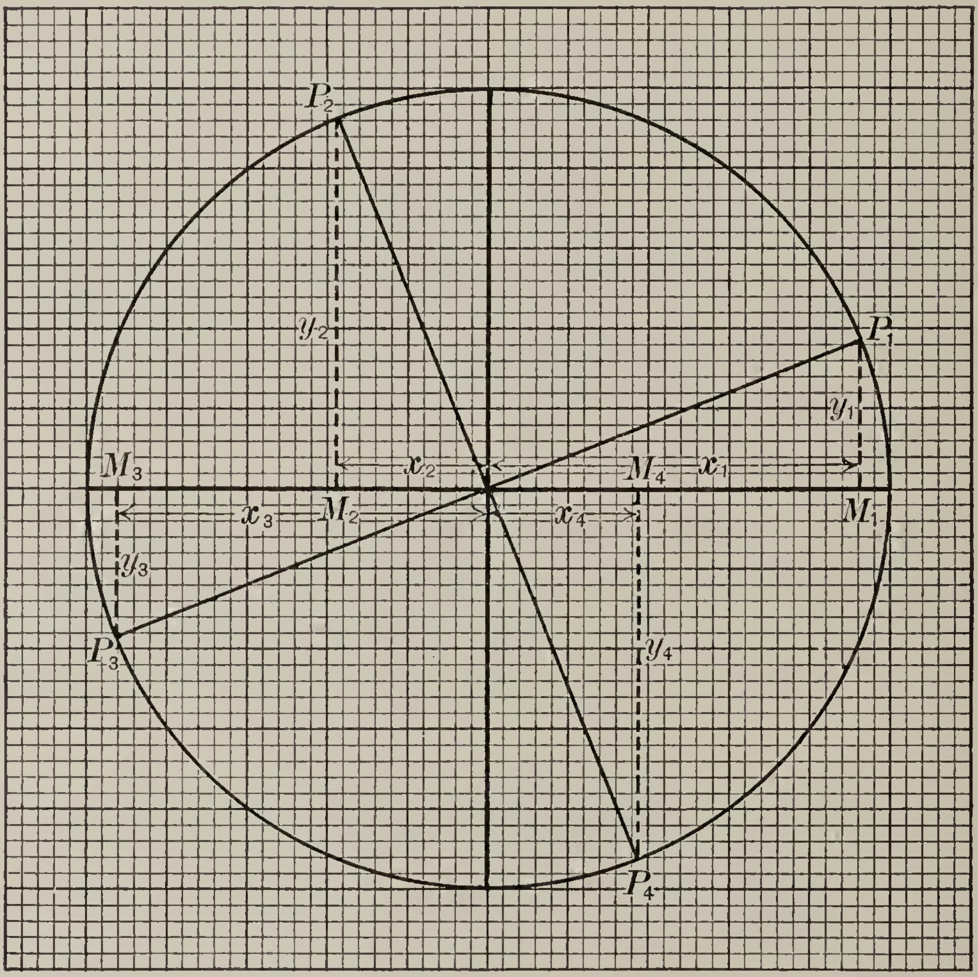
CHAPTER XIV

THE CIRCLE

1. Formulas. —

$$x^2 + y^2 = r^2. \qquad \begin{cases} x - h = r\cos\theta, \\ y - k = r\sin\theta. \end{cases} \quad \text{Parametric equations of the circle.}$$

$$(x - h)^2 + (y - k)^2 = r^2.$$



$$OM_1^2 + M_1P_1^2 = OP_1^2, \quad OM_2^2 + M_2P_2^2 = OP_2^2, \quad OM_3^2 + M_3P_3^2 = OP_3^2,$$
$$OM_4^2 + M_4P_4^2 = OP_4^2.$$

$$x_1^2 + y_1^2 = r^2\,;\ x_2^2 + y_2^2 = r^2\,;\ x_3^2 + y_3^2 = r^2\,;\ x_4^2 + y_4^2 = r^2\,;\ x^2 + y^2 = r^2.$$

For any point $P(x, y)$ on a circle of radius 10, center the origin, we have the relation,

$$x^2 + y^2 = 100,$$

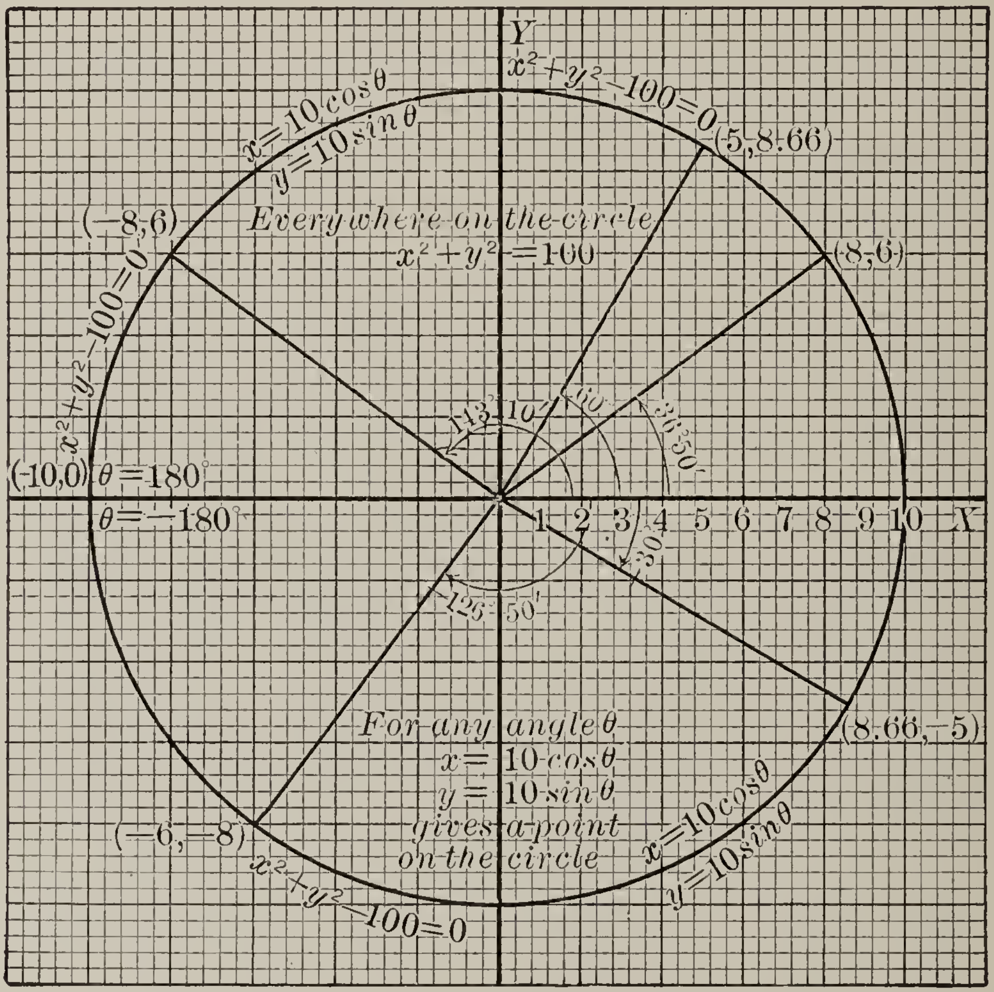
which is the equation then of a circle of radius 10 and center at the origin (0, 0). This equation is obtained directly from the distance formula; $x^2 + y^2 = 100$ expresses the fact that the distance of the point (x, y) from the point (0, 0) is 10; any point (x, y) which satisfies this equation is at a distance 10 from (0, 0) and any point at a distance of 10 units from O (0, 0) satisfies this equation. The locus of this equation, then, is the circle of radius 10 and center (0, 0).

The formula may readily be verified on the figure; take P any point on the circle, drop PM a perpendicular to the x-axis. Then $OM^2 + MP^2 = OP^2$, in any one of the four triangles, representing any possible position of P. Herein OM and MP must be regarded initially as positive quantities, since the formulas of plane geometry were applied only to positive lengths. However OM, as a positive length $= x$, where the negative sign is taken for points in II and III and $MP = y$, where the negative sign is taken in III and IV, whence substituting in $\overline{OM}^2 + \overline{MP}^2 = \overline{OP}^2$ you have $x^2 + y^2 = 10^2$. For a circle of radius r the equation $x^2 + y^2 = r^2$ is satisfied by any point $P(x, y)$ which is upon the circle, for OP will equal r, and every point which satisfies the equation evidently lies on the circle. Hence, by definition, the locus of $x^2 + y^2 = r^2$ is the circle of center (0, 0) and of radius r, for every point on the circle satisfies this equation and every point which satisfies the equation lies upon the circle. These two conditions must be fulfilled in order that any given curve may be designated as the locus of a given equation. In other words, the given curve must include all points whose coördinates satisfy the equation and must exclude all whose coördinates do not satisfy the given relation.

Similarly the two equations:

$$x = 10 \cos \theta,$$
$$y = 10 \sin \theta,$$

give for every value of θ, called a parameter, the coördinates of a point which lies upon the circle. The locus of this pair of equations is the circle of radius 10. Thus the ten values of $\theta = 0°$, $30°$, $45°$, $60°$, $90°$, $120°$, $150°$, $180°$, $210°$, and $330°$, give the ten points, $(10, 0)$, $(5\sqrt{3}, 5)$, $(5\sqrt{2}, 5\sqrt{2})$, $(5, 5\sqrt{3})$, $(0, 10)$, $(-5, 5\sqrt{3})$, $(-5\sqrt{3}, 5)$, $(-10, 0)$, $(-5\sqrt{3}, -5)$, and $(+5\sqrt{3}, -5)$, which lie upon the circle. Intermediate values



Circle of radius 10 ; units are eighths of an inch

$$x^2 + y^2 = 100, \text{ or } \begin{cases} x = 10 \cos \theta. \\ y = 10 \sin \theta. \end{cases}$$

can readily be obtained using the tables of sines and cosines.

The two equations together constitute the equations of the circle in parametric form, a type of equation of particular importance in applied mathematics.

If desired, we may eliminate θ as follows: squaring and adding gives

$$x^2 + y^2 = 100 (\cos^2 \theta + \sin^2 \theta),$$

or $$x^2 + y^2 = 100 \text{ (since } \sin^2 \theta + \cos^2 \theta = 1),$$

a relation independent of θ. But for many purposes it is more convenient to keep the equations in parametric form.

For the distance from any point $C(h, k)$ to a point $P(x, y)$ we have found the formula $d = \sqrt{(x-h)^2 + (y-k)^2}$; all points (x, y) which satisfy this equation for a given value of d, and for (h, k) a fixed point, lie upon a circle of which (h, k) is the center and d is the radius; no point not on the circle satisfies this equation.

$(x - h^2) + (y - k)^2 = r^2$ is then the equation of a circle of center (h, k) and radius r. Any equation which can be put into this form represents a circle, for it expresses the fact that the distance from any point (x, y) whose coördinates satisfy the given equation, to the fixed point (h, k) is constant and equal to r.

In parametric form, the two equations representing the circle with center (h, k) and radius r are written:

$$x - h = r \cos \theta.$$
$$y - k = r \sin \theta.$$

If θ is given values the corresponding values of x and y determine points upon the circle $(x - h)^2 + (y - k)^2 = r^2$.

Illustrative problem. — Find the equation of the circle of radius 5; center $(3, -7)$.

By the distance formula, taking (x, y) as any point on the circle,

$$(x - 3)^2 + (y + 7)^2 = 25,$$

or

$$x^2 - 6x + y^2 + 14y - 33 = 0.$$

In parametric form the equations of this circle are,

$$x - 3 = 5 \cos \theta.$$
$$y + 7 = 5 \sin \theta.$$

2. Reduction to standard form. — Any equation of the type,

$$x^2 + y^2 + 2Gx + 2Fy + C = 0,$$

or

$$Ax^2 + Ay^2 + 2Gx + 2Fy + C = 0,$$

represents a circle. The center and radius are determined by completing the square, as in the illustrative problem below. If the expression for the radius is zero, the circle reduces to a point; if it is negative the circle is imaginary.

Illustrative problem. — Find the center and radius of the circle,

$$2x^2 + 2y^2 + 6x - 7y - 15 = 0.$$

This equation represents a circle since it can be put into the form of a circle, as indicated herewith:

$$2(x^2 + 3x) + 2(y^2 - \tfrac{7}{2}y) = 15.$$
$$2(x^2 + 3x + \tfrac{9}{4}) + 2(y^2 - \tfrac{7}{2}y + \tfrac{49}{16}) = 15 + \tfrac{9}{2} + \tfrac{49}{8}.$$
$$2(x + \tfrac{3}{2})^2 + 2(y - \tfrac{7}{4})^2 = \tfrac{205}{8}.$$
$$(x + \tfrac{3}{2})^2 + (y - \tfrac{7}{4})^2 = \tfrac{205}{16} \text{ (or 12.81)}.$$

This equation states that the point (x, y) is at the distance $\frac{\sqrt{205}}{4}$ from the point $(-\frac{3}{2}, \frac{7}{4})$; this equation represents a circle with the center $(-\frac{3}{2}, \frac{7}{4})$ and radius $\frac{\sqrt{205}}{4}$ or $\frac{14.32}{4}$ or 3.58.

PROBLEMS

Find the equations of the following circles:

1. Center $(3, -4)$, radius 5.

2. Center $(0, 0)$, radius 10.

3. Center $(-4, 0)$, radius 4.

4. Center $(-6, 6)$, radius 6.

5. Center $(-6, -8)$, radius 10.

6. Draw the circle of radius 10, center $(0, 0)$ and estimate carefully its area on the coördinate paper.

Find the centers and radii of the following circles; time yourself; the eight problems should be completed numerically within 12 minutes.

7. $x^2 + y^2 - 12x + 12y + 36 = 0.$

8. $x^2 + y^2 - 12x + 12y - 36 = 0.$

9. $x^2 + y^2 - 39 = 0.$

10. $x^2 - 10x + y^2 - 39 = 0.$

11. $2x^2 + 2y^2 - 6x - 8y - 19 = 0.$

12. $2x^2 + 2y^2 - 5x + 7y - 15 = 0.$

13. $3x^2 + 3y^2 - 15x + 17y + 9 = 0.$

14. $x^2 + 6x + y^2 - 10 = 0.$

15. Draw the graphs of the preceding 8 circles, using only one or two sheets of graph paper; time yourself, keeping a record of the time.

16. Given
$$x = 5 \cos \theta,$$
$$y = 5 \sin \theta,$$
locate 16 points on the curve, using the values $\sin 0° = 0$, $\sin 30° = .5$, $\sin 45° = .707$, $\sin 60° = .866$ for these and related angles.

17. Given
$$x = 5 + 5 \cos \theta,$$
$$y = -3 + 5 \sin \theta,$$
locate 16 points on this circle.

18. When $\theta = 37°, 43°, 62°, 80°$, and $85°$ find x and y in the preceding problems.

19. Through what point on the circle $x^2 + y^2 = 25$ does the radius which makes an angle arc tan 2 with OX, pass?

The Rialto in Venice

A famous circular arch, 95 feet wide by 25 feet high.

3. To find the intersection of a line with a circle. Tangents. — The intersections of the circle, $x^2 + y^2 = 100$, with any line as $y = x + 5$, are represented by the solutions of the two equations regarded as simultaneous. Six problems are given here.

1. $x^2 + y^2 = 100$,
$y = x$.

2. $x^2 + y^2 = 100$,
$y = x + 5$.

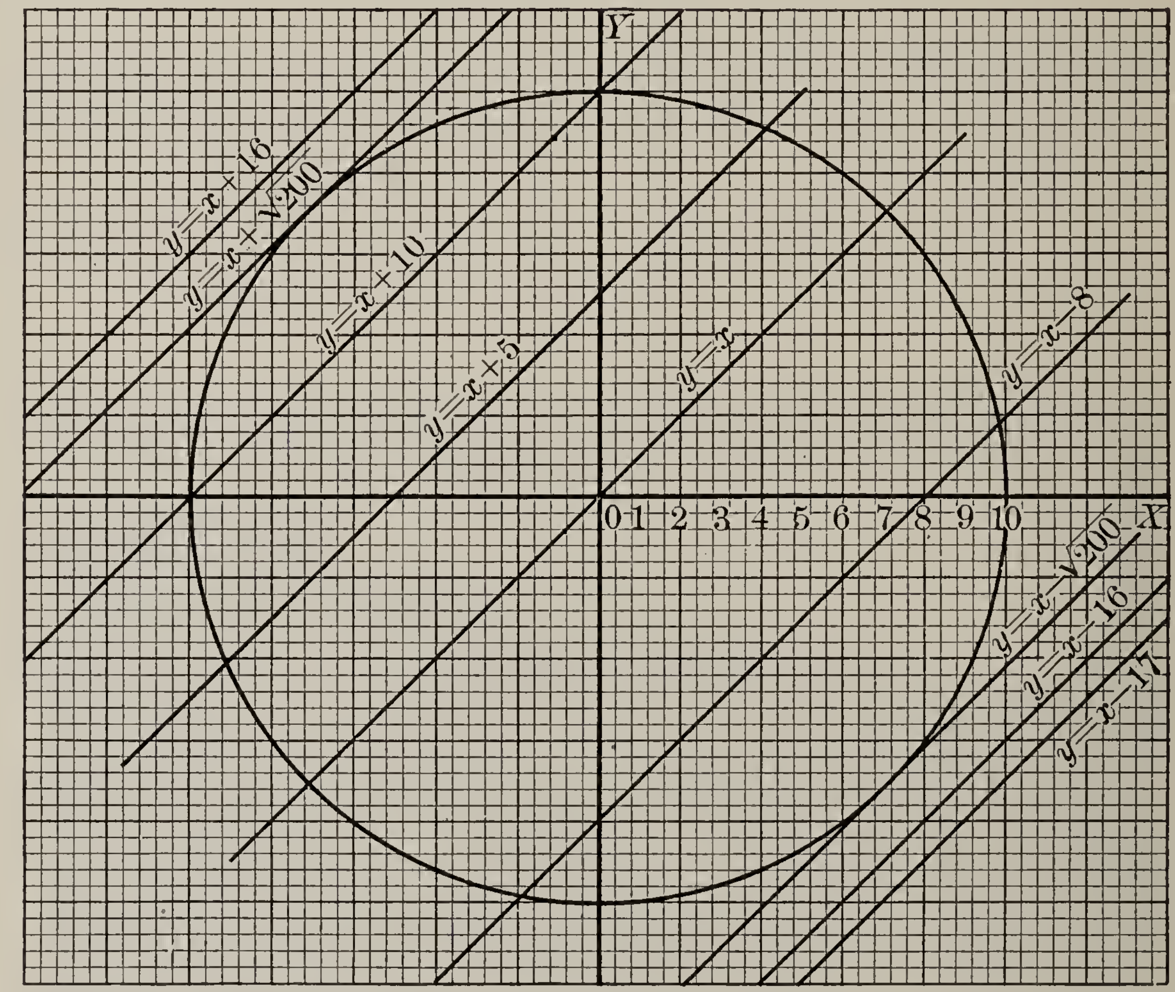
3. $x^2 + y^2 = 100,$
$y = x + 10.$

4. $x^2 + y^2 = 100,$
$y = x + 16.$

5. $x^2 + y^2 = 100,$
$y = x - 8.$

6. $x^2 + y^2 = 100,$
$y = x + k.$

Solving, by substitution in each of the six cases indicated above:



Graphical solution, determining intersections of the circle, $x^2 + y^2 = 100$, with various lines of slope 1

1. gives $2x^2 = 100$, $x^2 = 50$, $x = \pm\sqrt{50} = \pm 7.07$, $y = \pm 7.07$;
2. gives $2x^2 + 10x + 25 = 100$, $x^2 + 5x - 37.5 = 0$;

$$x = -2.5 \pm \sqrt{6.25 + 37.5} = -2.5 \pm 6.61$$
$$= +4.11 \text{ or } -9.11,$$
$$y = 9.11 \text{ or } -4.11;$$

3. gives $x^2 + 10x = 0$, $x = 0$ or -10 (by factoring, simplest),
$y = 10$ or 0;

4. gives $2x^2 + 32x + 156 = 0$, $x^2 + 16x + 78 = 0$,

$$\begin{aligned} x &= -8 + \sqrt{64 - 78} \\ &= -8 + \sqrt{-14} \\ &= \text{imaginary values, not any scalar values of } x; \end{aligned}$$

5. gives $2x^2 - 16x - 36 = 0$, $x^2 - 8x - 18 = 0$, $x = 4 \pm \sqrt{34}$

$$\begin{aligned} &= 4 \pm 5.83 \\ &= 9.83, \text{ or } -1.83, \\ y &= 1.83 \text{ or } -9.83; \end{aligned}$$

6. gives $2x^2 + 2k \cdot x + (k^2 - 100) = 0$,

$$\begin{aligned} x &= \frac{-k \pm \sqrt{k^2 - 2(k^2 - 100)}}{2} \\ &= -\frac{k}{2} \pm \tfrac{1}{2}\sqrt{200 - k^2}. \end{aligned}$$

Very evidently the solutions of 1 to 5 are all included under the solution 6, as special cases.

Geometrically the lines of slope 1 are divided by the circle into three classes, viz. (*a*) those which cut the circle in two distinct points; (*b*) those which do not cut the circle; and (*c*) those which are tangent to the circle, or cut the circle in two coincident points.

Evidently lines in 1, 2, 3, and 5 belong to the first class; the line in 4 to the second class. To determine the tangents one must find the value of k for which $200 - k^2 = 0$, as only when $200 - k^2 = 0$ are two points whose abscissas are given by $-\frac{k}{2} + \frac{1}{2}\sqrt{200 - k^2}$ and $-\frac{k}{2} - \frac{1}{2}\sqrt{200 - k^2}$ coincident. In this case, $k = \pm 14.14$, the lines $y = x \pm 14.14$ are tangent to the circle $x^2 + y^2 = 100$. The abscissa of the point of tangency is ∓ 7.07, since it equals $-\frac{k}{2}$.

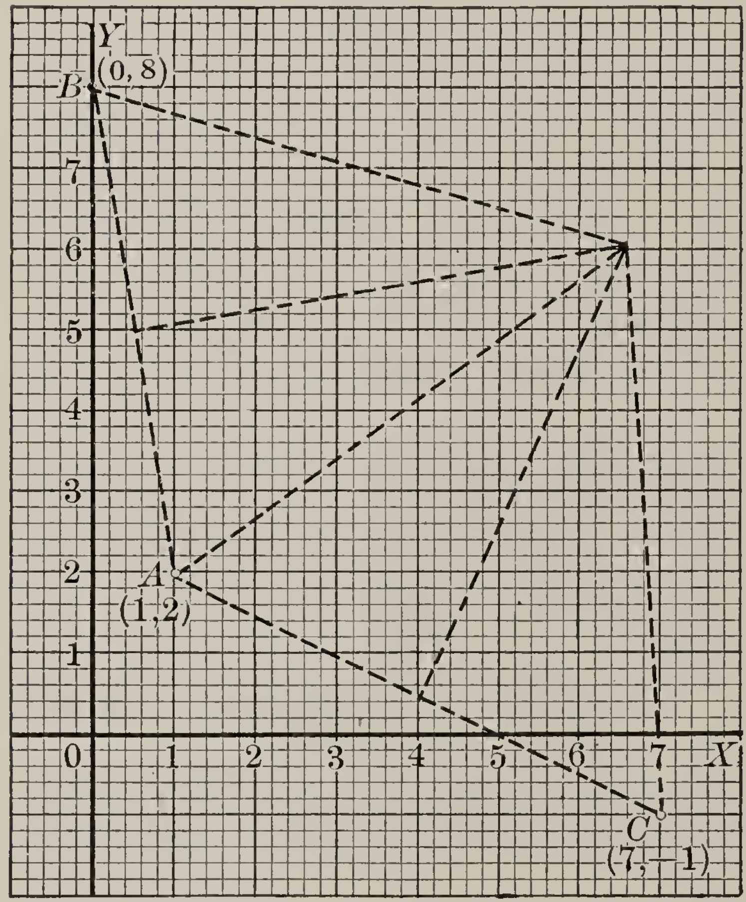
RULE. — *To find the tangent with given slope to a given circle write the equation of the family of lines of the given slope,* $y = mx + k$, *and solve for the points of intersection with the circle; get the condition that the two points of intersection should be coincident. This gives the value of* k *for which the line* $y = mx + k$ *is tangent to the given circle.*

NOTE. — The method will apply to any curve of the second degree.

4. Circles satisfying given conditions. — To find the equation of a circle which satisfies given conditions it is necessary to use the analytic formulas which we have derived combined with the geometric properties of a circle. In general *call* (h, k) or (x, y) the center of the circle and r the radius; sketch the lines and points which are given and indicate roughly the probable position of the desired circle; solve the problem geometrically if possible, or indicate the solution, and express the geometrical facts in algebraical language by using the preceding formulas.

5. Illustrative problems. — Find the equation of the circle through A $(1, 2)$, B $(0, 8)$, and C $(7, -1)$, (1) using the distance formula, (2) using the perpendicular bisector of the line joining two points, (3) using the general equation $(x-h)^2+(y-k)^2=r^2$ which may represent any circle, and (4) using the general equation

$$Ax^2 + Ay^2 + 2\,Gx + 2\,Fy + C = 0.$$



Circle through three points

Determination of center by perpendicular bisectors of chords.

(1) Call the center $P(h, k)$, then $PA=PB$, $PB = PC$, and $PA=PC$.

The distance from A to P equals the distance from B to P, whence by the distance formula,

$$\text{I.} \qquad \sqrt{(h-1)^2+(k-2)^2} = \sqrt{(h-0)^2+(k-8)^2}\,;$$

Similarly,

II. $$\sqrt{(h-1)^2+(k-2)^2}=\sqrt{(h-7)^2+(k+1)^2}$$

expresses analytically the fact that $PA = PC$; and

III. $$\sqrt{(h-0)^2+(k-8)^2}=\sqrt{(h-7)^2+(k+1)^2}$$

that $PB = PC$.

Since equation III is derivable from I and II, it adds nothing new; any two of these equations are sufficient to determine (h, k) the center. Squaring in each and combining terms we obtain from I,

$$12\,k - 2\,h - 59 = 0,$$

a straight line which is the locus of all points equidistant from A and B; and from II,

$$4\,h - 2\,k - 15 = 0.$$

Solving, we obtain the one point which is equidistant from A, B, and C,

$h = 6.77.$

$k = 6.05.$

$r = \sqrt{(6.77-1)^2+(6.05-2)^2} = \sqrt{(33.29+16.40)} = \sqrt{49.69}$
$= 7.05.$

The circle is $(x-6.77)^2+(y-6.05)^2=(7.05)^2$.

(2) The center of the circle is the intersection of the perpendicular bisectors of the sides; finding the slopes of the sides, the mid-points, the slope of the perpendicular to each side, the equations of the perpendicular bisectors of AB and AC are found (point-slope) to be

$$12\,y - 2\,x - 59 = 0.$$
$$4\,x - 2\,y - 15 = 0.$$

Examination shows that these are precisely in x and y the equations obtained in our first solution in h and k and from this point the solution proceeds as in (1). The student should explain the reason for this.

(3) I. $$(x-h)^2+(y-k)^2=r^2$$

is the equation of any circle, center (h, k), radius r.

Substituting in this equation (1, 2), (0, 8), and (7, − 1), gives,

II. $$(1-h)^2+(2-k)^2=r^2.$$

III. $$(0-h)^2+(8-k)^2=r^2.$$

IV. $$(7-h)^2+(-1-k)^2=r^2.$$

V. III − II $$-2\,h+12\,k-59=0.$$

VI. II − IV $$-48+12\,h+3-6\,k=0$$
$$\text{or } 4\,h-2\,k-15=0.$$

V and VI are seen to be in h and k precisely the equations solved in method (1), and in method (2) for x and y as variables.

(4) I. $$Ax^2 + Ay^2 + 2\,Gx + 2\,Fy + C = 0.$$

Substitute in this equation (1, 2), (0, 8), and (7, – 1) and solve for the values of G, F, and C in terms of A.

II. $$A + 4\,A + 2\,G + 4\,F + C = 0.$$

III. $$64\,A + 16\,F + C = 0.$$

IV. $$49\,A + A + 14\,G - 2\,F + C = 0.$$

V. II – III $$-59\,A + 2\,G - 12\,F = 0.$$

VI. II – IV $$-45\,A + 12\,G + 6\,F = 0$$

$$\text{or } -15\,A - 4\,G + 2\,F = 0.$$

These are the same equations in $-\dfrac{G}{A}$ and $-\dfrac{F}{A}$, regarded as the unknowns, as appeared above in h and k.

6. Tangency conditions. — If a circle is to be tangent to a given line the distance formula (normal form) from a point to a line may be used; if a circle to be found is to be tangent to a given circle, then the radius sought, plus or minus the given radius, must be equal numerically to the distance from center to center, according as the circles are tangent externally or internally.

7. Circle through the intersection of two circles. —

I. $x^2 + y^2 + 10\,x = 0$, a circle of radius 5, center $(-5, 0)$.

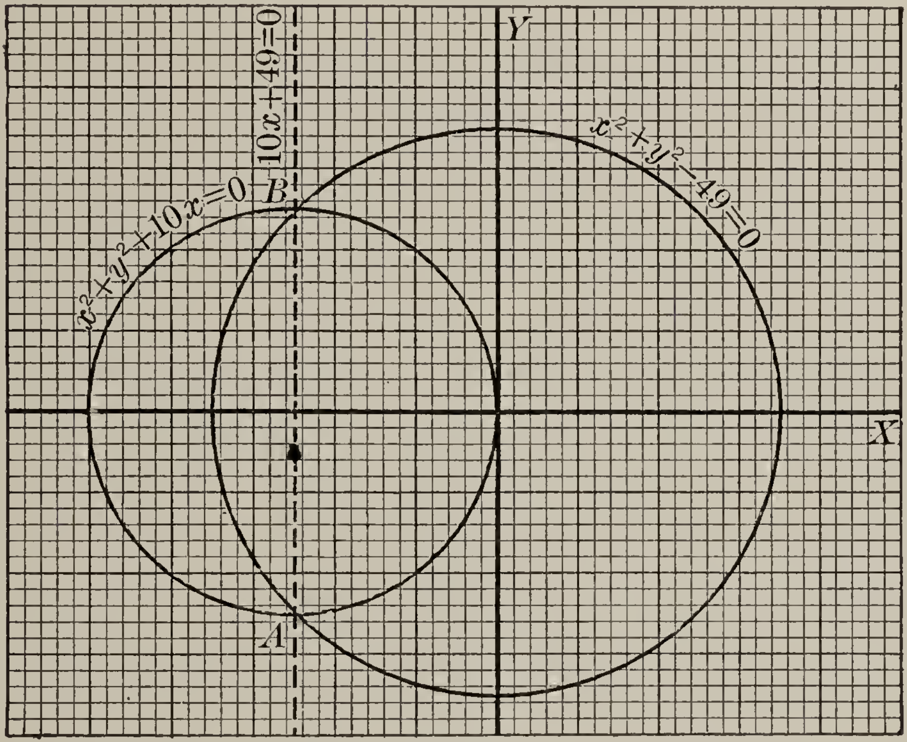
II. $x^2 + y^2 - 49 = 0$, a circle of radius 7, center $(0, 0)$.

III. $(x^2 + y^2 + 10\,x) + k(x^2 + y^2 - 49) = 0.$

The third equation is satisfied by the points of intersection of curves I and II, for all values of k (see page 83). For all constant values of k, III may be written

$$(1 + k)x^2 + (1 + k)y^2 + 10\,x - 49\,k = 0,$$

and the form shows that this represents a circle. To determine the circle through the intersections of I and II, and any other given point substitute the coördinates in III, and solve for k; since a circle is determined by the three points, it is easily seen that every circle through the two points of intersection of the given circle is included in the family of circles, $(1+k)x^2+(1+k)y^2+10x-49k=0$. The method of determining k to have the circle pass through some other given



Common chord of two circles or radical axis

point is precisely the same as in the similar problem with straight lines (page 83).

For $k=-1$, this equation reduces to the linear equation representing the common chord of the family of circles; whether two given circles intersect or not, this line, whose equation is obtained by eliminating x^2+y^2 between the two given equations, is called the radical axis of the two circles.

8. Geometrical property of the radical axis. —

$(x-h)^2+(y-k)^2$ is the square of the distance from (x, y) to the center of any circle; $(x-h)^2+(y-k)^2-r^2$ is the square of the length of the tangent to the circle from any point outside the circle of center (h, k), radius r.

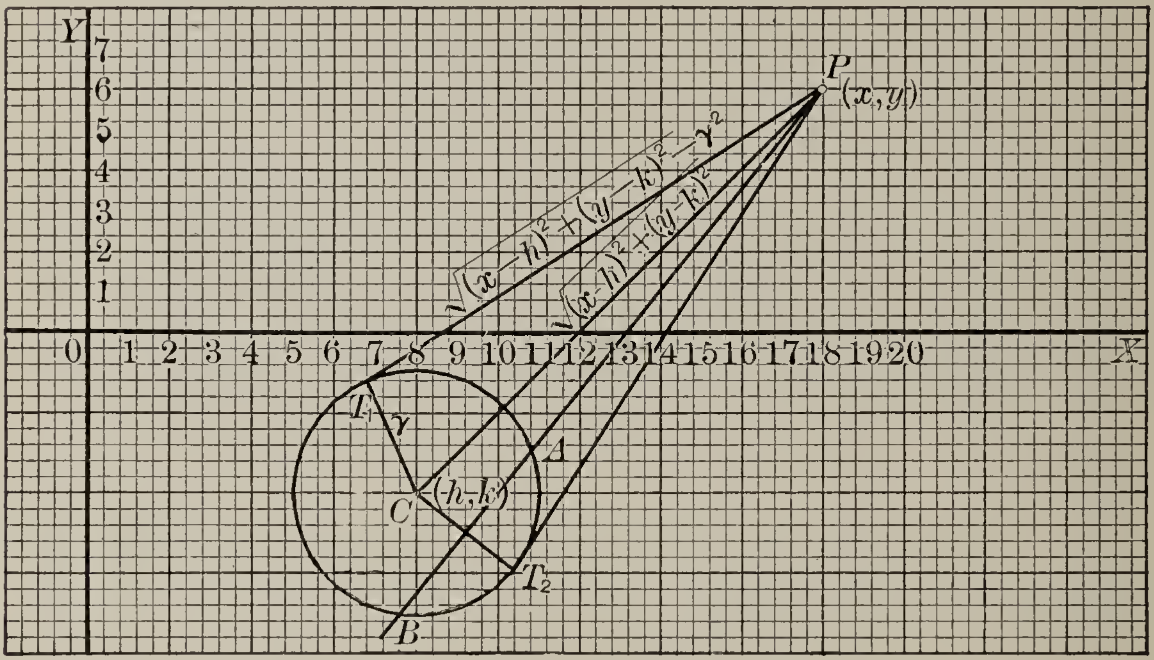
$x^2+y^2+2Gx+2Fy+C$ is the square of the length of the tangent from any point (x, y) to the circle whose equation is $x^2 + y^2 + 2\,Gx + 2\,Fy + C = 0$, since the left-hand member is identical with the left-hand member when written in this form:

$$(x + G)^2 + (y + F)^2 - (G^2 + F^2 - C) = 0.$$

Note that if any secant PAB is drawn through $P(x, y)$ then $PA \cdot PB = PT^2$; hence the expression

$$x^2 + y^2 + 2\,Gx + 2\,Fy + C$$

gives the product of the two distances along any straight line from the point $P(x, y)$ on the line to the circle. There is a correspondence to the normal form of a straight line, since the left-hand member there also represents a distance.



Distances from a point to a circle

On any secant through P, PAB, $PA \times PB$ is constant.

$$PA \times PB = \overline{PT_1}^2 = (x - h)^2 + (y - k)^2 - r^2.$$

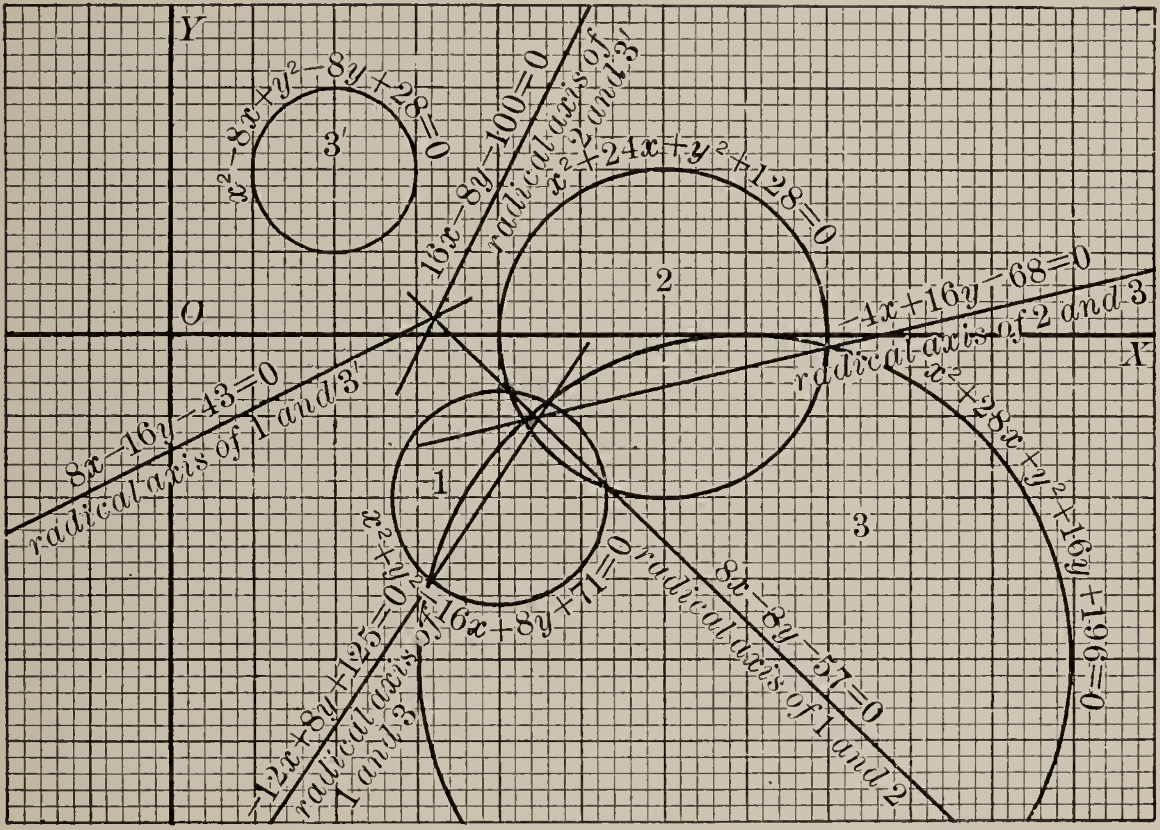
$$x^2 + y^2 + 2\,G_1x + 2\,F_1y + C_1 = x^2 + y^2 + 2\,G_2x + 2\,F_2y + C_2$$

is an equation which is satisfied by any point from which tangents drawn to the two circles

$$x^2 + y^2 + 2\,G_1x + 2\,F_1y + C_1 = 0,$$
$$x^2 + y^2 + 2\,G_2x + 2\,F_2y + C_2 = 0,$$

are equal in length. Hence, the radical axis is the locus of points from which the tangents drawn to the two circles are equal in length.

9. Radical center of three circles. — Given three circles, each of the three pairs of circles which may be formed from the three has a radical axis; the three radical axes pass through a common point, as may be easily shown by Sec. 4, Chapter V.



Radical axes and radical centers

Radical center of the three circles, 1, 2, and 3.
Radical center of the three circles, 1, 2, and 3′.

a. $x^2 + y^2 + 2\,G_1x + 2\,F_1y + C_1 = 0.$

b. $x^2 + y^2 + 2\,G_2x + 2\,F_2y + C_2 = 0.$

c. $x^2 + y^2 + 2\,G_3x + 2\,F_3y + C_3 = 0.$

d. $(a - c)$ $2(G_1 - G_2)x + 2(F_1 - F_2)y + C_1 - C_2 = 0$
radical axis of *a* and *b*.

e. $(b - c)$ $2(G_2 - G_3)x + 2(F_2 - F_3)y + C_2 - C_3 = 0$
radical axis of *b* and *c*.

f. $(d+e)$ or $(a-c)$ $2(G_1-G_3)x+2(F_1-F_3)y+C_1-C_3=0.$

Since $d+e=0$ gives a straight line through the intersection of d and e, and since $d+e=0$ gives the radical axis of a and c, the latter line passes through the intersection of the two former radical axes.

10. Limiting forms of the circle equation. —

$(x-h)^2+(y-k)^2=r^2$ represents a real circle when r^2 is positive.

$(x-h)^2+(y-k)^2=0$ represents a point circle; the only real point which satisfies this equation is the point (h, k).

$(x-h)^2+(y-k)^2=-r^2$, r a real quantity, represents an imaginary circle; no real point satisfies this equation, since every real value of x and y makes $(x-h)^2$ positive and $(y-k)^2$ positive.

PROBLEMS ON THE CIRCLE

1. Find the center and give radius to 1 decimal place of each of the following circles; plot; find the three radical axes and the radical center.

a. $x^2+y^2+6x-8y-16=0.$

b. $3x^2+3y^2-8x+15y-7=0.$

HINT. $3(x^2-\frac{8}{3}x \quad)+3(y^2+5y \quad)=7$; complete squares inside parentheses and note that 3 times the quantity added within each of the parentheses must be added on the right.

c. $x^2+y^2-6x-8=0.$

2. Plot the following two circles and determine their common chord; what is its length?

a. $x^2+y^2-10x-100=0.$

b. $x^2+y^2+10y-100=0.$

3. Write the equation of the family of circles

a. With center on x-axis, passing through the origin.

b. With center on y-axis, passing through the origin.

c. Passing through the origin.

d. With center on $3x - 4y - 5 = 0$, radius 5.

NOTE. $3h - 4k - 5 = 0$.

4. What limitation is imposed upon the coefficients A, G, F, and C in
$$Ax^2 + Ay^2 + 2Gx + 2Fy + C = 0,$$

a. if the circle passes through (0, 0)? (1, 1)?

b. if the circle has its center on the axis of x? y-axis?

c. if the circle is tangent to the x-axis? y-axis? tangent to $x - 3 = 0$?

d. if the circle is tangent to $x - y - 5 = 0$?

5. Find the equations of the circles through the following three points:

a. (0, 0), (6, 0), (0, 8).

b. (1, 5), (− 3, 1), (7, − 3).

c. (0, 0), (8, 2), (15, − 3); use two different methods.

6. Find circle tangent to $3x + 4y - 25 = 0$, and passing through (2, 3) and (5, 1). Note the two solutions.

7. Find the radical axis of each of the three pairs of circles
$$x^2 + y^2 - 6x - 8y - 10 = 0,$$
$$x^2 + y^2 - 20x + 50 = 0,$$
$$2x^2 + 2y^2 + 8x + 6y - 25 = 0.$$
Find the radical center. Plot.

8. Find the tangents of slope 2 to the first circle in 7; find the normal and the point of tangency.

9. Find the circle of radius 5 tangent to the line whose equation is $4x - 3y - 9 = 0$ at (3, 1).

10. Find for what value of r the line $4x - 3y - 9 = 0$ is tangent to $x^2 + y^2 - r^2 = 0$. Two methods. Find the point of tangency.

11. Find to one decimal place the points of intersection of the circle $x^2 + y^2 - 20x + 50 = 0$ with the line $y = 2x - 12$. Plot.

12. Use the trigonometric functions to find points of intersection of

$$\begin{cases} x^2 + y^2 = 100, \\ y = 2\,x. \end{cases}$$

Note that $\tan\theta = 2$, where θ is the slope-angle of the line.

13. Use trigonometric functions to find k, when $y = 2\,x + k$ is tangent to the circle $x^2 + y^2 = 100$. Draw figure; note that $\tan\theta = -\frac{1}{2}$ where θ is the slope-angle of the normal.

14. Plot the circle $\begin{cases} x = 3 + 10\cos\theta, \\ y = -5 + 10\sin\theta. \end{cases}$

15. Plot the circle $\begin{cases} x = 8\sin\theta, \\ y = 8\cos\theta. \end{cases}$

Note that θ is here the angle made with the y-axis by any radius.

16. Find the equation of the complete circle of the circular arch of the Rialto, referred to the horizontal water line and the axis of symmetry of the arc as axes. The arch is 95 feet wide by 25 feet high.

17. Find the equation of the circle of which the arch of the Rocky River Bridge, 280 feet by 80 feet, is an arc, referred to a tangent at the highest point of the arc as x-axis and the perpendicular at the point of tangency as y-axis. Determine the lengths of vertical chords between the arc and the x-axis, spaced at intervals of forty feet.

18. Prove that the radical axis of any two circles is perpendicular to their line of centers.

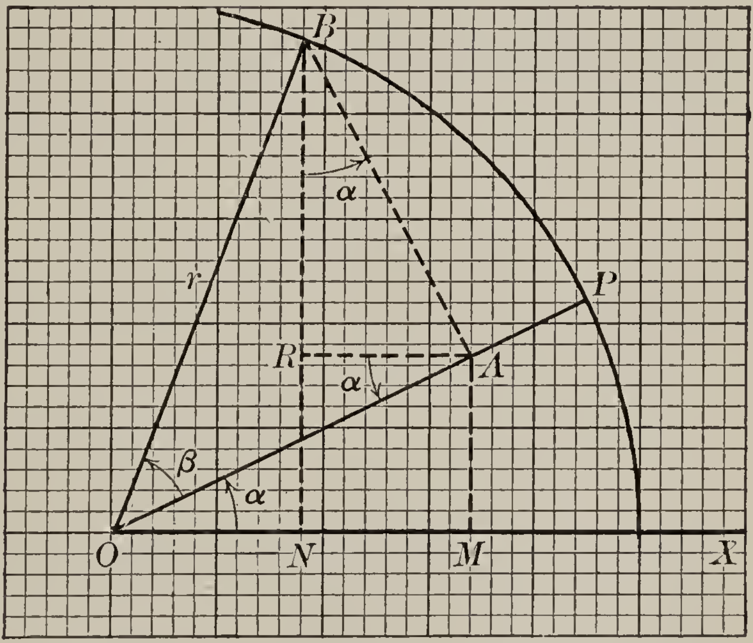
19. If a circle is tangent to a given line and to a given circle, what conditions must the coördinates (h, k) of its center satisfy?

CHAPTER XV

ADDITION FORMULAS

1. Functions of the sum and difference of two angles. — The formulas for $(a+b)^2$ and $(a-b)^2$ are illustrations of addition formulas frequently of fundamental importance in mathematical work. Thus $10^x \cdot 10^y = 10^{x+y}$ is an addition formula leading to the whole theory of logarithms, which revolutionized computation processes. The question arises as to addition formulas in the case of the trigonometric functions after the functions have been defined. Just as the exponent formula $10^{x+y} = 10^x \cdot 10^y$, which was first proved for positive integers, is extended to hold for all values of x and y, so the formulas which are established for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ when α and β are acute angles will be found to hold for all real values of α and β.

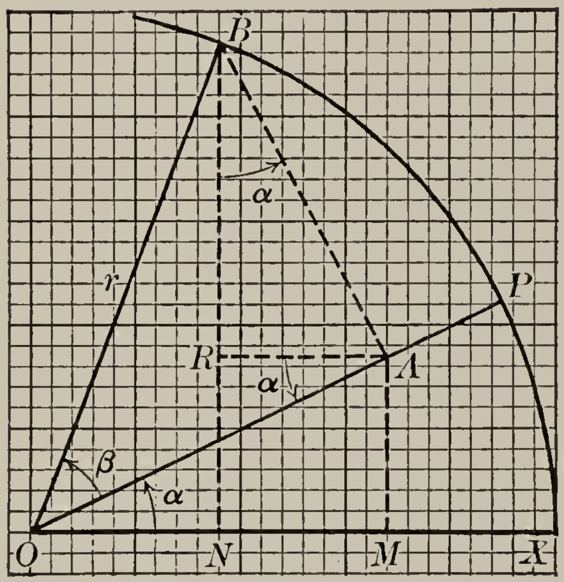
2. Geometrical derivation of sin $(\alpha+\beta)$ and cos $(\alpha+\beta)$; α and β acute and $\alpha+\beta<90°$. — Given α and β, two acute angles whose sum is less than 90°, to find $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ in terms of $\sin\alpha$, $\cos\alpha$, $\sin\beta$, and $\cos\beta$.



$\angle XOP = \alpha$; $\angle POB = \beta$; $\angle XOB = \alpha + \beta$; $OB = r$

On the figure let α and β be two positive acute angles whose sum is less than 90°, taken, for convenience, distinctly different from each other. Let OP make the angle α with OX, and OB make the angle β with OP, and thus $\alpha+\beta$ with OX. From B drop perpendiculars BA to OP and BN to OX; from A on OP drop a perpendicular AM to OX; from A draw a parallel to OX cutting BN at R.

On the figure, noting that OB is taken as r, we have the following evident relations:



$$AB = r\sin\beta;\quad RB = AB\cos\alpha$$
$$= r\cos\alpha\sin\beta;$$

$$OA = r\cos\beta;\quad AM = OA\sin\alpha$$
$$= r\sin\alpha\cos\beta;$$

$$NM = RA = AB\sin\alpha$$
$$= r\sin\alpha\sin\beta;$$

$$OM = OA\cos\alpha = r\cos\alpha\cos\beta.$$

$$\sin(\alpha+\beta) = \frac{NB}{r} = \frac{MA+RB}{r}$$

$$= \frac{r\sin\alpha\cos\beta + r\cos\alpha\sin\beta}{r}.$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Similarly, $\cos(\alpha+\beta) = \frac{ON}{r} = \frac{OM-NM}{r}$

$$= \frac{r\cos\alpha\cos\beta - r\sin\alpha\sin\beta}{r},$$

whence $$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Having established these formulas geometrically for $\alpha+\beta$ when $0<\alpha<90°$, $0<\beta<90°$, and $\alpha+\beta<90°$, it now remains to establish that these formulas hold for *all* angles α and β, including negative angles. This extension is made by employing the theorems of Section 12, Chapter VII.

3. Generalization for any two acute angles. —

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

First we will show that when α and β are *any* two acute angles the two formulas established above when $\alpha + \beta < 90°$ continue to hold. The extension to *any acute* angles requires that we prove these formulas to be true further (a) when $\alpha + \beta = 90°$, and (b) when $\alpha + \beta < 90°$.

Proof. (a) If $\alpha + \beta = 90°$, $\beta = 90° - \alpha$, whence

$\sin\beta = \sin(90° - \alpha) = \cos\alpha$; $\cos\beta = \cos(90 - \alpha) = \sin\alpha$.

The two formulas then give, by substitution,

$$\sin(\alpha + \beta) = \sin 90° = \sin^2\alpha + \cos^2\alpha = 1,$$
$$\cos(\alpha + \beta) = \cos 90° = \cos\alpha\sin\alpha - \sin\alpha\cos\alpha = 0.$$

The sine of 90° is 1, and the cosine of 90° is 0; hence our formulas continue to hold even when $\alpha + \beta = 90°$.

(b) $\alpha + \beta > 90°$. Take the complements of α and β to be respectively x and y, whence $x = 90° - \alpha$ and $y = 90° - \beta$. Evidently $x + y$ will be less than 90°, by the same amount that $\alpha + \beta$ exceeds 90°. Further, since $x = 90° - \alpha$ and $y = 90° - \beta$,

$$\sin x = \cos\alpha, \qquad \sin y = \cos\beta,$$
$$\cos x = \sin\alpha, \quad \text{and} \quad \cos y = \sin\beta.$$

Now, $\sin(\alpha + \beta) = \sin(90° - x + 90° - y)$

$= \sin(180° - \overline{x + y}) = \sin(x + y)$, since

$\sin(180° - \theta) = \sin\theta$.

Since $x + y < 90°$, $\sin(x + y) = \sin x \cos y + \cos x \sin y$, as established above; making the substitutions for $\sin x$, $\cos y$, $\cos x$, and $\sin y$, we have $\sin(\alpha + \beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta$.

Q. E. D.

Similarly,

$\cos(\alpha + \beta) = \cos(180° - \overline{x + y}) = -\cos(\overline{x + y})$, since $\cos(180° - \theta) = -\cos\theta$, for any angle θ. But x and y are acute angles, whose sum is less than 90°;

therefore
$$\cos(x+y) = +\cos x \cos y - \sin x \sin y$$
$$= +\sin\alpha \sin\beta - \cos\alpha\cos\beta.$$

Now $\cos(\alpha+\beta) = -\cos(x+y)$,

or
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \qquad \text{Q. E. D.}$$

4. Extension of the formulas for sin (α + β) and cos (α + β) to all angles without restriction. — To show that these formulas hold for all angles it is necessary now to show that if either α or β is increased by 90° the formulas continue to hold provided that they hold for α and β.

Thus given
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$
we wish to show that $\sin(\alpha+y)$ and $\cos(\alpha+y)$ are given by similar formulas, when $y = \beta + 90°$.
$$\sin(\alpha+y) = \sin(\alpha+\beta+90°) = \cos(\alpha+\beta),$$
$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$
but $\sin y = \cos\beta$ and $\cos y = -\sin\beta$, whence substituting,
$$\sin(\alpha+y) = \cos\alpha\sin y + \sin\alpha\cos y. \qquad \text{Q. E. D.}$$

Similarly for $\cos(\alpha+y)$, we find $\cos\alpha\cos y - \sin\alpha\sin y$; since α and β enter symmetrically in the above formulas this proof establishes that α also could be increased or, by an entirely analogous procedure, decreased by 90°, with the same formulas for the new values.

This establishes the formulas for any two angles α and β whatever. For since the formulas have been proved above to hold for any two acute angles α and β, the formulas hold for any obtuse angle and any acute angle since y, the obtuse angle, may be regarded as $90° + \beta$. This establishes the formulas for any angle in I and any angle in II; now increase α by 90°, thus establishing the formula for any two obtuse angles. Continuing in this way α can be any angle in any quadrant, I to IV, and β also an angle in any quadrant whatever, and the formulas continue to be true.

After these formulas are established for all positive angles up to 360°, another method of procedure to establish the formulas for all positive and negative angles is to note that any integral multiple of 360°, $k \cdot 360°$ with k a positive or negative integer, can be added to any angle without changing the functions of the angle involved in our formulas. Thus if $-\beta$ is any negative angle, numerically less than 360°, the functions of $\alpha+(-\beta)$ are the same as the functions of $\alpha+(360°-\beta)$ which is the sum of two positive angles; but the functions of $360°-\beta$ are the same as those of $-\beta$ and after application of the formula the 360° can be dropped. In other words in these formulas any integral multiple of 360° can be added at pleasure and also dropped at pleasure, and in this way the formulas are established for all angles.

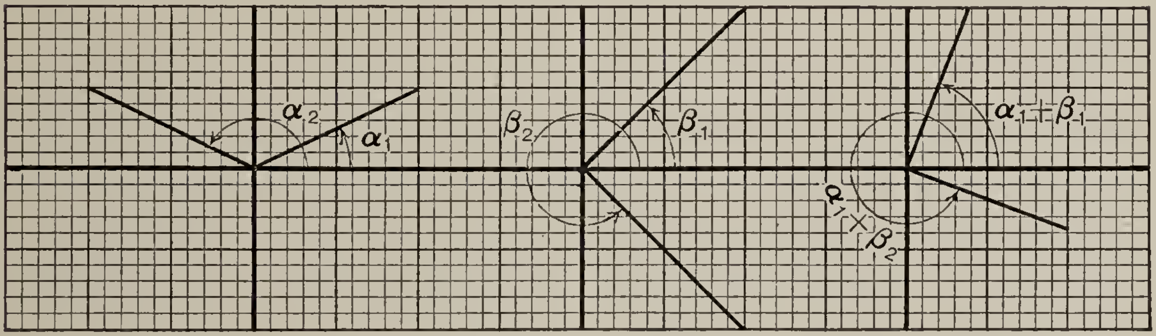
Illustrative problem. — Given $\sin\alpha = .45$, $\cos\beta = .68$, find $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$.

$\sin\alpha = .45$; α can be in I or II since $\sin(180° - \alpha) = \sin\alpha$.

$\cos\alpha = \pm\sqrt{1 - .45^2} = \pm\sqrt{.7975} = \pm .893.$

$\cos\beta = .68$; β can be in I or IV since $\cos(-\beta) = \cos\beta$.

$$\sin\beta = \pm\sqrt{1-.68^2} = \sqrt{(.32)(1.68)} = \pm .16 \times \sqrt{21}$$
$$= \pm .16 \times 4.58 = \pm .733.$$



sin α = .45 determines either α₁ or α₂

cos β = .68 determines either β₁ or β₂

There are strictly four problems, solved as follows:

α in I, β in I. $\qquad$ α in I, β in IV.

$\sin\alpha = .45.$

$\cos\alpha = +.893.$

$\cos\beta = +.68.$

$$\sin\beta = +.733.$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

$$\sin(\alpha+\beta) = .45\times .68 + .893\times .733 = .306 + .655 = .961.$$

$$\cos(\alpha+\beta) = .893\times .68 - .45\times .733 = .607 - .330 = .277.$$

$$\sin\beta = -.733.$$

$$\sin(\alpha+\beta) = .45\times .68 - .893\times .733.$$

$$\sin(\alpha+\beta) = .306 - .655 = -.349.$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

$$\cos(\alpha+\beta) = .893\times .68 + .45\times .733 = .607 + .330 = .937.$$

The two columns represent two solutions which have the three central values, sin α, cos α, and cos β, in common.

The student is expected to complete the solution, beginning as follows:

α in II, β in I. α in II, β in IV.

$$\sin\alpha = .45.$$

$$\cos\beta = .68.$$

$$\cos\alpha = -.893.$$

$$\sin\beta = +.733. \qquad \sin\beta = -.733.$$

In general work only one case, indicating which solution is given.

5. Historical note. — The formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ are closely allied to Ptolemy's theorem (c. 150 A.D.) that in any inscribed quadrilateral the product of the diagonals is equal to the sum of the products of the opposite sides. If a, b, c, and d are the sides, in order around the quadrilateral, and e and f the diagonals, $ef = ac + bd$; in the Greek trigonometry employing chords this theorem plays the same rôle that the formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ play in the trigonometry employing sines and cosines. A great French mathematician, Viète (1540–1603), the first to use generalized coefficients in algebraic equations, was the first to give these formulas, as $\sin(2\alpha+\beta)$ and $\cos(2\alpha+\beta)$ in terms of $\alpha+\beta$ and α; the modern form appeared in 1748 in the work of the Swiss mathematician Euler.

PROBLEMS

1. Given $\alpha = 30°$, $\beta = 45°$, find $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$. Check by tables.

2. Given $\alpha = 60°$, $\beta = 45°$, find $\sin 105°$, and $\cos 105°$. Check by the preceding problem, and explain the check.

3. Given $\sin\alpha = \frac{3}{4}$, and $\sin\beta = \frac{5}{13}$, find $\sin(\alpha+\beta)$, when α and β are both acute; find $\sin(\alpha+\beta)$ when α and β are both obtuse; when α is obtuse, β acute.

4. Given α and β acute angles, $\sin\alpha = .351$, $\cos\beta = .652$, find $\sin(\alpha+\beta)$ by the formula and check with the tables.

5. Given $\sin 18° = .3090$, $\cos 18° = .9511$, find $\sin 36°$.

6. Given $\sin 18° = .3090$, find $\sin 78°$.

7. Using the results of problem 1 for $\sin 75°$ and $\cos 75°$ with the data of problem 5, find $\sin 93°$ and $\cos 93°$; thus find $\sin 3°$ and $\cos 3°$.

8. What are $\sin(45°+\alpha)$ and $\cos(45°+\alpha)$ in terms of α?

9. Find $\sin(60°+\alpha)$ in terms of $\sin\alpha$ and $\cos\alpha$.

6. The formulas for sin (α − β) and cos (α − β). — If β is a negative angle, $\alpha-\beta$ comes directly under the $\alpha+\beta$ formula as $\alpha+(-\beta)$; if β is any positive angle greater than 360°, β can be reduced to less than 360° by subtracting 360° (or some multiple of 360°) without affecting the functions of $\alpha-\beta$ or of β; if β is positive and less than 360°, the functions of $\alpha-\beta$ will be the same as the functions of $\alpha+(360°-\beta)$, since this simply adds one complete revolution to $\alpha-\beta$. Hence

$$\sin(\alpha-\beta) =$$

$$\sin(\alpha+\overline{360°-\beta}) = \sin\alpha\cos(360°-\beta) + \cos\alpha\sin(360°-\beta)$$
$$= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta).$$

$$\cos(\alpha-\beta) = \cos\alpha\cos(360°-\beta) - \sin\alpha\sin(360°-\beta)$$
$$= \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta).$$

Substituting in these formulas, $\cos(-\beta) = \cos\beta$, and $\sin(-\beta) = -\sin\beta$ we obtain the subtraction formulas:

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta,$$
$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

7. Tangent formulas. —

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}; \quad \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$

Since $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$

and $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$

for all angles α and β, without restriction, it follows that

$$\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta},$$

for all angles α and β.

Dividing numerator and denominator of the right-hand expression by $\cos\alpha\cos\beta$, we have

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$$

Similarly, $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$

8. Functions of double an angle. — The formulas for $\sin(\alpha + \beta)$, $\cos(\alpha + \beta)$, and $\tan(\alpha + \beta)$ hold if $\beta = \alpha$, whence

$$\sin(2\alpha) = \sin(\alpha + \alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha = 2\sin\alpha\cos\alpha,$$
$$\sin(2\alpha) = 2\sin\alpha\cos\alpha.$$

Similarly, $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha.$
By division and simplification, or directly from $\tan(\alpha + \beta)$,

$$\tan(2\alpha) = \frac{2\tan\alpha}{1 - \tan^2\alpha}.$$

Note that whether α be regarded as positive or negative, *i.e.* as obtained by positive or negative rotation, as $+\alpha$, or $+\alpha - 360°$, 2α has the same terminal line as $2\alpha - 720°$.

PROBLEMS

NOTE. — See the preceding list of problems, and use numerical values as there computed.

1. Given $\alpha = 45°$, $\beta = 30°$, find $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$, checking by the tables.

2. Given $\alpha = 60°$, $\beta = 45°$, find $\sin(\alpha - \beta)$ and compare with problem 1. Find $\tan(\alpha + \beta)$, $\tan(\alpha - \beta)$, and $\tan 2\alpha$.

3. Given $\sin\alpha = \frac{3}{4}$ and $\sin\beta = \frac{5}{13}$, find $\sin(\alpha - \beta)$ when α and β are acute; find $\sin(\alpha - \beta)$ when α and β are both obtuse. Explain the result; find $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$ when α is obtuse and β is acute. Interpret. Find $\tan(\alpha - \beta)$, $\tan(\alpha + \beta)$, and $\tan 2\alpha$ for α and β in I.

4. Given α and β acute, $\sin\alpha = .351$ and $\cos\beta = .652$, find $\sin(\alpha - \beta)$ and check by the tables. Find $\tan(\alpha - \beta)$.

5. Given $\sin 18° = .3090$, $\cos 18° = .9511$, and $\sin 15°$ from problem 1, find $\sin 3°$ and $\cos 3°$.

6. Find $\sin 42°$ as $\sin(60° - 18°)$.

7. Express $\sin(60° - \alpha)$ and $\cos(60° - \alpha)$ in terms of functions of α.

8. Find the value of $\sin(60° + \alpha) - \sin(60° - \alpha)$.

9. Find the value of $\cos(45° + \alpha) + \cos(45° - \alpha)$.

10. Show that $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$.

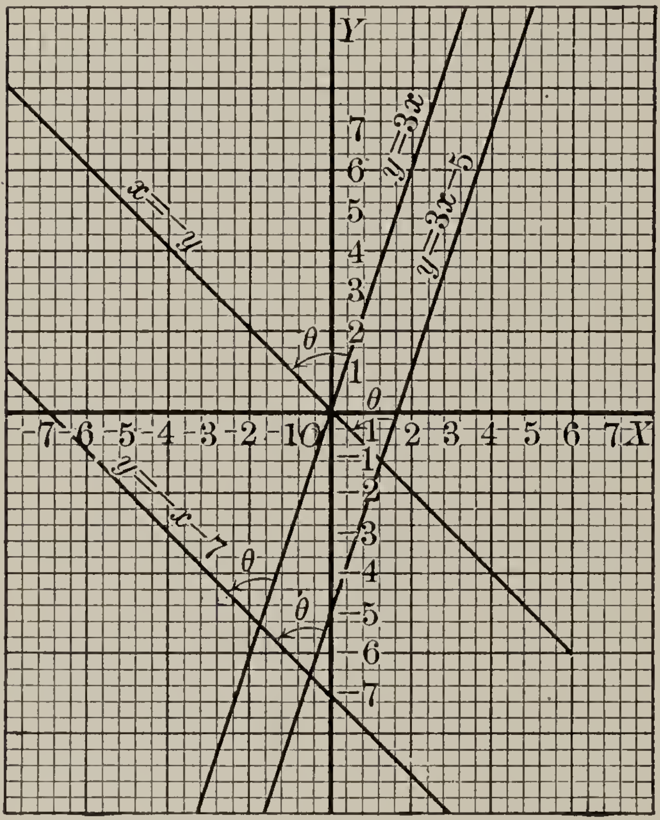
11. Find a value of $\cos(\alpha + \beta)\cos(\alpha - \beta)$, similar to the preceding.

12. Given $\tan\alpha = 1.4$, find $\tan 2\alpha$.

13. Given $\cos 2\alpha = .63$, find $\sin\alpha$ and $\cos\alpha$; are there two solutions?

14. Given that one line cuts the x-axis at an angle α such that $\tan\alpha = 3$, and another line cuts the x-axis at an angle β such that $\tan\beta = \frac{1}{2}$, find the tangent of the angle between the two lines by assuming that they intersect on the x-axis. Check by using the tables to find the slope angles of these lines.

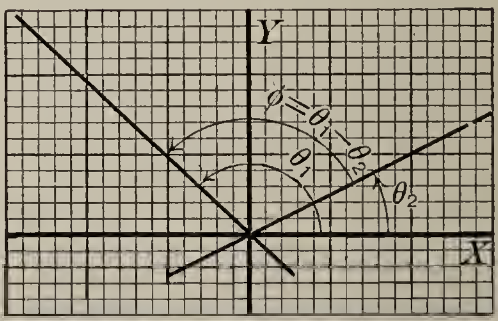
9. The tangent of the angle between two lines. — Given any two lines as $y = 3x - 5$, $y = -x - 7$, it is evident by plane geometry that the angle between them is the same as the angle between $y = 3x$, $y = -x$, lines parallel to these given lines through the origin. The word "between" implies no distinction as to priority of either line; thus the angle may be taken as either a positive or negative acute angle, or the corresponding supplementary angle. Thus if the lines were inclined to each other at 30°, the angle might be considered as $+30°$, $-30°$, $+150°$, or $-150°$; the tangent of the angle would then have the value given by the expression $\pm \frac{\sqrt{3}}{3}$.



Angle between two lines

Parallel lines through the origin make the same angle.

To distinguish between the two lines we may say that we wish the angle from the line of slope $+3$ to the line of slope -1, or in the general case, from the line of slope m_2 to the line of slope m_1; by analogy with our use in defining the angle which a line makes with the x-axis, when we say the angle which the line $y = -x$ makes with $y = 3x$ we mean the angle obtained by revolving the line whose slope is



Angle between two lines

$\phi = \theta_1 - \theta_2,\ \theta_1 > \theta_2.$

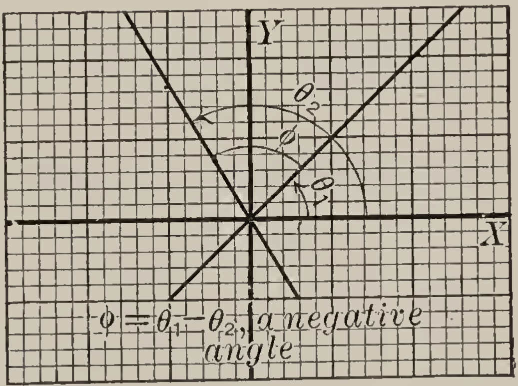
3 so as to make it coincide with the line whose slope is -1. Calling the angle whose tangent is 3 (written $\tan^{-1} 3$ or arc tan 3, meaning the angle whose tangent is 3), θ_2, and the angle whose tangent is -1, θ_1, we find that the angle ϕ from the θ_2 line to the θ_1 line is $\phi = \theta_1 - \theta_2$.

$$\tan \phi = \tan (\theta_1 - \theta_2) = \frac{\tan \theta_1 - \tan \theta_2}{1 + \tan \theta_1 \tan \theta_2} = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

If the two lines are parallel the angle is 0, hence $\tan \phi = 0$, and $m_1 - m_2 = 0$, or $m_1 = m_2$, as anticipated; if the lines are perpendicular $\tan \phi$ becomes infinitely large, and for finite values of m_1 and m_2 (excluding lines parallel to the axes), the denominator $1 + m_1 m_2 = 0$, or $m_2 = -\frac{1}{m_1}$, *i.e.* the slope of a perpendicular is the negative reciprocal of the slope of the given line. When one line is parallel to the y-axis, its slope m_2 (or m_1) is infinite, but the angle between the two lines can be obtained by dividing numerator and denominator of $\tan \phi$ by m_2 (or m_1), giving $\tan \phi = \dfrac{\dfrac{m_1}{m_2} - 1}{\dfrac{1}{m_2} + m_1}$, or $-\dfrac{1}{m_1}$ when m_2 approaches infinity, for the tangent of the angle made by a given line with the y-axis.

$\tan \phi = \dfrac{m_1 - m_2}{1 + m_1 m_2}$ gives the angle from the m_2 line to the m_1 line.

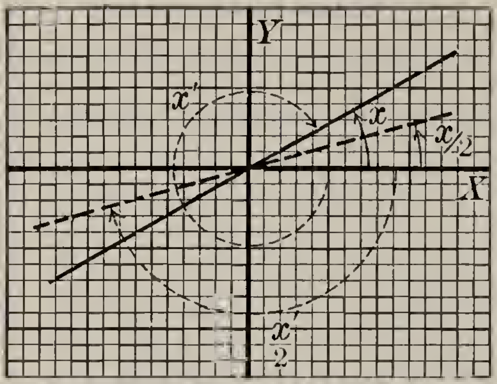
If $\theta_2 > \theta_1$, ϕ is negative, but the formula $\phi = \theta_1 - \theta_2$ still holds.



$\phi = \theta_1 - \theta_2,\ \theta_2 > \theta_1$

10. Functions of half an angle. $\cos (2\alpha) = \cos^2 \alpha - \sin^2 \alpha$ for all values of α.

Substitute x for 2α, and $\frac{x}{2}$ for α.



Half-angle relations

$$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}.$$

$$1 = \cos^2 \frac{x}{2} + \sin^2 \frac{x}{2},$$

$$1 + \cos x = 2 \cos^2 \frac{x}{2},$$

whence

$$\cos \frac{x}{2} = \pm \sqrt{\tfrac{1}{2}(1 + \cos x)}.$$

Similarly, $$\sin \frac{x}{2} = \pm \sqrt{\tfrac{1}{2}(1 - \cos x)}.$$

$$\tan \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}} = \pm \sqrt{\frac{(1 - \cos x)(1 - \cos x)}{1 - \cos^2 x}},$$

+ if x is in I or II, and − if x is in III or IV; the formula

$$\tan \frac{x}{2} = \frac{1 - \cos x}{\sin x}$$ is one in which

the $\sin x$ takes care of the algebraic sign; and so also

$$\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x},$$ both

by rationalization.

Note that if x is regarded as a positive angle, less than 360°, $\sin \frac{x}{2}$ is +; but the same position of the terminal line is obtained by $x \pm 360°$; $\frac{x}{2}$ and $\frac{x}{2} + 180°$ have sine and cosine opposite in sign, but $\tan\left(\frac{x}{2} \pm 180°\right) = \tan \frac{x}{2}$. Since $\cos(-x) = \cos(x)$ it must be stated whether x is in I or IV, or in II or III; *i.e.* $\cos x$ alone does not locate the angle x.

If x in I is regarded as a positive angle, $\frac{x}{2}$ is + acute, and $\sin \frac{x}{2}$ and $\cos \frac{x}{2}$ are positive; if x in I is regarded as a negative reflex angle, $\frac{x}{2}$ is negative obtuse, and $\sin \frac{x}{2}$ and $\cos \frac{x}{2}$ are both negative; in either case $\tan \frac{x}{2}$ is positive. Similarly if x is taken in II, III, or IV, the formula takes care of all positions, proper account being taken of the algebraic sign of the radicals.

PROBLEMS

Find the angle between the following lines:

1. $y = 3x - 5$, and $y = -x - 7$.

2. $y = 3x - 5$, and $y = x - 7$.

3. $2y - 3x - 7 = 0$, and $3y + 4x - 5 = 0$.

4. $3y = 5x - 5$, and $y = 8x - 10$.

5. $3y = 5x - 5$, $x = 5$.

6. $3y = 5x - 7$, $y = 5$.

7. In the preceding 6 problems, find the tangent of the angle made by the first line with the second line, *i.e.* the tangent of the angle obtained by rotating the second line until it coincides with the first. Why is it that the sense of this rotation is immaterial?

8. In the above problems check by finding from the tables the trigonometric angles involved.

9. Find the pencil of parallel lines making an angle of 30° with each of the lines in problem 1; find the one of the family through $(-3, 5)$.

10. Find the pencil of lines making an angle of 45° with each of the lines in problem 3; find the particular one through the origin.

11. Find the pencil of lines making an angle of 90° with each of the lines in problem 4.

12. Given $\sin 30° = .5000$, $\cos 30° = .8660$, and $\tan 30° = .5774$, find $\sin 15°$, $\cos 15°$, and $\tan 15°$.

13. Find $\sin 7\frac{1}{2}°$, $\cos 7\frac{1}{2}°$, $\tan 7\frac{1}{2}°$, using half-angle formulas.

14. Given $\sin 45° = \cos 45° = .7071$, find $\sin 22\frac{1}{2}°$, $\cos 22\frac{1}{2}°$, and $\tan 22\frac{1}{2}°$.

15. Use $\sin(\alpha - \beta)$ formula to obtain $\sin 7\frac{1}{2}°$ and $\cos 7\frac{1}{2}°$, from the functions of 30° and $22\frac{1}{2}°$. Compare with problem 13.

16. Given $\sin 18° = .3090$ and $\cos 18° = .9511$, find $\sin 12°$ and $\cos 12°$.

17. From the functions of 12°, compute the functions of 6°, and then the functions of 3° and of $1\frac{1}{2}°$, using half-angle formulas.

18. Compute $\sin 1\frac{1}{2}°$ and $\cos 1\frac{1}{2}°$ by the difference formulas taking $1\frac{1}{2}°$ as $7\frac{1}{2}° - 6°$.

19. Compute the functions of $\frac{3}{4}°$ from the functions of 1°.

20. Find by interpolation $\sin 1°$ and $\cos 1°$ from the computed values of the functions of $\frac{3}{4}°$ and $1\frac{1}{2}°$. Compare with the tabular values.

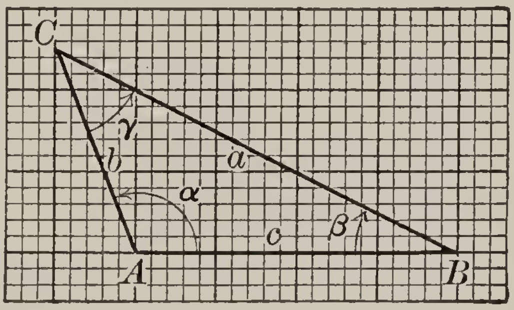
21. Make a table of values of the sine, from 0 to 45° increasing by $1\frac{1}{2}°$ intervals.

CHAPTER XVI

TRIGONOMETRIC FORMULAS FOR OBLIQUE TRIANGLES

1. General statement. — Employing elementary theorems of plane geometry it is possible to construct any triangle when given the three sides, or two sides and an angle, or one of the three sides together with two of the angles; in trigonometry the corresponding problem is the numerical solution, not simply the graphical, of the types of triangles mentioned. The trigonometric solution which has been given of the different types of right triangles, with unknown parts, can be applied to effect the trigonometric solution of any oblique triangle; but in general, these methods do not give convenient formulas for computation. As the general triangle is fundamental in surveying (note the term "triangulation"), in astronomical work, and in many problems in physics, more convenient formulas than those given by right triangles are a practical necessity.

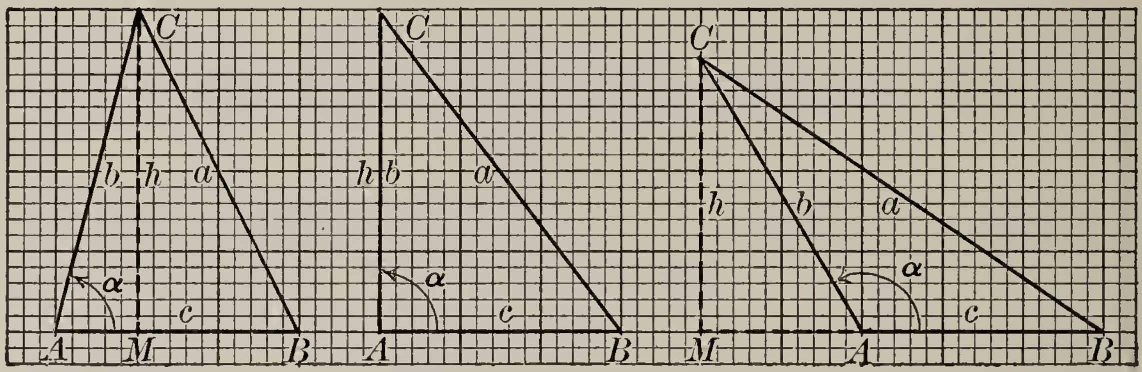
In general the laws and formulas of plane trigonometry connect directly with propositions of plane geometry; the effort is to express the interdependence of the angles and sides in the form of equations involving the trigonometric functions of the angles.



A, B, C; α, β, γ; a, b, c

The vertices of any triangle being lettered A, B, C, it is convenient to designate the corresponding angles at these vertices by α, β, and γ, respectively, or by A, B, and C, if no confusion of meaning is possible; the sides opposite A, B, and C are designated by a, b, and c, respectively.

2. Cosine law. — If the two sides of a triangle are given, the third or variable side, opposite the angle α, between the two given sides, evidently changes as α changes. Let b and c remain fixed. Let M be the foot of the perpendicular from C upon AB; then $AM = b \cos \alpha$, for any angle α when the direction AB is taken as positive. Further in every position



$a^2 = b^2 + c^2 - 2\,bc \cos \alpha$

$$MB = AB - AM = c - b \cos \alpha,$$

for in every position $AM + MB = AB$.

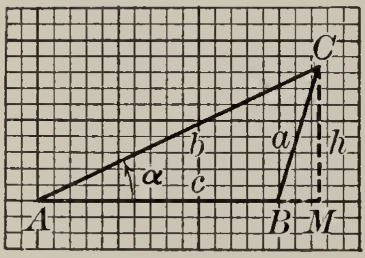
The altitude $MC = h = b \sin \alpha$.

Hence, $\overline{BC}^2 = \overline{MB}^2 + \overline{MC}^2$

$$= (c - b \cos \alpha)^2 + (b \sin \alpha)^2$$

$$= c^2 - 2\,bc \cos \alpha + b^2(\cos^2 \alpha + \sin^2 \alpha).$$

$$a^2 = b^2 + c^2 - 2\,bc \cos \alpha.$$



M falling outside B
Formula unchanged

All limitations upon α are removed by the different types of figures. Hence for any angle α,

$$a^2 = b^2 + c^2 - 2\,bc \cos \alpha$$

gives the length of the side a, opposite α, in terms of the other two sides and α. Since a and α may represent any side and the opposite angle of any given triangle, b and c being the other two sides, our formula may be stated as follows:

The square of any side of a triangle is equal to the sum of the squares of the other two sides less twice their product into the cosine of the including angle.

Or, *The cosine of any angle equals the difference between the sum of the squares of the two including sides and the square of the side opposite, divided by twice the product of the including sides.*

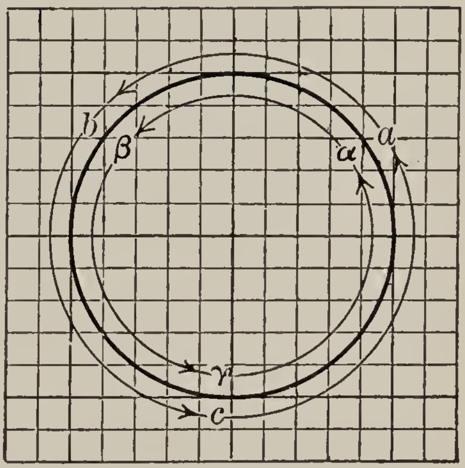
If a, b, and c are the sides of any triangle, with α, β, and γ the corresponding opposite angles, we have the following relationships:

$$a^2 = b^2 + c^2 - 2\,bc \cos \alpha,$$
$$b^2 = c^2 + a^2 - 2\,ac \cos \beta,$$
$$c^2 = a^2 + b^2 - 2\,ab \cos \gamma;$$

or

$$\cos \alpha = \frac{b^2 + c^2 - a^2}{2\,bc}.$$
$$\cos \beta = \frac{c^2 + a^2 - b^2}{2\,ac}.$$
$$\cos \gamma = \frac{a^2 + b^2 - c^2}{2\,ab}.$$

3. Cyclic interchange. — Any formula which has been derived, without imposing any limitations upon a, b, c, α, β, or γ, connecting a, b, c, and trigonometric functions of the angles α, β, and γ, will continue to hold if a and b and, at the same time α and β, are interchanged; or if $\left\{\begin{matrix} a \\ \alpha \end{matrix}\right.$ are changed to $\left\{\begin{matrix} b \\ \beta \end{matrix}\right.$, $\left\{\begin{matrix} b \\ \beta \end{matrix}\right.$ to $\left\{\begin{matrix} c \\ \gamma \end{matrix}\right.$, and $\left\{\begin{matrix} c \\ \gamma \end{matrix}\right.$ to $\left\{\begin{matrix} a \\ \alpha \end{matrix}\right.$; such changes effect simply a re-lettering of the figure. The change of a into b, b into c, and c into a is called a cyclic interchange of the letters a, b, and c. Note that cyclic interchange gives the second formula from the first, and the third from the second.



Cyclic interchange

In the figures, α in the first is chosen as an acute angle, but this limitation is removed by deriving the same formulas for α a right angle and for α an obtuse angle; c is taken as longer than b, but interchanging b and c in our derived formula leaves the formula unchanged; assuming b and c equal would involve no change whatever in our proof; and if b is assumed greater than c, a fourth figure can be drawn in which M falls beyond B on AB produced; but the formula $a^2 = b^2 + c^2 - 2\,bc \cos \alpha$ remains the same, as the student may easily verify.

PROBLEMS

In the following problems use .866, .707, and .500 for the cosines of 30°, 45°, and 60°, respectively.

1. Given $b = 140$, $c = 230$, $\alpha = 60°$, compute a. Refer back to the section on extraction of square root, page 23.

2. Compute a when $\alpha = 30°$ and 45°, 90°, 120°, 135°, 180°, when $b = 140$, $c = 230$.

3. Given $a = 155$, $c = 234$, $\beta = 35°$, compute b. What changes in b are effected by changes of $\pm 10'$ in β?

4. Given $a = 155$, $c = 234$, compute β when $b = 172$. What is the maximum change in β which an error of $\pm \frac{1}{2}$ unit in a, b, and c could introduce, β being computed to minutes? Take $155\frac{1}{2}$, $234\frac{1}{2}$ with $171\frac{1}{2}$; take $154\frac{1}{2}$, and $234\frac{1}{2}$, with $172\frac{1}{2}$. Note that $(155\frac{1}{2})^2$ and $(154\frac{1}{2})^2$ differ from $(155)^2$ by about 155; similarly with the other values; if the squares are found by logarithms it is well to look up log 155.5 and log 154.5 at the same time as log 155, etc.

5. In problem 1, find $\cos \beta$, and then β, taking for a the value obtained there.

6. In problem 1, find $\cos \gamma$ and γ, using the computed value of a. Check by summing β and γ with the given angle.

7. Given $a = 200$, $b = 150$, $c = 300$, find α. What change in α would a change of ± 1 in a effect? Suppose that a, b,

and c are given only to two significant figures, *i.e.* a is between 195 and 205, b is between 145 and 155, c is between 295 and 305, compute α and discuss limiting values.

8. Compute the third side in the following 5 problems, using logarithms for squaring; time yourself in the exercise. Fifty minutes should be ample time for the 5 problems; devise a convenient form and use it in each example.

a. Given $a = 366$, $b = 677$, $\gamma = 15° \, 10'$.

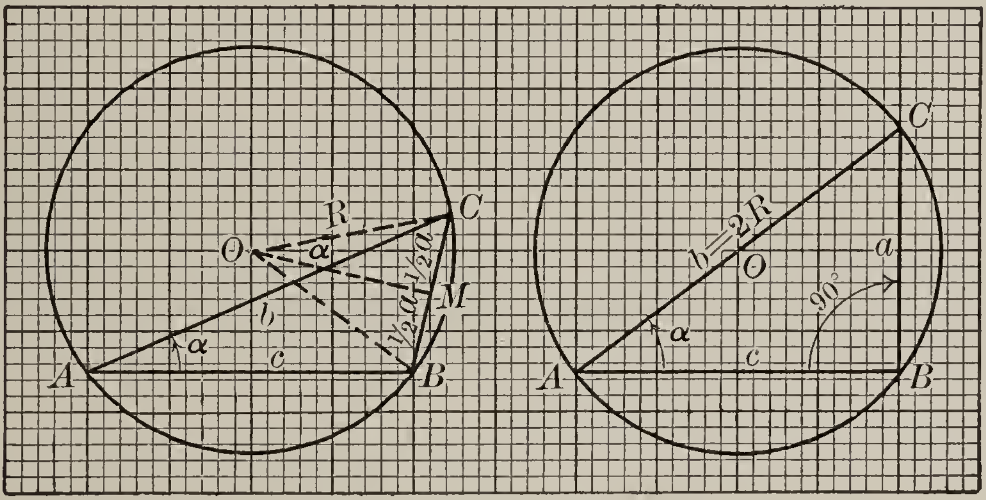
b. Given $a = 423$, $c = 288$, $\beta = 35° \, 15'$.

c. Given $b = 627$, $c = 816$, $\alpha = 100° \, 41'$.

d. Given $a = 635$, $c = 341$, $\beta = 67° \, 38'$.

e. Given $c = 184$, $b = 295$, $\alpha = 130° \, 54'$.

4. Sine law. — A circle may be circumscribed about any triangle; let the radius be designated by R. The figure shows



Sine law: $\dfrac{a}{\sin \alpha} = \dfrac{b}{\sin \beta} = \dfrac{c}{\sin \gamma}$

that if A is an acute angle, $\sin \alpha = \dfrac{\frac{1}{2} a}{R} = \dfrac{a}{2R}$; if α is 90°, this formula is still true, as a equals $2R$, and the formula gives $\sin 90° = 1$; if α is obtuse, the figure gives $\sin (180° - \alpha) = \dfrac{a}{2R}$, whence $\sin \alpha = \dfrac{a}{2R}$.

Therefore without any limitation whatever upon α,

$$\sin\alpha = \frac{a}{2R};$$

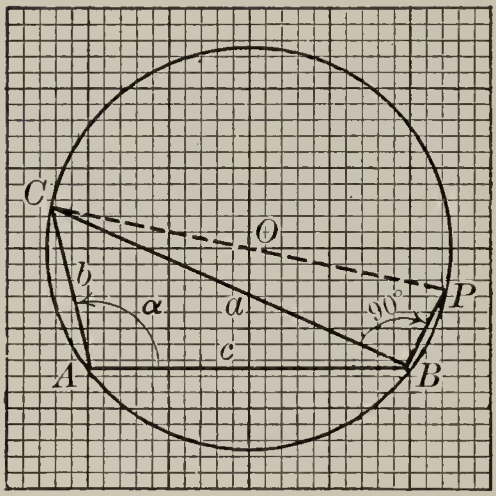
interchange of letters gives

$$\sin\beta = \frac{b}{2R},$$

and $$\sin\gamma = \frac{c}{2R}.$$

Whence

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R.$$



Sine law: α obtuse

This formula states that in any triangle *the ratio of the side opposite any angle to the sine of that angle is constant,* and this ratio is numerically equal to the diameter of the circumscribed circle.

Further, $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$, or the ratio of the sine of any angle to the side opposite is constant.

Note that if $2R$ is regarded as the chord of 180° of the circle in which the triangle ABC is inscribed, the proposition states, in effect, that in any circle any chord is proportional to the sine of the inscribed angle which intercepts the arc of the chord.

The formula may be stated:

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} = \frac{\sin 90^\circ}{2R} = \frac{\sin 30^\circ}{R} = \frac{\sin k^\circ}{\text{chord }(2k^\circ)},$$

all of the chords being chords of the circle circumscribed about the triangle. The ratio of the sine of any central angle in a circle to the chord of double the angle can readily be shown to be constant, $\frac{1}{2R}$.

5. The sine law historically. — The sine law was discovered by an Arabic (Persian, by birth) mathematician, Nasir al-Din, at-Tusi, who lived 1201–1274 A.D. To him we owe the first systematic treatise on plane trigonometry, an achievement made possible by the combination of the Greek trigonometry using chords with the Hindu trigonometry employing sines. To Europeans the sine law was communicated by the great German mathematician and astronomer, Regiomontanus, in his work on trigonometry, *De Triangulis*, the first published systematic treatise; it was published at Nuremberg in 1539, many years after the death of Regiomontanus, who lived 1436–1476.

PROBLEMS

1. Given $a = 150$, $b = 200$, $\alpha = 30°$, find $\sin \beta$ using natural functions.

2. Given $a = 150$, $\alpha = 30°$, $\beta = 45°$, find b, using natural functions.

3. Given $a = 150.4$, $b = 214.3$, $\alpha = 31° 10'$, find $\sin \beta$ employing logarithms.

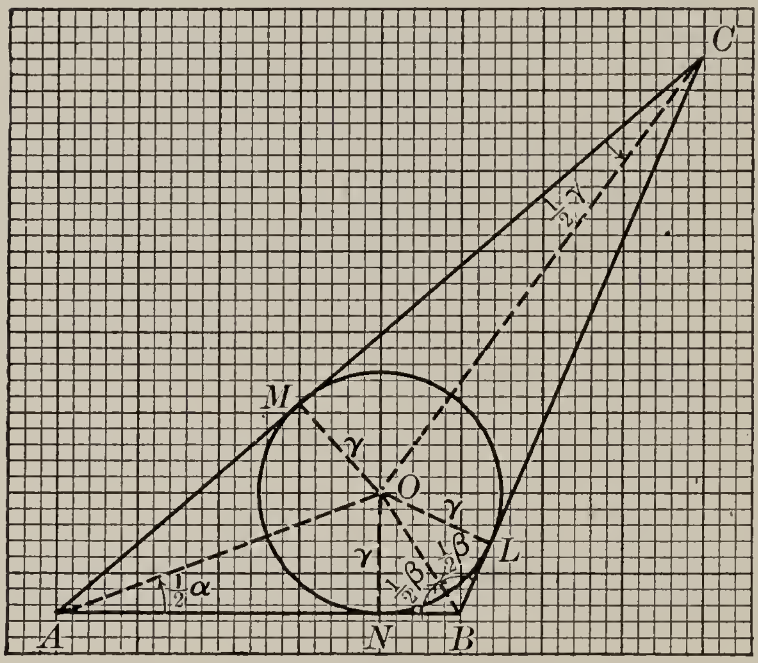
4. Given $a = 150.4$, $\alpha = 31° 10'$, $\beta = 44° 16'$, find b by logarithmic computation.

5. In the formula, $a^2 = b^2 + c^2 - 2bc \cos \alpha$, substitute the values as given in problem 1 and solve for c. Note that there are two solutions. What is the explanation?

6. Time yourself in solving the following set of 6 problems, applying the sine law; make a type form of solution and use it in each problem. Thirty minutes should be sufficient for the 6 problems.

a. Given $a = 366$, $b = 677$, $\alpha = 15° 10'$. Find $\sin \beta$ and β.
b. Given $a = 423$, $c = 288$, $\gamma = 35° 15'$. Find $\sin \alpha$ and α.
c. Given $a = 627$, $\alpha = 100° 11'$, $\beta = 43° 15'$. Find b.
d. Given $b = 816$, $\beta = 67° 18'$, $\gamma = 34° 9'$. Find c.
e. Given $c = 635$, $\beta = 130° 14'$, $\alpha = 20° 12'$. Find b.
f. Given $b = 284$, $\alpha = 40° 10'$, $\beta = 35° 15'$. Find c.

6. Half-angle formulas. — As the circumscribed circle has yielded a formula of great value trigonometrically the inscribed circle may be examined trigonometrically with the hope of a similar result.



$OL = OM = ON = r$, radius inscribed circle

The bisectors of the three angles of the triangle meet in a point which is the center of the inscribed circle; let this circle be drawn and let L, M, N, be the points of tangency, then

$$AM = AN,$$
$$BL = BN,$$
$$CL = CM,$$

being tangents from an exterior point. Evidently the six segments mentioned make the perimeter, $2s$, of our triangle; $2s = a + b + c$; adding above we have that

$$AM + BL + CL = AN + BN + CM = s,$$

but $BL + CL = a$, and $BN + AN = c$;

whence $AM = s - a$, $CM = s - c$, and similarly

$$BN = BL = s - b$$

$$ON = OL = OM = r.$$

$$\tan\frac{1}{2}\alpha = \frac{r}{s-a}; \quad \tan\frac{1}{2}\beta = \frac{r}{s-b}; \quad \tan\frac{1}{2}\gamma = \frac{r}{s-c}.$$

7. Area. — In the preceding section the area of the given triangle is easily determined in terms of r and s, for the area equals the sum of the three triangles on a, b, and c as bases, each having the altitude r. $\therefore A = \frac{1}{2} r(a + b + c) = rs$.

However, if the three sides are given, this formula does not enable us to determine r without using a further formula to determine A.

$$A = \tfrac{1}{2}\,bc \sin\alpha = \tfrac{1}{2}\,ac \sin\beta = \tfrac{1}{2}\,ab \sin\gamma.$$

This type of formula for the area is applicable when two sides and the included angle of a triangle are given or found.

$A = \frac{1}{2}\,bc \sin\alpha$ can be combined with $\cos\alpha = \dfrac{b^2 + c^2 - a^2}{2\,bc}$ in such a way as to eliminate α, giving A in terms of the three sides.

$$\begin{aligned}
A^2 = \tfrac{1}{4}\,b^2c^2 \sin^2\alpha &= \tfrac{1}{4}\,b^2c^2\,(1 - \cos^2\alpha)\\
&= \tfrac{1}{4}\,b^2c^2\,(1 - \cos\alpha)(1 + \cos\alpha)\\
&= \frac{b^2c^2}{4}\left(1 - \frac{b^2 + c^2 - a^2}{2\,bc}\right)\left(1 + \frac{b^2 + c^2 - a^2}{2\,bc}\right)\\
&= \frac{b^2c^2}{4}\left(\frac{a^2 - b^2 + 2\,bc - c^2}{2\,bc}\right)\left(\frac{b^2 + 2\,bc + c^2 - a^2}{2\,bc}\right)\\
&= \frac{b^2c^2[a^2 - (b - c)^2][(b + c)^2 - a^2]}{16\,b^2c^2}\\
&= \frac{(a - b + c)(a + b - c)(b + c - a)(b + c + a)}{2 \cdot 2 \cdot 2 \cdot 2}.
\end{aligned}$$

But $\dfrac{a + b + c}{2} = s$, and $\dfrac{a - b + c}{2} = \dfrac{a + b + c}{2} - b = s - b$, etc.

The above formula for A^2 may be written,

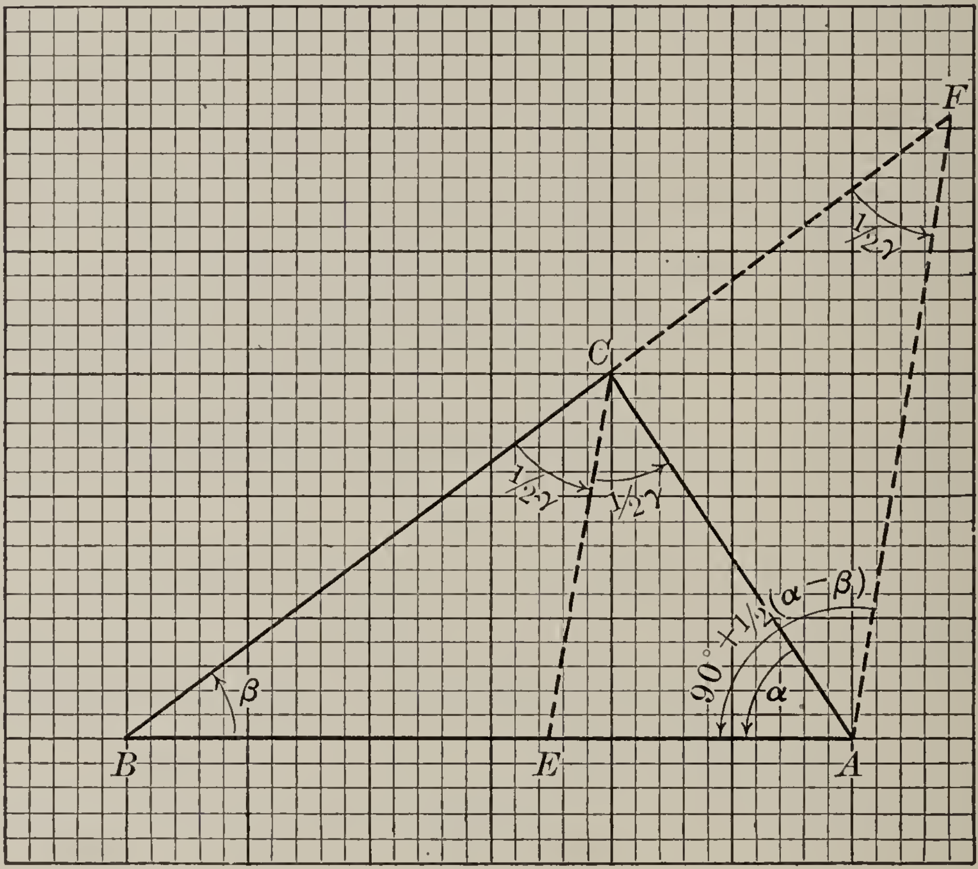
$$A^2 = s(s - a)(s - b)(s - c)$$

$$A = \sqrt{s(s - a)(s - b)(s - c)}.$$

Further $A = rs$, whence $r = \dfrac{A}{s} = \sqrt{\dfrac{(s - a)(s - b)(s - c)}{s}}.$

This value of r is employed with the half-angle tangent formulas of the preceding article to determine the angles of a triangle when the three sides are given.

8. Newton's check formula. — A formula which involves all of the sides and all of the angles of an oblique triangle is particularly desirable as a check formula to be used upon the results obtained by direct application of the sine law or in a solution obtained by right triangles. Such a formula was devised by Sir Isaac Newton and appeared in his *Arithmetica universalis* of 1707; our proof follows the lines of that by Newton.



Let ABC be any triangle;
from C draw the bisector CE of the angle ACB or γ;
extend BC to F, making $CF = CA = b$;
AF is parallel to CE, by plane geometry;
angle CFA = angle $BCE = \frac{1}{2}\gamma$.

Now angle $BAF = \alpha + \frac{1}{2}\gamma = 90° - \frac{1}{2}(\alpha - \beta)$,
since $\frac{1}{2}\alpha + \frac{1}{2}\beta + \frac{1}{2}\gamma = 90°$.

Applying the sine law to the triangle BAF, we have the desired formula:

$$\frac{a+b}{c} = \frac{\cos\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}\gamma}.$$

By drawing the bisector of the exterior angle, and drawing a parallel from A, a second useful check formula is obtained:

$$\frac{a-b}{c}=\frac{\sin\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}\gamma}.$$

Noting that $\frac{1}{2}\gamma=90^\circ-\frac{1}{2}(\alpha+\beta)$, division of the second equation by the Newtonian, member for member, gives

$$\frac{a-b}{a+b}=\frac{\tan\frac{1}{2}(\alpha-\beta)}{\tan\frac{1}{2}(\alpha+\beta)};$$

this symmetrical formula is known as the tangent law.

Cyclical interchange gives in each one of the above two corresponding formulas.

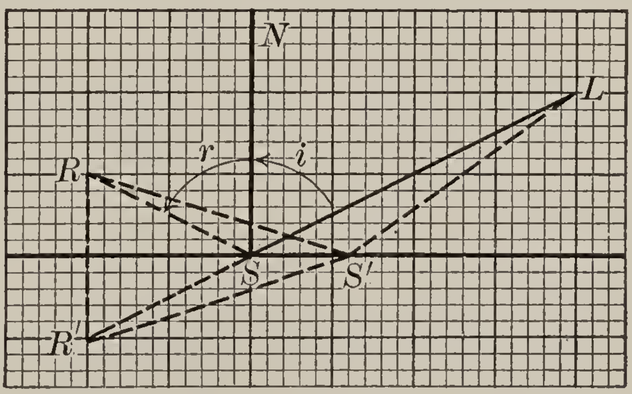
9. Historical note. — The formula $A=\sqrt{s(s-a)(s-b)(s-c)}$ was first given by Hero of Alexandria, first century A.D., a teacher of mathematics and mechanics in what was probably a kind of technical school at Alexandria in Egypt; it is called Hero's formula.

An extension of this formula is given by Bhaskara, a Hindu mathematician of about 1000 A.D. Bhaskara's formula gives the area of any quadrilateral which is inscribable in a circle, *i.e.* with the opposite angles supplementary, as

$$A=\sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

The triangle may be regarded as a special case with $d=0$.

10. Reflection and refraction of light. — Rays of light, like rays of heat and sound and electric rays of various types, travel in straight lines from the source. Rays of light emanating from the sun travel in nearly parallel rays, since the point of convergence, the source at the sun, is at so great a distance from the earth. A ray of light which meets a polished plane surface, a mirror, is reflected at an angle which is such

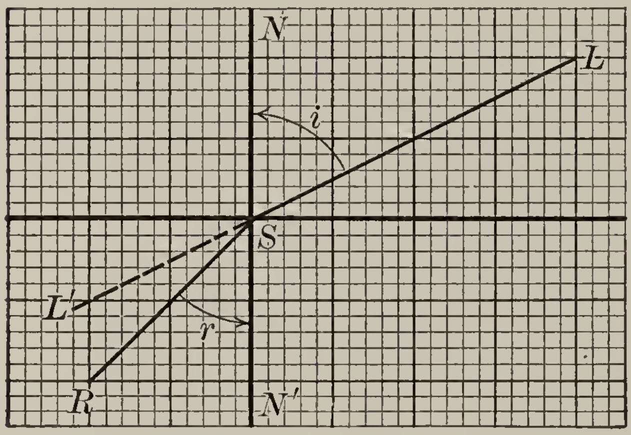


Reflected ray travels shortest path

as to make the total path from the source (L) to the reflecting surface and then to a second position (R) the shortest possible. LSR is the shortest distance from L to S to R if the angle of incidence i, made by the original ray with the normal, to the surface at S where the ray strikes, is equal to the angle of reflection r. Evidently $LSR = LSR'$; the straight line joining L to R', a point symmetrically situated to R with respect to the polished surface, is shorter than any other line, for any other broken line $LS'R = LS'R'$ is greater than the straight line LSR' and hence greater than LSR.

If the ray of light meets, not a polished surface but some transparent medium, other than that in which the ray is traveling, the ray of light is not continued in the same straight line in which it starts but it is broken, or refracted, continuing on its path in a straight line which makes a different angle with the normal than does the original, incident ray. It is found by physical experiments that the angle of refraction, the angle of the refracted ray with the normal, bears a simple relation to the angle of incidence, $\frac{\sin i}{\sin r} = k$, wherein k depends upon the nature of the two media through which the light is passing. Thus for a ray of light passing from air, a rarer light medium, to the denser water the value of k is $\frac{4}{3}$,

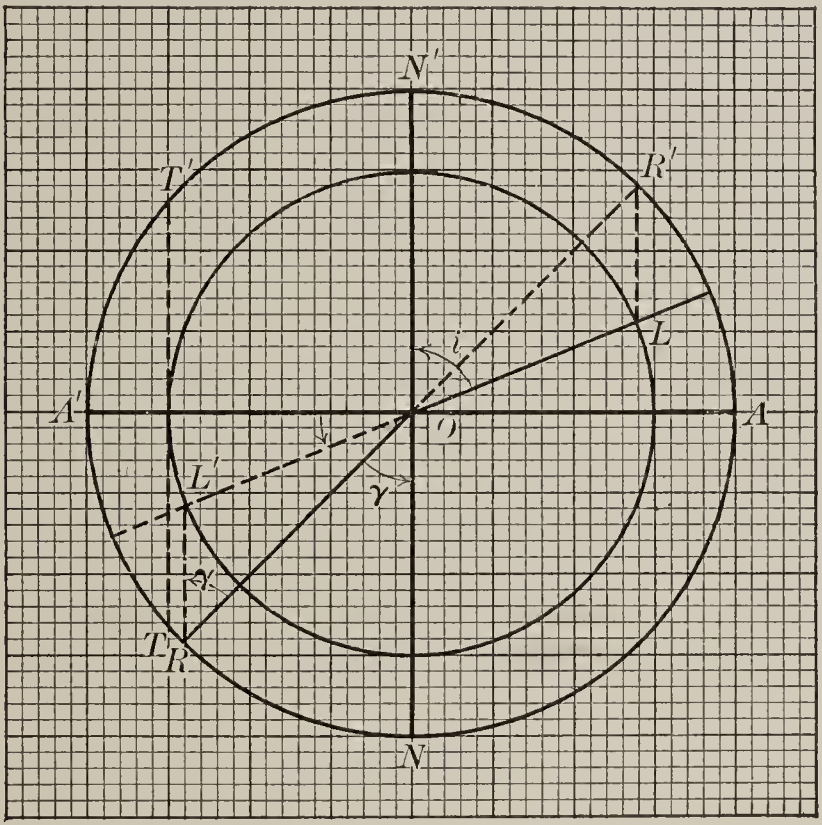
$$\frac{\sin i}{\sin r} = \frac{4}{3}.$$



Refracted ray of light

A student who thoughtfully examines this formula will be reminded of the sine law, which does indeed give a very simple construction for the refracted ray when the constant k is known.

Let two concentric circles be drawn whose radii bear to each other the ratio, $\frac{4}{3}$, of the index of refraction. In the figure the ratio is taken $\frac{4}{3}$, the index of refraction for light from air to water. Extend LO, the incident ray, to L', cutting the circle of smaller radius. From L' drop a line parallel to the normal $N'O$ to cut the larger circle in R. Connecting R with O gives the refracted ray. In the triangle $OL'R$, the $\angle OL'R = 180° - i$, and the $\angle ORL' = \angle r$, of refraction; by the sine law



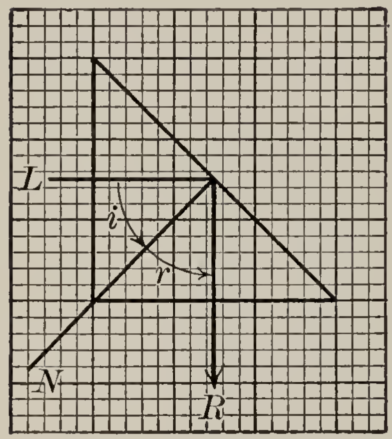
LO is the incident ray; OR is the refracted ray

$$\frac{\sin(180° - i)}{\sin r} = \frac{\sin i}{\sin r} = \frac{4}{3}.$$

From water to air the index is $\frac{3}{4}$, it being found that if the refracted ray is replaced by an original ray, this new ray in the second medium will be refracted along OL, the path of the incident ray with which we started.

The construction for the refracted ray in air for a ray of light emanating from the water, RO, is entirely similar to the preceding. RO is extended to R' on the larger circle. From

R' a parallel $R'L$ is drawn to the normal to cut the smaller circle. OL is the refracted ray.



Glass reversing prism Index of refraction $\frac{3}{2}$

Angle of incidence, 45°. For any angle i greater than α, $\sin \alpha = \frac{2}{3}$, the beam is reflected.

Evidently if $\sin r = \frac{3}{4} \sin i$, $\sin r$ is always less than $\frac{3}{4}$. If a ray of light starts from any point within the arc $A'T$ wherein T is the intersection of the vertical tangent to the smaller circle with the larger circle it cannot be refracted into the air at O, and the whole light is reflected at O. This property of the light rays is utilized in certain spectroscopic work. Thus in the case of a glass prism, index of refraction $\frac{3}{2}$, if light strikes the plane surface at an angle of incidence greater than 41° 48′, since $\sin 41° \, 48' = .6666$, or $\frac{2}{3}$, all the light will be reflected; this type of prism is used in projecting lanterns.

PROBLEMS

1. Given $a = 9$, $b = 14$, $c = 19$, find the area of the triangle, using Hero's formula.

2. Given $a = 9$, $b = 14$, $c = 19$, find α, using the cosine law.

3. Given $a = 9$, $b = 14$, $c = 19$, find the area by the formula $A = \frac{1}{2} bc \sin \alpha$.

4. Given $a = 9.34$, $b = 14.31$, $c = 19.27$, find the area by Hero's formula, using logarithms.

5. Given $a = 9.34$, $b = 14.31$, $c = 19.27$, find α by the cosine law, and then find the area using the formula involving $\sin \alpha$.

6. In the two triangles above find r, the radius of the inscribed circle, using $r \cdot s = A$.

7. In the two angles above find $\tan \frac{1}{2} \alpha$, $\tan \frac{1}{2} \beta$, and $\tan \frac{1}{2} \gamma$, using the half-angle formulas. Find the angle sum in each case.

8. Draw circles with radii two inches and three inches and show how to construct the refracted rays of light passing from air into glass at angles of incidence of 30°, 45°, and 60°.

9. For what angle will a ray of light passing from glass into water be reflected, and not refracted? The index of refraction of light passing from glass into water is $\frac{8}{9}$. Draw the figure.

10. Find the angle of refraction of rays of light passing from air into water, $k = 1.33$, when the angles of incidence are $31° 15'$, $37° 18'$, $44° 25'$, $67° 10'$, $83° 15'$. For which of these angles is the course of the ray changed by the greatest amount?

11. Suppose the rays in problem 10 to pass from air into glass, solve for the angles of refraction.

12. Construct two of the figures in both problems 10 and 11, and check graphically the results obtained above.

CHAPTER XVII

SOLUTION OF TRIANGLES

1. Solution of triangles given two angles and one side: $c\alpha\beta$ type. — With surveying instruments the simplest method of locating the distances from two fixed points to a third inaccessible point is to determine the length AB and the angles α and β, at A and B respectively, wherein A and B are two points from which C is visible. Using the sine law,

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma},$$

we select the equation $\frac{a}{\sin\alpha}=\frac{c}{\sin\gamma}$,

or $\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}$, since in each of these only one unknown quantity appears. The third equation $\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}$, not independent of the other two, is used as a partial check upon the computed values. As a more complete check use Newton's formula,

$$\frac{a+b}{c}=\frac{\cos\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}\gamma}.$$

This form of triangle appears in the classical problem, whose solution by plane geometry is ascribed to one of the seven wise men of Greece, Thales of Miletus, sixth century B.C. The problem is familiar to the surveyors, being used in determining distances across a stream, or to an inaccessible point. The astronomer has the same problem in locating the distance of fixed stars using two observations, at different points in the earth's orbit, of the angle made by lines from the earth to the star and to the sun; for simplicity, the two points of observation may be considered as taken at the extremities of the diameter of the earth's path.

In locating batteries by the sound waves this type of triangle is employed; two or three observers at different points can locate an enemy battery by this method within a radius of fifty feet or thereabouts.

2. Type form of solution: $c\alpha\beta$ type. — The form of the solution is important; follow the given form closely.

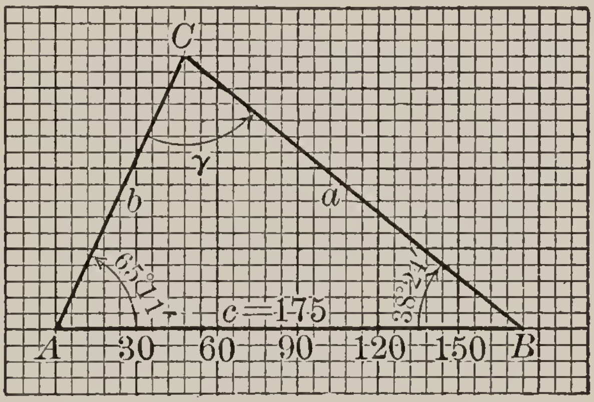
Given $\alpha = 65° 11'$, $\beta = 38° 24'$, $c = 175$ feet. Find a and b.

$a = \dfrac{c \sin \alpha}{\sin \gamma}$ (written from the formula $\dfrac{a}{\sin \alpha} = \dfrac{c}{\sin \gamma}$, which should not be set down).

$$b = \frac{c \sin \beta}{\sin \gamma}; \text{ check } b = \frac{a \sin \beta}{\sin \alpha},$$

$$A = \frac{1}{2} bc \sin \alpha = \frac{1}{2} \frac{c^2 \sin \alpha \sin \beta}{\sin \gamma}.$$

$$\begin{aligned} \alpha &= 65° 11' \\ \beta &= 38° 24' \\ \gamma &= 76° 25' \end{aligned}$$



Two angles and a side given

$$\begin{aligned}
\log c &= 2.2430 \\
+ \log \sin \alpha &= \underline{9.9580 - 10} \\
& 12.2010 - 10 \\
- \log \sin \gamma &= \underline{9.9876 - 10} \\
\log a &= 2.2134 \\
a &= \underline{163.4} \\
\log a &= 2.2134 \\
+ \log \sin \beta &= \underline{9.7932 - 10} \\
& 12.0066 - 10 \\
- \log \sin \alpha &= \underline{9.9580 - 10} \\
\log b &= 2.0486 - 10
\end{aligned}$$

But log $b = 2.0486$ by above computation, which checks.

$$\begin{aligned}
\log c &= 2.2430 \\
+ \log \sin \beta &= \underline{9.7932 - 10} \\
& 12.0362 - 10 \\
- \log \sin \gamma &= \underline{9.9876 - 10} \\
\log b &= 2.0486 \\
b &= \underline{111.8} \\
\log c^2 &= 4.4860 \\
+ \log \sin \alpha &= 9.9580 - 10 \\
+ \log \sin \beta &= \underline{9.7932 - 10} \\
& 14.2372 - 10 \\
- \log \sin \gamma &= \underline{9.9876 - 10} \\
\log 2A &= 4.2496 \\
2A &= 17{,}760 \\
A &= 8880
\end{aligned}$$

The check which we have used is only partial as an error in γ or $\sin\gamma$ would be carried through the work without showing up in the check. The Newtonian formula gives a real check upon the computation.

Check. $$\frac{a+b}{c}=\frac{\cos\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}\gamma}.$$

$$\begin{array}{rll}
a+b=275.2 & \log(a+b) & =2.4396\\
 & \log c & =\underline{2.2430}\\
 & & .1966\\
\alpha-\beta=26^\circ\,47' & \log\cos\frac{1}{2}(\alpha-\beta) & =9.9880\\
\gamma=76^\circ\,25' & \log\sin\frac{1}{2}\gamma & =\underline{9.7914}\\
 & & .1966 \text{ which checks.}
\end{array}$$

Notes. — The whole form of solution is placed on paper before the logarithms are inserted. Place the given angles in vertical column and obtain the third angle by noting the angle which added to the given angles makes 180°; thus, here note first that to complete 11′ and 24′ to 1° takes *25′*. Add this 1° to the 8° and 5°, the units of our given angles, making 14°; complete by *6*°, which is written in its proper place, to 20°. Carry the 2 tens, to the tens, making 11 tens, or 110°, requiring 7 tens (written in the proper place) to complete to 180°.

Look up $\log c$, *i.e.*, log 175, writing this immediately in all places where it occurs; for the area, it is simpler to calculate $2A$ and divide by two than to divide by subtracting log 2 in the work. $\log c^2 = 2\log c$, which is set down in its place. Finish, as far as possible, with the logs of numbers before taking up the logs of trigonometric functions. log sin 65° 11′, log sin 38° 24′, and log sin 76° 25′ should be found in the order in which they occur in the tables, to avoid useless thumbing back and forth; any value found should be immediately inserted wherever it occurs in the form.

PROBLEMS

1. Prove the sine law by using perpendiculars dropped from a vertex to the opposite side.

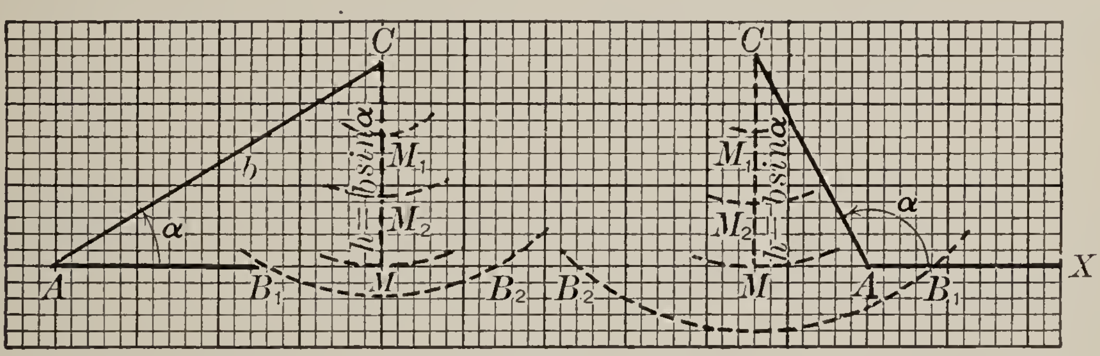
2. Given $c=350.4$, $\alpha=36^\circ\,14'$, $\beta=100^\circ\,24'$, find b and a, by the sine law.

3. Given $a=.03504$, $\alpha=36^\circ\,14'$, $\beta=100^\circ\,24'$, find b and c, by the sine law.

4. Solve completely the following 5 triangles; take the time of your solutions; write the complete form of solution for each problem, in turn, before inserting any logarithms. The five problems should be completed within one hour and 20 minutes using the rough check by the sine law. As a separate exercise check all by Newton's formula, timing yourself.

a.	$a = 627$	$\alpha = 100° 11'$	$\beta = 43° 15'$
b.	$b = 816$	$\beta = 67° 18'$	$\gamma = 34° 09'$
c.	$c = 635$	$\beta = 130° 14'$	$\alpha = 20° 12'$
d.	$b = 284$	$\alpha = 40° 10'$	$\beta = 35° 15'$
e.	$a = 366$	$\alpha = 15° 10'$	$\beta = 95° 14'$

3. Given two sides and the angle opposite one: *abα* type. — Given b, α, and a to construct the triangle geometrically. AC is laid off of length b and the line AX is drawn so as to make $\angle CAX = \alpha$. Since a must lie opposite to α, a is taken as radius and with C as center an arc is swung to cut the side AX. Since the shortest distance from C to AX is the length of the perpendicular CM, if a is given less than this perpendicular there is no solution. If a is given equal to the perpendicular there is one solution; if a is greater than the perpendicular the arc cuts AX in two points, but unless $a < b$ the one point of intersection to the left of A will not represent a solution. The perpendicular is of length $b \sin \alpha$; if α



Given two sides and the angle opposite one

The side opposite the given angle must always be greater than, or equal to, the corresponding altitude.

is equal to or greater than 90°, there will be one solution if $a > b$, and none if $a \leq b$, for the greater angle lies opposite the greater side. By plane geometry then, we have the following scheme, indicating whether one solution, two solutions, or no solutions are possible.

$\alpha \geq 90°$, $a \leq b$, no solution.
$\alpha \geq 90°$, $a > b$, one solution.
$\alpha < 90°$, $a < b \sin \alpha$, no solution.
$\alpha < 90°$, $a = b \sin \alpha$, one solution.
$\alpha < 90°$, $b \sin \alpha < a < b$, two solutions.
$\alpha < 90°$, $a > b$, one solution.

Trigonometrically, by our formulas, we would arrive at these facts, but a student who is not able to observe the geometrical relationships is not likely to be able to interpret the trigonometric formulas. When the sine of an angle is given, the angle may be either in I or II, α or $180° - \alpha$ if α is either angle which satisfies the relationship. Then,

$$\frac{\sin \beta}{b} = \frac{\sin \alpha}{a}, \text{ gives } \sin \beta = \frac{b \sin \alpha}{a};$$

if $a < b \sin \alpha$, $\sin \beta$ will be greater than 1 and there is no angle satisfying the relationship; if $a > b$, $\alpha > \beta$ (greater angle, greater side opposite), and only the acute angle β can be taken; if $a < b$, both values of β can be taken.

PROBLEMS

1. Given $\alpha = 30°$, $a = 150$, $b = 60$, 70, 75, 100, 150, 180, and 200 respectively; draw the figures and determine the number of solutions in each case. Solve for β in each case where it is possible.

2. Given $\alpha = 90°$, $a = 150$, $b = 75$, 100, 150, 200. Discuss.

3. Given $a = 150$, $b = 75$; $\alpha = 20°$, 30°, 45°, 60°, 80°, 90°, 120°, 150°. Discuss the solutions, geometrically and trigonometrically.

4. Solve the following eight problems, having one or two solutions, and time yourself. Use the following form of solution. The eight problems should be completed within one hour.

a. Given $a = 366$, $b = 677$, $\alpha = 15° 10'$; solve for β.

$$\sin\beta = \frac{b \cdot \sin\alpha}{a}.$$

$$\begin{aligned}
\log b &= \\
+\log\sin\alpha &= \underline{\qquad\qquad} \\
\\
-\log a &= \underline{\qquad\qquad} \\
\log\sin\beta &= \\
\beta_1 &= \\
\beta_2 &=
\end{aligned}$$

or simply $\beta =$, if there is only one solution.

b. Given $a = 423$, $c = 288$, $\gamma = 35° 15'$; find α.

c. Given $b = 376$, $c = 804$, $\gamma = 68° 20'$; find β.

d. Given $b = 650$, $a = 830$, $\alpha = 98° 56'$; find β.

e. Given $a = 67.2$, $c = 40.4$, $\gamma = 24° 49'$; find α.

f. Given $b = .0188$, $c = .0196$, $\gamma = 100° 14'$; find β.

g. Given $a = 504.2$, $c = 1763$, $\alpha = 12° 39'$; find γ.

h. Given $b = 3{,}245{,}000$, $c = 2{,}488{,}000$, $\beta = 80° 28'$; find γ.

4. Type form: *aba* type with two solutions. —

Form of solution when two solutions are found.

Given $a = 187$, $b = 235$, $\alpha = 37° 15'$.

$$\sin\beta = \frac{b\sin\alpha}{a};\quad c = \frac{a\sin\gamma}{\sin\alpha};\quad \textit{check},\quad c = \frac{b\sin\gamma}{\sin\beta}.$$

Or Newton's *check formula*,

$$\frac{b + a}{c} = \frac{\cos\frac{1}{2}(\beta - \alpha)}{\sin\frac{1}{2}\gamma}.$$

$$\begin{aligned}
\log b &= \quad 2.3711 \\
+\log\sin\alpha &= \underline{\quad 9.7820 - 10} \\
&\quad 12.1531 - 10 \\
-\log a &= \underline{\quad 2.2718 \qquad\quad} \\
\log\sin\beta &= \quad 9.8813 - 10
\end{aligned}$$

$$\begin{aligned}
\beta_1 &= 49^\circ\, 33' \\
\alpha &= 37^\circ\, 15' \\
\gamma_1 &= 93^\circ\, 12' \\
\log a &= 2.2718 \\
+ \log \sin \gamma_1 &= \underline{9.9993 - 10} \\
& 12.2711 - 10 \\
- \log \sin \alpha &= \underline{9.7820 - 10} \\
\log c_1 &= 2.4891 \\
c_1 &= 308.4 \\
\log b &= 2.3711 \\
+ \log \sin \gamma_1 &= \underline{9.9993 - 10} \\
& 12.3704 - 10 \\
- \log \sin \beta_1 &= \underline{9.8813 - 10} \\
\log c_1 &= 2.4891
\end{aligned}$$

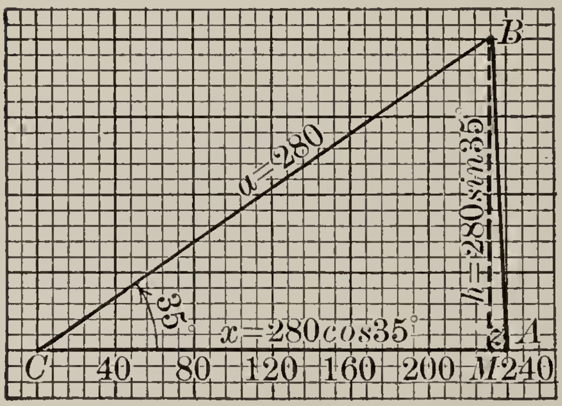
$$\begin{aligned}
\beta_2 &= 130^\circ\, 27' \\
\alpha &= 37^\circ\, 15' \\
\gamma_2 &= 12^\circ\, 18' \\
\log a &= 2.2718 \\
+ \log \sin \gamma_2 &= \underline{9.3284 - 10} \\
& 11.6002 - 10 \\
- \log \sin \alpha &= \underline{9.7820 - 10} \\
\log c_2 &= 1.8182 \\
c_2 &= 65.8 \\
\log b &= 2.3711 \\
+ \log \sin \gamma_2 &= \underline{9.3284 - 10} \\
& 11.6995 - 10 \\
- \log \sin \beta_2 &= \underline{9.8813 - 10} \\
\log c_2 &= 1.8182
\end{aligned}$$

Compare with values for $\log c_1$ (and $\log c_2$) found above.

$$\begin{aligned}
2\,A_1 &= ab \sin \gamma_1 \\
\log a &= 2.2718 \\
+ \log b &= 2.3711 \\
+ \log \sin \gamma_1 &= \underline{9.9993 - 10} \\
\log 2\,A_1 &= 4.6422 \\
2\,A_1 &= 43870 \\
A_1 &= 21935
\end{aligned}$$

$$\begin{aligned}
2\,A_2 &= ab \sin \gamma_2 \\
\log a &= 2.2718 \\
\log b &= 2.3711 \\
\log \sin \gamma_2 &= \underline{9.3284 - 10} \\
\log 2\,A_2 &= 3.9713 \\
2\,A_2 &= 9360 \\
A_2 &= 4680
\end{aligned}$$

5. Given two sides and the included angle: type $ab\gamma$. — The method of solution here given is the solution by right triangles, since that involves no new formula and no greater amount of computation than the common solution employing a new tangent formula; the tangent formula is given in Section 8 of the preceding chapter.



Solution by right triangles.

Given $a = 280$, $\gamma = 35^\circ$,

$b = 240$, $\alpha =$

$\beta =$

$h = a \sin 35^\circ$,

$x = a \cos 35^\circ$,

$z = b - a \cos 35^\circ$.

$$\tan \alpha = \frac{h}{z} = \frac{a \sin 35^\circ}{b - a \cos 35^\circ}; \quad c = \frac{a \sin \gamma}{\sin \alpha}; \quad \textit{check},\ c = \frac{b \sin \gamma}{\sin \beta}.$$

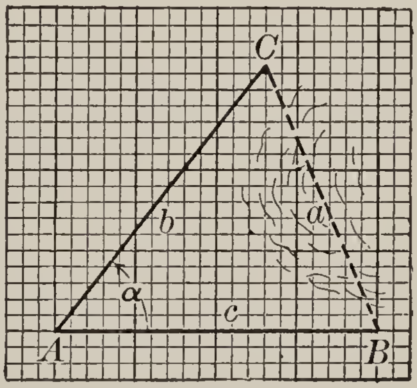
$$\begin{aligned}
2\,A &= ab \sin \gamma \\
\log a &= 2.4472 \\
+ \log \sin 35^\circ &= \underline{9.7586 - 10} \\
\log h &= 2.2058 \\
- \log z &= \underline{1.0253} \\
\log \tan \alpha &= 1.1805 \\
\alpha &= 86^\circ\ 13' \\
\beta &= 58^\circ\ 47' \\
\log a &= 2.4472 \\
+ \log \sin \gamma &= \underline{9.7586 - 10} \\
& \ 2.2058 - 10 \\
- \log \sin \alpha &= \underline{9.9990 - 10} \\
\log c &= 2.2068 \\
c &= 161.0 \\
\log b &= 2.3802 \\
+ \log a &= 2.4472 \\
+ \log \sin \gamma &= \underline{9.7586 - 10} \\
\log 2\,A &= 4.5860 \\
2\,A &= 38550 \\
A &= 19270
\end{aligned}$$

$$\begin{aligned}
\log a &= 2.4472 \\
+ \log \cos 35^\circ &= \underline{9.9134 - 10} \\
\log x &= 2.3606 \\
x &= 229.4 \\
z &= 10.6
\end{aligned}$$

Check.

$$\begin{aligned}
\log b &= 2.3802 \\
+ \log \sin \gamma &= \underline{9.7586} \\
& \ 12.1388 \\
- \log \sin \beta &= \underline{9.9321} \\
\log c &= 2.2067
\end{aligned}$$

This problem also occurs frequently in surveying. It permits the determination of the direction and length of a tunnel through a mountain by means of the location of some point from which both ends of the proposed tunnel are visible. The distance and direction from a given point to a second point, past some barrier, are determined by this method. Thus the distance from *B* to *C* through woods can be found, if some point can be located from which both *C* and *B* are visible. The distance and direction to invisible points are constantly needed in artillery fire; another method of finding distance and direction of the target is that of finding an observation point from which both the gun and the target, invisible at the gun, are visible to an observer.



Distance across a barrier

PROBLEMS

Using the form of solution above, solve for the side opposite the given angle.

1. Given $a = 3846$, $b = 4977$, $\gamma = 38° 10'$. Find c.

2. Given $b = 4.832$, $c = 8.973$, $\alpha = 108° 56'$. Find a.

3. Given $a = .0485$, $c = .0682$, $\beta = 58° 38'$. Find b.

4. Using the form of solution given above, find the side opposite the given angle in the following five problems; time yourself, and compare with the time for the same five problems solved by the cosine law (page 255).

a. Given $a = 366$, $b = 677$, $\gamma = 15° 10'$.
b. Given $a = 423$, $c = 288$, $\beta = 35° 15'$.
c. Given $b = 627$, $c = 816$, $\alpha = 100° 41'$.
d. Given $a = 635$, $c = 341$, $\beta = 67° 38'$.
e. Given $c = 184$, $b = 295$, $\alpha = 130° 54'$.

6. **Discussion of checks and methods.** — The procedure by logarithms as with physical measurements involves numerous approximations. That two values of $\log c$, in the check and in the solution, or by two different methods of solution, do not agree precisely is a frequent result of correct computation. However, the disagreement will be within certain well-defined limits, depending entirely upon the range (number of places) of the logarithm tables which are used both for numbers and for angles; the tabular difference should be noted, mentally, and any discrepancy between check and computation should be examined as to its effect upon the value of the computed quantity. Thus no error in our computation (page 273) accounts for the difference between $\log c = 2.2068$, and $\log c = 2.2067$, nor does this here affect the value of c. However the angle of 86° 13′ is so near to 90° that the tangent grows very rapidly; the tabular difference here is large, and might easily affect our result, through the inevitable inaccuracy of ordinary interpolation in this neighborhood.

7. To determine the angles of a triangle when the three sides are given : *abc* type. —

$$\tan\frac{1}{2}\alpha = \frac{r}{s-a}$$

$$\tan\frac{1}{2}\beta = \frac{r}{s-b}$$

$$\tan\frac{1}{2}\gamma = \frac{r}{s-c}$$

$a =$

$b =$

$c =$

$2s =$

$s =$

$s - a =$

$s - b =$

$s - c =$

Check by noting sum of $s - a$, $s - b$, $s - c$ which equals s.

$\log r =$

$-\log(s-a) =$ ________

$\log\tan\frac{1}{2}\alpha =$

$\frac{1}{2}\alpha =$

$\alpha =$

$\log r =$

$-\log(s-c) =$ ________

$\log\tan\frac{1}{2}\gamma =$

$\frac{1}{2}\gamma =$

$\gamma =$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$s = \frac{a+b+c}{2}$$

$$A = \sqrt{s(s-a)(s-b)(s-c)} = r \cdot s$$

Check. $\alpha =$

$\beta =$

$\gamma =$ ________

$\alpha + \beta + \gamma =$

$\log(s-a) =$

$+\log(s-b) =$

$+\log(s-c) =$ ________

$-\log s =$ ________

$\log r^2 =$

$\log r =$

$\log r =$

$-\log(s-b) =$ ________

$\log\tan\frac{1}{2}\beta =$

$\frac{1}{2}\beta =$

$\beta =$

$\log r =$

$\log s =$

$\log A =$

$A =$

Complete the solution of a problem of this type, form as above, taking $a = 4320$, $b = 6840$, and $c = 8630$.

TIMING EXERCISES

1. Employing the form of solution as above, solve the following four problems for the angles α, β, and γ, timing yourself; write down the complete form necessary for the solution of each problem before using the logarithm table.

a. Given $a = 320$, $b = 640$, $c = 580$.

b. Given $a = 44.8$, $b = 76.2$, $c = 70.4$.

c. Given $a = 4.49$, $b = 8.87$, $c = 9.13$.

d. Given $a = .0624$, $b = .0688$, $c = .0731$.

2. Solve the following four problems for α, β, and γ, taking note of the time required.

a. Given $a = 320.4$, $b = 640.6$, $c = 580.4$.
b. Given $a = 3482$, $b = 7461$, $c = 5395$.
c. Given $a = 1.835$, $b = 2.346$, $c = 3.045$.
d. Given $a = 1.43 \times 10^{-6}$, $b = 2.34 \times 10^{-6}$, $c = 2.87 \times 10^{-6}$.

3. How would the solution of problem 2 *d* be changed if a, b, and c were given as 1.43, 2.34, and 2.87, respectively?

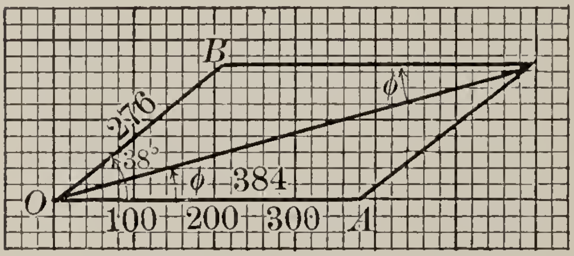
PROBLEMS

Type problems. Solve for the other parts; make a note of the time required for the solution of each problem.

	a	b	c	α	β	γ	A
1.	8294	6788		33° 15′			
2.	8294	6788			33° 15′		
3.	8294	6788				33° 15′	
4.	8294	6788	9645				
5.	8206			33° 15′	67° 25′		
6.	8206				33° 15′	67° 25′	
7.	8206	6009		133° 15′			
8.		356	235	64° 10′			
9.	.03267		.05431			63° 40′	

10. 83×10^6, 67×10^6, 54×10^6.

11. Given two forces of magnitude 384 and 276 acting at an angle of 38° with each other. Find the angle which the resultant makes with the larger force and the magnitude of the resultant. Note that the problem is simplified by regarding the line of the larger force as an axis of reference; the second force adds to this a component 276 cos 38°; the vertical component is 276 sin 38°; $\tan \phi = \dfrac{276 \sin 38^\circ}{384 + 276 \cos 38^\circ}$; the magnitude of the resultant, r, is by the sine law,



Resultant of two forces

$$\frac{r}{\sin(180° - 38°)} = \frac{276}{\sin \phi},$$

whence $$r = \frac{276 \sin 38°}{\sin \phi};$$

or $$r = \sqrt{(276 \sin 38°)^2 + (384 + 276 \cos 38°)^2},$$

as hypotenuse of a right triangle; or

$$r = \sqrt{(276)^2 + (384)^2 - 2 \times 384 \times 276 \times \cos 38°},$$

by the cosine law.

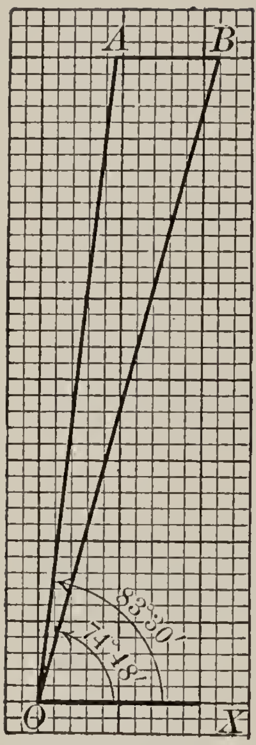
12. Given two forces of magnitude 684 and 450, acting at an angle of 64°. Find the resultant in magnitude and direction, graphically and trigonometrically.

13. From an aëroplane moving horizontally at rate of 120 miles per hour (176 feet per second) a bullet is shot at right angles to the path of the aëroplane with a velocity of 2800 feet per second. What is the resultant velocity in magnitude and direction?

14. A cylindrical trough of horizontal length 20 feet and with the ends semicircles of radius 2 feet each, contains how many gallons of water for 1 foot in depth, for $1\frac{1}{2}$ feet, for 2 feet?

15. A typical oil tank is 30 feet long and has a diameter of 8 feet. Compute the volume in barrels (see page 94) for each foot of depth. Do not carry beyond tenths of a barrel.

16. What angle does $y = 2x + 12$ make with the x-axis? What angle does $3y - 4x - 20 = 0$ make with the x-axis? Find the area and the other angle of this triangle, formed by the two lines and the x-axis, by trigonometrical methods. Find the angle between the two lines by the formula, $\tan \phi = \frac{m_1 - m_2}{1 + m_1 m_2}$ and check. Find the area by analytical methods to check upon the trigonometrical solution.



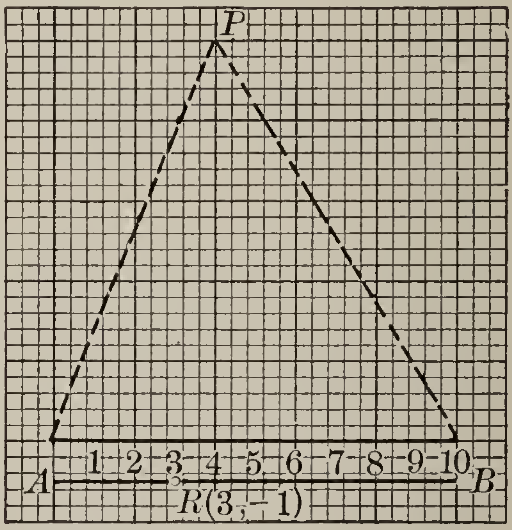
Moving aëroplane

17. In the figure AB represents the movement in 30 seconds of an aëroplane moving parallel to OX at rate of 120 miles per hour; $\angle AOX$ is measured as 83° 30′ and $\angle BOX$ is measured as 74° 48′; find the distance OB and the position of the aëroplane at the end of the next 30 seconds if it continues on its present course.

18. In problem 17 discuss the percentage effect of an error of 1° in angle AOX and in angle AOB, respectively.

19. Given that observers A and B at the ends of a battleship 340 feet long observe an object O at angles of 106° 30′ and 72° 48′ respectively with the line AB. Find BO and AO. Solve this problem also graphically.

20. A and B are observation stations on the shore, 10 miles apart and may be assumed to be on an east and west line; R, the battery, is three miles east of A and one mile south. Given that a battleship P is observed from A at an angle of 68° 12′ and from B at an angle of 59° 02′ with AB; find the coördinates of P; find the range from R and the angle made by RP with the east and west line; find the distance using the formula for the distance between two points. This problem is solved in actual practice graphically on large plotting boards. Using $\frac{1}{2}$ inch to the mile, how closely could you approximate the distance?

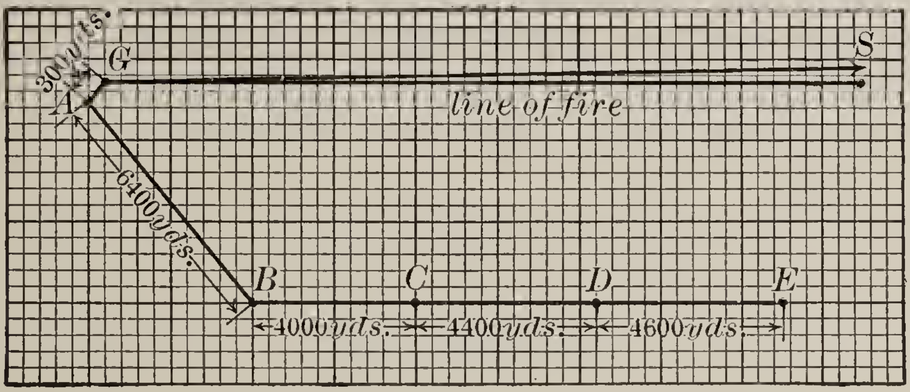


Shore battery observations

21. In the preceding problem, suppose that the observations reported at the end of 1 minute are from A, 68° 08′, and from *B*, 59° 12′, locate the direction of movement of the ship.

22. If the battleship of the two preceding problems is 600 feet long and broadside to *A*, what angle does it subtend at *A*?

23. When guns are tested at Sandy Hook or other proving grounds, the actual range for any angle of elevation of the gun is obtained by coincident observation of the splash of the shell by several observers located in towers along a line roughly parallel to the line of fire of the projectile. The shell is loaded with a slight charge of high explosive in order to give a splash of some magnitude. For convenience in our figure, we have assumed the towers on a north and south line spaced as indicated; the height of the tower is regarded as negligible. Given that observers at *A*, *B*, *C*, *D*, and *E* observe the angle of the splash with the north and south line, the azimuth angle, as 2° 20′, 21° 24′, 27° 16′, 41° 35′, and 68° 54′, find the distance and direction of *S*, the splash, from *G*, the gun, which is 400 yards due east of *A*. Note that any two observers give the position of *S*; the substantial agreement of three observers is taken as sufficient. Compare the solutions. Determine the coördinates of *S* with respect to a horizontal axis through *BCDE* and a vertical axis through *AG*. Assuming that the gun was pointed south the deflection,



Observation towers at proving grounds

to the east here, is termed the drift; it is due to wind and other causes; determine this angle.

It is not essential that the towers be in a straight line; the distance and direction of lines between adjacent towers is carefully measured. A large plotting board is used to obtain a graphical solution.

24. Employing the general form of solution, adapted to each case, solve the following triangles for the unknown sides or angles, making note of the time required:

a. Given $a = 44.82$, $b = 76.24$, $c = 70.48$.
b. Given $a = 366.5$, $b = 677.9$, $\gamma = 15° 11'$.
c. Given $b = 376.3$, $c = 804.8$, $\gamma = 68° 27'$.
d. Given $b = 816.4$, $\beta = 67° 17'$, $\gamma = 34° 9'$.
e. Given $a = 915.5$, $\gamma = 90°$, $\alpha = 32° 3'$.

London Bridge

Five pure elliptical arches; central one 152 feet by 37 feet 10 inches, and the others 140 feet by 37 feet 2 inches.

CHAPTER XVIII

THE ELLIPSE

1. Parametric equations of an ellipse. — The parametric equations of the circle, with center at the origin

$$x = r \cos \theta,$$
$$y = r \sin \theta,$$

and of the circle with center at (h, k)

$$x - h = r \cos \theta,$$
$$y - k = r \sin \theta,$$

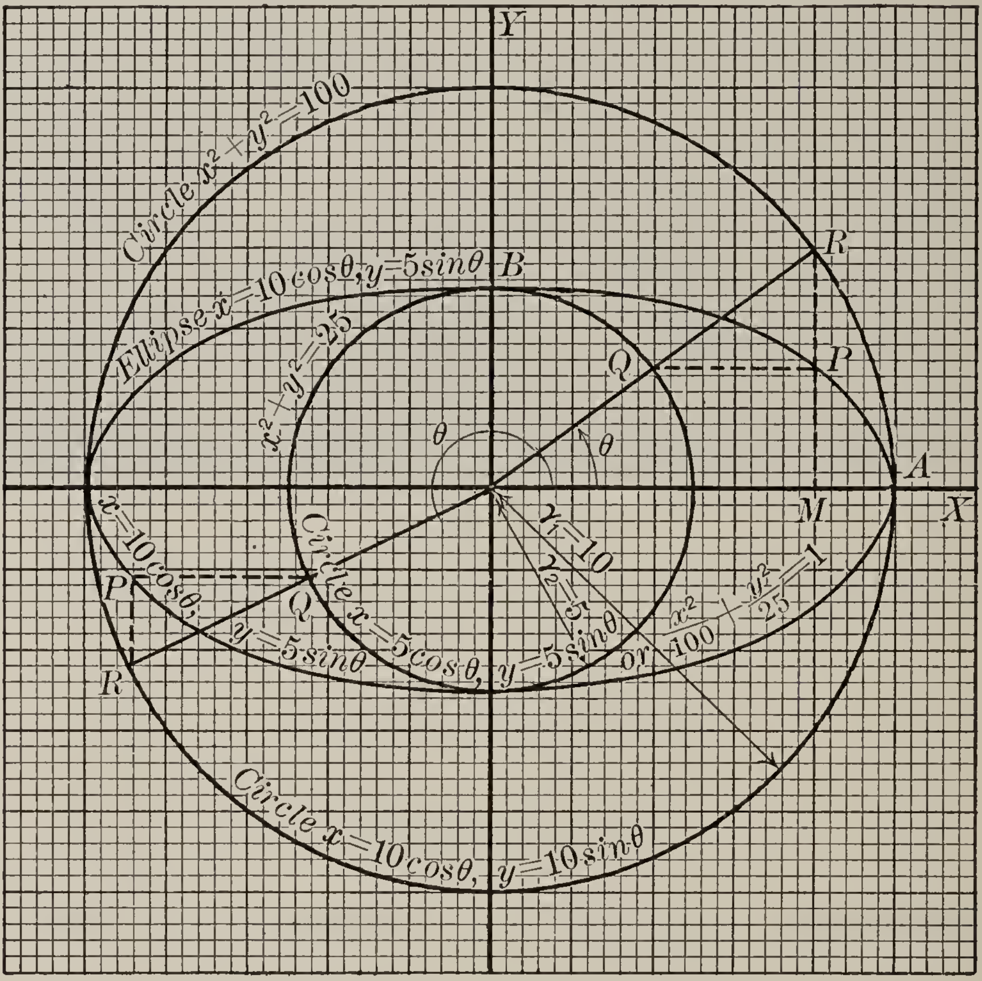
if modified to read,

$$x = r_1 \cos \theta,$$
$$y = r_2 \sin \theta,$$

and

$$x - h = r_1 \cos \theta,$$
$$y - k = r_2 \sin \theta,$$

respectively, give a curve which is closely related to the circle. This curve is called the ellipse; r_1 and r_2 are called the major and minor semi-axes of the ellipse, and the circles obtained with radii r_1 and r_2 are called the major and minor auxiliary circles of the ellipse, assuming r_1 greater than r_2.



An ellipse with its major and minor auxiliary circles

$$\begin{matrix} x = r_1 \cos\theta \\ y = r_1 \sin\theta \end{matrix} \text{ gives the larger circle.} \qquad \begin{matrix} x = r_2 \cos\theta \\ y = r_2 \sin\theta \end{matrix} \text{ gives the smaller circle.}$$

$$\begin{matrix} x = r_1 \cos\theta \\ y = r_2 \sin\theta \end{matrix} \text{ gives the ellipse.}$$

P, Q, and R are called corresponding points.

Eliminating θ between the two parametric equations of the ellipse gives

$$\frac{x^2}{r_1{}^2} + \frac{y^2}{r_2{}^2} = \cos^2\theta + \sin^2\theta = 1.$$

Writing, as is customary, a and b for r_1 and r_2 this equation becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

If the ellipse had the center (h, k) and the two values a and b corresponding to r_1 and r_2, the equation of the ellipse would be written

$$(x-h)=a \cos \theta,$$
$$(y-k)=b \sin \theta,$$

in parametric form; and

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1,$$

in so-called standard form. If we assume, as we have above, that a is the radius of the larger circle, we would have for an ellipse placed vertically,

$$x-h=b \cos \theta,$$
$$y-k=a \sin \theta,$$

or

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1,$$

as the equation of an ellipse whose major axis is vertical. Note particularly that the terminal side of the angle θ does not pass, in general, through the point on the ellipse which corresponds to that angle, but the angle θ is made by a line passing through the corresponding points on the major and minor auxiliary circles.

PROBLEMS

1. Construct the ellipse

$$x=10 \cos \theta,$$
$$y=6 \sin \theta,$$

by finding the points on the ellipse given by $\theta=0°$, $30°$, $45°$, $60°$, $90°$, and the corresponding points in the other quadrants.

2. Construct the ellipse

$$x=10 \cos \theta,$$
$$y=6 \sin \theta,$$

using corresponding points on the major and minor auxiliary circles to locate points on the ellipse.

3. How does the ellipse

$$x - 3 = 10 \cos \theta,$$
$$y + 2 = 6 \sin \theta,$$

differ from the preceding ellipse?

4. Locate 10 points on the ellipse

$$x - 3 = 6 \cos \theta,$$
$$y + 2 = 10 \sin \theta,$$

and draw the curve.

5. Prove that a is the largest value and b the smallest of a line from O to any point on the ellipse. This gives the reason for the names, major and minor axes.

6. Show that every chord of the ellipse through O is bisected; O is the center of the curve.

7. The five arches of the London Bridge are of true elliptical shape; the central arch has the width 152 feet ($= 2a$) and the height 37 feet 10 inches ($= b$). Find the equation and plot 12 points on this arch; b may be taken as 38 feet. The adjacent arches are of length 140 feet and height 37 feet 2 inches. Write the equations. In each case take the major axis as x-axis and minor axis as y-axis.

2. Properties of the ellipse. — The equations

$$x = a \cos \theta,$$
$$y = b \sin \theta,$$

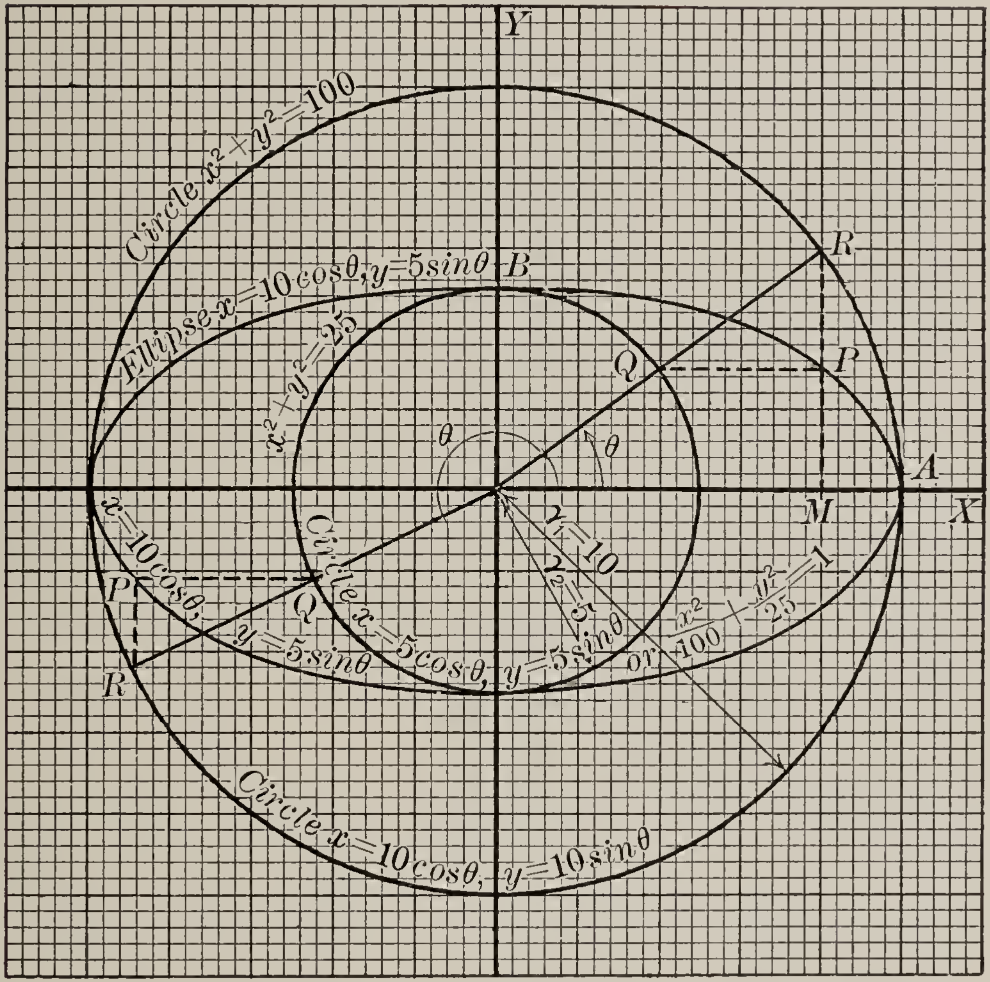
wherein $a > b$, can represent any ellipse whatever, when the axes of the ellipse are taken as the axes of reference. The geometrical peculiarities of this ellipse will be characteristic of all ellipses, provided, of course, that no limitation is placed upon a and b (except that a may be taken as greater than b).

Each ordinate of the ellipse

$$x = a \cos \theta,$$
$$y = b \sin \theta,$$

is a constant proportional part of the corresponding ordinate in the major auxiliary circle,

$$x = a\cos\theta,$$
$$y = a\sin\theta.$$



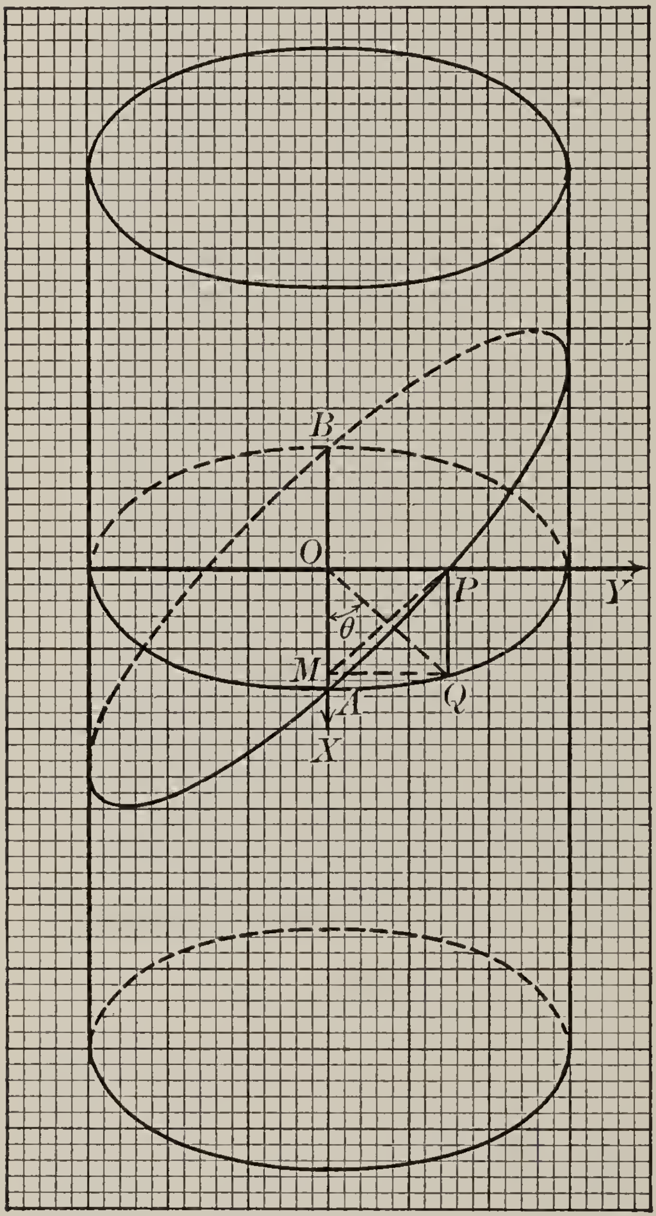
An ellipse with its major and minor auxiliary circles

$\left.\begin{array}{l} x = r_1\cos\theta \\ y = r_1\sin\theta \end{array}\right\}$ gives the larger circle. $\left.\begin{array}{l} x = r_2\cos\theta \\ y = r_2\sin\theta \end{array}\right\}$ gives the smaller circle.

$\left.\begin{array}{l} x = r_1\cos\theta \\ y = r_2\sin\theta \end{array}\right\}$ gives the ellipse.

P, Q, and R are called corresponding points.

Take any point $(a\cos\theta,\ b\sin\theta)$ on the ellipse, the corresponding ordinate in the circle is $y_c = a\sin\theta$; but $y_e = b\sin\theta = \frac{b}{a}(a\sin\theta) = \frac{b}{a}\cdot y_c$. Conversely, if the ordinate of any point on a given curve is always a constant proportional part of the ordinate of a corresponding point on a given circle, for equal values of the abscissa, the curve is an ellipse; the ratio need not be less than unity as the figure clearly indicates, a vertical ellipse being represented when the ratio is greater than unity. This property of this ellipse is equally true of any ellipse, replacing the terms ordinate and abscissa, where given or implied, by perpendiculars to the major and minor axes, respectively, of the curve.



Elliptical section of a circular cylinder

If the major auxiliary circle is rotated about OX as an axis through an angle α, $\cos \alpha = \frac{b}{a}$, the projection upon the plane of the original position will be the ellipse

$$x = a \cos \theta,$$
$$y = b \sin \theta;$$

evidently the x of any point on the projected curve may be taken as the x of a point on the projecting circle, or $x = a \cos \theta$. The ordinate on the projected curve is in a constant ratio, $\cos \alpha = \frac{b}{a}$, to the ordinate on the circle. Further any plane section of a cylinder with circular base is an ellipse, for with the same abscissas, the ordinates of the curve of section bear a constant ratio $\frac{1}{\cos \alpha}$ to the corresponding ordinates on the circular base.

Let the plane cut the axis of the cylinder in O, and let BOX, on the diagram, be the intersection of the cutting plane with the plane of the circular section through O (parallel to the base of the cylinder). Note that angle PMQ is constant.

Similarly if a circular cone is cut by a plane cutting all the elements of the cone, the section formed is an ellipse; the geometrical proof is rather too complicated to give here but an analytical proof is indicated in Chapter XXXII, Section 6. The ancient Greeks studied the properties of the ellipse entirely from the point of view of the curve as a plane section of a cone. The Greek theoretical work concerning the properties of conic sections made it possible for Kepler to discover that the path of the earth about the sun is an ellipse, and for Newton to formulate the law of gravitation.

The properties mentioned are intimately connected with the applications of the ellipse in engineering problems.

From the property of the ordinates it follows that the area of an ellipse is $\frac{b}{a} \cdot \pi a^2 = \pi ab.$ $A = \pi ab.$

The proof is strictly by a "limit process," and may be made reasonably evident by dividing the semi-axis into 25, 50, 100, $\cdots$ equal parts and drawing two series of rectangles on these equal subdivisions about the corresponding ordinates to fall entirely within (or entirely without) the ellipse and the circle, respectively. As the subdivisions are increased in number the one series of rectangles has the area of the quarter-ellipse as a limit; it differs from this area never by an area as great as the rectangle of height b and base one subdivision; so also the sum of the second series of rectangles has the quarter-circle as a limit, never differing from it by an area as great as the rectangle of height a and base one subdivision; but the sum of the series of smaller rectangles always equals $\frac{b}{a}$ times the sum of the series of larger rectangles since the bases are equal and the altitude of any one of the smaller is $\frac{b}{a}$ times the altitude of the corresponding one of the larger; evidently, then, the limits of these sums are in the same ratio. In the diagram above the ruling of the paper divides the semi-major axis into 25 parts, and gives the rectangles.

The Colosseum in Rome

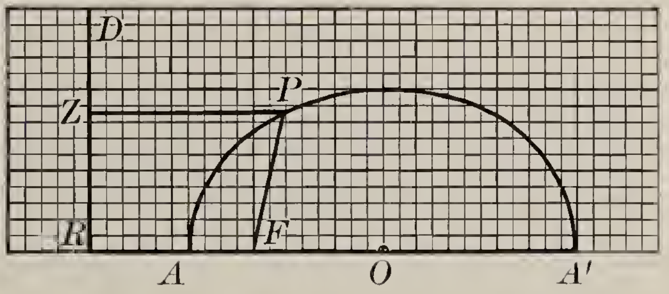
A famous elliptical structure, 615 feet by 510 feet, by 159 feet high; the arena is an ellipse 281 by 177 feet.

A rectangle of dimensions 2 a and 2 b may be circumscribed about the ellipse. The sides of this rectangle are tangents at the vertices of the ellipse; the middle lines parallel to the sides are the axes of the ellipse; the horizontal sides of this rectangle cut the major auxiliary circle in points which correspond to four symmetrically located points on the ellipse;

$$\sin\theta = \pm\frac{b}{a},\ \cos\theta = \pm\sqrt{1-\frac{b^2}{a^2}} = \pm\frac{\sqrt{a^2-b^2}}{a}, \text{ whence}$$

$x = \pm\sqrt{a^2-b^2}$, $y = \pm\frac{b^2}{a}$, are the coördinates of these points on this ellipse. The points on the major axes $(\pm\sqrt{a^2-b^2}, 0)$ are the foci of the ellipse, and enjoy special properties with respect to points on the ellipse. The sun is at one focus of the path of the earth.

3. Standard definition of the ellipse. — The locus of a point which moves so that its distance from a fixed point called the focus is in a constant ratio, e, less than 1, to its distance from a fixed line called the directrix, is an ellipse.



$PF = e \cdot PZ$ defines the ellipse

The constant e is called the eccentricity.

Let the fixed point be F and $D'D$ the fixed line. Through F drop a perpendicular to the directrix $D'D$ meeting it in R and use this line as x-axis. By definition, then, the ellipse will be the locus of a point which moves so that $PF = e \cdot PZ$, wherein PZ is the perpendicular from P to the directrix. Two points of our curve will be found to lie upon the x-axis; the two points are the points A and A' which divide the segment FR internally and externally in the ratio e (taken as $\frac{2}{3}$ on our figure). Take the middle point of $A'A$ as the origin, designating $A'A$, which is a fixed length dependent upon FR and e, by $2a$. Then $OA = OA' = a$.

$$\begin{aligned} AF &= e \cdot AR. \\ A'F &= e \cdot A'R. \\ AF + A'F &= e(AR + A'R). \\ 2a &= e(AR + OA' + OR). \\ &= e(OA + AR + OR) \\ &= 2e \cdot OR, \text{ whence} \end{aligned}$$

$$OR = \frac{a}{e}.$$

$$\begin{aligned} A'F - AF &= e(A'R - AR). \\ 2\,OF &= 2\,ae. \\ OF &= ae. \end{aligned}$$

$$F \text{ is } (-ae, 0) \text{ and } D'D \text{ is } x + \frac{a}{e} = 0;$$

$PF = e \cdot PZ$, taking P as (x, y), gives in analytical language, or formulas,

$$\sqrt{(x+ae)^2+(y-0)^2} = e \cdot \left(x+\frac{a}{e}\right) = ex + a.$$

$$x^2 + a^2e^2 + y^2 = e^2x^2 + a^2.$$

$$x^2(1-e^2) + y^2 = a^2(1-e^2).$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1.$$

Let $$b^2 = a^2(1-e^2),$$

whence $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{1}$$

But this equation is satisfied by every point

$$x = a \cos \theta,$$
$$y = b \sin \theta,$$

determined as we have indicated above, and conversely; our two definitions are equivalent to each other.

The form of equation (1) shows that the curve is symmetrical with respect to the two lines which we have chosen as axes; *i.e.* since x and $-x$ give precisely the same equation to determine y, the curve is symmetrical with respect to the y-axis; similarly since y and $-y$ give precisely the same values for x, the curve is symmetrical with respect to the x-axis. The coördinate axes are axes of symmetry of the curve.

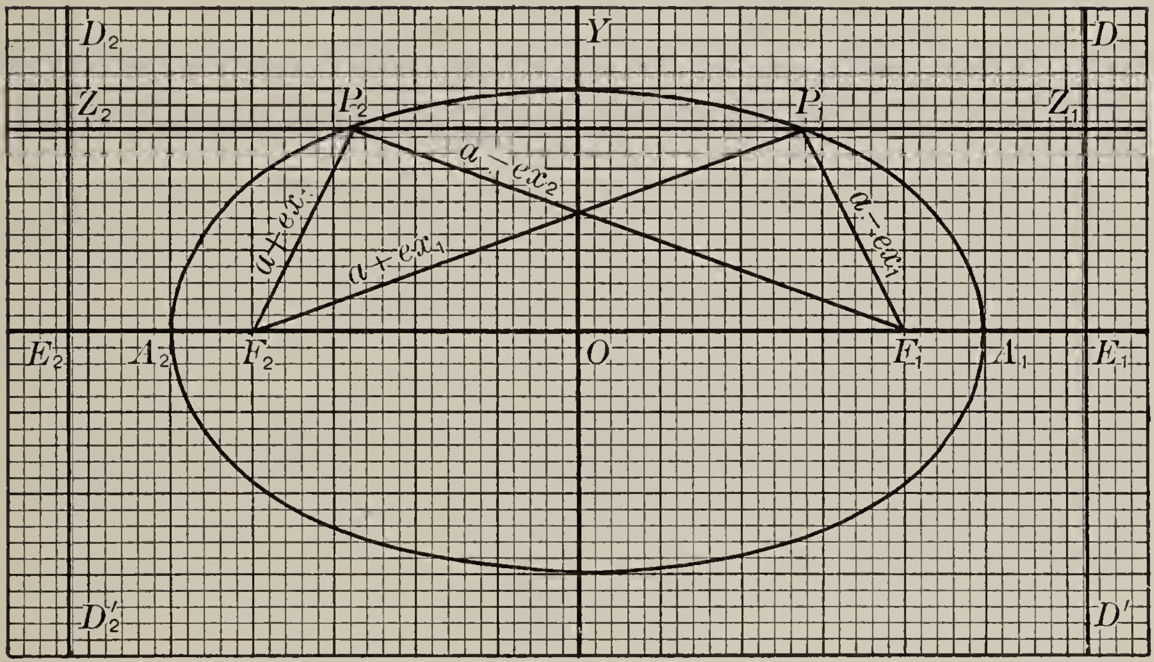
By symmetry with respect to the y-axis we mean that if the right half of the curve is folded over on the y-axis as an axis, *i.e.* revolved about it as an axis (or axle) through an angle of 180°, the two sides will coincide throughout. Hence corresponding to F_1 and $D'D$, there is another focus F_2 and a corresponding directrix $D_2D'_2$, enjoying precisely the same properties with regard to the curve as F_1 and $D'D$.

By symmetry, $P_1F_1 = P_2F_2$, $P_1Z_1 = P_2Z_2$,

and since $P_1F_1 = e \cdot P_1Z_1$, $P_2F_2 = e \cdot P_2Z_2$.

Similarly, $P_2F_1 = P_1F_2$, $P_2Z_1 = P_1Z_2$,

and, since $P_2F_1 = e \cdot P_2Z_1$, $P_1F_2 = e \cdot P_1Z_2$.



Symmetry of the ellipse with respect to its axes

4. Sum of the focal distances constant. — Taking any point $P_1(x_1, y_1)$ on the ellipse, the focal distances P_1F_1 and P_1F_2 are equal to $a - ex_1$ and $a + ex_1$, respectively.

For $P_1F_1 = e \cdot P_1Z = e\left(x_1 - \frac{a}{e}\right) = ex_1 - a$, which is a negative distance since P_1 and O are upon the same side of the line $D'D$. Similarly $P_1F_2 = e \cdot P_1Z_2 = e \cdot \dfrac{x_1 + \frac{a}{e}}{-1} = -(ex_1 + a)$. As positive values P_1F_1 and P_1F_2 are $a - ex_1$, and $a + ex_1$; these may also be derived by the formula for the distance between two points, noting that (x_1, y_1) is on the curve;

$$PF_1 + PF_2 = a - ex_1 + a + ex_1 = 2a.$$

5. Right focal chord. — When $x = ae$,

$$\frac{a^2e^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$$y^2 = b^2(1 - e^2) = \frac{b^4}{a^2}, \text{ since } b^2 = a^2(1 - e^2),$$

$$y = \pm\frac{b^2}{a}, \text{ giving } \frac{2\,b^2}{a} \text{ as right focal chord.}$$

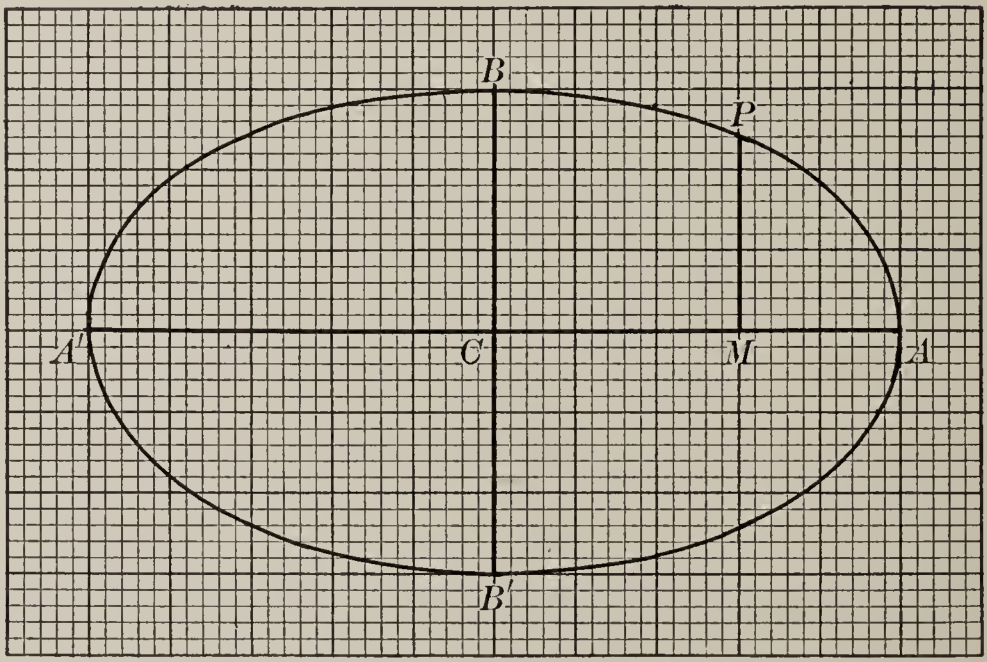
The value $y = \frac{b^2}{a}$ is the value of y obtained in the parametric form when $\sin\theta = \frac{b}{a}$, $y = b\sin\theta = \frac{b^2}{a}\cdot$

$$x = a\cos\theta = \pm\, a\sqrt{1 - \frac{b^2}{a^2}} = \pm\sqrt{a^2 - b^2} = \pm\, ae,$$

showing that the focus as we have now defined it coincides with the focus as first defined.

6. Standard forms of the ellipse. — The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ may be interpreted geometrically,

$$\frac{\overline{CM}^2}{\overline{CA}^2} + \frac{\overline{MP}^2}{\overline{CB}^2} = 1,$$



in which CM and MP represent the distances cut off from the center on the major and minor axes of the ellipse by the perpendiculars to the axes from any point on the ellipse; CA and CB are the lengths of the semi-major and semi-minor axes.

Given a horizontal ellipse with center at (h, k) and axes a and b, the relation

$$\frac{\overline{CM}^2}{\overline{CA}^2}+\frac{\overline{MP}^2}{\overline{CB}^2}=1$$

becomes

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1, \text{ since}$$

$$CM = x - h \text{ and } MP = y - k.$$

Similarly a vertical ellipse with center (h, k) and axes a and b, respectively, has the equation

$$\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1, \text{ since}$$

CM is here $y - k$ and MP is $x - h$.

Type forms : $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$; $\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$.

$$b^2 = a^2(1 - e^2);$$

foci of $\left\{\begin{array}{l}\text{horizontal}\\\text{vertical}\end{array}\right.$ ellipse on the line $\left\{\begin{array}{l}y-k=0\\x-h=0\end{array}\right.$

at a distance $ae=\sqrt{a^2-b^2}$, from the center (h, k);

right focal chords extending a distance $\frac{b^2}{a}\left\{\begin{array}{l}\text{vertically}\\\text{horizontally}\end{array}\right.$

from the foci on either side of the major axis $\left\{\begin{array}{l}y-k=0,\\x-h=0.\end{array}\right.$

7. Limiting forms of the ellipse equations. —

$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ represents an ellipse.

$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=0$ represents a point ellipse or two imaginary straight lines through (h, k); the only real point which satisfies this equation is the point (h, k).

$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=-1$ represents an imaginary ellipse; only imaginary values of x and y can make the sum of the two squares on the left equal to -1.

The equation of any ellipse with axes parallel to the coördinate axes may be written,

$$\frac{(x-x_1)^2}{a^2}+\frac{(y-y_1)^2}{b^2}=k, \text{ wherein}$$

ka^2 and kb^2 represent the squares of the semi-axes. As k approaches zero the ellipse diminishes in size, and when $k=0$ the equation represents the point (x_1, y_1); when k becomes negative the ellipse becomes imaginary.

8. Illustrative problems. —

a. Plot the ellipse $4x^2+16x+9y^2-18y-75=0$.

First write in form to complete the squares,

$$4(x^2+4x \qquad)+9(y^2-2y \qquad)=75.$$

Complete squares:

$$4(x^2+4x+4)+9(y^2-2y+1)=75+16+9.$$
$$4(x+2)^2+9(y-1)^2=100.$$

Write in standard form:

$$\frac{(x+2)^2}{25}+\frac{(y-1)^2}{\frac{100}{9}}=1.$$

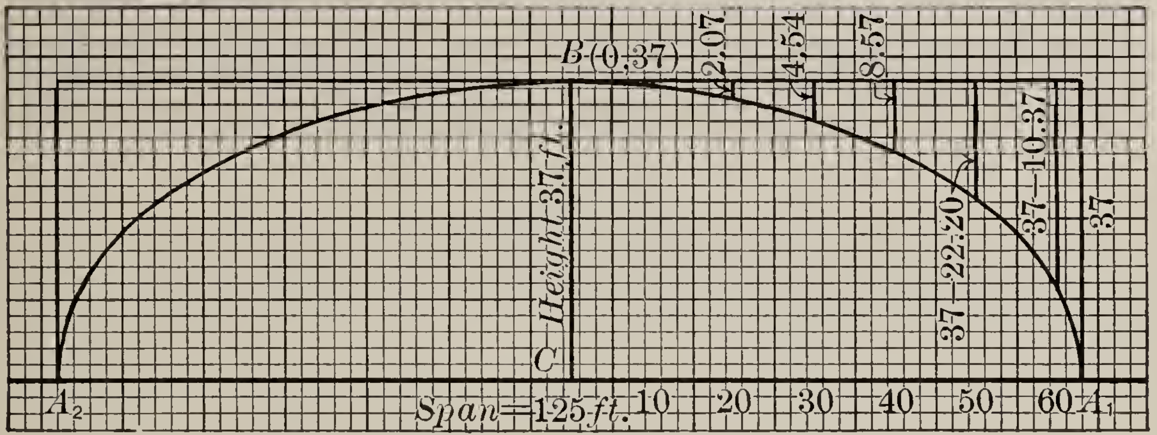
$$C(-2,1)\ ;\ a^2=25\ ;\ b^2=\frac{100}{9};\ ae=\sqrt{25-\frac{100}{9}}=\sqrt{\frac{125}{9}}=\frac{11.2}{3}$$

$$\frac{b^2}{a}=\frac{100}{45}=\frac{20}{9}.$$

Plot the center, extremities of major and minor axes; foci; extremities of right focal chords; at least one further point, obtained from the original equation and selected so as to give a point approximately midway between the extremity of a focal chord and the corresponding extremity of the minor axis; by symmetry three other points are obtained. In this case $x=0$ gives desirable further points.

$$\begin{aligned} y &= \frac{9 \pm \sqrt{81+675}}{9} \\ &= 1 \pm \tfrac{1}{9}\sqrt{756} \\ &= 1 \pm \tfrac{1}{9}(27.5) \\ &= 1 \pm 3.06 \\ &= 4.06 \text{ or } -2.06. \end{aligned}$$

b. Plot the elliptical arch of a bridge, arch 125 feet wide and 37 feet high. Plot points for every 10 feet of the span; and compute to $\frac{1}{10}$ of 1 foot.



Elliptical arch, 125 foot span by 37 feet high

Scale, 1 inch to 40 feet. Horizontal measurements are from the middle of a quarter-inch square.

$$\frac{x^2}{(62.5)^2} + \frac{y^2}{37^2} = 1$$

$$a^2 = (62.5)^2\ ;\ b^2 = 37^2\ ;\ ae = \sqrt{(62.5)^2 - 37^2}\ ;\ \frac{b^2}{a} = \frac{37^2}{62.5}.$$

$$y^2 = 37^2\left(1 - \frac{x^2}{62.5^2}\right).$$

$$y^2 = 37^2 - \left(\frac{37}{62.5}\right)^2 \cdot x^2.$$

Here compute 37^2 and $\left(\frac{37}{62.5}\right)^2$, and multiply the latter by $x^2 = 100$, 400, 900, 1600, 2500, and 3600; extract the square root of the difference; use four-place logarithms.

$$\begin{aligned}
\log 37 &= 1.5682\\
\log 62.5 &= \underline{1.7959}\\
\log \text{quot.} &= 9.7723 - 10\\
\log \text{quot.}^2 &= 9.5446 - 10\\
\left(\frac{37}{62.5}\right)^2 &= \ .3504
\end{aligned}$$

$$\begin{aligned}
y_{10}^2 &= 1369 - 35.04\ ;\ y_{10} = +\ 36.51\\
y_{20}^2 &= 1369 - 140.16\ ;\ y_{20} = +\ 34.93\\
y_{30}^2 &= 1369 - 315.36\ ;\ y_{30} = +\ 32.46\\
y_{40}^2 &= 1369 - 560.64\ ;\ y_{40} = +\ 28.43\\
y_{50}^2 &= 1369 - 876.0\ \ ;\ y_{50} = +\ 22.20\\
y_{60}^2 &= 1369 - 1261.44\ ;\ y_{60} = +\ 10.37
\end{aligned}$$

Exercise. — Draw the major auxiliary circle on half-inch coördinate paper, and compute corresponding ordinates in the preceding arch as $\frac{37}{62.5}$ of the ordinates on the circle; *e.g.* for $x = 10$, find y graphically on the circle and multiply by $\frac{37}{62.5}$.

PROBLEMS

1. Plot the ellipse $9x^2 + 36x + y^2 - 6y = 0$; what are the coördinates of the foci?

2. Plot one quarter of the ellipse $\frac{x^2}{147^2} + \frac{y^2}{59^2} = 1$.

3. Plot an elliptical arch, width 233 feet, height 73 feet, plotting at least 10 points spaced at 20-foot intervals from the center. These are the dimensions of the arch of the Walnut Lane Bridge in Philadelphia; the arch is approximately an ellipse.

4. Plot the upper half of an ellipse, giving a vertical elliptical arch 13 feet $1\frac{1}{2}$ inches high, and 9 feet $2\frac{1}{2}$ inches wide. This represents the upper portion of an elliptical sewer used in the city of Philadelphia.

5. What limitation is there upon the values of A and B if the equation, $Ax^2 + By^2 + 2Gx + 2Fy + C = 0$, is to represent an ellipse?

6. Put the following equations in standard form, completing the square first and reducing to standard form by division. Time yourself.

a. $4x^2 + 9y^2 - 8x + 36y = 0.$
b. $3x^2 + 24x + y^2 - 6y - 43 = 0.$
c. $5x^2 - 17x + 10y^2 - 100 = 0.$
d. $5x^2 + 12y^2 - 117 = 0.$
e. $3x^2 - 24x + 4y^2 + 16y - 52 = 0.$

7. Plot the preceding five ellipses, choosing an appropriate scale. Plot the extremities of major and minor axes, the

extremities of the right focal chords, and one other point obtained by computation, together with the three points symmetrical to the computed point.

8. Determine a^2 and b^2 to one decimal place, in the following three ellipses:

a. $17x^2 + 43y^2 = 397$.
b. $5x^2 - 17x + 10y^2 - 35y = 0$.
c. $7(x-2)^2 + 3(y-3)^2 = 39$.

9. In the three ellipses immediately preceding determine ae, $\frac{b^2}{a}$, and $\frac{a}{e}$ to one decimal place.

10. In each ellipse of problem 8 determine x when $y = 2$.

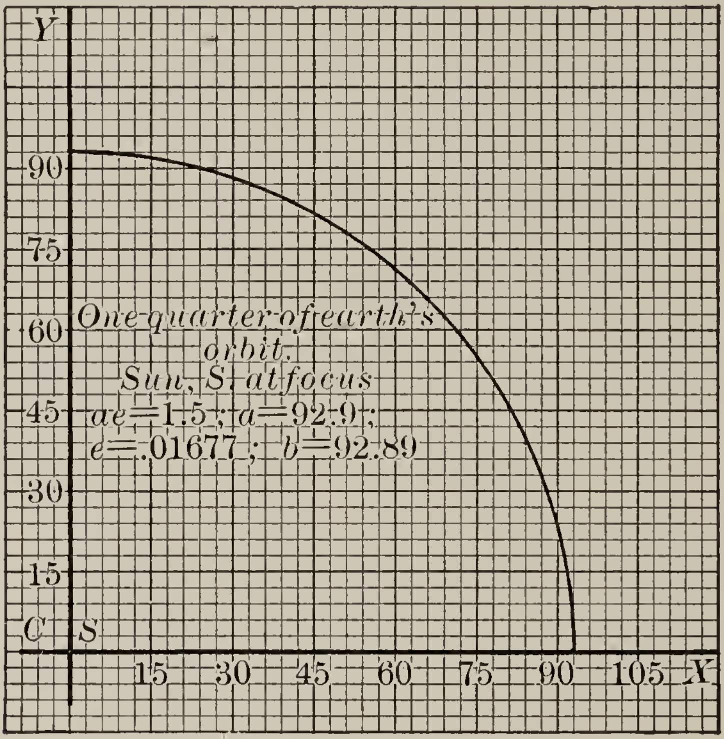
11. Using the data of problems 8, 9, and 10, plot the three ellipses of problem 8.

12. In the ellipse $\frac{x^2}{100} + \frac{y^2}{36} = 1$, find the foci. What is the distance of the point whose abscissa is $+3$ from each focus? of the point whose abscissa is 4, 5, 6, 7, x_1?

13. Put the following equations in standard form and discuss the curves:

a. $x^2 - 6x + y^2 + 8y - 10 = 0$.
b. $x^2 - 6x + 4y^2 + 8y + 11 = 0$.
c. $x^2 - 6x + 4y^2 + 8y + 13 = 0$.
d. $x^2 - 6x + 4y^2 + 8y + 14 = 0$.
e. $x^2 - 6x + 4y^2 + 8y + k = 0$.

14. The path of the earth about the sun is an ellipse with eccentricity .01677; this may be taken as $\frac{1}{60}$ in the following computations. If the major axis of the earth's orbit is 185.8 million miles, determine the focal distance, *i.e.* from the sun to the center of the path; determine also the minor axis. If a scale of one-half inch to fifteen million miles is taken, at what distance will the point representing the sun be from the center of the path? What will be the difference in length between major and minor axes?



The path of the earth about the sun

The distance of the sun from the center of the ellipse is represented by $\frac{1}{20}$ of an inch, on this diagram.

9. Tangent to ellipse of given slope. — (See page 225.) To find the tangent of slope 2, $y = 2x + k$ is solved as simultaneous with the ellipse equation. An equation whose roots are the abscissas of the two points of intersection is found and the condition is used that the two points of intersection be coincident.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$$y = 2x + k.$$

$$b^2x^2 + a^2(4x^2 + 4kx + k^2) - a^2b^2 = 0.$$

$$x^2(b^2 + 4a^2) + 4a^2kx + a^2k^2 - a^2b^2 = 0.$$

$$x = \frac{-2a^2k \pm \sqrt{4a^4k^2 - (b^2 + 4a^2)(a^2k^2 - a^2b^2)}}{b^2 + 4a^2}$$

$$= \frac{-2a^2k \pm \sqrt{a^2b^4 + 4a^4b^2 - a^2b^2k^2}}{b^2 + 4a^2}$$

$$= \frac{-2a^2k \pm \sqrt{a^2b^2(b^2 + 4a^2 - k^2)}}{b^2 + 4a^2}.$$

Put $b^2 + 4a^2 - k^2 = 0.$

$$k^2 = b^2 + 4a^2,$$

$$k = \pm\sqrt{b^2 + 4a^2},$$

$$y = 2x \pm \sqrt{b^2 + 4a^2}$$

are the two tangents of slope 2.

Similarly the tangent of slope m is obtained by solving

(1) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ and}$$

(2) $$y = mx + k$$

as simultaneous, and writing the condition for equal roots. Clearing (1) of fractions,

$$b^2x^2 + a^2y^2 - a^2b^2 = 0;$$

substituting from (2),

$$b^2x^2 + a^2(m^2x^2 + 2kmx + k^2) - a^2b^2$$
$$= (b^2 + a^2m^2)x^2 + 2ka^2mx + (a^2k^2 - a^2b^2) = 0.$$

$$x = \frac{-ka^2m \pm \sqrt{k^2a^4m^2 - (b^2 + a^2m^2)(a^2k^2 - a^2b^2)}}{b^2 + a^2m^2}.$$

Putting the discriminant equal to zero,

$$k^2 = b^2 + a^2m^2.$$

$$k = \pm\sqrt{b^2 + a^2m^2}, \text{ whence}$$

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

are the two tangents of slope m to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This method of obtaining the tangent applies to any curve given by an equation of the second degree.

10. Focal properties of the ellipse. — The perpendicular from the focus $(ae, 0)$ on any tangent of slope m meets it in the point, whose coördinates are found by solving the equations of these lines as simultaneous.

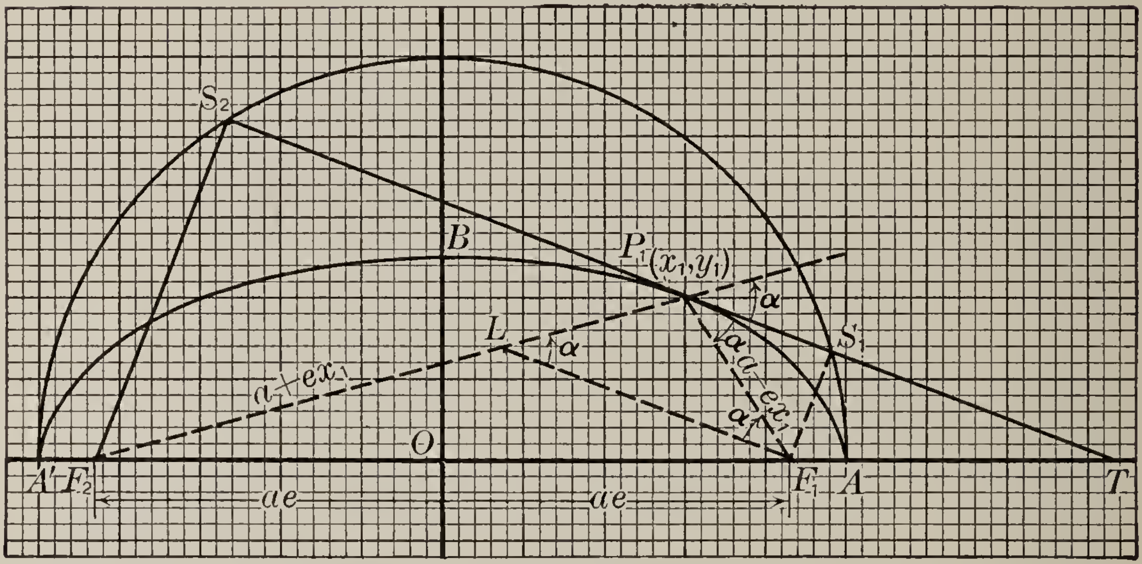
The equations, $y = mx + \sqrt{a^2m^2 + b^2}$, of the tangent,

and $y - 0 = -\frac{1}{m}(x - ae)$, of the perpendicular,

may be written

$$y - mx = \sqrt{a^2m^2 + b^2}.$$

$$my + x = ae.$$



The perpendicular from the focus upon any tangent to an ellipse meets it on the major auxiliary circle

The point of intersection satisfies both these equations; further it satisfies the equation obtained by squaring and adding both members of each of these equations:

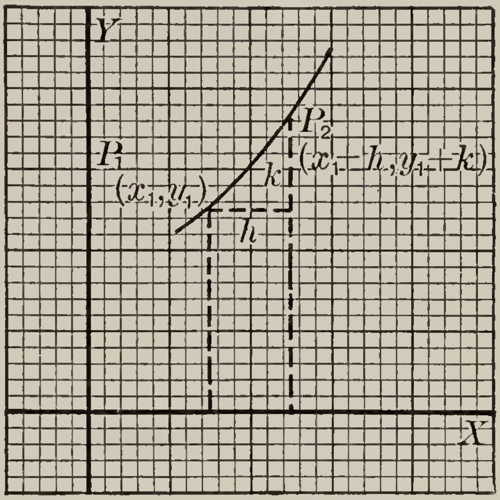
$$\begin{aligned}(1 + m^2)y^2 + (1 + m^2)x^2 &= a^2m^2 + b^2 + a^2e^2\\ &= a^2m^2 + b^2 + a^2 - b^2\\ &= a^2(1 + m^2).\end{aligned}$$

$$x^2 + y^2 = a^2;$$

hence the point of intersection of the perpendicular from the focus on any tangent lies on the major auxiliary circle.

Note that the above demonstration applies equally well to the perpendicular from the other focus $(-ae, 0)$ and equally well to the other tangent of slope m, $y = mx - \sqrt{a^2m^2 + b^2}$.

11. Tangent to an ellipse at a point $P_1(x_1, y_1)$ on the ellipse. — The method outlined is general, being applicable to any algebraic curve. The point $P_1(x_1, y_1)$ on the curve, considered as fixed during the discussion, is joined to a neighboring point $P_2(x_1 + h, y_1 + k)$ on the curve, which second point is then made to approach (x_1, y_1) *along the curve.* The slope of the chord joining P_1 to P_2, $\frac{k}{h}$, is found not to change indefinitely, but is found to approach a definite limiting value as h and k approach zero, *i.e.* as P_2 approaches P_1 along the curve. This limiting position of the chord is the tangent at P_1; this line can be shown in the case of the ellipse (or any curve given by an equation of the second degree) to cut the curve in two coincident points at (x_1, y_1) and in no other point.



$\frac{k}{h}$ is the slope of the chord P_1P_2

The method is applied in parallel columns

to a general problem and to a particular problem.

Tangent to the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad\qquad \frac{x^2}{25} + \frac{y^2}{4} = 1$$

at $P_1(x_1, y_1)$ on the ellipse. at $P_1(3, \frac{8}{5})$

Take the second point

$P_2(x_1 + h, y_1 + k)$ on curve $P_2(3 + h, \frac{8}{5} + k)$ on curve

Substituting,

$$b^2(x_1 + h)^2 + a^2(y_1 + k)^2 - a^2b^2 = 0.$$

$$4(3 + h)^2 + 25(\tfrac{8}{5} + k)^2 - 100 = 0.$$

$$b^2x_1^2 + 2\,b^2hx_1 + b^2h^2 + a^2y_1^2 + 2\,a^2ky_1 + a^2k^2 - a^2b^2 = 0.$$

But $b^2x_1^2 + a^2y_1^2 - a^2b^2 = 0.$

$$36 + 24\,h + 4\,h^2 + 64 + 80\,k + 25\,k^2 - 100 = 0.$$

Subtracting (and canceling)

$$2\,b^2hx_1 + b^2h^2 + 2\,a^2ky_1 + a^2k^2 = 0. \qquad 24\,h + 4\,h^2 + 80\,k + 25\,k^2 = 0.$$

$$\therefore\ k(a^2k + 2\,a^2y_1) = -h(2\,b^2x_1 + b^2h). \qquad k(80 + 25\,k) = -h(24 + 4\,h).$$

$$\frac{k}{h} = -\frac{2\,b^2x_1 + b^2h}{2\,a^2y_1 + a^2k}. \qquad \frac{k}{h} = -\frac{24 + 4\,h}{80 + 25\,k}.$$

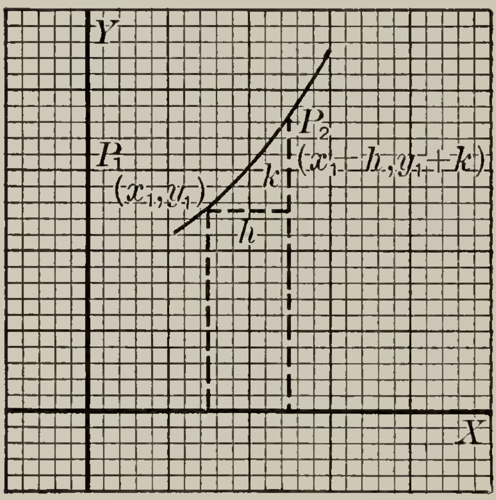
The chord P_1P_2 is given by:

$$y - y_1 = \frac{k}{h}(x - x_1). \qquad y - \frac{8}{5} = \frac{k}{h}(x - 3).$$

Since P_2 is on the curve, the chord equation may be written:

$$y - y_1 = -\frac{2\,b^2x_1 + b^2h}{2\,a^2y_1 + a^2k}(x - x_1). \qquad y - \frac{8}{5} = -\frac{24 + 4\,h}{80 + 25\,k}(x - 3).$$

Let P_2 approach P_1 along the curve; h and k both approach 0, *i.e.* can be made just as small as you please. Thus if h is made .01 in our numerical problem, k will be about $-.003$, which can be obtained by solving the quadratic



$\frac{k}{h}$ is the slope of the chord P_1P_2

$$25\,k^2 + 80\,k + 24\,h + 4\,h^2 = 0,$$

for k. It is evident that the constant here may be regarded as $24\,h + h^2$ and that as $h \doteq 0$, this constant approaches zero, and one root of k approaches zero.

Note that the second value of k approaches $-2\,y_1$, and corresponds to the fact that the given value of x, $x_1 + h$, is the abscissa of two points, on the upper and lower parts of the curve, respectively.

Since h and k both approach zero,

$$2\,b^2x_1 + b^2h \doteq 2\,b^2x_1 \qquad 24 + 4\,h \doteq 24$$

and $\qquad\qquad$ and

$$2\,a^2y_1 + a^2k \doteq 2\,a^2y_1 \qquad -80 + 25\,k \doteq -80$$

and the slope of the chord approaches more and more nearly, and as near as you may please to make it, by taking h (and thus k) small enough,

$\dfrac{-2b^2x_1}{2a^2y_1}.$	$-\dfrac{24}{80}.$

Hence the chord approaches a limiting position, given by

$y-y_1=-\dfrac{2b^2x_1}{2a^2y_1}(x-x_1),$	$y-\dfrac{8}{5}=-\dfrac{24}{80}(x-3),$

which may be written

$b^2x_1x+a^2y_1y=a^2y_1^2+b^2x_1^2$	$y-1.6=-.3(x-3)$
$=a^2b^2.$	or $y-1.6=-.3(x-3).$
$b^2x_1x+a^2y_1y-a^2b^2=0$	$10y+3x-25=0.$
or $\dfrac{x_1x}{a^2}+\dfrac{y_1y}{b^2}=1.$	$40y+12x-100=0.$

By precisely this method, step for step, the tangent to

$$Ax^2+By^2+2Gx+2Fy+C=0, \text{ is found to be}$$

$$Ax_1x+By_1y+G(x+x_1)+F(y+y_1)+C=0.$$

PROBLEMS

1. Find the tangents of slope $+2$ and of slope -3 to each of the ellipses in problem 6 of the preceding set of problems; the five problems, tangents of slope $+2$, should take not to exceed 30 minutes; note that after substituting $2x+k$ for y it is better procedure, surer and quicker, to combine terms by inspection rather than to expand each binomial before combining. This means to pick out the terms containing x^2, for example, and write the sum of these coefficients directly.

2. Draw at least three of the tangents of slope $+2$ in the preceding exercise, and three of slope -3, each to its conic as previously drawn. Find the point of tangency algebraically and graphically.

3. Without plotting the ellipse itself plot 12 tangents of slope 0, 1, 2, 3, 4, 5, 6, and 10, and of slope $-\frac{1}{2}$, -1, -3, and -6 to the ellipse $9x^2 + 25y^2 = 900$; note that these give a fair outline of the ellipse.

4. In each of the ellipses of problem 8, page 297, find the tangents at the point whose ordinate is 2, employing the results of problem 10 of the same set, and using the formula, $Ax_1x + By_1y + G(x + x_1) + F(y + y_1) + C = 0$.

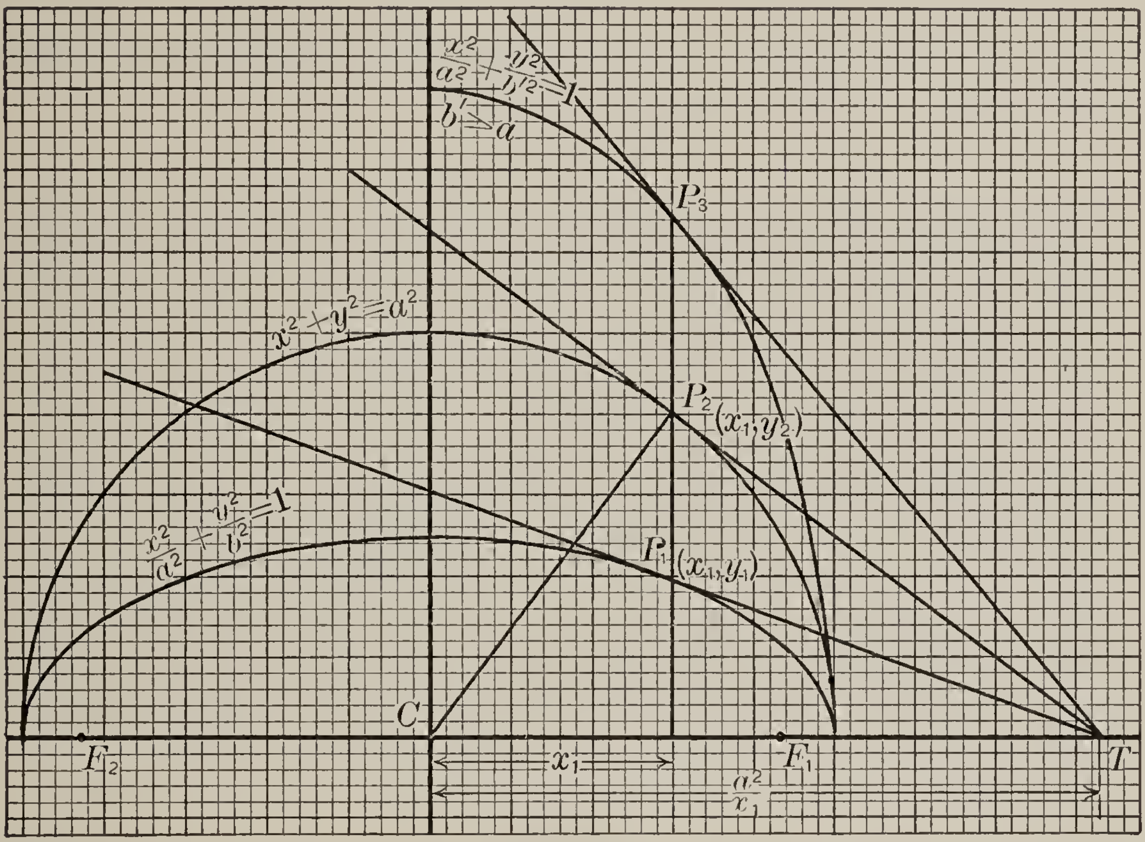
5. Derive by the method outlined in section 11 of this chapter the tangent to the ellipse $9x^2 + 25y^2 = 900$ at the point (8, 3.6) which is on the curve; at (– 6, 4.8) on the curve.

6. In the ellipse $9x^2 + 25y^2 = 900$, verify that the perpendicular from the focus upon any tangent to the ellipse meets it on the major auxiliary circle. Note that the converse is also true. This gives a method for drawing the tangent to an ellipse from a point outside the ellipse; explain.

12. The tangent to an ellipse at a point on the ellipse constructed from the tangent to the auxiliary circle. — Let (x_1, y_1) be any point on the ellipse; the tangent is $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$; the x-intercept, x_i, of this tangent is $\frac{a^2}{x_1}$, obtained by solving $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$ as simultaneous with $y = 0$. Evidently $x_i = \frac{a^2}{x_1}$ depends only upon x_1 and a, not involving b or y_1. Hence this value would be unchanged if b were taken equal to a. The tangent to the major auxiliary circle $x^2 + y^2 = a^2$ at (x_1, y_2) on the circle is $x_1x + y_2y = a^2$, and the intercept of this tangent on the x-axis is also $\frac{a^2}{x_1}$. This gives the following rule for drawing a tangent to an ellipse at any point on the ellipse:

Construct the major auxiliary circle to the given ellipse; find the point P_2 on the circle having the abscissa of the given point; at the point P_2 construct the tangent to the circle, cutting the

x-axis at T; the line joining T to P_1 on the ellipse is the tangent to the ellipse.



Tangent to an ellipse constructed from the tangent to the auxiliary circle a the corresponding point

Note that even if b is greater than a, this construction gives the tangent, but the circle $x^2+y^2=a^2$ is then the minor auxiliary circle of the given ellipse.

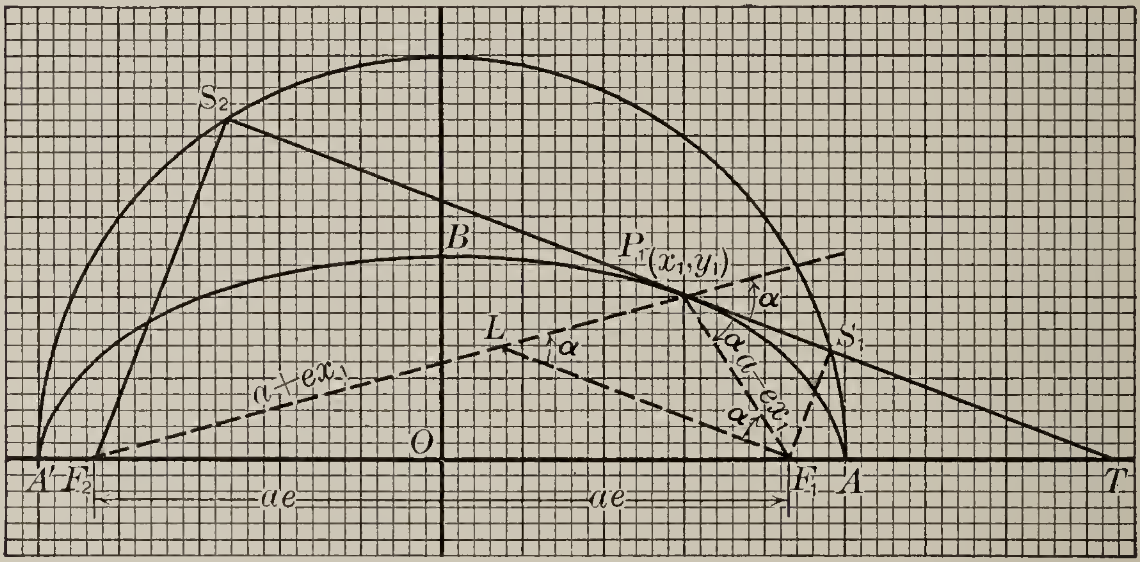
13. The tangent to an ellipse bisects the angle between the focal radii to the point of tangency. — Let $\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1$ be the tangent at (x_1, y_1) which intersects the x-axis at $T\left(\frac{a^2}{x_1}, 0\right)$.

$$OT=\frac{a^2}{x_1};\quad F_2T=\frac{a^2}{x_1}+ae=\frac{a}{x_1}(a+ex_1).$$

$$F_1T=\frac{a^2}{x_1}-ae=\frac{a}{x_1}(a-ex_1).$$

Hence,
$$\frac{F_2T}{F_1T}=\frac{a+ex_1}{a-ex_1}=\frac{P_1F_2}{P_1F_1}.$$

$\therefore P_1T$ bisects the exterior angle of the triangle $F_2P_1F_1$, since it divides the opposite side into segments proportional to the adjacent sides. The normal bisects the interior angle between the two focal radii; if the normal be drawn, each focal radius makes the same angle with it.



Tangent to an ellipse constructed from the focal radii to the point of tangency

Any ray of light or sound striking a reflecting surface is reflected in the plane of the normal to the surface and the original ray in such a way as to make the angle of incidence (*i.e.* between normal and original ray) equal to the angle of reflection. Hence rays starting from F_1 in our figure converge at F_2. This is the principle of "whispering galleries," in which the rays of sound starting from a point F_1 converge at another point F_2, making audible at F_2 whispers at F_1; at intermediate points the conversation may not be audible as there is no reënforcement by convergence.

PROBLEMS

1. In the ellipse $9x^2 + 25y^2 = 900$, give the two graphical methods for drawing the tangent at the point (6, 4.8) on the ellipse.

2. In the ellipse $25x^2 + 9y^2 = 900$, give the graphical methods for drawing a tangent at the point $(-4.8, 6)$ on the ellipse.

3. Construct the three ellipses:

$$\frac{x^2}{100} + \frac{y^2}{36} = 1.$$

$$\frac{x^2}{100} + \frac{y^2}{100} = 1.$$

$$\frac{x^2}{100} + \frac{y^2}{144} = 1.$$

Draw the tangent to each of these ellipses at the point whose abscissa is $+6$; find the equation of each of these tangents and prove that they intersect on the x-axis.

4. Find the equations of the two focal radii to the point (6, 4.8) on the ellipse $\frac{x^2}{100} + \frac{y^2}{36} = 1$; find the bisectors of the angles between these focal radii; find the bisector of the angle which does not include the origin, and prove that it coincides with the tangent at (6, 4.8) to the ellipse.

5. If an elliptical arch is to be in the form of the upper half of an ellipse, find the equation and plot ten points, given that the width of the arch is to be 100 feet and the height is to be 40 feet.

6. If the preceding arch is to have the dimensions as given, but is to be constructed as the upper quarter of a vertical ellipse, find the equation of the curve. Note that you have a point (50, 40) which is to satisfy the equation of the curve which can be written with only the denominator of x^2 as unknown. Compare this arch with the preceding one as to beauty of design.

7. Find the lengths of the ten vertical chords of the arch in problem 5, dropped from the tangent at the top of the arch and equally spaced horizontally.

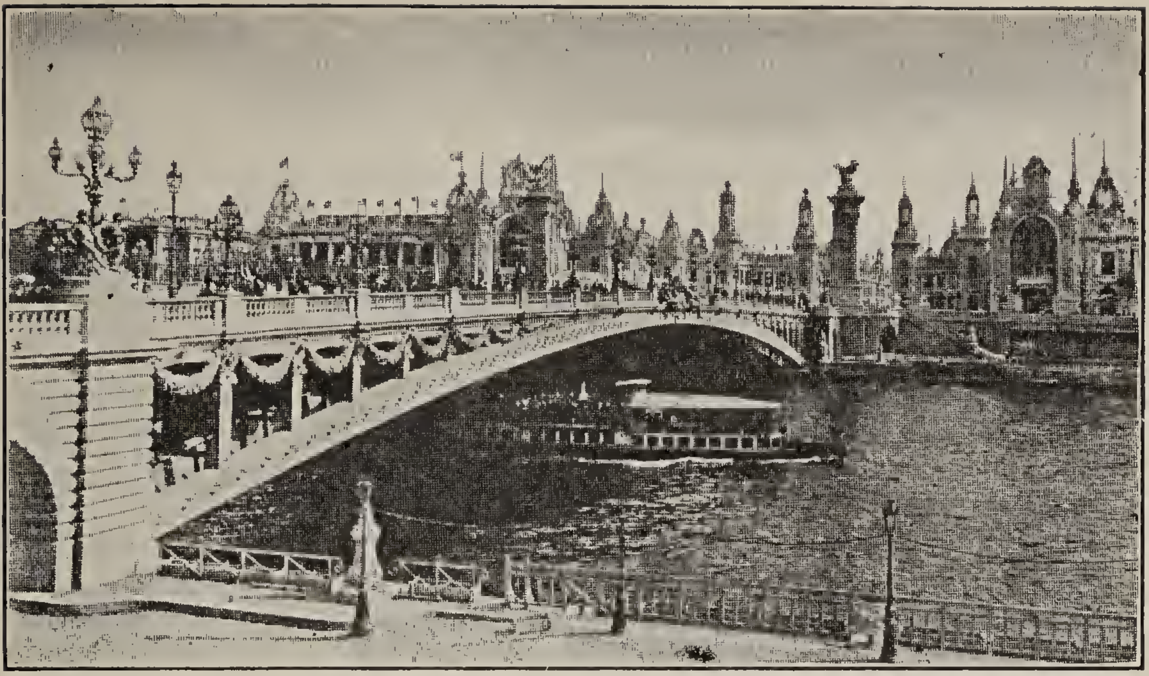
8. By the method of article 13, find the tangent to the curve given by the equation, $xy = 15$, at the point (3, 5) on the curve.

9. By the method of article 9, find the tangent of slope -2 to the curve given by the equation, $xy = 15$.

10. Write the equations of the three ellipses of problem 8, page 297, in parametric form.

11. In the ellipse, $\frac{x^2}{100^2} + \frac{y^2}{30^2} = 1$, find where lines from (50, 30) inclined to the horizontal axis at angles of 15°, 30°, 45°, and 60° respectively, cut the ellipse. Find the lengths of these lines from (50, 30). See problem 5.

12. If in the ellipse of problem 6, supporting chords are drawn diagonally between parallel vertical chords, computed in problem 7, each from the upper point of the right-hand chord to the lower point of the left-hand chord (on the right side of the ellipse), compute the lengths of these chords.



From Tyrrell's *Artistic Bridge Design*

Alexander III Bridge in Paris

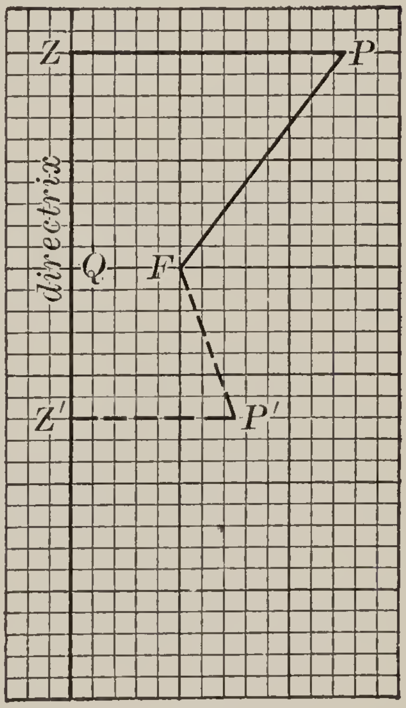
A parabolic arch ; span, 107.6 m.; rise, 6.75 m.; width, 40 m.

CHAPTER XIX

THE PARABOLA

1. Definition. — The ellipse has been defined (page 289) as the locus of a point which moves so that its distance from a fixed point, the focus, is in a constant ratio less than one to its distance from a fixed line, the directrix. If this constant ratio is taken equal to one, the curve generated by the moving point is called a parabola.

$PF = e \cdot PZ$, $e < 1$, defines an ellipse.
$PF = PZ$ defines a parabola.
$PF = e \cdot PZ$, $e > 1$, defines a hyperbola.



F, the focus; *ZZ′*, the directrix

A point equidistant from *F* and *ZZ′* moves on a parabola.

2. Equation of the parabola. — Through the focus draw the perpendicular *FQ* to the directrix ; take this line as *x*-axis. Take *FQ*, which is constant, as $2a$. On the axis chosen only one point is found which is on the given curve; the mid-point O, dividing the segment QF in the ratio 1 to 1, is such that its distance from F, the focus, equals its distance from the directrix. Through O take a perpendicular to OX as the y-axis. Evidently F is the point $(a, 0)$, and the directrix is the line $x + a = 0$.

Take $P(x, y)$ any point which is on the given curve, *i.e.* any point equally distant from F and from the line, giving

$$PF = PZ.$$

$PF = \sqrt{(x-a)^2 + y^2}$, distance between two points.

$PZ = x + a$, distance from a point to a line.

Note $\dfrac{x+a}{-1}$ gives the distance as negative; but it is not necessary to take account of the sign as in the simplification this expression is squared.

Equating, $PF = PZ$, gives

$$\sqrt{(x-a)^2 + y^2} = x + a.$$

$$x^2 - 2\,ax + a^2 + y^2 = x^2 + 2\,ax + a^2.$$

$$y^2 = 4\,ax.$$

3. Right focal chord. — When $x = a$, $y = \pm 2\,a$, giving the total length of the right focal chord as $4\,a$; the coefficient of x in $y^2 = 4\,ax$ represents the length of the right focal chord.

4. Geometrical properties. — The curve is symmetrical with respect to the x-axis, for, assigning any value to x, you find for y two values $\pm\sqrt{4\,ax}$; numerically equal but opposite in sign, or lying symmetrically placed with respect to the OX-line, which consequently is here the axis of the curve.

The y-axis, $x = 0$, is tangent to this curve since, solving,

$$y^2 = 4\,ax,$$
$$x = 0, \text{ as simultaneous,}$$

gives $y^2 = 0$; y equals zero twice, or the two points of intersection of $x = 0$ with $y^2 = 4\,ax$ are coincident. The point

(0, 0) is the point on the axis of symmetry, $y = 0$, which corresponds to itself; this point is called the vertex. The line $y = 0$ cuts the curve in only one finite point, given by $y = 0$, and $x = 0$; the other point of intersection of $y = 0$ with the curve is at an infinite distance.

Given a as positive, negative values of x lead to imaginary values of y. Hence all points on the curve lie to the right of the tangent at the vertex. As x increases without limit, y increases also without limit. This curve extends, we may say, to infinity. In plotting a parabola, the vertex, the extremities of the right focal chord, and at least two other points, should be plotted.

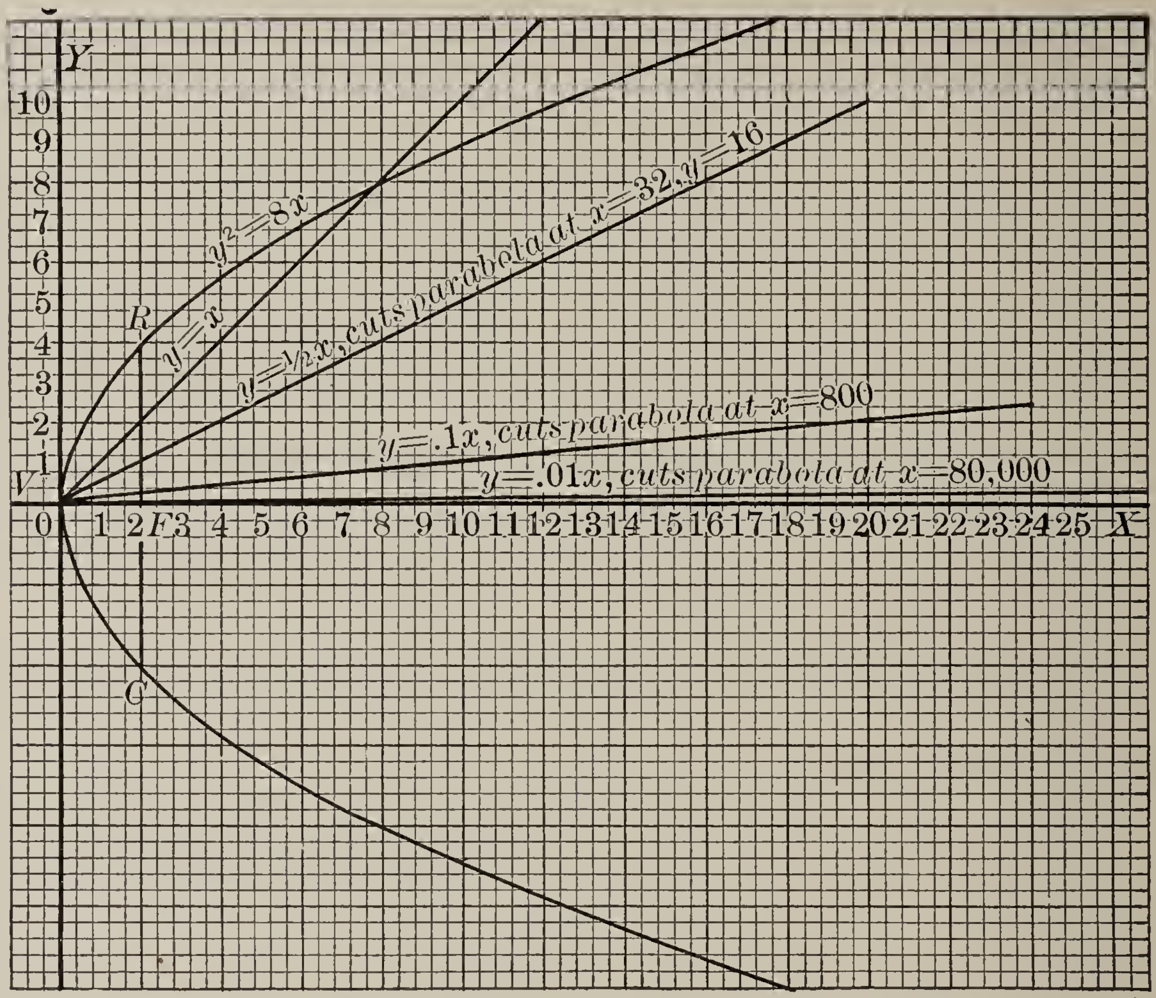
The quantity $y^2 - 4ax$ is evidently negative for points inside the curve, zero for points on the curve, and positive for points outside the curve.

5. Finite points and the infinitely distant point on the parabola. — To plot carefully $y^2 = 8x$, note that $4a = 8$, whence $a = 2$; indicate the vertex and, 2 units to the right, the focus; 4 units $(2a)$ above and below the focus locate the extremities of the right focal chord which has the length $4a$; take values of x at appropriate distances from the vertex and focus to give the portion of the parabola desired; in $y^2 = 8x$, $x = 0, 1, 2, 3, 4, 6$, and 8 give sufficient points to plot the curve for our purposes.

The parabola $y^2 = 8x$ is intersected by the line $y = x$ at (0, 0) and at (8, 8); $y = \frac{1}{2}x$ cuts this parabola at $x = 0$ and at $x = 32$; $y = .1x$ cuts at $x = 800$, $y = 80$. These values are obtained by solving the equation $y^2 = 8x$ as simultaneous with each of the linear equations.

Solving,
$$y = .01x$$
$$y^2 = 8x$$

gives $.0001x^2 = 8x$, $x = 0$ or $x = 80{,}000$. If one centimeter is taken as 1 unit, $y = .01x$ cuts the parabola $y^2 = 8x$ at the vertex and at a distance of 80,000 cm., nearly $\frac{1}{2}$ mile, from the vertex. As the line $y = mx$ moves nearer and nearer to the



A line parallel to the axis of a parabola cuts it in a point infinitely distant

axis $y=0$, the second point of intersection moves off farther and farther, without limit. This is the meaning of the expression that the axis of the parabola, and by similar reasoning any line parallel to the axis, "cuts the curve at an infinite distance."

The methods given above for plotting $y^2=8x$ apply to any parabola $y^2=4ax$.

PROBLEMS

Plot the following parabolas carefully:

1. $y^2=4x$, from $x=0$ to $x=8$.
2. $y^2=x$, from $x=0$ to $x=10$.
3. $y^2=12x$, from $x=0$ to $x=12$.
4. $y^2=\frac{1}{16}x$, from $x=0$ to $x=100$.

5. $s = 16t^2$, or $t^2 = \frac{1}{16}s$, taking OS as the horizontal axis and taking $\frac{1}{4}$ inch to represent 10 units of s (distance in feet) and $\frac{1}{4}$ inch to represent 1 unit of t (time in seconds). This gives the distance fallen from rest in time t by a freely falling body.

6. Find the intersections of $y = x$, $y = \frac{1}{3}x$, and $y = \frac{1}{10}x$ with the curves of problems 1, 2, and 3.

7. Solve $s = 800t$ with $s = 16t^2$; $s = 800t$ is the space covered by a body moving with uniform velocity 800 units per second. What is the physical meaning of the values obtained for the point of intersection?

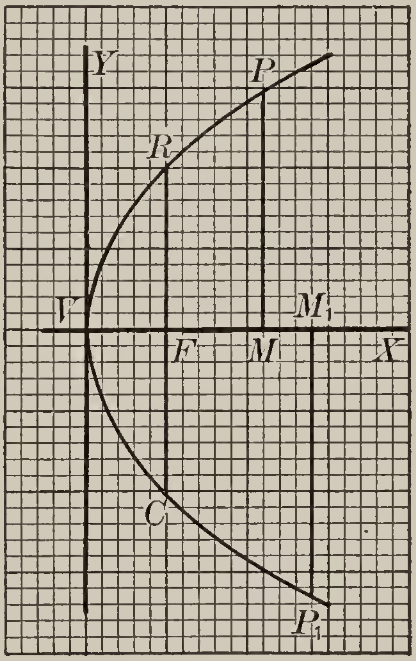
8. A mass rotated on a cord exerts a force of tension on the cord, $F = \frac{m \cdot v^2}{r}$. Given $m = 1$ pound, and $r = 10$ feet; draw the graph for velocities of 1 to 100 feet per second, taking F on the horizontal axis. Compute the corresponding number of revolutions per minute for $v = 10$, 20, 50, and 100 feet per second. What is the relation between v and n, where n is the number of revolutions per minute? Indicate on the vertical scale a second scale giving n.

6. Geometrical interpretation of $y^2 = 4ax$.

The equation $y^2 = 4ax$ may be interpreted geometrically as follows:

$$\overline{MP}^2 = 4a \cdot \overline{VM},$$

the square of the perpendicular from any point on a parabola to the axis is equal to the rectangle formed by the length cut off on the axis from the vertex by the perpendicular, with a constant line of length $4a$, the length of the right focal chord.

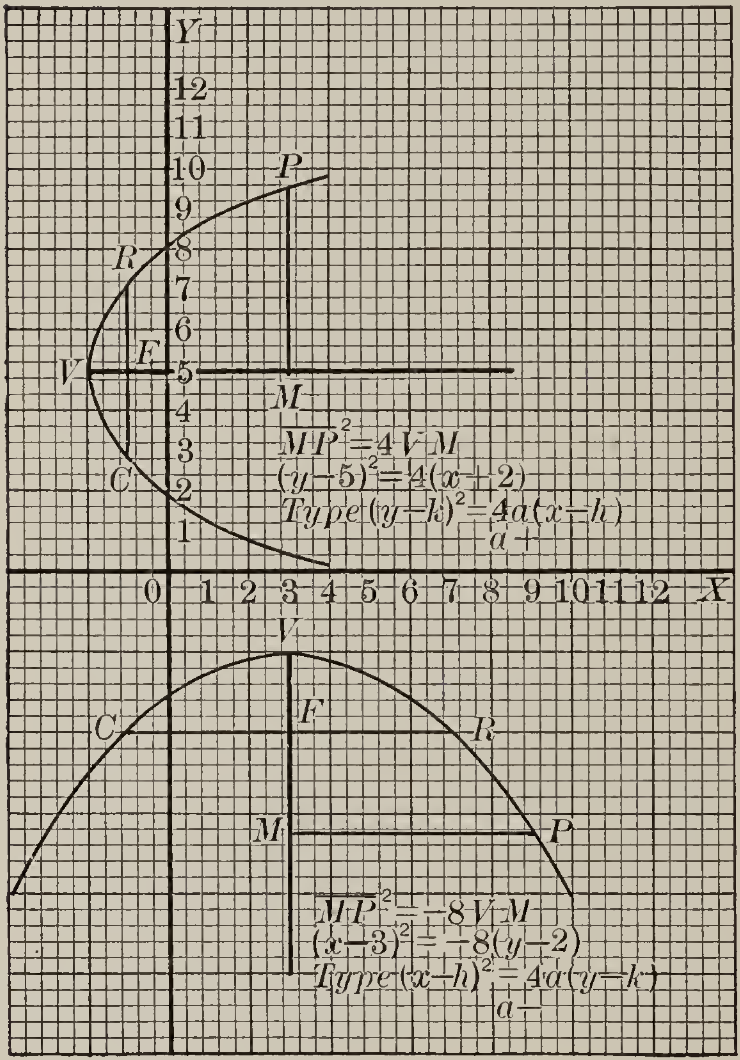


The square on PM equals the rectangle with VM and RFC as sides

7. Standard and limiting forms. — Given that the axis of a parabola is parallel to one of the coördinate axes, the relation,

$$\overline{MP}^2 = 4\,a \cdot VM$$

leads to 2 (or 4) standard forms of the parabola.



A horizontal and a vertical parabola

Thus in the upper figure a parabola is drawn, having $V(-2, 5)$ as vertex, $4\,a = 4$, axis parallel to the x-axis and opening to the right; the relation

$$\overline{MP}^2 = 4\,a \cdot VM$$

leads to

$$(y - 5)^2 = 4(x + 2).$$

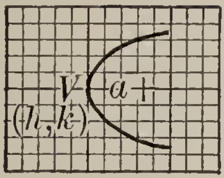
Were V the point (h, k), VM would be $x - h$ since it is the distance from a point whose abscissa is h to a point whose abscissa is x; similarly MP is $y - k$; now when the curve opens to the right VM is positive; hence the equation is $(y - k)^2 = 4\,a(x - h)$, with a positive. Were the axis parallel to the x-axis, but the curve opening to the left, the equation would be:

$$(y - k)^2 = 4\,a(x - h), \text{ with } a \text{ negative.}$$

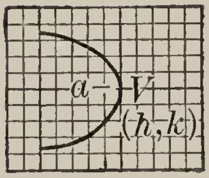
Similarly the curve on our figure which opens down is given by the equation $(x - 3)^2 = - 8(y - 2)$.

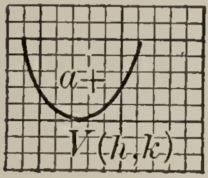
In the general case, $(x - h)^2 = 4\,a\,(y - k)$ has $V(h, k)$ as vertex; the axis is $x - h = 0$ and is parallel to the y-axis;

the curve opens up when a is positive and down when a is negative.

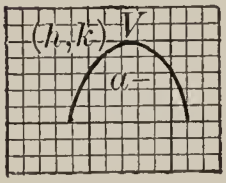


$$(y-k)^2 = 4\,a(x-h)$$





$$(x-h)^2 = 4\,a\,(y-k)$$



Standard forms of the parabola equation

As a approaches 0 in $y^2 = 4\,ax$, the parabola approaches more and more nearly to coincidence with the x-axis; two coincident straight lines constitute a limiting form of the parabola. As a becomes larger, the parabola approaches the y-axis.

8. Tangent of slope *m*.

$$y^2 = 4\,ax.$$

$$y = mx + k.$$

$$m^2x^2 + 2\,kmx + k^2 - 4\,ax = 0.$$

$$x = \frac{(2\,a - km) \pm \sqrt{(2\,a - km)^2 - m^2k^2}}{m^2},$$

whence, since $\Delta = 0$, $\qquad k = \frac{a}{m}.$

$y = mx + \frac{a}{m}$ is the tangent of slope m to $y^2 = 4\,ax$;

$\left(\frac{a}{m^2}, \frac{2\,a}{m}\right)$ is the point of tangency.

9. Tangent from an external point. — For any given point (x_1, y_1) outside the curve two values of m will be found for which $y = mx + \frac{a}{m}$ will pass through (x_1, y_1); hence there are, in general, two tangents which pass through a given point outside the curve.

$$x_1m^2 - y_1m + a = 0,$$

$$m = \frac{y_1 \pm \sqrt{y_1^2 - 4\,ax_1}}{2\,x_1}.$$

For points inside the curve, $y_1^2 - 4\,ax_1$ is negative, and there are no tangents.

10. Tangent at a point $(x_1,\ y_1)$ on the parabola. — By the method of article 9 of the preceding chapter the tangent to the parabola, $y^2 = 4\,ax$, at $(x_1,\ y_1)$ on the curve is found to be

$$y_1y = 2\,a\,(x + x_1).$$

Similarly the tangent to

$$By^2 + 2\,Gx + 2\,Fy + C = 0$$

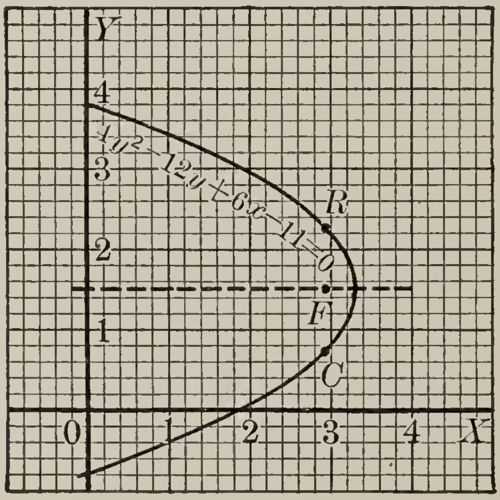
is found to be

$$By_1y + G(x + x_1) + F(y + y_1) + C = 0\,;$$

and with the x^2 term present, Ax_1x replaces in the above expression the term By_1y.

11. Illustrative example. — Put in standard form and plot carefully

$$4\,y^2 - 12\,y + 6\,x - 11 = 0,$$



$$4(y^2 - 3\,y \qquad) = -\,6\,x + 11.$$

$$4(y^2 - 3\,y + \tfrac{9}{4}) =$$

$$-\,6\,x + 11 + 9 = -\,6\,x + 20,$$

completing the square inside the parenthesis.

$$4(y - \tfrac{3}{2})^2 = -\,6(x - \tfrac{20}{6})$$

$$= -6(x - \tfrac{10}{3}).$$

$$(y - \tfrac{3}{2})^2 = -\tfrac{3}{2}(x - \tfrac{10}{3}).$$

$V(\frac{10}{3}, \frac{3}{2})$; type $a-$; axis $y - \frac{3}{2} = 0$;

$$4\,a = -\,\tfrac{3}{2};\ a = -\,\tfrac{3}{8}.$$

Plot V, F, RFC, and the further point where $y = \frac{7}{2}$;

$$(\tfrac{7}{2} - \tfrac{3}{2})^2 = 4 = -\,\tfrac{3}{2}(x - \tfrac{10}{3});$$

$$x - \tfrac{10}{3} = -\,\tfrac{8}{3};\ x = \tfrac{2}{3}.$$

Draw a smooth curve tangent to $x = \frac{10}{3}$, at $V(\frac{10}{3}, \frac{3}{2})$ through the points which are plotted; here it would be well to find from the original equation the intercepts on the axes.

PROBLEMS

Put in standard form and plot:

1. $4y^2 - 12y + 6x - 11 = 0.$
2. $4x^2 - 12x - 6y - 11 = 0.$
3. $4x^2 - 12x + 6y - 11 = 0.$
4. $y^2 - 6y - 8x - 5 = 0.$
5. $(y-3)^2 = 8x + 11.$
6. $y^2 = 6x + 11.$
7. $y^2 = \frac{1}{16}x.$
8. Solve graphically, to 1 decimal place,

$$x^2 + y^2 = 25$$

$y^2 = 8x$, by drawing both graphs to the same axes.

Put in standard form and plot the equations obtained in the two following problems:

9. The formula for the height of a bullet shot vertically upward with a velocity of 800 feet per second, $s = 800t - 16t^2$.

10. The formula for the time of beat, in seconds, of a pendulum is $t^2 = \frac{\pi^2}{g^2} \cdot l$, taking $g = 980$ and l measured in centimeters; taking $g = 32$, l must be measured in feet.

Compute corresponding values of t and l by logarithms, correct to 3 significant figures. Would the diagram be changed if g is taken as 982 instead of 980? At sea level on the equator $g = 978.1$ cm./sec.²; at Washington, 980.0; at New York, 980.2; at London, 981.2; or in feet/sec.² 32.09, 32.15, 32.16, and 32.19, respectively.

11. The Hell-Gate steel arch bridge in New York is one of the largest arch bridges in the world. See the illustration, p. 353. The lower arc of the arch is a parabola, 977.5 feet as span and 220 feet as height of the arch. Write the equation of the arc, taking as x-axis the tangent at the vertex of the parabola and as y-axis the axis of the parabola. Compute $4a$ to 1 decimal place. The roadway is 130 feet above the

base of the arch; compute the length of the roadway between the parabolic arcs. There are 23 panels or openings, spaced 42.5 feet apart at the centers; compute the vertical lengths to the roadway from the arc of the parabola, also to one decimal place. Compute the approximate length of the parabolic arch itself by computing the lengths of the twenty-three chords on the parabola; note that only 12 computations are necessary; do not carry beyond tenths of a foot, as hundredths would have little significance. Locate the focus and the directrix of this parabola.

12. Engineers use the following method for constructing a parabolic arch; show that it is correct. Suppose that it is desired to construct a parabolic arch of width 100 feet and height 30 feet; a rectangle 50 by 30 is drawn and the right-hand side is divided into 10 (or n) equal parts which are joined to the upper left-hand vertex of the rectangle (and parabola) by radiating lines; the upper horizontal side is also divided into 10 (or n) equal parts and ordinates are drawn at these points; corresponding lines intersect at points on the parabolic arch desired. Of the lines drawn the second (or rth) ordinate to the right of the vertex corresponds to the second (or rth) radiating line drawn from the vertex to the second (or rth) point of division from the top, on the right-hand side.

13. If an ordinary automobile headlight reflector is cut by a plane through its axis the section is a parabola having the light center as focus. If the dimensions of the headlight are 10 inches in diameter by 8 inches deep, locate the focus.

14. Locate the focus of a parabolic reflector, 6 inches in diameter and 4 inches deep; 5 inches deep; 6 inches deep.

15. The cable of a suspension bridge whose total weight is uniformly distributed over the length of the bridge takes the form of a parabola. Assuming that the cable of the Brooklyn bridge is a parabola, which it is approximately, find the equation in simplest form; the width between cable suspension points is about 1500 feet and the vertex of the curve is 140 feet, approximately, below the suspension points.

16. The Kornhaus Bridge over the Aar at Berne, Switzerland, has for central arch a parabola; the span is 384 feet and the height of the arch is 104 feet. If there are vertical columns spaced 24 feet apart, determine the length of these columns, assuming that the roadbed is 30 feet above the vertex of the parabola. If the floor of the roadbed is on a 2.7 per cent grade, determine the difference in elevation between the center of the bridge and the ends.

17. The parabolic reflector at the Detroit Observatory, University of Michigan, has a diameter, which corresponds to arch span, of 37.5 inches; the focal length of the mirror is 19.1 feet, from vertex to focus. Determine the height of the arch (or the depth of the reflector); determine the equation of the parabolic curve. The rays from a sun or star which strike this surface parallel to the axis of the parabola converge at the focus. What is the slope of this mirror at the upper point of the mirror? At the point whose abscissa is $\frac{1}{3}$ inch?

18. To draw a tangent to a parabola from an external point you can proceed as follows: take the external point as center, the focal distance as radius, and describe an arc cutting the directrix; from the point of intersection draw a line parallel to the axis; the intersection point with the parabola is the point of tangency. Prove this method.

19. Find the tangents of slope $+\frac{1}{2}$ and $-.3$ to each of the parabolas in exercises 1 to 7; time yourself.

20. Find the tangent to each of the parabolas in exercises 1 to 7 at the point on each parabola whose abscissa is $+2$; the exercise should be completed within thirty minutes. Show the geometrical method of working one of these problems.

21. Find the tangents to the parabola in problem 1 from the point (2, 10) outside the parabola. Describe a geometrical method of working this problem after the graph of the parabola is drawn.

22. Plot to scale with the dimensions given the fundamental parabola of the Alexander III Bridge.

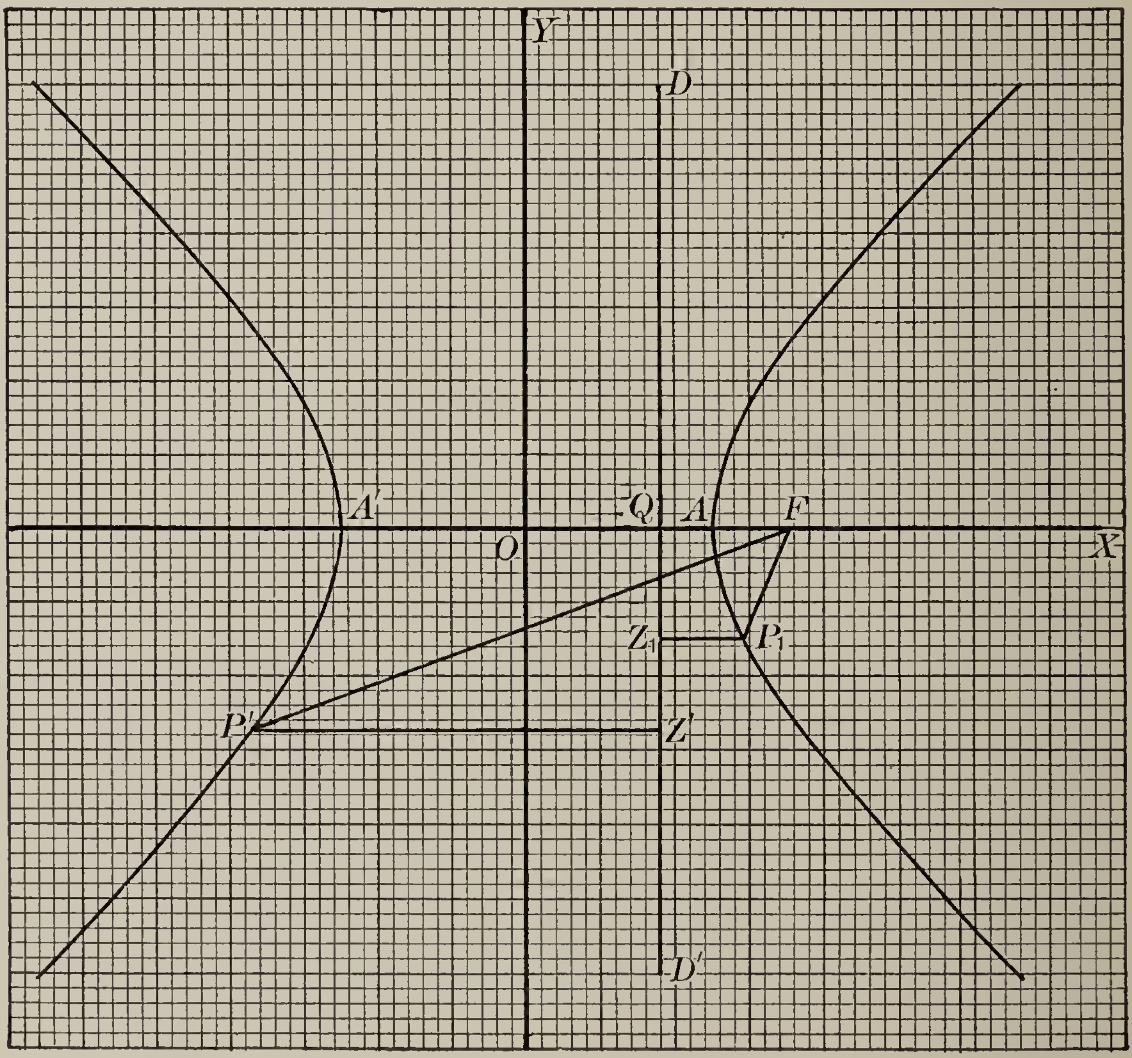
CHAPTER XX

THE HYPERBOLA

1. Definition and derivation of the equation. — (See ellipse, page 289, and parabola, page 309.)

$$PF = e \cdot PZ, \qquad e > 1.$$

Take FX perpendicular to $D'D$ as the x-axis, intersecting the given directrix at Q.



Let A and A' divide the segment QF internally and externally in the ratio e ($\frac{3}{2}$ in the figure). The mid-point of

AA', O is taken as the origin and the perpendicular through this point to $X'X$ as the y-axis, $OA = OA' = a$.

Precisely as in the ellipse,

$$AF = e \cdot AQ,$$

$$A'F = e \cdot A'Q;$$

whence

$$AF + A'F = e \cdot (AA').$$

$$AF + OA' + OF = 2\,ae,$$

$$2\,OF = 2\,ae,$$

$$OF = ae,$$

and, by subtraction,

$$OQ = \frac{a}{e}.$$

F is $(ae, 0)$; $D'D$ is $x - \frac{a}{e} = 0$.

The relation $PF = e \cdot PZ$ gives the equation,

$$\sqrt{(x - ae)^2 + y^2} = e\left(x - \frac{a}{e}\right).$$

$$x^2 + a^2e^2 + y^2 = e^2x^2 + a^2.$$

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2).$$

Up to this point the work is practically identical with the work in the case of the ellipse; here, however, $1 - e^2$ is negative, since $e > 1$. Hence we write this equation,

$$x^2(e^2 - 1) - y^2 = a^2(e^2 - 1).$$

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1.$$

Let

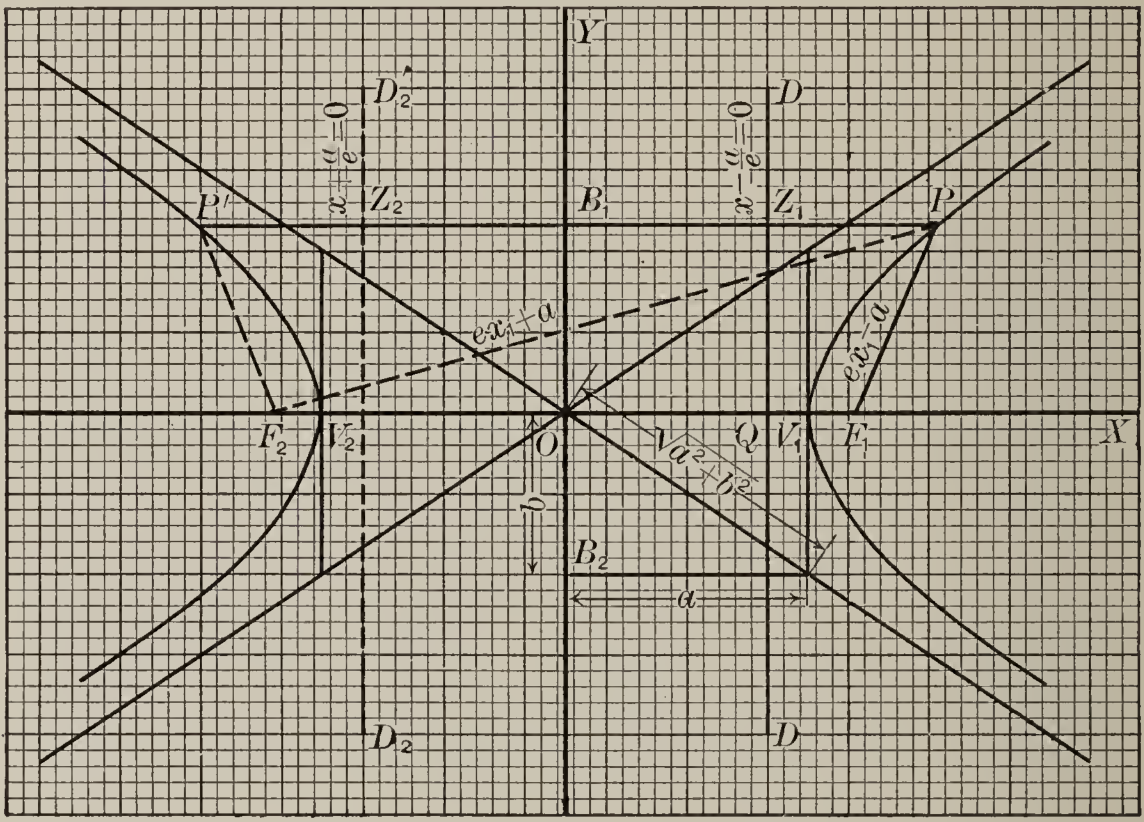
$$b^2 = a^2(e^2 - 1).$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

2. Geometrical properties of the hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. — Since y and $-y$ lead to the same values of x, the curve is symmetrical with respect to the x-axis; since x and $-x$ lead to the same values of y, the curve is symmetrical with respect to the y-axis. Hence the intersection of the two axes is the center of the curve.

Solving for y, $$y = \pm \frac{b}{a}\sqrt{x^2 - a^2};$$

this expression shows that the curve is symmetrical with respect to the x-axis, for any value of x gives two values of y equal in value but opposite in sign. Since any value of x numerically less than a gives imaginary values of y, the curve lies wholly outside the region bounded by $x + a = 0$ and $x - a = 0$. When $x = \pm a$, $y = 0$; these are self-corresponding points on the horizontal axis of symmetry; these points are called the vertices. When $y = 0$, x is imaginary; the vertical axis of symmetry does not intersect the curve.



Symmetry of the hyperbola

Solving for x, $x = \pm \frac{a}{b}\sqrt{y^2 + b^2}$; the curve is symmetrical with respect to the y-axis; every real value of y gives two corresponding real values of x, symmetrically placed with respect to the y-axis; the vertical axis of symmetry does not cut the curve in real points. As y increases in value, without limit, so do the two corresponding values of x increase in value, numerically without limit.

Since there is this vertical axis of symmetry it is evident, precisely as in the ellipse, that there is a second focus, $F_2(-ae, 0)$, and a corresponding directrix, $x + \frac{a}{e} = 0$.

The axis which cuts the curve is called the principal axis; the other axis is called the conjugate axis. The lines $x = \pm a$ are tangent to the curve at the vertices. See page 310.

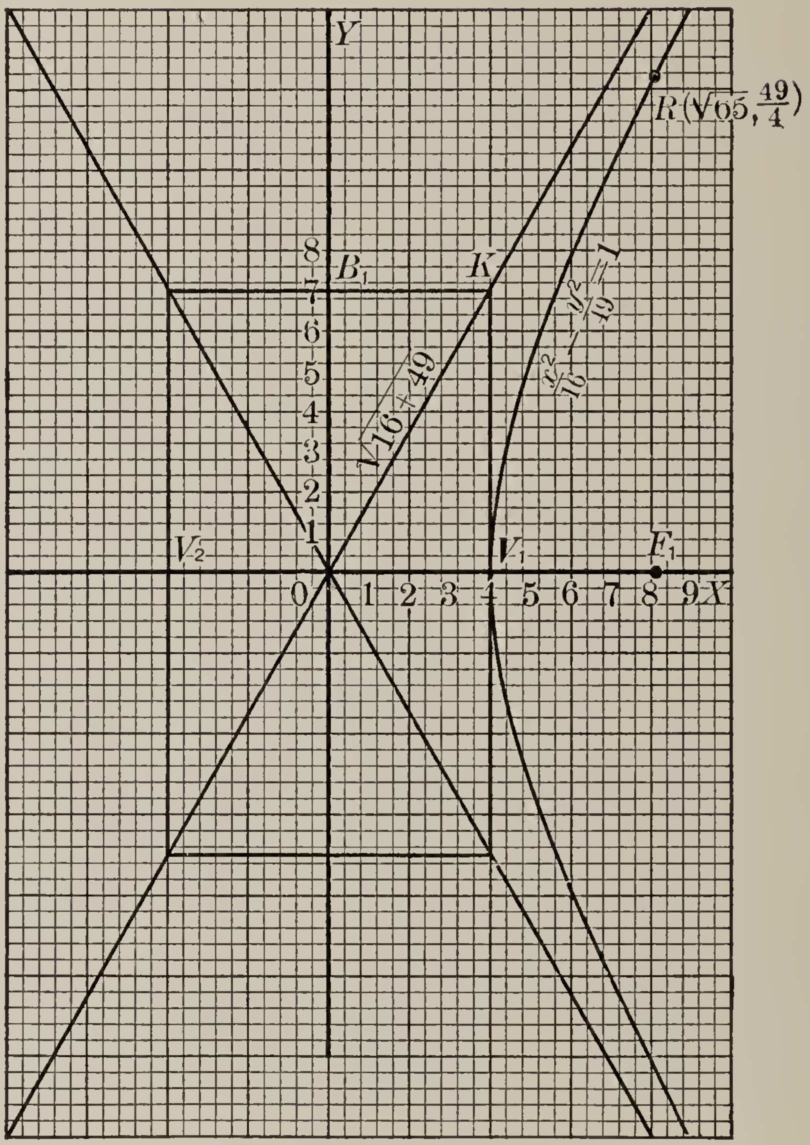
3. Right focal chords. — The foci are the points $(ae, 0)$ and $(-ae, 0)$; when $x = \pm ae$, $y = \pm \frac{b}{a}\sqrt{a^2e^2 - a^2} = \pm b\sqrt{e^2 - 1}$; but since $b^2 = a^2(e^2 - 1)$, these values of y equal $\pm \frac{b^2}{a}$; each right focal chord is of length $\frac{2\,b^2}{a}$, and is constructed by erecting at the focus lines perpendicular to the principal, or transverse, axis of length $\frac{b^2}{a}$ on each side of the axis.

The foci are at a distance $\pm ae$ from the center; now $b^2 = a^2(e^2 - 1)$ gives $ae = \sqrt{a^2 + b^2}$, which is the length of the diagonal of a rectangle of sides a and b.

4. Finite and infinitely distant points on the hyperbola. — Given to plot the hyperbola $\frac{x^2}{16} - \frac{y^2}{49} = 1$, note that $a^2 = 16$, $b^2 = 49$; $49 = 16(e^2 - 1)$, whence $e^2 - 1 = \frac{49}{16}$, $e^2 = \frac{65}{16}$, and

$$ae = \sqrt{49 + 16} = \sqrt{65}; \quad \frac{b^2}{a} = \frac{49}{4}.$$

It will be found convenient to draw the rectangle having O as center and extending 4 units to the right and left of O and 7 units above and below. The half-diagonal of this rectangle has the length $\sqrt{65}$, and may be used to locate on the principal axis the two foci.



Asymptotes and one branch of the hyperbola $\frac{x^2}{16} - \frac{y^2}{49} = 1$

The line $y = x$ cuts the curve at $\frac{x^2}{16} - \frac{x^2}{49} = 1$, $\frac{33\,x^2}{16 \times 49} = 1$;

$$x = \pm \frac{28}{\sqrt{33}} = \pm \frac{28\sqrt{33}}{33} = \pm 4\ 88.$$

The line $y = mx$ cuts the curve in two points, whose abscissas are given by

$$\frac{x^2}{16} - \frac{m^2x^2}{49} = 1, \text{ or } x^2 = \frac{49 \times 16}{49 - 16\,m^2}; \; x = \pm \frac{28}{\sqrt{49 - 16\,m^2}}.$$

As $16\,m^2$ approaches nearer and nearer to 49 these two points of intersection move farther and farther off; when $49-16\,m^2=0$, $m=\pm\frac{7}{4}$, the two points of intersection of each of these lines with the hyperbola move off to an infinite distance; the lines $y=\frac{7}{4}x$ and $y=-\frac{7}{4}x$ are called asymptotes of this hyperbola, intersecting the curve in two coincident points both at an infinite distance.

In the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the two lines $y=\frac{b}{a}x$ and $y=-\frac{b}{a}x$, or $\frac{x}{a}-\frac{y}{b}=0$ and $\frac{x}{a}+\frac{y}{b}=0$ are asymptotes; note that $\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{a}+\frac{y}{b}\right)$ gives the left-hand member of the equation of the hyperbola, when the right hand is unity.

5. Illustrative problems. — Plot the hyperbola

$$\frac{x^2}{16}-\frac{y^2}{49}=1.$$

The rectangle of sides 8 and 14, parallel to x- and y-axes respectively, is plotted with its center at the origin. As noted above, the diagonals of this rectangle give the distance from the center on the transverse axis, here horizontal, of the foci; the diagonals extended are the asymptotes, and to these lines the curve approaches more and more nearly as the curve recedes towards infinity. The right focal chords have the total length $\frac{2\,b^2}{a}$; plot $\frac{49}{4}$ vertically above and below the foci. Take $x=6$, this gives another point between vertex and right focal chord;

$$\frac{y^2}{49}=\frac{36}{16}-1\,;\; y=\pm\,7\sqrt{1.25}=\pm\,7(1.12)=\pm\,7.84\,;$$

take $x=10,\ y=\pm\,7\sqrt{5.25}=\pm\,7(2.29)=\pm\,16.03.$

Plot also the symmetrical points.

PROBLEMS

1. Plot the hyperbola $\frac{x^2}{49}-\frac{y^2}{16}=1.$

2. Plot the hyperbola $\frac{x^2}{100}-\frac{y^2}{100}=1.$

This type of hyperbola, $a=b$, is called an equilateral or rectangular hyperbola. Why?

3. Plot the hyperbola $\frac{x^2}{37} - \frac{y^2}{59} = 1$, computing values required to one decimal place.

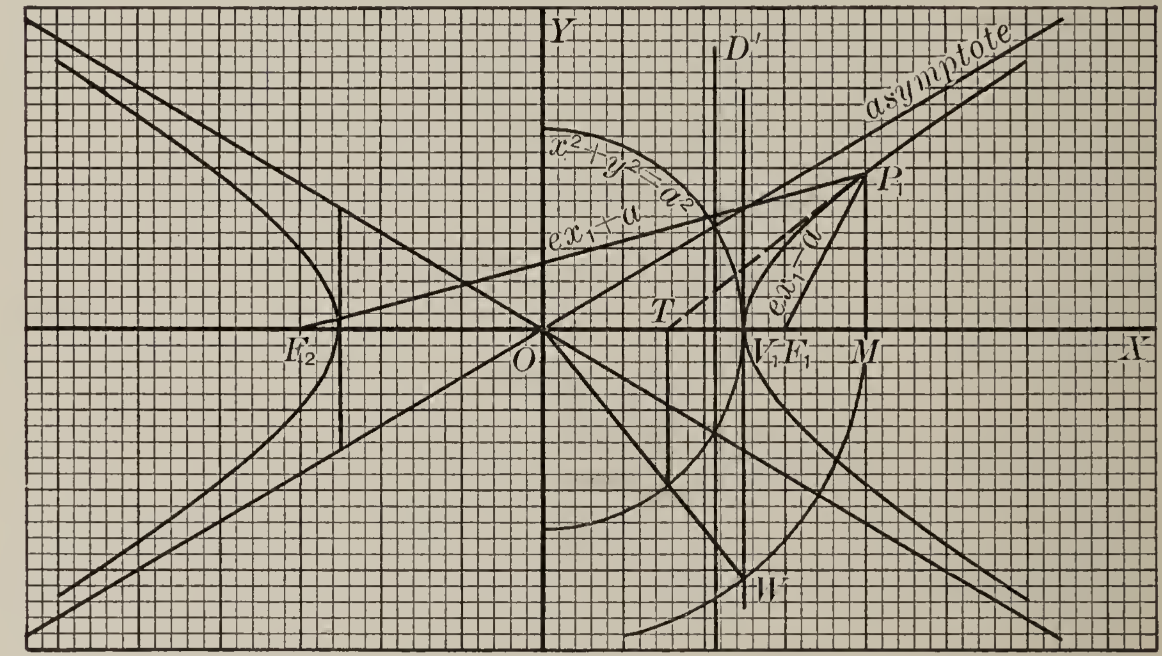
4. The equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ may be put in parametric form,

$$x = a \sec \theta$$
$$y = b \tan \theta$$

Noting that $\sec \theta$ is $\frac{1}{\cos \theta}$, find by using logarithms five points on each of the above hyperbolas. Check on the graphs drawn. What is the geometrical significance of θ?

5. Find the equations of the asymptotes of each of the preceding hyperbolas and find to 1′ the angle of inclination to the horizontal axis.

6. Compare the right-hand branch of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$, with the parabola $y^2 = \frac{81}{16}(x - 4)$; this parabola has the same vertex and passes through the extremities of the right focal chord. Prove this. Do these curves coincide in other points?



Focal distances to P_1 (x_1, y_1) on the hyperbola

6. Difference of the focal distances constant. — Designate the right focus by F_1 and the corresponding directrix by DD', and the left focus by F_2, having $D_2D'_2$ as the corresponding directrix. Then the focal distances PF_1 and PF_2 are $ex_1 - a$ and $ex_1 + a$, respectively; the difference, $PF_2 - PF_1 = 2a$, is constant.

The hyperbola may be defined as a curve generated by a point which moves so that the difference of its distances from two fixed points is constant.

7. Standard forms of the equation of the hyperbola. —

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,$$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1.$$

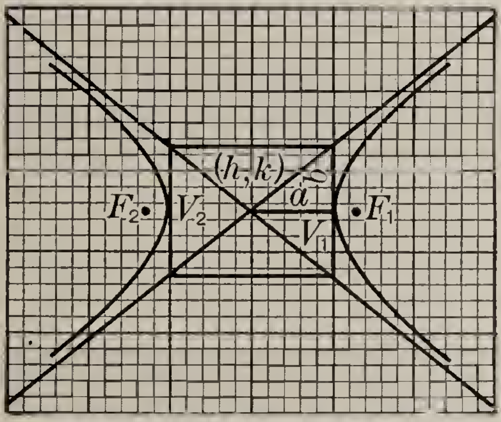
Precisely as in the ellipse, article 6, Chapter 18, the equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, may be interpreted

$$\frac{\overline{OM}^2}{\overline{OA}^2} - \frac{\overline{MP}^2}{\overline{OB}^2} = 1.$$

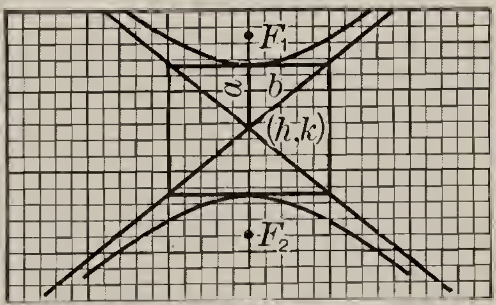
For any hyperbola whose axes of symmetry are parallel to the coördinate axes we obtain, from this relation, the equations

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1, \qquad \text{and} \qquad \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1,$$

for a horizontal hyperbola, for a vertical hyperbola.



Horizontal hyperbola



Vertical hyperbola

8. Conjugate hyperbolas and limiting forms of the hyperbola equation. — Given any hyperbola,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

the lines

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$$

represent the asymptotes; the equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$$

represents a vertical hyperbola about the same rectangle and having the same asymptotes. Any two hyperbolas so related are called conjugate hyperbolas.

Illustration.

$\frac{(x-3)^2}{16}-\frac{(y+2)^2}{49}=1$ and $\frac{(x-3)^2}{16}-\frac{(y+2)^2}{49}=-1$ are conjugate hyperbolas; the second is written in standard form $\frac{(y+2)^2}{49}-\frac{(x-3)^2}{16}=1$, wherein the focal distances $\pm ae$ from the center, and the distances $\pm\frac{a}{e}$ of the directrices from the center $(3,-2)$ are obtained, regarding a^2 as 49 and b^2 as 16.

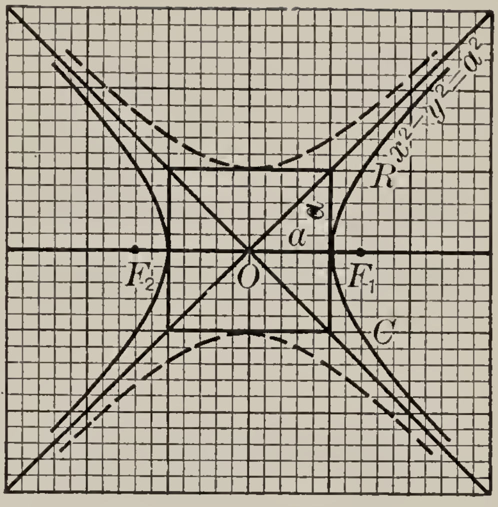
The asymptotes of these conjugate hyperbolas are given by the equation $\frac{(x-3)^2}{16}-\frac{(y+2)^2}{49}=0$, or by the equivalent in separate factors,

$$\frac{x-3}{4}-\frac{y+2}{7}=0 \text{ and } \frac{x-3}{4}+\frac{y+2}{7}=0.$$

The equation $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=0$, representing two real straight lines, is a limiting form of the hyperbola equation, $\frac{(x-h)^2}{a^2}-\frac{(y+k)^2}{b^2}=m$. As m approaches zero, $\frac{b}{a}$ remains constant and the hyperbola approaches more and more nearly the two straight lines $\frac{x-h}{a}-\frac{y-k}{b}=0$ and $\frac{x-h}{a}+\frac{y-k}{b}=0$.

9. The equilateral or rectangular hyperbola. — The hyperbola $\frac{x^2}{a^2}-\frac{y^2}{a^2}=1$ is called an equilateral hyperbola since $b=a$; it is also called a rectangular hyperbola as the asymptotes are at right angles to each other.

Since $b^2=a^2(e^2-1)$, the value of e in an equilateral hyperbola is $\sqrt{2}$ or 1.414; for $e>\sqrt{2}$, $b^2>a^2$; for $e<\sqrt{2}$, $b^2<a^2$.



The equilateral or rectangular hyperbola

10. Illustrative problem. — Put the equation

$$4x^2+16x-9y^2-18y-75=0$$

in standard form and plot the curve.

$$4(x^2+4x\qquad)-9(y^2+2y\qquad)=75$$
$$4(x^2+4x+4)-9(y^2+2y+1)=75+16-9$$
$$4(x+2)^2-9(y+1)^2=82$$
$$\frac{(x+2)^2}{20.5}-\frac{(y+1)^2}{9.11}=1.$$

The center is at $(-2,\ -1)$; the hyperbola is of horizontal type;

$$a^2=20.5 \text{ and } a=4.53\ ;\ b^2=9.11 \text{ and } b=3.02\ ;$$

$$ae=\sqrt{20.5+9.11}=\sqrt{29.61}=5.44\ ;\ \frac{b^2}{a}=\frac{9.11}{4.53};$$

further convenient points are given by $x=5$; substituting in the original is easiest, giving

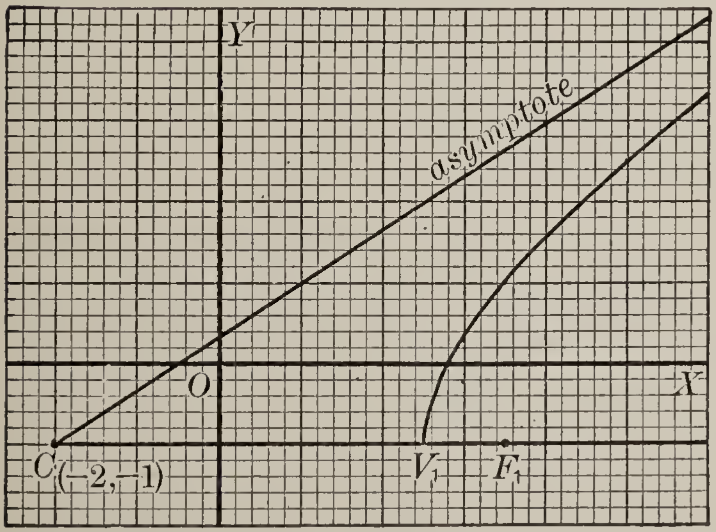
$$9y^2+18y-105=0\ ;$$

$$y^2+2y-\tfrac{105}{9}=0\ ;$$

$$y=-1\pm\sqrt{1+\tfrac{105}{9}}$$

$$=-1\pm\tfrac{1}{3}(10.68)$$

$$=-1\pm 3.56.$$



Upper portion of right-hand branch of the given hyperbola

PROBLEMS

1. Put the equation

$$4x^2 + 16x - 9y^2 - 18y + 107 = 0,$$

in standard form and plot.

2. Plot one quarter of the hyperbola

$$\frac{x^2}{147^2} - \frac{y^2}{59^2} = 1.$$

3. Plot a hyperbolic arch, width 200 feet, height 60 feet, as part of a rectangular hyperbola. Assume the equation $y^2 - x^2 = a^2$, and note that $(100, a + 60)$ is on the curve.

4. What limitation is there upon the values of A and B, if the equation $Ax^2 + By^2 + 2Gx + 2Fy + C = 0$ represents a hyperbola?

5. Any equation of the form

$$xy = k,$$

or

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = k,$$

has for its locus a hyperbola; the lines obtained by equating the left-hand member to zero are the asymptotes. Plot the hyperbolas $xy = 10$ and $(x - 3y)(x - 4y) = 50$.

6. Put the following equations in standard form, completing the square first and reducing to standard form by division.

a. $4x^2 - 9y^2 - 8x + 36y = 0.$

b. $3x^2 + 24x - y^2 + 6y - 43 = 0.$

c. $5x^2 - 17x - 10y^2 + 100 = 0.$

d. $5x^2 - 12y^2 - 117 = 0.$

e. $3x^2 - 24x - 4y^2 - 16y - 52 = 0.$

7. Plot the preceding five hyperbolas, choosing an appropriate scale. Plot the extremities of the conjugate axes; plot the rectangle and its diagonals; plot the extremities of the right focal chords; plot at least one further point, properly chosen to give the form of the curve, and its symmetrical points with respect to the axes. It is desirable to plot at least two of these curves completely; the remaining curves need be sketched only in the first quadrant.

8. Determine a^2 and b^2 to one decimal place in the following three hyperbolas:

a. $17x^2 - 43y^2 = 397.$

b. $5x^2 - 17x - 10y^2 - 35y = 0.$

c. $7(x-2)^2 - 3(y-3)^2 = 39.$

9. In the three hyperbolas immediately preceding determine ae, $\frac{b^2}{a}$, and $\frac{a}{e}$ to one decimal place.

10. In each hyperbola of problem 8 determine x when $y = 2$.

11. Using the data of the three preceding problems, plot the three hyperbolas of problem 8.

12. In the hyperbola $\frac{x^2}{64} - \frac{y^2}{36} = 1$, find the coördinates of the foci. What is the distance of the point whose abscissa is 12 from each of the foci? of the points whose abscissas are 10, 11, 15? State the general form for this distance.

13. Put the following equations in standard form and discuss the curves represented by these equations:

a. $x^2 - 6x - y^2 - 6y = 0.$

b. $x^2 - 6x - 4y^2 - 8y + 7 = 0.$

c. $\frac{(x-3)^2}{25} - \frac{(y+2)^2}{36} = 0.$

CHAPTER XXI

TANGENTS AND NORMALS TO SECOND DEGREE CURVES

1. The general quadratic in x and y. — The general equation of the second degree in x and y is written,

$$Ax^2 + 2\,Hxy + By^2 + 2\,Gx + 2\,Fy + C = 0.$$

The equations of the circle, parabola, ellipse, and hyperbola are special types of this general equation. Since none of these standard forms, representing curves of the second degree with axes of symmetry parallel to the coördinate axes, have an xy term, we find it convenient to discuss the general equation, with $H = 0$, or

$$Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0.$$

This represents one of the curves — circle, ellipse, parabola, or hyperbola — mentioned above, or some limiting form of the same, including pairs of lines and imaginary types. It can be shown that

$$Ax^2 + 2\,Hxy + By^2 + 2\,Gx + 2\,Fy + C = 0$$

represents no new curve; simply one of the above-mentioned types turned at an angle to the coördinate axes.

2. General equation of the second degree represents a conic section. — Given a right circular cone, it can be shown by the geometrical methods of Euclidean geometry, that the section which is made with the surface of the cone by any plane is one of the curves above mentioned; thus a plane parallel to the base cuts the cone in a circle, or in a point circle if through the vertex.

The cone is conceived as the whole surface determined by the straight line elements of the cone produced to infinity.

A plane which runs parallel to only one of the elements cuts the cone in a parabola, or in two coincident lines if the plane passes through an element and a tangent to the circular base of the cone.

A plane which cuts all the elements in finite points cuts the cone in an ellipse; this is a point ellipse when the plane passes through the vertex of the cone.

A plane which cuts the cone parallel to the plane of two elements cuts it in a hyperbola; if the plane passes through the vertex the hyperbola reduces to two straight lines.

3. Historical note on conic sections. — The fundamental properties of conic sections were discovered by Greek mathematicians nearly two thousand years before the invention of analytical geometry which was perfected by Descartes and Fermat, French mathematicians of the seventeenth century. A treatise on conics was written by Euclid (c. 320 B.C.), but it was entirely superseded a century later by a treatise by Apollonius (c. 250 B.C.) of Perga, whose treatise included most of the fundamental properties which we discuss. The properties of the parabola connected directly with focus and directrix are not included in the eight books (chapters) on conic sections by Apollonius, nor was the directrix of the central conics employed by him. Pappus of Alexandria (c. 300 A.D.), almost the last of the Greek mathematicians of any note, included these in his *Mathematical Collections.*

The Greek mathematicians were interested in these curves for the pure geometrical reasoning involved. That the paths of the planets were conics they did not know; nor did they know any practical applications of these conics. However, the fact that Greek mathematicians had studied these properties made it possible for John Kepler and Isaac Newton to establish the laws of movement of the planets in the universe in which we live. The men mentioned and Nicolas

Copernicus, who reasserted the heliocentric theory of the universe, were all thoroughly versed in the pure geometry of the Greeks; their new theories were built directly upon this foundation of pure geometry.

4. Tangent of slope *m* to a second degree curve. — Any line $y = mx + k$ cuts a curve given by an equation of the second degree in two real points, or in two imaginary points, or in two coincident points. The abscissas of the points of intersection are given by the quadratic in x obtained by substituting $y = mx + k$ in the equation of the curve; the two equations are solved as simultaneous equations. The condition for tangency is that the two points of intersection of the line with the curve shall be coincident; this will be the case when the two roots of the quadratic in x, *i.e.* the two values of the abscissas of the points of intersection, are equal.

When the intersections of a line with a quadratic curve are given by a linear, instead of a quadratic, equation, the meaning is that one point of intersection has moved off to an infinite distance. As the coefficient of the square term of a quadratic approaches zero one root becomes larger and larger without limit; see page 98.

Parabola, $y^2 = 4\,ax,$

$$y = mx + k.$$

Solving, $$x = \frac{-(km - 2\,a) \pm \sqrt{4\,a(a - mk)}}{m^2}.$$

For equal roots, or coincident points, $k = \dfrac{a}{m}.$

A. $y = mx + \dfrac{a}{m}$ is tangent to $y^2 = 4\,ax$ at $\left(\dfrac{a}{m^2},\ \dfrac{2\,a}{m}\right).$

Ellipse, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, and the line $y = mx + k.$

Solving, $$x = \frac{-\,a^2km \pm \sqrt{a^2b^2(a^2m^2 + b^2 - k^2)}}{b^2 + a^2m^2}.$$

B. $y = mx \pm \sqrt{a^2m^2 + b^2}$ is tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ at } x = \mp \frac{a^2m\sqrt{a^2m^2 + b^2}}{a^2m^2 + b^2},\ y = \frac{\pm\, b^2}{\sqrt{a^2m^2 + b^2}}.$$

For every value of m there are two real tangents to an ellipse. Similarly

C. $$y = mx \pm \sqrt{a^2m^2 - b^2}$$

is tangent to the hyperbola,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ at } x = \pm \frac{a^2m}{\sqrt{a^2m^2 - b^2}}.$$

For values of $|\,m\,| > \frac{b}{a}$, there are two real tangents to a hyperbola; for $|\,m\,| < \frac{b}{a}$ the tangents are imaginary; for $m = \pm \frac{b}{a}$, there is only one tangent and its point of tangency is at an infinite distance, or, as noted before, the lines $y = \pm \frac{b}{a}x$ are asymptotes of the curve.

The method of this article is employed in deriving tangents; the equations given under *A*, *B*, and *C* above are used mainly in proving geometrical properties of these curves. Note that if these equations are used as formulas, they apply only to curves of the type given; $y = mx + \frac{a}{m}$ gives the tangent only to a parabola of the form $y^2 = 4\,ax$ (a may be positive or negative). Similarly, $y = mx \pm \sqrt{a^2m^2 + b^2}$ gives the tangent of slope m only to the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ (a^2 \text{ may be less than } b^2).$$

PROBLEMS

Find tangents of slope 2 and of slope -3 to the following three curves. Follow the method of article 4.

1. $x^2+y^2-10x=0$; find the points of tangency.

2. $3y^2-4x-6y=0$; find also the normal of slope $-\frac{1}{2}$.

3. $x^2+3y^2-4x-6y=0$; find the diameter joining the two points of tangency.

4. $xy-25=0$. Find the tangents of slope -2, and the points of tangency. Find the tangent of slope m. For what values of m are the tangents imaginary? Plot 10 points on this curve. Where do all points of this curve lie? Find the intersections of $y=.01x+5$ with this curve.

5. Find the tangents at the extremities of the right focal chord of $y^2=8x$; where do they intersect? Similarly in $y^2=4ax$. Is this true of any parabola? Explain.

6. Find the tangents at the extremities of either right focal chord of

$$\frac{x^2}{25}+\frac{y^2}{9}=1;$$

where do they intersect? Similarly with

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ at } \left(ae, \frac{b^2}{a}\right).$$

Where do these tangents intersect? What change is necessary to prove this property for the hyperbola? Explain.

7. Find the perpendicular from the focus of $y^2=8x$ to the tangent of slope 2; where do they intersect? Similarly for the tangent of slope m. State what you have found as a property of any parabola.

8. In the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$, find the perpendicular from the focus to the tangent of slope 2; find the point of intersection; note that it is a point on the circle, $x^2+y^2=25$. Prove that

the same is true of the perpendicular from the focus upon any tangent of slope m. Prove that in the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

the perpendicular from the focus upon any tangent meets it on the major auxiliary circle.

9. Follow the directions of problem 8 with the hyperbola,

$$\frac{x^2}{16}-\frac{y^2}{9}=1,$$

and

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

making necessary changes.

10. Find the angle between $y=5x-7$ and the parabola $y^2=8x$.

The angle between a straight line and a curve is defined to be the angle between the straight line and a tangent to the curve at the point of intersection.

Note. — Solve for the points of intersection; write the equation of tangent at each point of tangency; find the angle between each tangent and the line $y=5x-7$. See article 8, chapter 15. Check by finding the slope angle of the lines $y=5x-7$ and of the tangent lines.

11. Find the angle between $y=2x-5$ and $x^2+y^2=100$, at each point of intersection. Check as in problem 10.

5. Tangent at a point (x_1, y_1) on a curve given by an algebraic equation. — On any curve a line joining a point P_1 to a point P_2 is called a secant; obviously this secant cuts the curve in two distinct points. If P_2 approaches P_1 along the curve, the secant changes, approaching more and more nearly, in general, a definite limiting line. The limiting position of the secant is called the tangent to the curve at P_1.

The analytical method of obtaining this limiting line is as follows:

Take P_1 (x_1, y_1) any point on the curve;

Take P_2 as $(x_1 + h, y_1 + k)$ also on the curve; the chord P_1P_2 has the equation $y - y_1 = \frac{k}{h}(x - x_1)$.

Find the value of $\frac{k}{h}$ conditioned by the fact that P_2 and P_1 both lie on the curve, by substituting (x_1, y_1) and $(x_1 + h, y_1 + k)$ in the given equation and subtracting, member for member.

This value of $\frac{k}{h}$ will be found, in general, to have a definite limiting value as k and h approach zero; this limiting value is the slope of the tangent.

The method outlined applies to any curve given by an algebraic equation.

$y^2 = 4\,ax$; $P_1(x_1, y_1)$ on curve; $P_2(x_1 + h,\ y_1 + k)$ on curve;

$$y - y_1 = \frac{k}{h}(x - x_1), \text{ chord joining } P_1P_2.$$

$(y_1 + k)^2 = 4\,a(x_1 + h)$, or $y_1^2 + 2\,ky_1 + k^2 = 4\,ax_1 + 4\,ah$, since P_2 is on curve.

$$y_1^2 = 4\,ax_1, \text{ since } P_1 \text{ is on curve.}$$

$$2\,ky_1 + k^2 = 4\,ah, \text{ by subtraction.}$$

$\frac{k}{h} = \frac{4\,a}{2\,y_1 + k}$ gives the slope of the chord joining P_1 to P_2.

Let h approach 0, k also approaches 0, but $\frac{k}{h}$ always equals $\frac{4\,a}{2\,y_1 + k}$ and this value approaches more and more nearly to $\frac{4\,a}{2\,y_1}$ or $\frac{2\,a}{y_1}$ as a limit; this limit is the slope of the tangent.

$$y - y_1 = \frac{2\,a}{y_1}(x - x_1) \text{ is the tangent to}$$

$$y^2 = 4\,ax \text{ at } (x_1, y_1) \text{ on curve.}$$

This equation may be simplified,

$$y_1y - y_1^2 = 2\,ax - 2\,ax_1,$$

$$y_1y = 2\,ax + y_1^2 - 2\,ax_1; \text{ but } y_1^2 = 4\,ax, \text{ whence}$$

$$y_1y = 2\,a(x + x_1), \text{ which is the tangent equation.}$$

By precisely this method, the tangent to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ at } (x_1, y_1)$$

on curve has been found, in section 11, chapter 18, to be

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1.$$

The tangent to

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ at } (x_1, y_1) \text{ on curve is } \frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1.$$

The tangent to

$$Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0, \text{ at } (x_1, y_1) \text{ on curve is}$$

$$Ax_1x + By_1y + G(x + x_1) + F(y + y_1) + C = 0.$$

The tangent to

$$Ax^2 + 2\,Hxy + By^2 + 2\,Gx + 2\,Fy + C = 0, \text{ at } (x_1,y_1) \text{ on curve is}$$

$$Ax_1x + H(x_1y + y_1x) + By_1y + G(x + x_1) + F(y + y_1) + C = 0.$$

All the preceding are embraced in the last formula, as special cases. The final form should be remembered and used as a formula.

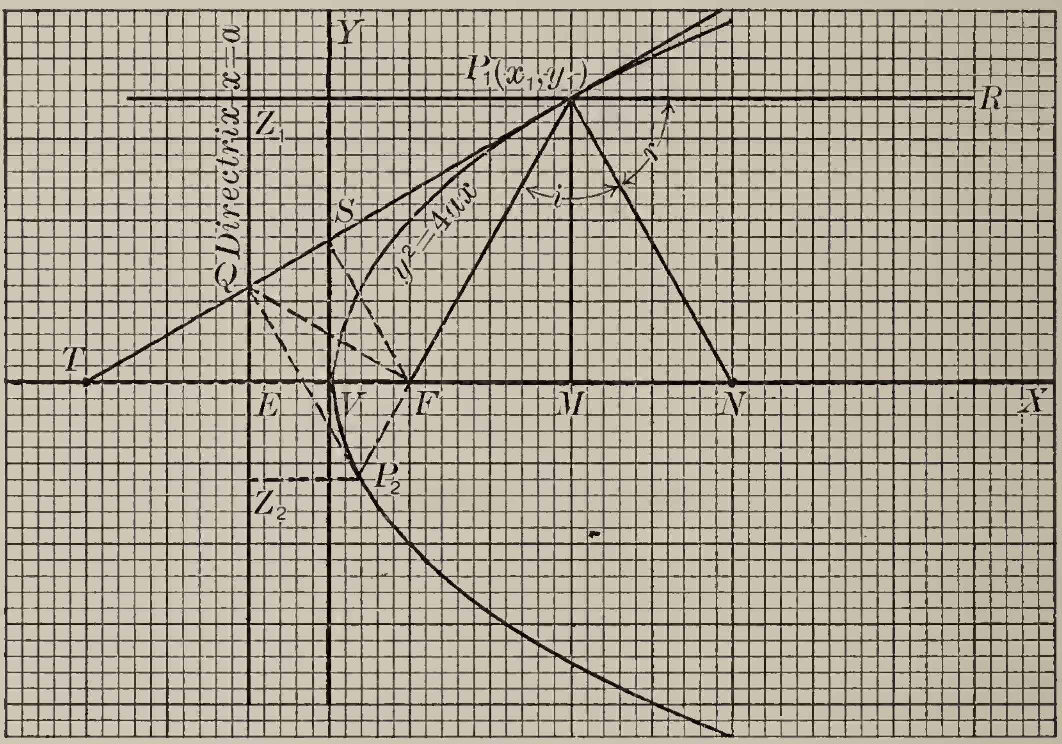
The above special forms for the equations of the tangents to conics given by equations in standard form may be used to derive tangential properties of these curves. Some of these properties are touched upon in the problems below and will recur in the next chapter.

6. Tangential properties of the parabola. —

$$y^2 = 4\,ax, \text{ any parabola } P_1VP_2.$$

$y_1y = 2\,a(x + x_1)$, tangent at $P_1(x_1, y_1)$ on curve, cutting the axis at T, the directrix at Q, the vertex tangent at S.

$y - y_1 = -\dfrac{y_1}{2\,a}(x - x_1)$, normal P_1N, cutting the axis at N.



Tangential properties of the parabola

T and N are obtained as the x-intercept of tangent and normal, respectively;

$$T \text{ is } (-x_1, 0)\,;\ N \text{ is } (x_1 + 2\,a, 0).$$

Hence $VT = VM$, the tangent cuts off from the vertex on the axis the same distance that the perpendicular from the point of tangency on the axis cuts off from the vertex. This gives a simple method of drawing a tangent to a parabola: drop the perpendicular P_1M to the axis; extend the axis from the vertex, making $VT = VM$, the length of the intercept; P_1T is the tangent.

$$P_1F = P_1Z = x_1 + a.$$
$$TF = x_1 + a = FN.$$

Hence $\angle FTP_1 = \angle FP_1T$, base angles of an isosceles $\triangle$.

Now $\angle FTP_1 = \angle TP_1Z$, alternate interior angles of parallel lines, etc. $\angle FP_1T = \angle TP_1Z$, *i.e.* the tangent bisects the angle between a focal chord and a line parallel to the axis; the normal P_1N bisects the supplementary angle FP_1R, making $\angle FP_1N = \angle NP_1R$.

Further S is the mid-point of TP_1 (since VSY is parallel to P_1M and bisects the side TM).

$\therefore$ FS is perpendicular to TP_1; the perpendicular from the focus to any tangent meets it on the vertex tangent.

QF being drawn, $\triangle QFP_1 = \triangle QZP_1$.

Hence QFP_1 is a right angle.

Extend P_1F to cut the parabola at P_2; draw P_2Z_2 to the directrix, and P_2Q.

$$\triangle P_2Z_2Q = \triangle P_2FQ.$$

Hence P_2Q bisects the angle Z_2P_2F and is the tangent.

Further $\angle P_2QP_1$ is a right angle, since it is half of the straight angle about Q.

Summary of tangential properties of the parabola

1. The tangent bisects the angle between the focal radius and a line parallel to the axis; the normal bisects the inscribed angle between the focal radius and a line parallel to the axis through the point of tangency.

2. The perpendicular from the focus of any parabola on any tangent meets it on the vertex tangent.

3. Tangents at the extremity of a focal chord meet on the directrix, and at right angles.

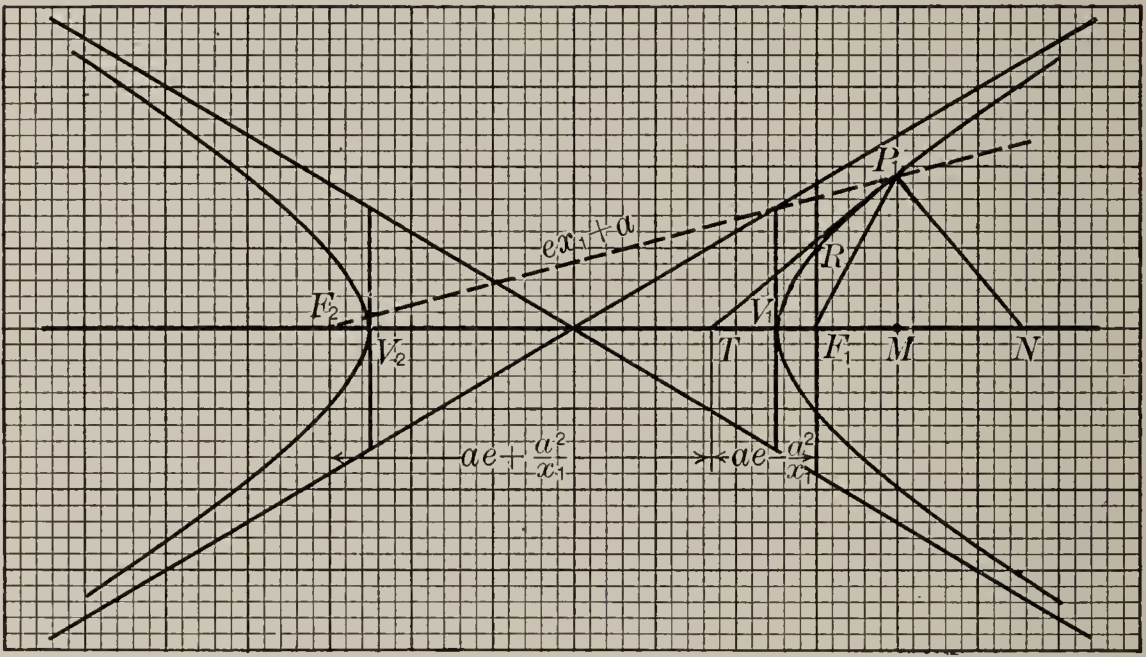
4. The focal chord is perpendicular to the line joining the focus to the intersection on the directrix of the tangents at the extremities of the focal chord.

7. Tangential properties of the ellipse and hyperbola. —

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ any ellipse,} \qquad \frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \text{ any hyperbola.}$$

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1, \qquad \frac{x_1x}{a^2}-\frac{y_1y}{b^2}=1,$$

tangent at (x_1, y_1) on curve, cutting the principal axis in the point T.



Tangential properties of the hyperbola

The tangent bisects the angle between focal radii to the point of tangency.

$$y-y_1=+\frac{a^2y_1}{b^2x_1}(x-x_1); \qquad y-y_1=-\frac{a^2y_1}{b^2x_1}(x-x_1),$$

giving the normal at x_1, y_1 on curve.

$$T \text{ is } \left(\frac{a^2}{x_1}, 0\right); \ N \text{ is } (e^2x_1, 0).$$

In the ellipse,	In the hyperbola,
$F_1T=\frac{a^2}{x_1}-ae$; $F_1N=ae-e^2x_1$.	$F_1T=ae-\frac{a^2}{x_1}$; $F_1N=e^2x_1-ae$.
$F_2T=ae+\frac{a^2}{x_1}$; $F_2N=ae+e^2x_1$.	$F_2T=ae+\frac{a^2}{x_1}$; $F_2N=e^2x_1+ae$.

The lengths F_1N and F_2N are seen to be proportional to the lengths P_1F_1 and P_1F_2.

$$\frac{P_1F_1}{P_1F_2}=\frac{F_1N}{F_2N}, \text{ since } \frac{a-ex_1}{a+ex_1}=\frac{ae-e^2x_1}{ae+e^2x_1};$$

Similarly $$\frac{P_1F_1}{P_1F_2}=\frac{F_1T}{F_2T}, \text{ since } \frac{a-ex_1}{a+ex_1}=\frac{\frac{a^2}{x_1}-ae}{\frac{a^2}{x_1}+ae}.$$

If a line from the vertex of a triangle divides the opposite side into segments proportional to the adjacent sides, the line bisects the angle of the triangle; hence the tangent and normal at any point on the ellipse and hyperbola bisect internally and externally the angle between the two focal radii to the point.

Another method of constructing the tangent is to construct $\frac{a^2}{x_1}$. In the ellipse the major auxiliary circle $x^2+y^2=a^2$ is drawn and the tangent at $P_3(x_1, y_2)$ on this circle $x_1x+y_2y=a^2$ has the intercept $\frac{a^2}{x_1}$; draw the tangent at (x_1, y_2) to the circle, cutting the X-axis at T; connect T with P_1 on the ellipse.

In the hyperbola $x_1>a$, so that this construction cannot be used; from $M(x_1, 0)$ on the X-axis a tangent to the circle $x^2+y^2=a^2$ intersects it at a point U whose abscissa is $\frac{a^2}{x_1}$, since

$$\frac{OT}{a}=\frac{a}{x_1}.$$

Summary of tangential properties of the ellipse, hyperbola, and parabola, regarding the parabola as having a second focus at an infinite distance on its axis of symmetry.

1. The tangent to an ellipse, hyperbola, or parabola bisects the angle between the focal radii to the point of tangency.

2. The perpendicular from the focus upon any tangent meets it on the circle having the center of the conic as center,

and passing through the principal vertices. In the parabola this circle has an infinite radius and so reduces to the tangent line at the vertex of the parabola.

3. Tangents at the extremities of a focal chord meet on the directrix.

4. The focal chord is perpendicular to the line joining the focus to the intersection on the directrix of tangents at the extremities of the given focal chord.

PROBLEMS

1. Find the tangent to the curve $xy = 25$ at the point $(5, 5)$ by the method of article 5.

2. Find the tangent to $x^2 = 8y$ at the point $(5, \frac{25}{8})$ by the method of article 5.

3. Find the tangents to the curves in problems 1–3 of the preceding set of problems at a point (x_1, y_1) on each curve by the general formula.

4. Find the tangent to the curve $x^2 = 8y$ at a point (x_1, y_1) on the curve; note that (x_1, y_1) satisfies the equation of the given curve; find a second equation which the point (x_1, y_1) must satisfy if the tangent obtained is to pass through $(5, 3)$ which is not on the curve; solve the two equations as simultaneous and thus obtain the point of tangency of a tangent from $(5, 3)$ to the given curve.

5. Find tangent and normal to the curve $x^2 - 10x - 8y - 5 = 0$, at the point whose abscissa is 2; find the tangent of slope -2 to this curve.

6. What tangential property of parabolic curves makes them useful in reflectors? Explain. Prove the property.

7. Write the equation of a hyperbola having the foci and vertices of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ as vertices and foci, respectively; find where these curves intersect; write the equation of a tangent to each of the curves at one of the points of intersection; discuss these lines.

8. Write the equation of the tangent at a point (x_1, y_1) to each of the following curves; use the general formula; time yourself.

a. $4x^2 - 6x + 9y^2 + 5y = 0.$

b. $4x^2 - 6x - 9y^2 - 5y = 0.$

c. $4x^2 - 6x - 5y = 0.$

d. $4x^2 + 6xy + 9y^2 - 6x - 5y = 0.$

e. $\frac{x^2}{16} + \frac{y^2}{25} = 1$; it is not necessary to clear of fractions as $\frac{1}{16}$ and $\frac{1}{25}$ can be thought of as co-efficients of x^2 and y^2.

f. $x^2 - 6x - 4y^2 - 8y + 7 = 0.$

9. Find the tangents to the first three curves in the preceding exercise at the points where these curves cut the x-axis.

10. Find one point on each of the curves of the eighth problem and write the equation of the tangent at that point.

From Tyrrell, *History of Bridge Engineering*

Elliptical arch bridge at Hyde Park on the Hudson

The span is 75 feet and the rise is 14.7 feet. Note that the reflection completes the ellipse.

CHAPTER XXII

APPLICATIONS OF CONIC SECTIONS

1. General. — Numerous applications of the conic sections, viz., circle, ellipse, parabola, and hyperbola, have been indicated in the problems given under the discussion of each curve. In general it is the tangential properties of the curves and the further geometrical peculiarities of these curves that make them so widely and so variously useful. The fact that simple geometrical properties are connected with curves given by algebraic equations of the first and second degrees in two variables seems to imply a certain harmony in the universe of algebra and geometry.

2. Laws of the universe. — In 1529 the Polish astronomer-mathematician, Copernicus (1473–1543), rediscovered and restated the fact, known to ancient Greeks, that the sun is the center of the universe in which we live; he conceived the planets to move about the sun in circular orbits. About a century later the great German astronomer, Kepler (1571–1630), was able to establish the following laws of the universe:

1. The orbits of the planets are ellipses with the sun at one focus.

2. Equal areas are swept out in equal times, by radii from the sun to the moving planet.

3. The square of the time of revolution of any planet is proportional to the cube of its mean distance from the sun; *i.e.* $\frac{T_1^2}{T_2^2}=\frac{d_1^3}{d_2^3}$, if T_1 and T_2 are the periodic times of two planets, and d_1 and d_2 the diameters of their respective orbits.

Kepler's work was made possible by that of all his predecessors, particularly the Greek mathematicians who had so thoroughly discussed the properties of the conic sections, and further by the work of the Dane, Tycho Brahe (1546–1601), whose refined observations gave the necessary data.

Newton (1642–1727) completed the work of systematizing the laws of motion in the universe in which we live, showing that the attraction of any two bodies for each other is inversely proportional to the square of their distance apart and directly proportional to their masses. Newton showed further that this assumption leads to the elliptical motion in the case of the sun and any planet.

The paths of comets which pass but once are known to be parabolas, or possibly hyperbolas with eccentricity close to 1.

3. Projectiles. — The first approximation to the path of a projectile is a parabola. Indeed for low velocities, below 1000 feet per second, the path is almost parabolic even with air resistance. The parametric equations of the path of a projectile shot horizontally with a velocity of 1000 feet per second, neglecting air resistance, are, in terms of t, the number of seconds of flight, as follows:

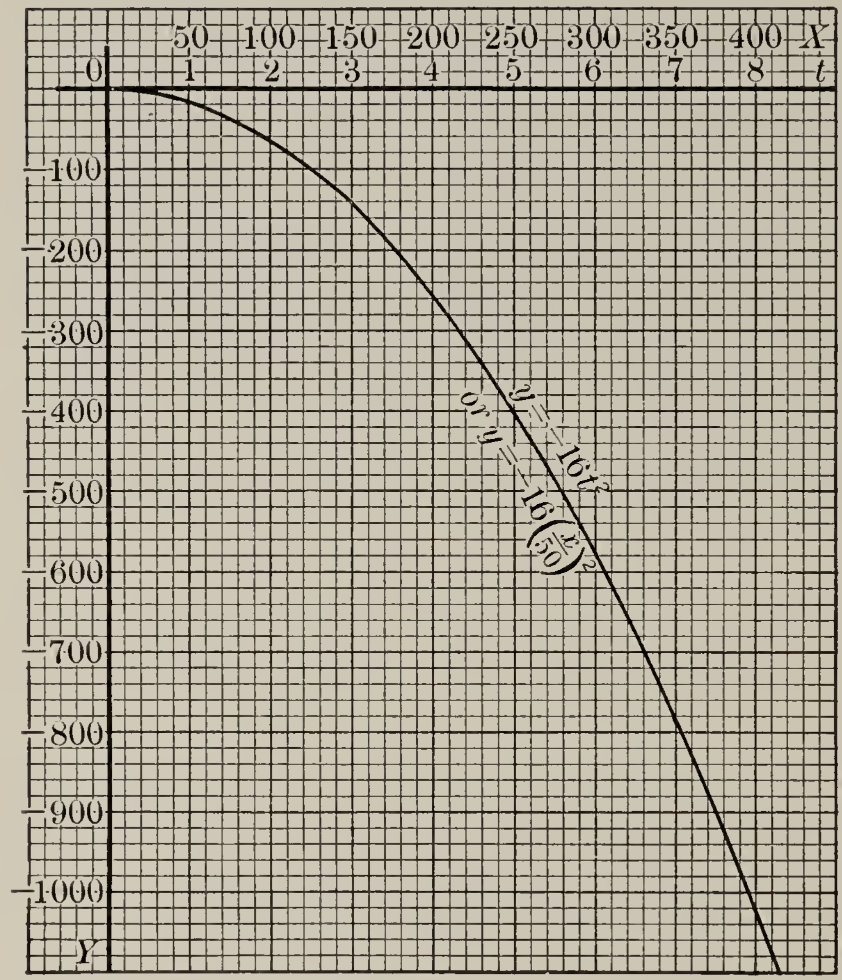
$$x = 1000\,t,$$
$$y = -16\,t^2.$$

When a projectile is shot at an angle α with the horizonal, we have shown that there is a horizontal component of velocity, $v \cos \alpha$, and a vertical component of velocity, $v \sin \alpha$. The equations of the path of this projectile shot from the ground as x-axis are as follows:

$$x = v_0 \cos \alpha \cdot t,$$

$$y = v_0 \sin \alpha \cdot t - 16\,t^2.$$

Problem. — Find the path of a projectile thrown with a velocity of 50 feet per second horizontally from the top of a tower 1000 feet high.



Path of projectile shot horizontally from a tower 1000 feet high; initial velocity of 50 feet per second

$$x = 50\,t,$$

$$y = -16\,t^2,$$

constitute the parametric equations of the path, the axes being taken through the top of the tower. Giving to t values $t = 1, 2, 3, \cdots 8$, these equations determine the position of the projectile after t seconds.

$t = 0, 1, 2, 3, 4, 5, 6, 7, 8$ determines the following points upon the parabola:

(0, 0) (50, − 16) (100, − 64) (150, − 144) (200, − 256) (250, − 400) (300, − 576) (350, − 784) and (400, − 1024).

Since $x = 50\,t$, $y = -16\,t^2$, $y = -16 \cdot \left(\frac{x}{50}\right)^2$, for all values of t.

$x^2 = -156.25\,y$ is the equation of the parabola in standard form; the coördinates (x_1, y_1) of any point obtained by substituting a given value for t in the parametric equations above will satisfy this equation since $t^2 = \left(\frac{x}{50}\right)^2 = -\frac{y}{16}$.

On any ordinary coördinate paper the curve $x^2 = -156.25\,y$ can be plotted only as $x^2 = -156\,y$.

The drawing shows very plainly that the projectile reaches the earth when $t = 7.9$ seconds, approximately; solving

$$-1000 = -16\,t^2 \quad (y = -16\,t^2,\ y = -1000),$$

gives

$$t^2 = 62.5,$$
$$t = 7.91^-.$$

The motion of a falling body is a special case of the equations above. $y = -16\,t^2$ gives the space in feet covered in time t seconds by a freely falling body, falling from rest.

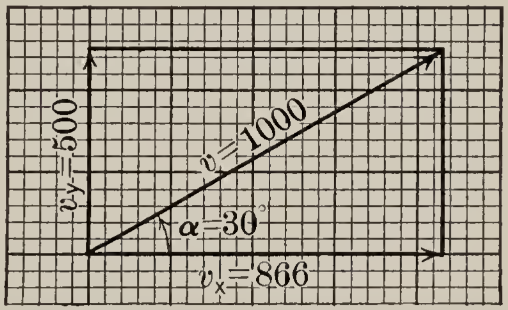
4. Illustrative problem. — For a bullet shot at an angle of 30° with a velocity of 1000 feet per second the equations are:

$$x = 866\,t,$$
$$y = 500\,t - 16\,t^2.$$

This bullet will, on a level plain, remain in the air until $y = 0$; solving gives the value for the time of flight. The range is given by inserting the value of t so found to find x.

The velocity of 1000 feet is equivalent to two separate velocities, one vertical of 500 feet per second, one horizontal of 866 feet per second. These are x and y components of the velocity. If no other force acted on this projectile, the path would be a straight line, given by

$$x = 866\,t,$$
$$y = 500\,t.$$



Vertical and horizontal components of a given velocity

But since gravity acts, diminishing the vertical velocity, the total y is given by

$$y = 500\,t - 16\,t^2,$$

the $-16t^2$ being due to the effect of gravity. The fall in 1 second due to gravity is independent of the upward motion.

The path is given by

$$x = 866\,t,$$
$$y = 500\,t - 16\,t^2.$$

When $y = 0$, the projectile is on the ground, the x-axis.

$t(500 - 16\,t) = 0$, $t = 0$, or $t = \frac{500}{16}$; the first value, $t = 0$, means simply that the projectile is shot from the ground. $t = \frac{500}{16} = 31\frac{1}{4}$, is the number of seconds the projectile is in the air. Finding x when $t = 31\frac{1}{4}$ gives the horizontal distance, or the range. Eliminating t gives the Cartesian form of the equation of the parabola

$$x = 866\,t, \text{ or } t = \frac{x}{866}, \text{ whence}$$
$$y = 500\frac{x}{866} - 16\left(\frac{x}{866}\right)^2, \text{ which reduces to}$$
$$(x - 13530)^2 = -46870(y - 3906).$$

The numerical work is somewhat tedious in such a problem, and it is indeed in most practical problems. The labor can be materially shortened by remembering that since the initial velocity is probably correct only to the second significant figure, correct here to hundreds of feet, and since y is taken as 32, instead of 32.2, an error of defect in the division of $\frac{2}{3}$ of 1 %, the error by excess in the quotient will be also $\frac{2}{3}$ of 1 %.

PROBLEMS

1. Of the planets Mercury is nearest to the sun. The mean distance of Mercury ($= a$) is 36 million miles; $e = .2056$; compute the equation of the orbit referred to the principal diameter as axis; find the distance of the sun from the center of the path.

2. Venus is the planet which is second in order of distance from the sun; the mean distance is 67.27 million miles; $e = .0068$, compute the equation and constants as in the preceding problem.

3. Compute the orbit of Mars and focal distance; mean distance is 141.7 million miles; $e = .0933$.

4. Knowing that the earth has a time of revolution of 365.256 (use 365.3) days and that its mean distance is 92.9 million miles, compute by Kepler's third law the times of revolution of the planets in the three preceding problems.

5. Discuss the maximum and minimum speed of the earth, assuming that the angular velocity is constant; note that the focal distance is 1.5 million miles, and the variation will depend upon the different lengths of the radius.

6. Assuming that the big gun which bombarded Paris had a range of 75 miles when pointed at an angle of 45°, find the initial velocity from the equations,

$$x = .707\,vt,$$
$$y = .707\,vt - 16.1\,t^2.$$

Insert in these equations $x = 75$ times 5280 and $y = 0$ and solve for v and t; the values obtained are the theoretical initial velocity in feet per second and the time of flight in seconds, neglecting air resistance.

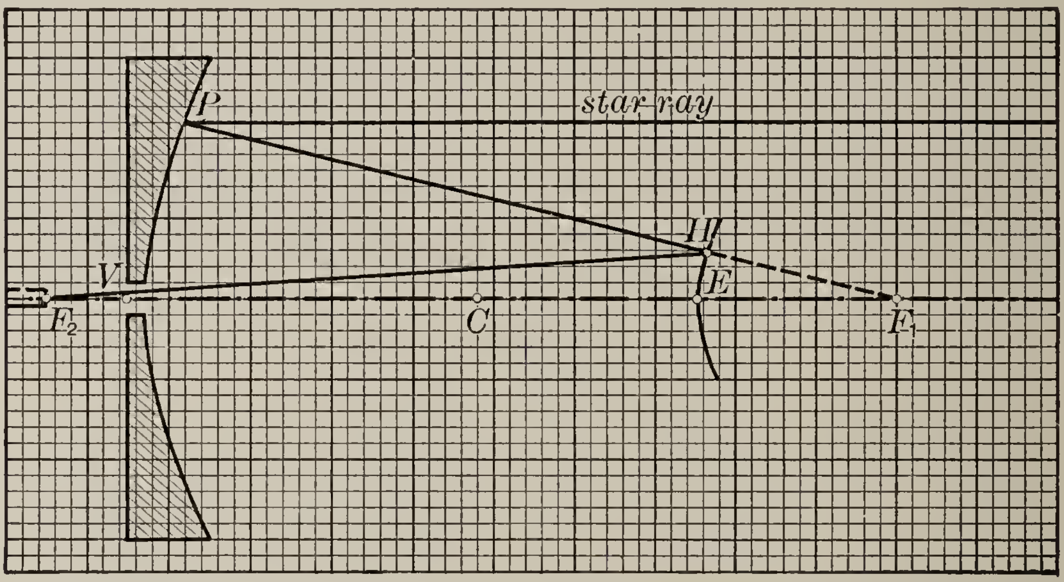
7. A body falls a distance of 10,000 feet; find the time of fall.

8. A body is thrown up vertically with a velocity of 100 feet per second; discuss the motion.

5. **Reflectors.** — The fact that the tangent to a parabola bisects the angle between a focal radius and a line parallel to the axis leads to diverse uses of the parabola. Rays of light from the sun or from a star meet a parabolic mirrored surface whose axis is directed towards the sun or star in rays parallel to the axis of the parabolic surface; these rays converge at the focus of the parabola and are by this means intensified.

In an automobile reflector and in searchlights the conditions are reversed; rays emanating from the central light at the focus are reflected in rays parallel to the axis.

A ray of light directed towards one focus of a hyperbolic surface striking the surface is reflected towards the other focus, since the tangent bisects the angle between the focal radii. This property is used by astronomers to re-focus the rays of light from the parabolic mirror at a point which does not lie between the parabolic mirror and the sun or star. An elliptical mirror beyond F_1 might be used for the same purpose.



Parabolic and hyperbolic reflectors at the Detroit Observatory

The curvature of the large parabolic mirror is greatly exaggerated on the diagram.

The parabolic mirror here pictured, is in use at the Detroit Observatory, Ann Arbor; the diameter of the mirror is 37.5 inches; the focal length is 19 feet; the focal length of the hyperbolic mirror used is 5 feet; the second focus of the hyperbola is 2 feet behind the vertex of the parabola and at this point, F_2, the rays are directed into a spectroscope.

Sound rays are entirely similar to light rays so far as reflecting properties are concerned. In an auditorium it is desired that the sound waves should be thrown out from the reflecting walls about the stage in parallel lines to all parts of the building; the reflecting surfaces have parabolic sections with the focus at the center of the stage.

This is the case in the Hill Auditorium at Ann Arbor, Michigan; axial sections of the hall made by planes are parabolic in form, having the focus at the center of the stage.

6. Architectural uses of conics. — The intimate connection between beauty of form and numerical relations is undoubtedly illustrated by the "golden section." The most satisfactory dimensions of a rectangle from an artistic standpoint are such, so it is accepted by those qualified to judge, that the longer dimension is, approximately, to the shorter as the shorter is to the difference between the two. In other words, if the width is given, the desired height is found by the "golden section," *i.e.* by dividing the line in extreme and mean ratio. Thus for width 40, the height x is found by solving the equation,

Hell Gate bridge, over the East River, New York City

The largest parabolic arch in the world, in one of the most beautiful bridges of the world; the arch has a span of 977.5 feet, height 220 feet.

$$\frac{40}{x}=\frac{x}{40-x};$$

this gives a quadratic equation for x. Note that if a square is cut off at one end of this rectangle a similar rectangle remains; so also if the square on the longer side is added to the rectangle a larger rectangle similar to the original one is formed.

We have found a certain connection apparently existing between simplicity of form and simplicity of algebraic equation. Thus the straight line is represented by the simplest algebraic equation in two variables, the first degree equation; the circle which is the simplest curved line to construct is represented by a particularly simple type of quadratic equation; to other types of quadratic equations in two variables correspond only three further curves, viz., ellipse, parabola, and hyperbola. That these second-degree curves, the conic sections, are artistically satisfactory is evident from the extended use which has been made of these forms by artists, ancient and modern.

The Williamsburg bridge over the East River, New York

Longest suspension bridge in the world; the parabolic arc of each 18-inch cable is 1600 feet in span by 180 feet in depth, width 118 feet; weight of the whole 1600-foot span is 8000 tons. Largest traffic of any bridge in the world.

In the construction of arches it is found that beauty of geometric form is intimately connected with simplicity of algebraic equation. The parabola and the ellipse have wide uses in construction not only because of beauty of form, but also because of purely mechanical adaptation to the stresses and strains caused by the weight of arch structures. A recognized authority[1] on bridge building, states that "arches must be *perfect* curves," and warns against the use of false ellipses.

The fact that in many of the greatest bridges of the world the pure ellipse and parabola appear so frequently is an indication of the wide acceptance of the theory that elliptical and parabolic arches are beautiful in form. The great Hell Gate Bridge of New York has for the main arch a true parabola (see problem 11, p. 317); London Bridge has five elliptical arches as the fundamental part of the sub-structure. Even the hyperbola has been used, but that only rarely. Let it be noted that partly because of the greater ease in design the circular arch is much more common, and the approximation to ellipse or parabola by using several circular arcs with different centers is also common.

No less than four distinct and different uses of the parabolic arc are found in the construction of bridges and trusses. The suspension bridge with a parabolic cable is one type; the parabolic arch with vertex below the roadway of the bridge is a second use; the parabolic arch intersecting the roadway is the third type; and the parabolic arch entirely above the roadbed, as a *truss*, is a fourth type.

Elliptical arches and less frequently parabolic are commonly used in the design of large foyers of theaters and in other large halls.

Parabolic and pure elliptic arch forms are used, although not as frequently as circular and horseshoe forms, in the design of sewers. Even complete perfect ellipses have been used (see problem 6, p. 356).

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[1] Mr. G. H. Tyrrell, of Evanston, Illinois, *Artistic Bridge Design*, Chicago, 1912.

PROBLEMS

1. Solve the quadratic in the preceding article and check by drawing a diagram.

2. Find the width, x, of a rectangle whose height is 40, such that 40 is the "golden section" of the width, giving beauty of form of the rectangle.

3. The Panther-Hollow Bridge, Pittsburg, has a parabolic arch, 360 feet in span with a rise of 45 feet. Draw the parabola. Assuming that the vertical chords are spaced every twenty feet and rise 15 feet above the vertex, find their lengths.

4. In the preceding problem the smaller arches leading to the bridge itself are probably elliptical. The width of these arches is 28 feet and the height of the arch proper about 8 feet. Draw these arches.

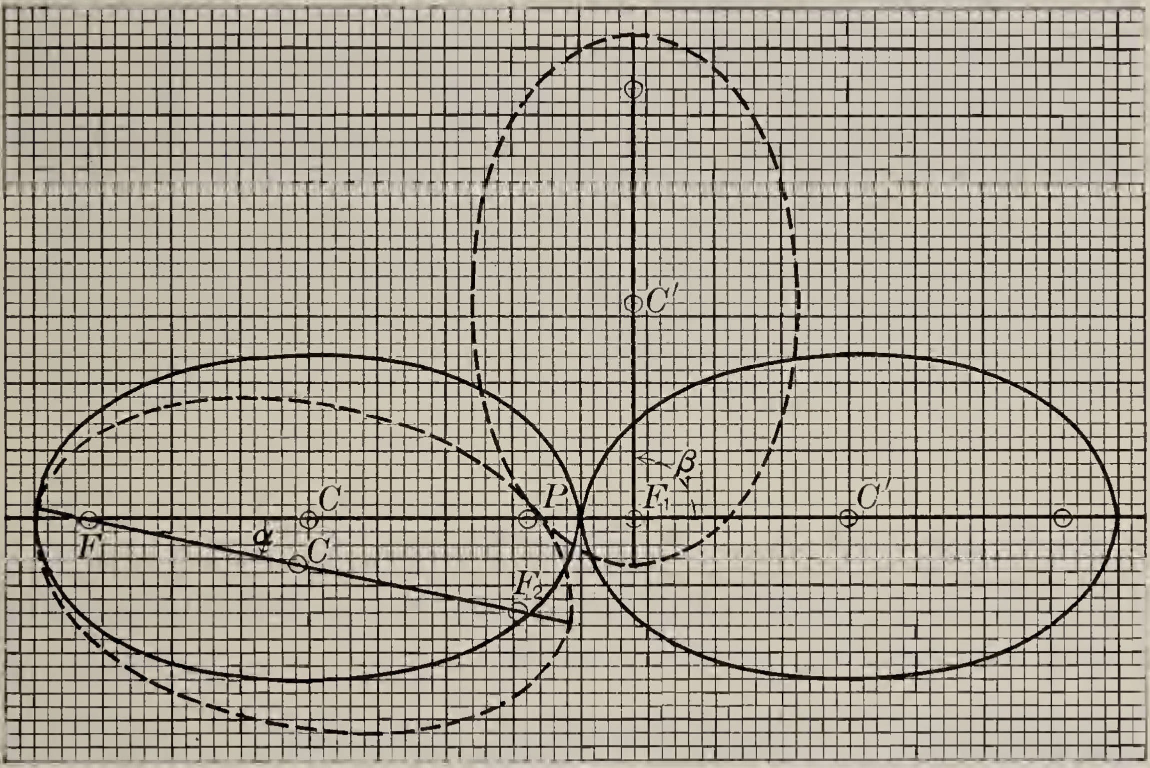
5. A parabolic sewer arch used in Harrisburg, Pa. (designed by J. H. Fuertes) has dimensions of 6 feet in width by 4 feet high. Construct ten points.

6. A vertical elliptical sewer in Chicago, Western Avenue sewer, constructed 1910, has dimensions 12 × 14 feet. Draw the figure.

7. Draw an elliptical and a parabolic arch, width 100 feet, height 30 feet; compare.

8. Draw to scale the parabolic arc of the Williamsburg suspension bridge, 1600 feet in span by 180 feet in depth. Find the equation in simplest form, choosing proper axes. Find the lengths of four vertical chords from cable to the tangent at the vertex of the arc.

7. Elliptical gears. — On machines such as shapers, planers, punches, and the like the actual movement during the operation of shaping, planing, or punching is desired to be slow and steady, and the return motion is desired to be much more rapid. Circular gears give a uniform motion, but elliptical gears permit the combination of slow effective movement with quick return. The two ellipses are of the same size and are mounted at the corresponding foci. In every position then the two ellipses will be in contact since the sum of the focal distances in either ellipse equals $2a$, always, and this also equals the distance between the two fixed foci which are on the axes of rotation.



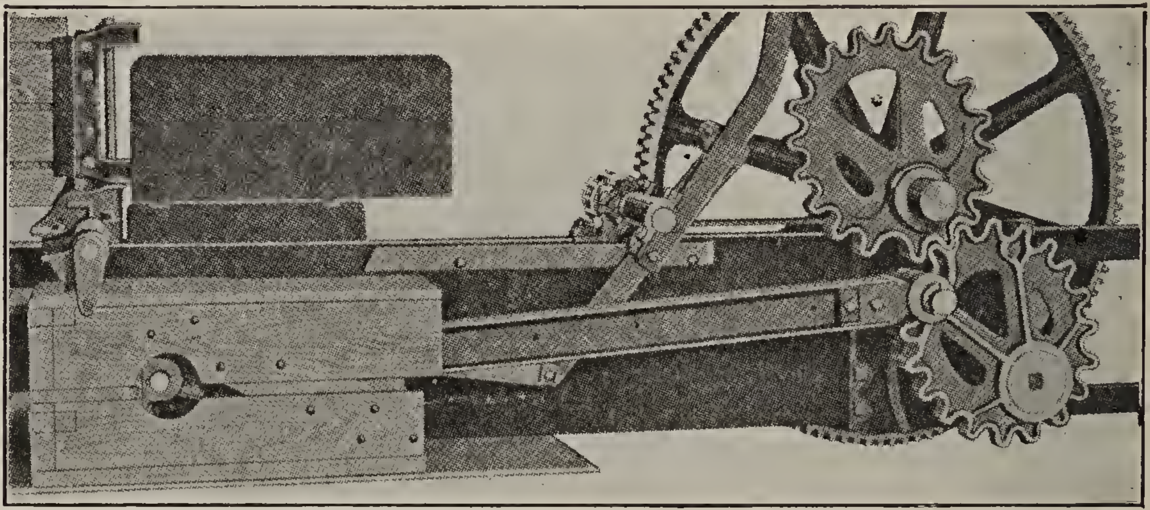
Two positions of elliptical gears, mounted on corresponding foci

Note that the right-hand ellipse swings through a large angle, in this position, as compared with the left-hand one.

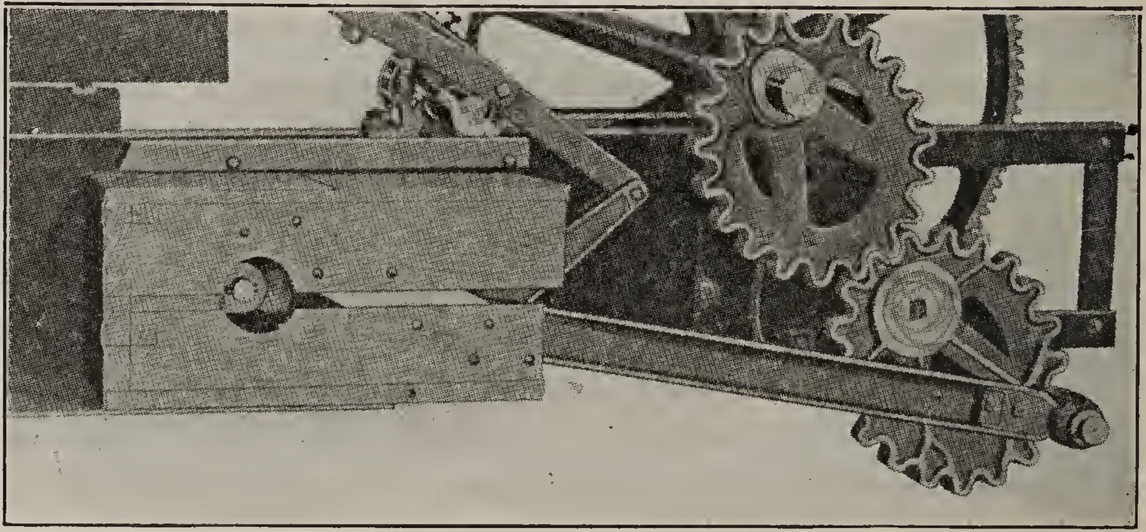
PROBLEMS

1. Draw three positions of two elliptical gears, each being an ellipse 6 inches by 10 inches. Determine maximum and minimum radii. When the ellipse is turned 5 times a minute, what is the fastest linear speed of a point on either ellipse? Note that it is given by using as radius the maximum radius, FP in our figure. Find the slowest speed.

2. In the Sandwich hay-press of our illustration the diameters of the ellipse of the elliptical gears are $21\frac{5}{16}$ inches and $18\frac{1}{16}$ inches; plot the graph and discuss maximum and minimum linear speed, given that the angular velocity is twenty to twenty-two revolutions per minute.



Elliptical gears on a hay press; the slow pressure stroke



Elliptical gears on a hay press; quick return motion

8. Applications in mechanics and physics. — The applications of the conics in mechanics and physics are very frequent. Thus the equation giving the period of a pendulum,

$$t = \pi\sqrt{\frac{l}{g}} \text{ or } t^2 = \frac{\pi^2 \cdot l}{g}$$

(see page 317) is the equation of a parabola, when g is taken as constant. Similarly the velocity of water flowing from a tube or over a dam depends upon the height or head of water above the level of the tube or dam; the relation is $v^2 = 64.4\,h$, where h is measured in feet; this also is a parabolic relation.

The bending-moment at any given section of a beam supported at both ends and uniformly loaded varies at different points on the beam, being greatest at the middle. These moments are computed graphically in the case of a bridge, being given by a so-called parabola of moments. This parabola for a bridge of length l, uniformly loaded with a weight of w per foot, is given by the equation,

$$M = \tfrac{1}{8}\,wl^2 - \tfrac{1}{2}\,w \cdot x^2,$$

wherein x is the distance from the center of the bridge. The parabola is plotted across the length l of the bridge, with the vertical ordinate at the mid-point representing the maximum moment.

Thus if a bridge is 100 feet wide and uniformly loaded 2 tons per foot, the moment at any point x distance from the center of the bridge is given by the formula

$$\begin{aligned} M &= \frac{2 \times 10^7}{8} - 1000\,x^2 \\ &= \tfrac{1}{4} \times 10^7 - 1000\,x^2. \end{aligned}$$

Draw the corresponding parabola, choosing appropriate units.

When a rotating wheel is stopped by the application of some force which reduces the velocity uniformly per second, the equations giving the number of revolutions before the wheel comes to rest correspond closely to the equations of motion of a body moving under the acceleration of gravity.

$$\theta = \omega_0 t - \tfrac{1}{2}\,kt^2,$$

θ represents numerically the angle covered in time t seconds, the body having an initial rotational speed of ω_0 revolutions per second and the velocity being retarded every second by k revolutions per second. Here again we have an equation between θ and t represented by a parabola. The time in which this body comes to rest is obtained by dividing the initial velocity by the uniform decrease in velocity per second, *i.e.* by the acceleration (or retardation).

The relation between pressure and volume of a perfect gas, temperature being constant, is given by the equation :

$$p \cdot v = k;$$

in words the volume is inversely proportional to the pressure. Plotting points gives points on a hyperbola of which the p-axis and the v-axis are the asymptotes.

Such illustrations could be multiplied, but many relations of this character, *e.g.* the ellipsoid of inertia, require considerable technical explanation which would go beyond the limits of this work.

9. Quadratic function. — The graph of the quadratic function,

$ax^2 + bx + c$, is the locus of the equation,

$$y = ax^2 + bx + c.$$

$$y = a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b^2 - 4ac}{4a^2}\right)$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}.$$

$$y + \frac{b^2 - 4ac}{4a} = a\left(x + \frac{b}{2a}\right)^2.$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{1}{a}\left(y + \frac{b^2 - 4ac}{4a}\right).$$

The graph of $y = ax^2 + bx + c$ is a parabola; $x + \frac{b}{2a} = 0$ is the axis. If a is positive the parabola opens up; the vertex is $V\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right)$; if $b^2 - 4ac$ is negative, the vertex is above the x-axis and no real value of x makes $y = 0$, since the graph does not cut the axis. If a is negative, the parabola opens down; if $b^2 - 4ac$ is negative, the vertex is below the x-axis and again the graph does not cut the x-axis. If $b^2 - 4ac = 0$ the graph is tangent to the x-axis. Evidently $b^2 - 4ac < 0$ is the condition that $ax^2 + bx + c = 0$ should have imaginary roots; $b^2 - 4ac = 0$ is the condition for equal roots; and $b^2 - 4ac > 0$ is the condition for real roots.

PROBLEMS

Plot to the same axes the graphs of the functions in problems 1, 2, and 3. Discuss.

1. $y = 5x^2 + 2x - 7$.

2. $y = 5x^2 + 2x + 7$.

3. $y = 5x^2 + 2x + \frac{1}{5}$.

4. If a wheel is rotating at the rate of 800 revolutions per second, and a force acting continuously reduces the speed each second by 40 revolutions per second, find the time in which it will stop, and the number of revolutions made during the retarded motion. Use the formula given in article 8.

5. Given that 10 cubic centimeters of air are subjected to pressure, at a pressure of 1 atmosphere the volume is 10, hence $pv = 10$ is the equation connecting volume and pressure. Plot the graph of this for values of p from $\frac{1}{4}$ atmosphere to 5 atmospheres' pressure.

6. Plot the parabola of moments for the Hell Gate Bridge, assuming a uniform loading of 2 tons per foot. Do not reduce tons to pounds, but use ton-feet as units of moment. The equation is $M = 490^2 - x^2$, taking 980 as the length of the bridge.

7. Plot the parabola of moments of the Panther-Hollow Bridge in problem 3 of the preceding exercise, assuming 2 tons per foot as loading. The equation is $M = 180^2 - x^2$.

8. Find the equation of a parabola whose focal length is 19 feet. Draw the graph to appropriate scale. This is the parabola which, revolved about its axis, gives the parabolic

reflector, previously mentioned, which is in use at the Detroit Observatory. (See page 352.)

Find the equation of a hyperbola which has the same focus as this parabola, the axis of the parabola as transverse axis, and the second focus on the axis at a distance of 2 feet on the other side of the vertex. This is the hyperbola which, revolved about its axis, gives the hyperbolic mirror mentioned.

The parabolic mirror has a diameter of 37 inches. Find the abscissa for the ordinate $\frac{18.5}{12}$, thus finding the depth of the mirror.

9. Plot the parabola $y^2 = 70.02\,x$. This is the parabola which is fundamental in the construction of the Hill Auditorium. (See page 352.) The plane of the floor cuts the side walls in this curve; so also the intersection of the ceiling and a plane passed vertically through the main aisle of the hall. In the plans the computations of ordinates for given abscissas are made to the thirty-second of an inch. Compute the focal ordinate. This is the radius of the circular arch over the stage. Compute the ordinates for $x = 21$, 26, 31, 51, and 71, and express in feet and inches.

10. The Italian amphitheaters are, in general, elliptical. The Colosseum in Rome (see illustration, page 288) is an ellipse with axes of 615 and 510 feet. Draw the graph to scale. On the same diagram and with the same center and axes of reference draw the arena, of which the dimensions are 281 feet by 177 feet, to the same scale. Note that the minor axis of the arena is almost the "golden section" of the major axis, *i.e.* 177 is approximately a mean proportional between 281 and 281 less 177. Find the mean and compare.

11. The bridge at Hyde Park (see illustration, page 346) is elliptical, with a span of 75 feet and an arch height of 14.7 feet. Draw this elliptical arch to scale.

CHAPTER XXIII

POLES, POLARS, AND DIAMETERS

1. Definition. — The straight line

$$Ax_1x + By_1y + G(x + x_1) + F(y + y_1) + C = 0$$

is called the *polar* of $P_1(x_1, y_1)$ with respect to the conic

$$Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0.$$

The point $P_1(x_1, y_1)$ is called the *pole* of the line.

2. Fundamental property of polar lines. — If the polar of $P_1(x_1, y_1)$ with respect to the given conic passes through $P_2(x_2, y_2)$, then reciprocally the polar of $P_2(x_2, y_2)$ will pass through $P_1(x_1, y_1)$.

This fundamental property of polar lines enables one to prove complicated geometrical theorems for conics with a minimum of machinery. The proof of the theorem is itself simple, for substituting in the polar of $P_1(x_1, y_1)$, the coordinates (x_2, y_2), we have that

$$Ax_1x_2 + By_1y_2 + G(x_2 + x_1) + F(y_2 + y_1) + C = 0,$$

if the polar of P_1 passes through P_2. However the polar of P_2 is, by definition,

$$Ax_2x + By_2y + G(x + x_2) + F(y + y_2) + C = 0,$$

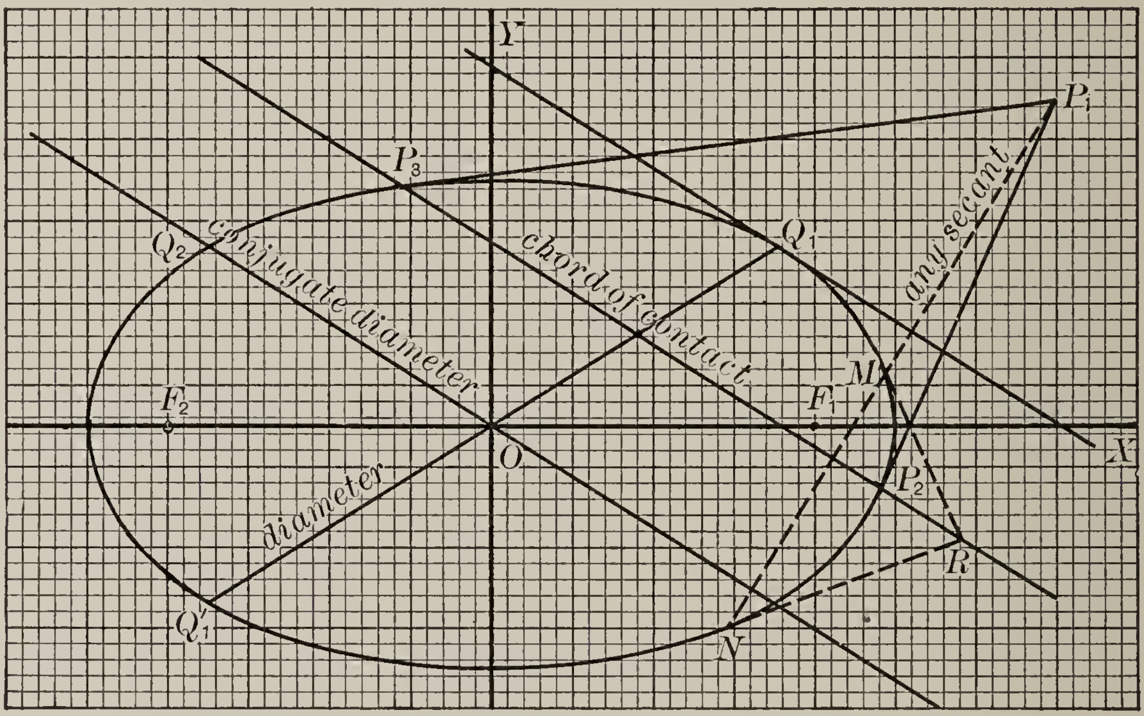
and substituting (x_1, y_1) gives precisely the preceding expression, which is of value 0; hence $P_1(x_1, y_1)$ is on the polar of $P_2(x_2, y_2)$.

3. Geometric properties of the polar. — If $P_1(x_1, y_1)$ lies on the curve, the polar is the tangent at that point. (See the preceding chapter.)

If $P_1(x_1, y_1)$ lies outside of the conic, the polar is the chord of contact of tangents from P_1.

Let $P_2(x_2, y_2)$ be the point of tangency of a tangent drawn from P_1;

by definition, the polar of P_2 is the tangent at $P_2(x_2, y_2)$;
by construction, the polar of P_2 passes through P_1;



The polar of any point outside a conic is the chord of contact

by the fundamental reciprocal property, since the polar of P_2 passes through P_1, the polar of P_1 will pass through P_2.

Similarly, calling $P_3(x_3, y_3)$ the other point of tangency, the polar of $P_1(x_1, y_1)$ will pass through P_3.

Since the polar of P_1 is a straight line and since it passes through the two points of tangency, it is the chord of contact joining these two points.

If $P_1(x_1, y_1)$ lies inside, or outside, or on the conic, the polar is the locus of the intersection $R(x', y')$ of tangents drawn at the extremities of any chord passing through P_1.

Let $R(x', y')$ be the intersection of tangents at the extremities of any secant;

then, by the construction, the secant drawn is the chord of contact of $R(x', y')$;

by construction, the polar of $R(x', y')$ passes through $P_1(x_1, y_1)$;

hence, by the fundamental reciprocal property, the polar of P_1 will pass through $R(x', y')$;

but the polar of P_1 is a straight line;

hence the locus of $R(x', y')$ is a straight line, the polar of $P_1(x_1, y_1)$.

By pure Euclidean geometry it is rather complicated to prove that the locus of the intersection of tangents at the extremities of all chords of a circle passing through a fixed point is a straight line. The above proves this property for every conic.

4. Diameter: definition and derivation. — The locus of the midpoints of a series of parallel chords in any conic is called a diameter of the conic.

The method, applicable to any equation of the second degree, is given for a special case. Find the diameter bisecting chords of slope 3 in the ellipse

$$9x^2 + 25y^2 = 900.$$

Let

$$y = 3x + k$$

represent any line of slope 3. Solve, as simultaneous, with

$$9x^2 + 25y^2 = 900, \text{ which represents}$$

the conic.

Substituting,

$$234x^2 + 150kx + k^2 - 900 = 0$$

is an equation whose roots are the abscissas, x_1 and x_2, of the two points of intersection.

Solving,

$$x_1 = \frac{-150k + \sqrt{(150k)^2 - 4(234)(k^2 - 900)}}{468}, \text{ and}$$

$$x_2 = \frac{-150k - \sqrt{(150k)^2 - 4(234)(k^2 - 900)}}{468}.$$

For the midpoint (x', y') of the chord,

$$x' = \frac{x_1 + x_2}{2} = \frac{-25k}{78};$$

the midpoint lies on the chord

$$y = 3x + k,$$

hence,

$$y' = 3x' + k = \frac{-75k}{78} + k = \frac{3k}{78} = \frac{k}{26}.$$

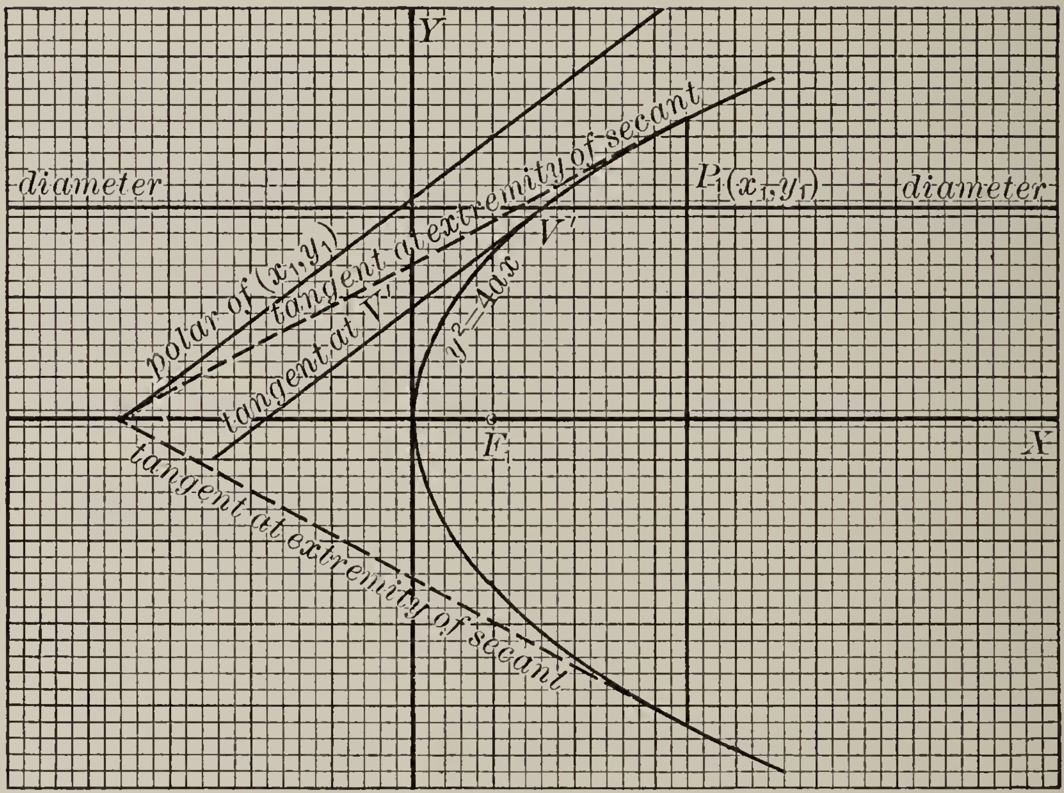
The middle point of any chord,

$$y = 3x + k,$$

is given by

$$x' = -\tfrac{25}{78}k,$$

$$y' = +\frac{k}{26},$$ and these two equations



Any diameter of a parabola is parallel to the axis of the parabola

constitute parametric equations of the locus of the midpoint. Eliminating k by solving for k and substituting (or by division here), we see that for every value of k the coördinates of the middle point satisfy the equation

$$y' = -\tfrac{3}{25}x'.$$

Hence the middle point is on the straight line

$$y = -\tfrac{3}{25}x.$$

By precisely similar reasoning, the diameter bisecting chords of slope m in the conic, given by

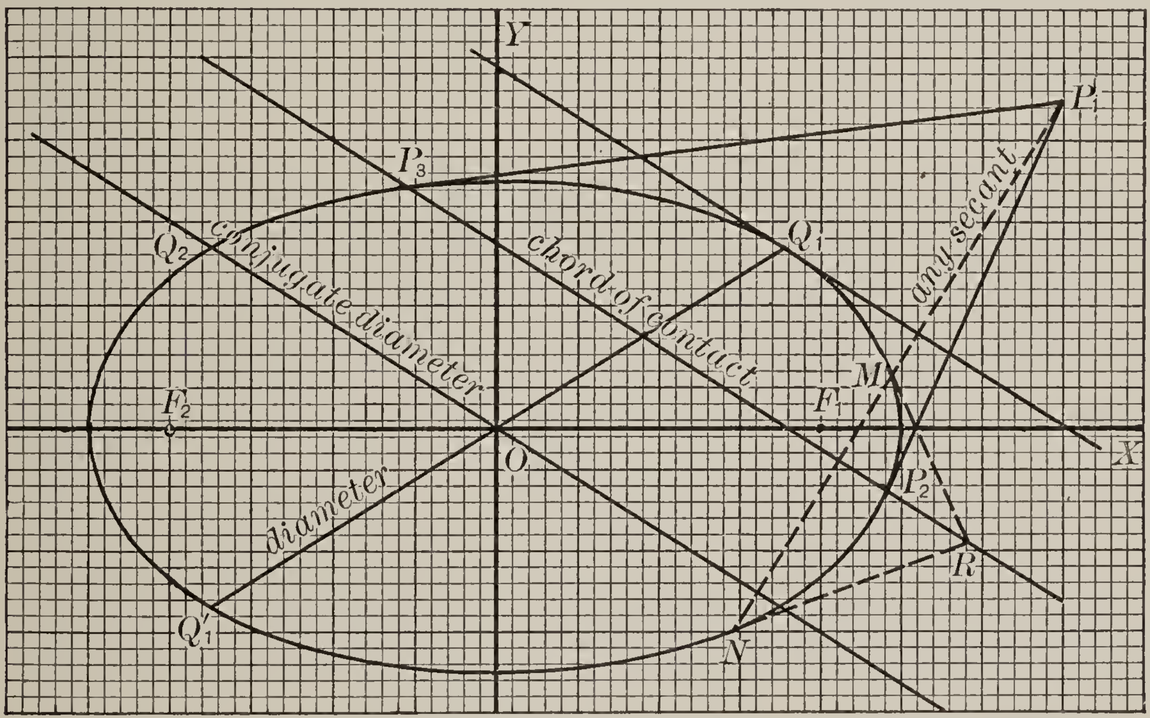
$$Ax^2 + By^2 + 2Gx + 2Fy + C = 0$$

is

$$Ax + mBy + G + mF = 0.$$

Applying this to the simplest standard forms of ellipse and hyperbola, we see that every diameter of a central conic passes through the center; applying to the parabola, $y^2 = 4ax$, we obtain $my - 2a = 0$. This shows that the diameter of *any* parabola is parallel to the axis of the parabola, for when a parabola is rotated or moved to any other position any diameter moves with the curve, preserving its position relative to the axis of the curve.

5. Reciprocal property of diameters in central conics. — In the conic $9x^2 + 25y^2 - 900 = 0$ above, the diameter bisecting chords of slope 3 has the slope $-\frac{3}{25}$. Now the diameter bisecting chords of slope $-\frac{3}{25}$ has the slope 3, for by substi-



Conjugate diameters; each bisects all chords parallel to the other

tution in the general formula, we have $9x - \frac{3}{25}(25)y = 0$, or $y = 3x$.

Each of these diameters bisects chords parallel to the other. These are called conjugate diameters. In precisely similar manner this property can be established in any ellipse or hyperbola for the diameter bisecting chords of slope m. Do this for the diameter of the general conic above.

PROBLEMS

1. What is the equation of the chord of contact (polar) of (10, 0) with respect to $9x^2 + 25y^2 = 225$? Solve this with the curve. This gives the points of tangency of tangents from (10, 0). Write the equation of the tangent at each of these points. This process illustrates an analytic method for finding a tangent from an external point to any conic,

$$Ax^2 + By^2 + 2Gx + 2Fy + C = 0. \quad \text{Explain.}$$

2. Find the equations of the tangents to the following conics from the points given; follow the method of problem 1; time yourself.

a. $y^2 - 8x = 0$, from $(-2, 6)$.
b. $y^2 - 8x - 6y - 10 = 0$, from $(-3, 5)$.
c. $x^2 + y^2 - 10x + 8y - 59 = 0$, from (18, 6).
d. $x^2 - 4y^2 - 10x + 8y - 59 = 0$, from (10, 5).
e. $x^2 + y^2 - 25 = 0$, from (1, 8) and from (2, 8).

3. Prove in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ that if the diameter is drawn through (x_1, y_1), the tangents at the extremities of this diameter are parallel to the polar of $P_1(x_1, y_1)$. Call the points on the diameter (x_2, y_2) and (x_3, y_3), and note that they lie on the ellipse. Write the equations of the different lines mentioned.

4. Prove the property mentioned in the preceding problem for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and for the parabola $y^2 - 4ax = 0$.

5. In problem 2, write the equations of the diameters bisecting chords of slope 2 and of slope $-\frac{1}{2}$, using the general formula for diameter.

6. In problem 5 write the equations of the conjugate diameters in the central conics.

7. Draw the circle $x^2 + y^2 - 36 = 0$; draw 5 secants through the point $(4, 3)$; draw the tangents at the two intersection points of each secant with the curve. The 5 points of intersection, one from each pair of tangents, should lie in a straight line. What theorem proves this?

8. Prove that the tangential parallelogram circumscribed at the ends of conjugate diameters of an ellipse $b^2x^2+a^2y^2=a^2b^2$ has a constant area. First show that the one end of the diameter conjugate to the diameter through $P_1(x_1, y_1)$ is $P_2\left(-\frac{ay_1}{b}, \frac{bx_1}{a}\right)$; find the equation of the tangent at $P_1(x_1, y_1)$; find the distance between $(0, 0)$ and (x_1, y_1); find the equation of OP_1; find the perpendicular distance from P_2 to OP_1; by multiplication show that the area of this quarter of the given parallelogram is constant.

9. Taking $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, which, by the preceding exercise, may be written $P_2\left(-\frac{ay_1}{b}, \frac{bx_1}{a}\right)$, as the extremities of a pair of conjugate diameters in the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, show that the sum of the squares of OP_1 and OP_2 equals $a^2 + b^2$.

Hint. — Reduce the expressions for OP_1^2 and OP_2^2 to common denominator, and use the fact that $P_1(x_1, y_1)$ is on the given ellipse.

10. In the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the conjugate diameter to the diameter through $P_1(x_1, y_1)$ on the hyperbola does not cut the curve itself. Prove this.

The extremities of the conjugate diameter are taken as the points in which the conjugate diameter intersects the conjugate hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$. With this definition the property of problem 8 can be proved to be true for the hyperbola. State the method. What modification would you expect so far as the property of problem 9 is concerned?

11. Prove that the polars of all points on a diameter of any conic are parallel, comparing with problems 3 and 4 above.

12. Show that tangents at the extremities of a series of parallel chords in any conic intersect on the corresponding diameter.

13. Prove that any point on a diameter of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the intersection of the polar of the point with the diameter divide the diameter length internally and externally in the same ratio.

Hint. — Take (x_1, y_1) and $(-x_1, -y_1)$ on the curve as the extremities of the diameter; take the point of the diameter as the point $\left(\frac{x_1+rx_1}{1-r}, \frac{y_1+ry_1}{1-r}\right)$ which divides the diameter externally in the ratio r; find the intersection point of the polar of this point with $y=\frac{y_1}{x_1}x$, and note that it is the same as the point which divides the line joining (x_1, y_1) to $(-x_1, -y_1)$ internally in the ratio r. The property holds for any conic.

If through any point a secant to a conic is drawn, the point and the intersection of the polar of the point with the secant divide the chord of the conic formed by the secant internally and externally in the same ratio. The proof is somewhat more complicated than that of the preceding special case.

14. Show that the tangential parallelogram to any central conic formed by the tangents at the extremities of a pair of conjugate diameters has its sides bisected by the points of tangency. An ellipse can be rather neatly inscribed in any parallelogram by drawing the ellipse tangent to the sides of the parallelogram at the midpoints.

CHAPTER XXIV

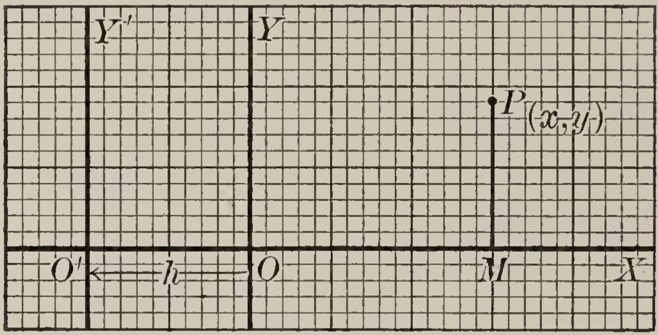
ALGEBRAIC TRANSFORMATIONS AND SUBSTITUTIONS

1. Transformation of coördinates. — For varied reasons it is sometimes found desirable to change the location of the coördinate axes with respect to a curve which is given by an equation involving variables. Usually this shifting of the axes is for the purpose of simplifying the discussion of the geometrical properties of the curve in question. Thus the ellipse has been given in the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, but the geometrical properties of the same curve are discussed with reference to the center (h, k) as origin, giving the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The axes may be subjected to a translation, giving new axes $O'X'$ and $O'Y'$ parallel to the old axes; or the axes may be turned through an angle α, giving new axes OX' and OY' about the old origin; the two motions can be combined, executing first the translation, usually, and then the rotation; it is possible also to shift to new axes inclined at an oblique angle to each other, but the formulas involved are too complicated for an elementary work.

2. Translation of axes. — Suppose the x-axis fixed and the y-axis moved parallel to itself to a new origin O' at distance $OO' = h$, from O. Take $P(x, y)$ as the coördinates of any point with reference to the original axes. Evidently, as the x-axis is unchanged, the y of this and every other point remains the same. Let M be the foot of the perpendicular from P to the x-axis; then by our fundamental property of the distances between three points on a directed line

$$OM = OO' + O'M.$$



Translation of axes

But $OM = x$, $OO' = h$, the distance either positive or negative OO'; while $O'M = x'$ by definition. Hence, whatever the position of $P(x, y)$, we have,

$$x = x' + h.$$

Similarly, if the x-axis is shifted parallel to itself by an amount k,

$$y = y' + k.$$

The two equations

$$x = x' + h,$$

$$y = y' + k,$$

transform any equation given with respect to any axes, to a set of parallel axes having the point (h, k) as origin.

3. Algebraic substitution in functions of one variable.

Theorem. — *Substitution of $x' + h$ for x in any algebraic equation of type $a_0x^n + a_1x^{n-1} + \cdots a_{n-1}x + a_n = 0$, n an integer, gives a new equation whose roots are h less than the roots of the old.*

The proof of this theorem depends directly upon the preceding article. The substitution $x = x' + h$ moves the y-axis h units, reducing the abscissas of all points by h if h is positive and increasing them by $-h$ if h is negative.

Illustration. — If the graph of $y = x^3 - 2x^2 - 18x + 24$ is plotted, the substitution $y = y$ and $x = x' + 4$ simply shifts the y-axis 4 units to the right, thus decreasing the numerical value of each root by 4.

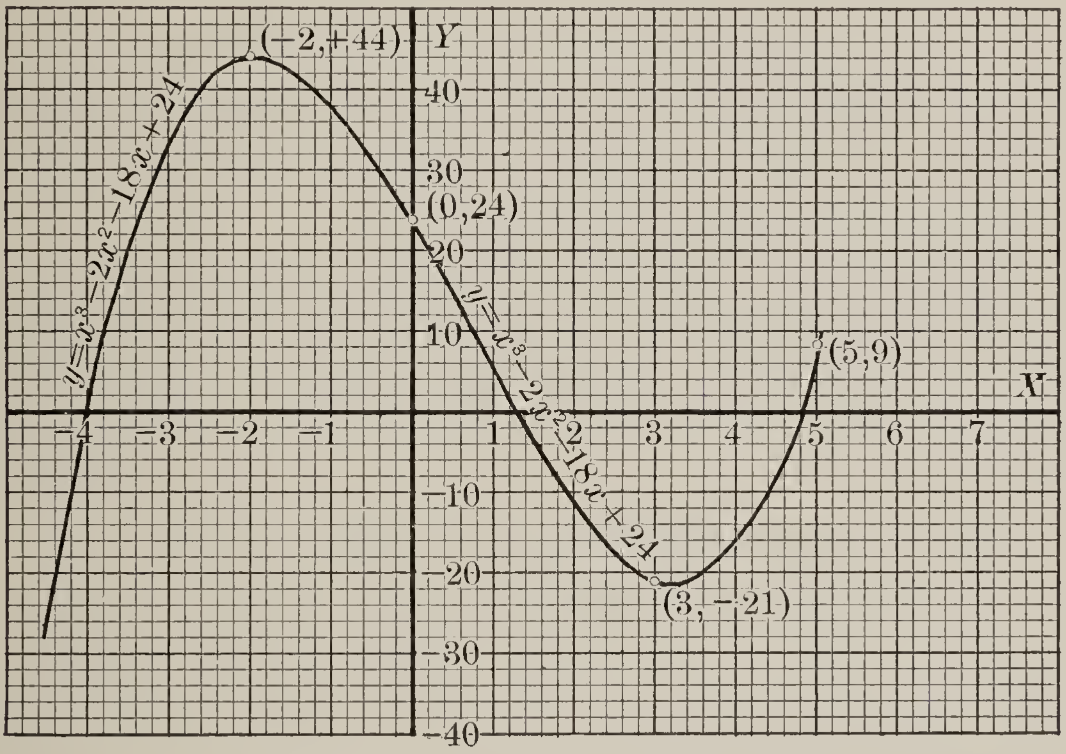
The new equation is

$$y = (x' + 4)^3 - 2(x + 4)^2 - 18(x' + 4) + 24 = x'^3 + 10x'^2 + 14x' - 16.$$

Now whatever number substituted for x makes $x^3 - 2x^2 - 18x + 24 = 0$, it is evident that 4 less substituted for x' will make

$$(x' + 4)^3 - 2(x' + 4)^2 - 18(x' + 4) + 24 = 0.$$

Graphically, of course, as we have indicated, the y-axis has been pushed 4 units towards the right, and the abscissa of each point of intersection of the curve with the x-axis has been reduced by 4.



Graph of $y = x^3 - 2x^2 - 18x + 24$

Similarly, in the general equation above, when $x' + h$ is substituted for x, whatever number a satisfies the original equation in x, $a - h$ will satisfy the new equation in x'.

Substitution of $x' + h$ for x in any algebraic equation forms a new equation in x' whose roots are h less than the roots of the given equation.

This type of substitution is used to facilitate the computation of roots of numerical algebraic equations.

A simple method of constructing the new equation in numerical equations will be explained below, in section 11 of the next chapter.

PROBLEMS

1. Show that the formulas of transformation given transform the equations of ellipse and hyperbola having (h, k) as center to the simpler form without first-degree terms.

2. Transform the equation $(y - 3)^2 = 8(x + 2)$ to the point $(3, -2)$ as new origin, new axes parallel to the old.

3. By translation of axes transform the equation

$$x^2 - 4xy - 6x + 8y - 10 = 0$$

into a new equation in which the first-degree terms are lacking.

4. Find the equation of the line $3y - 4x + 6 = 0$, referred to parallel axes through the point $(3, 2)$.

5. Compare the slope of a straight line referred to new axes by translation, with the slope referred to the old axes. Compare intercepts. Compare the slope of a tangent at a fixed point on any curve with respect to new and with respect to old axes.

6. Given the expression for the volume of 1000 cu. cm. of mercury at 0° C. when heated to $t°$ C., $v = 1000 + .0018\,t$ (see page 63), transform to parallel axes with the point $(t = 0, v = 1000)$ as new origin; find the new equation in v' and t. Does v' represent volume?

7. Given $v = 1054 + \frac{13\,t}{12}$, the velocity in feet per second of sound in air at $t°$ centigrade, transform to parallel axes with (32°, 1054) as new origin; discuss the equation.

8. The equation $x^3 - 2x^2 - 18x + 24 = 0$ (page 373) was found to have a root between $x = 4$ and $x = 5$; transform

$$y = x^3 - 2x^2 - 18x + 24 = 0$$

to parallel axes through $(0, 4)$ and the new equation in x' will have a root between 0 and 1. Compute this root to tenths by substitution.

9. The equation $x^3 - 2x^2 - 18x + 24 = 0$ has a further root between 1 and 2. Compute this root to one decimal place by the process explained in the preceding problem.

10. Find the roots of $2x^3 + 6x^2 - 10x - 8 = 0$, as in problems 8 and 9 by considering the graph of the equation $y = 2x^3 + 6x^2 - 10x - 8$, when referred to new axes. (See problem 4, page 99.)

11. Transform the following equations to parallel axes, having (h, k) as the new origin; determine (h, k) so that the terms of the first degree shall disappear.

a. $5x^2 + 4xy - y^2 - 8x - 5y - 10 = 0.$

b. $5x^2 + 4xy + y^2 - 8x - 5y - 10 = 0.$

c. $xy - 7x - 10y - 5 = 0.$

d. $4x^2 - 6x - y^2 - 8y - 10 = 0.$

12. Transform the following equations to parallel axes having (h, k) as origin. Can you determine (h, k) so that the terms of the first degree shall disappear? Why not? (See problem 13.)

a. $4x^2 - 6x - 8y - 10 = 0.$

b. $4x^2 + 4xy + y^2 - 8x - 5y - 10 = 0.$

13. Show that if an equation of the second degree contains no first-degree terms, the origin is the center of the curve by showing that if (x_1, y_1) is any point on the curve $(-x_1, -y_1)$ is also on the curve.

4. Rotation of axes. — The formulas for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$ give very neatly the relations which exist between the coördinates (x, y) of a point referred to the old axes and the coördinates (x', y') referred to the new axes. Take $P(x, y)$ any point referred to the original axes; let α be the angle of rotation through which the axes are turned; let β be the angle which the line OP makes with the x' or new x-axis. By section 4, chapter 15, for all values of α and β,

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta,$$

$$OP \cos(\alpha + \beta) = OP \cos\alpha \cos\beta - OP \sin\alpha \sin\beta.$$

But $OP \cos(\alpha + \beta) = x$; $OP \cos \beta = x'$; $OP \sin \beta = y'$; hence,

$$x = x' \cos \alpha - y' \sin \alpha.$$

Further, $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$.

Multiplying by OP, and substituting,

$$y = x' \sin \alpha + y' \cos \alpha.$$

These same relations might also have been obtained by projection; they hold for every position of the point P.

$$x = x' \cos \alpha - y' \sin \alpha,$$
$$y = x' \sin \alpha + y' \cos \alpha,$$

effects the rotation through the angle α, and refers any equation in two variables to new axes inclined at an angle α to the old axes.

5. Every second-degree equation in two variables represents a conic section. Proof. — To prove this theorem we need only to show that the equation

$$(1)\ Ax^2 + 2\,Hxy + By^2 + 2\,Gx + 2\,Fy + C = 0$$

can, by rotation of axes, be transformed to an equation of the type

$$(2)\ Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0.$$

Every equation of this latter type represents, as we have shown, either circle, ellipse, parabola, or hyperbola or some limiting form of one of these curves.

Substituting,

$$x = x' \cos \alpha - y' \sin \alpha,$$
$$y = x' \sin \alpha + y' \cos \alpha,$$

the equation $Ax^2 + 2\,Hxy + By^2 + 2\,Gx + 2\,Fy + C = 0$ becomes

$$\begin{aligned} A(x' \cos \alpha - y' \sin \alpha)^2 &+ 2\,H(x' \cos \alpha - y' \sin \alpha)(x' \sin \alpha + y' \cos \alpha) \\ &+ B(x' \sin \alpha + y' \cos \alpha)^2 + 2\,G(x' \cos \alpha - y' \sin \alpha) \\ &+ 2\,F(x' \sin \alpha + y' \cos \alpha) + C = 0. \end{aligned}$$

Collecting terms, we have,

$$\begin{aligned}&(A\cos^2\alpha+B\sin^2\alpha+2\,H\cos\alpha\sin\alpha)x'^2\\&\quad+(A\sin^2\alpha+B\cos^2\alpha-2\,H\cos\alpha\sin\alpha)y'^2\\&\quad+[-2\,A\cos\alpha\sin\alpha+2\,H(\cos^2\alpha-\sin^2\alpha)+2\,B\cos\alpha\sin\alpha]x'y'\\&\quad+(2\,G\cos\alpha+2\,F\sin\alpha)x'+(2\,F\cos\alpha-2\,G\sin\alpha)y'+C=0.\end{aligned}$$

Let $A'x'^2+2\,H'x'y'+B'y'^2+2\,G'x'+2\,F'y'+C=0$

represent this equation.

We wish to show that it is always possible to find an angle α for which H' becomes 0.

Setting $H'=0$, leads to the equation $\tan 2\,\alpha=\frac{2\,H}{A-B}$, noting that $\cos^2\alpha-\sin^2\alpha=\cos 2\,\alpha$ and $2\sin\alpha\cos\alpha=\sin 2\,\alpha$. Since H, A, and B are real numbers and since the tangent of an angle can have any value from negative to positive infinity, it follows that there is always some angle $2\,\alpha$ for which $\tan 2\,\alpha=\frac{2\,H}{A-B}$. There are in fact always two positive angles, less than 360°, $2\,\alpha$ and $2\,\alpha+180°$, which satisfy the given relationship. By turning through α, or $\alpha+90°$, one half of either of these angles, the equation $Ax^2+2\,Hxy+By^2+\cdots=0$, reduces to an equation of the type $A'x'^2+B'y'^2+\cdots=0$, with the coefficient of the $x'y'$ term equal to 0. The angle α of turning can always be selected as a positive acute angle, since if $\tan 2\,\alpha$ is positive, $2\,\alpha$ may be taken as an acute angle, and if $\tan 2\,\alpha$ is negative, $2\,\alpha$ may be taken as an obtuse angle of which the half-angle α will be acute.

Illustrative problem. — What angle of rotation will remove the xy term from $3\,x^2+6\,xy-5\,y^2=100$?

$$\tan 2\,\alpha=\frac{2\,H}{A-B}=\frac{6}{8},$$

$$\cos 2\,\alpha=\tfrac{4}{5};\ \cos\alpha=\sqrt{\tfrac{1}{2}(1+\cos 2\,\alpha)}=\frac{3}{\sqrt{10}},$$

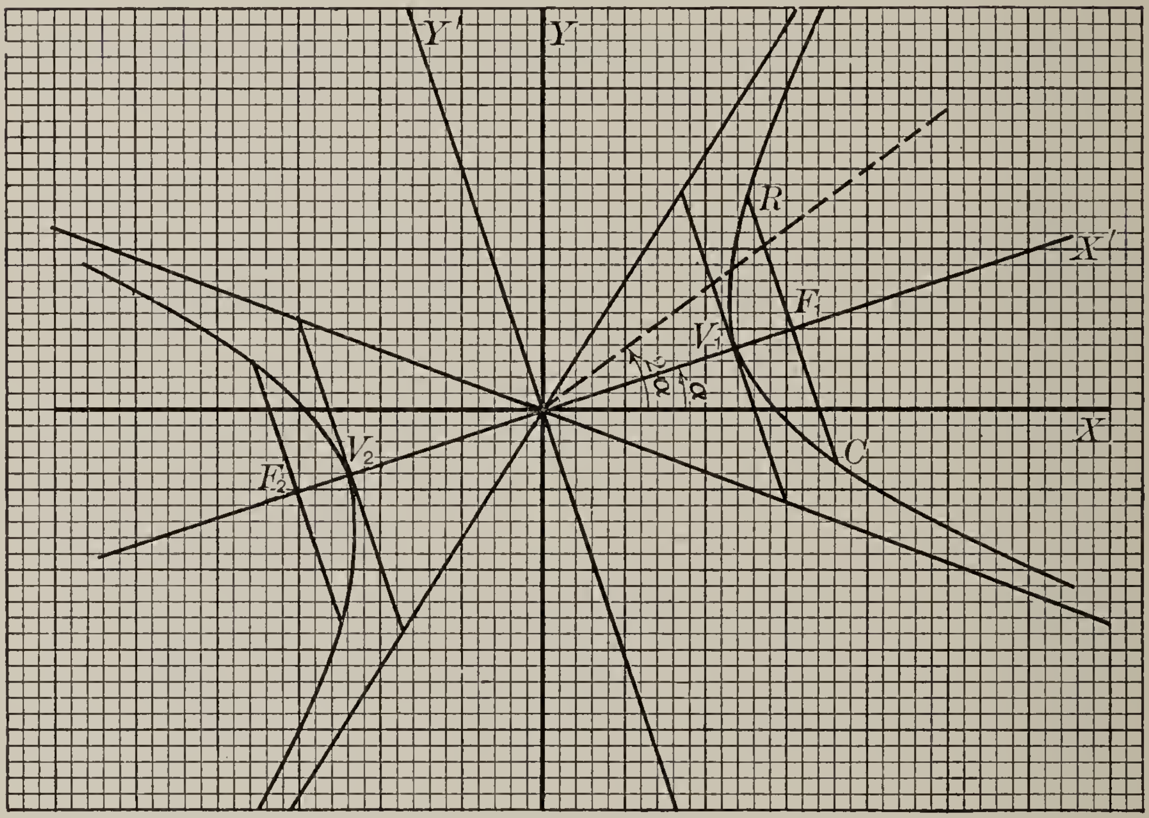
$$\sin\alpha=\sqrt{\tfrac{1}{2}(1-\cos 2\,\alpha)}=\frac{1}{\sqrt{10}}\cdot$$

We select α acute, as noted, hence the positive values of the radical are taken. The formulas of transformation become,

$$x = \frac{3}{\sqrt{10}}x' - \frac{1}{\sqrt{10}}y' = \frac{1}{\sqrt{10}}(3x' - y'),$$
$$y = \frac{1}{\sqrt{10}}x' + \frac{3}{\sqrt{10}}y' = \frac{1}{\sqrt{10}}(x' + 3y').$$

Substituting, we have,

$$\tfrac{3}{10}(3x' - y')^2 + \tfrac{6}{10}(3x' - y')(x' + 3y') - \tfrac{5}{10}(x' + 3y')^2 = 100.$$



The hyperbola $3x^2 + 6xy - 5y^2 = 100$, or $4x'^2 - 6y'^2 = 100$

In combining terms, do not write the expansion but preferably combine like terms by inspection.

Here the coefficient of x'^2 is $\frac{27}{10} + \frac{18}{10} - \frac{5}{10}$; of $x'y'$ the coefficient is $-\frac{18}{10} + \frac{48}{10} - \frac{30}{10}$, or 0, which checks; for y'^2 we have $\frac{3}{10} - \frac{18}{10} - \frac{45}{10}$. Our equation becomes,

$$4x'^2 - 6y'^2 = 100,$$
$$\frac{x'^2}{25} - \frac{y'^2}{16.67} = 1.$$

This curve is plotted with reference to the new axes, inclined at an angle α, $\tan \alpha = \frac{1}{3}$, to the x-axis. The coördinates of a point (x', y') on this curve, considered with respect to the new axes, satisfy the new equation

$\frac{x'^2}{25} - \frac{y'^2}{16.67} = 1$; when considered with reference to the old axes as (x, y), the coördinates satisfy the original equation. Thus the coördinates of the intersection with the original x-axis $(5.8, 0)$ satisfy the original equation; this point with reference to the new axes has the coördinates

$$x' = x \cos \alpha + y \sin \alpha = \frac{5.8 \times 3}{\sqrt{10}},$$

$$y' = -x \sin \alpha + y \cos \alpha = \frac{-5.8}{\sqrt{10}},$$

or $(5.5, -1.8)$. The values for (x', y') in terms of (x, y) can be conceived as obtained by rotating through the angle $-\alpha$.

PROBLEMS

1. Find the equation of the curve $xy - 7x + 3y - 15 = 0$ when referred to axes making an angle of 45° with the given axes. Note that α is 45°; $\sin \alpha = \cos \alpha = \frac{1}{\sqrt{2}}$; rationalize denominators after substituting. Plot the new axes at the angle indicated and plot the graph of the new equation, obtained by substitution, with reference to the new axes.

2. Find the equation of the curve

$$9x^2 + 24xy + 16y^2 - 6x - 15y = 0$$

with reference to axes making an angle arctan $\frac{4}{3}$ with the old axes. Note that $\sin \alpha = \frac{4}{5}$ and $\cos \alpha = \frac{3}{5}$; in substituting take the fraction $\frac{1}{5}$ as a factor in the value of both x and y and, after substituting, combine terms by inspection without writing each expansion separately.

3. Find the equation of the curve

$$59x^2 - 24xy + 66y^2 + 72x - 396y + 444 = 0,$$

when referred to new axes such that the new x-axis makes an angle whose tangent is $\frac{3}{4}$ with the old axis of abscissas.

4. In the equation $4xy - 8x + 10y + 7 = 0$ make the general substitutions which effect the turning of the axes through an angle α, and determine α so that the coefficient of the $x'y'$ term shall disappear.

6. Nature of the conic $Ax^2+2Hxy+By^2+2Gx+2Fy+C=0$.

A central conic is one which has a point which is such that every chord passing through this point is bisected. If this point be taken as origin of coördinates, it follows that if (x', y') is on the curve $(-x', -y')$ is also on the curve. A substitution, $x=x'+h$ and $y=y'+k$, which causes the terms of the first degree in our equation of the second degree to disappear gives the equation $Ax'^2+2Hx'y'+By'^2+C'=0$. Now whatever point (x', y') satisfies this equation $(-x', -y')$ will also satisfy the equation, and hence the new origin is the center of this conic.

The substitution $x=x'+h$ and $y=y'+k$ gives two linear expressions in h and k as coefficients of the new x' term and y' term, and these are set equal to zero and solved for h and k to determine the center.

$$2Ah+2Hk+2G=0$$

and

$$2Hh+2Bk+2F=0$$

are the two equations which determine the center.

If the two equations which serve to locate the center represent two parallel lines in h and k, the conic has no center and is a parabola. This condition is that $-\frac{A}{H}=-\frac{H}{B}$, or that $H^2-AB=0$. When $H^2-AB=0$, the terms $Ax^2+2Hxy+By^2$ form the square of a linear expression in x and y.

Further it is shown below that if $H^2-AB<0$, the conic is an ellipse, and if $H^2-AB>0$, the conic is a hyperbola. The conditions determining the nature of the general conic are as follows:

$$H^2-AB<0, \text{ ellipse,}$$
$$H^2-AB=0, \text{ parabola,}$$
$$H^2-AB>0, \text{ hyperbola.}$$

These are the conditions that there should be no points on the curve at infinity, one point at infinity, and two directions giving infinite points. They may be derived by substituting

$y = mx + k$ and determining values of m for which the quadratic has infinite roots; it follows that for these values of m the line $y = mx + k$ will meet the curve in points infinitely distant. For the ellipse the values of m will be imaginary, and $H^2 - AB < 0$; for the parabola the two values of m will coincide, and $H^2 - AB = 0$; for the hyperbola the two values of m will be real and different, representing the slopes of the two asymptotes, and $H^2 - AB > 0$.

A second and independent proof is given in the next article. There it is shown that the product $A'B'$ is positive when $H^2 - AB$ is negative; but when A' and B' are of the same sign the product is positive and the curve in x'^2 and y'^2, not involving $x'y'$, is an ellipse. Similarly the product $A'B'$ is negative when $H^2 - AB$ is positive, and the curve represented by the transformed equation is a hyperbola.

7. Central conics; abbreviated process of transformation. —

Substitution method. — Determine the center; transform to parallel axes with the center as new origin; determine α and substitute; plot with reference to the final axes.

Abbreviated method. — Determine the center (h, k); transform to (h, k) as new origin; determine A' and B' by solving as simultaneous the equations,

$$A' + B' = A + B,$$
$$- A'B' = H^2 - AB;$$

select the pair of values of A' and B' such that $A' - B'$ will have the same sign as H; plot the new equation with reference to new axes having the origin at the center determined and the axes inclined at an angle α with the old axes, α being such that $\tan 2\alpha = \dfrac{2H}{A - B}.$

Derivation of $A' + B' = A + B$; $- A'B' = H^2 - AB.$

$$A' = A\cos^2\alpha + B\sin^2\alpha + 2H\cos\alpha\sin\alpha.$$
$$B' = A\sin^2\alpha + B\cos^2\alpha - 2H\cos\alpha\sin\alpha.$$

By addition, $A' + B' = A + B.$

The proof that $-A'B' = H^2 - AB$ is somewhat long but not difficult. To the product $-A'B'$ add

$$H'^2 = [2H(\cos^2\alpha - \sin^2\alpha) - 2(A-B)\sin\alpha\cos\alpha]^2,$$

which does not alter the value since H' is taken to equal 0. The expressions will combine to $H^2 - AB$. The student would do well to verify at least one of the coefficients.

Since α is chosen as a positive acute angle, $A' - B'$ has the same sign as H, for $A' - B'$

$$\begin{aligned} &= (A-B)\cos 2\alpha + 2H\sin 2\alpha \\ &= 2H\left(\frac{A-B}{2H}\cos 2\alpha + \sin 2\alpha\right). \end{aligned}$$

Now $\sin 2\alpha$ is positive, and $\cos 2\alpha$ has the same sign as $\frac{2H}{A-B}$ and hence the product of $\cos 2\alpha$ by $\frac{A-B}{2H}$ is positive; hence $A' - B'$ is the product of $2H$ by the sum of two positive quantities and so is positive if H is positive and negative if H is negative.

The equations

$$A' + B' = A + B$$
$$-A'B' = H^2 - AB$$

enable us to determine A' and B' by solving these as simultaneous equations. Two solutions are found, and the solution is selected which makes $A' - B'$ have the same sign as H.

Only the new constant term, when transforming to (h, k) as new origin, offers any extended computation. This constant term

$$Ah^2 + 2Hhk + Bk^2 + 2Gh + 2Fk + C$$

may be written

$$h(Ah + Hk + G) + k(Hh + Bk + F) + Gh + Fk + C,$$

which reduces to $Gh + Fk + C$, since the other two expressions within parentheses were set equal to zero to determine the center.

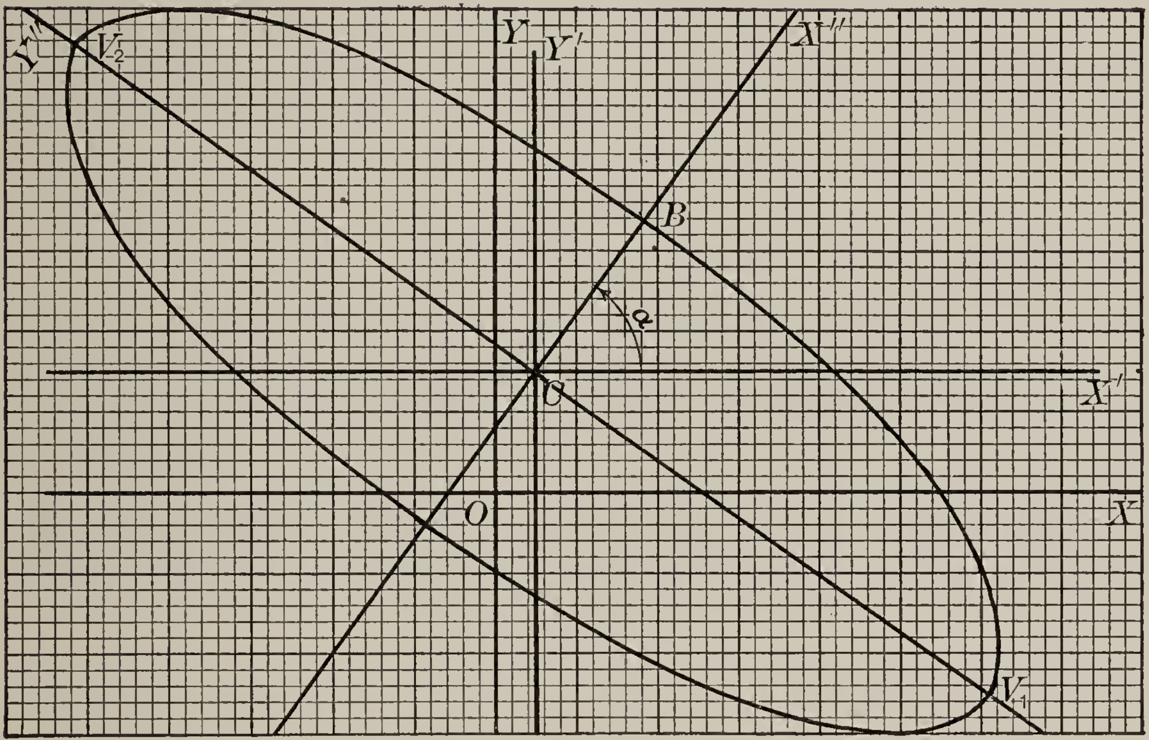
Illustrative problem. — Find center, axes, and plot the conic,

$$3x^2 + 6xy + 5y^2 - 12x - 18y - 24 = 0.$$

Substituting $(x' + h, y' + k)$ and selecting the coefficients of x' and y', to set equal to zero,

$$6h + 6k - 12 = 0,$$
$$6h + 10k - 18 = 0.$$

Solving, $k = \frac{3}{2}$, $h = +\frac{1}{2}$.



The ellipse $3x^2 + 6xy + 5y^2 - 12x - 18y - 24 = 0$

or $3x'^2 + 6x'y' + 5y'^2 - \frac{81}{2} = 0$, or $7.16x''^2 + .84y''^2 = 40.5$

C', the new constant, $Gh + Fk + C = -6 \cdot \frac{1}{2} - 9 \cdot \frac{3}{2} - 24 = -\frac{81}{2}$. The new (x', y') equation is

$$3x'^2 + 6x'y' + 5y'^2 - \frac{81}{2} = 0. \quad \text{Note that } \tan 2\alpha = \frac{6}{-2} = -3.$$

$$A' + B' = 8,$$
$$-A'B' = 9 - 15 = -6.$$

Solving by substitution,

$$-A'(8 - A') = -6,$$
$$A'^2 - 8A' + 6 = 0; \quad A' = 4 \pm \sqrt{10}.$$
$$B' = 4 \mp \sqrt{10}.$$

$A' - B'$ has the same sign as H; hence the upper algebraic signs are taken, $A' = 7.16$, $B' = .84$.

Our final equation is

$$7.16\,x''^2 + .84\,y''^2 = 40.5$$

$$\frac{x''^2}{5.66} + \frac{y''^2}{48.3} = 1$$

$$\frac{x''^2}{(2.38)^2} + \frac{y''^2}{(6.95)^2} = 1.$$

Some computation is unavoidable, and, in general, in practical applications the results are rarely expressible in small and convenient integers.

PROBLEMS

1. Find the center, axes, and plot the conic,

$$5\,x^2 - 6\,xy + 3\,y^2 + 12\,x - 6\,y - 30 = 0.$$

2. Plot the following conics by turning the axes through an angle α, $\tan 2\,\alpha = \frac{2\,H}{A - B}$, so as to eliminate the xy term, and thus obtain an equation to plot which can be put in standard form.

a. $4\,x^2 + 4\,xy + y^2 - 6\,x + 8\,y - 12 = 0.$

b. $x^2 - 4\,xy + y^2 + 2\,x - 10\,y - 11 = 0.$

c. $41\,x^2 + 24\,xy + 34\,y^2 - 26\,x - 32\,y - 171 = 0.$

d. $4\,xy - 3\,y^2 - 7\,x - 10\,y - 15 = 0.$

3. Apply the abbreviated method explained in section 6 to the central conics in the preceding problem; compare the numerical work involved by the two methods.

4. Find five points on the first and second conics in problem 2 by giving values to x and computing the corresponding values of y.

5. Find the intercepts with the coördinate axes of each of the conics in problem 2 and verify your graphical construction by these points.

6. In each of the conics of problem 2 find the points of intersection with the line $y = mx + b$; determine the values of m for which one of the points of intersection should be at an infinite distance. In the case of the hyperbolas real values of m will be found; substitute in turn each of these values for m and determine for what value of b the second point of intersection will move off to an infinite distance. This determines the two asymptotes. Explain.

7. Apply the abbreviated method to the discussion of the following central conics, having the origin as center:

$$
\begin{aligned}
&a.\quad x^2 + 2\,xy + 4\,y^2 = 16.\\
&b.\quad 4\,x^2 - 6\,xy - 3\,y^2 = 10.\\
&c.\quad 2\,x^2 - 4\,xy - y^2 = -\,9.\\
&d.\quad 5\,x^2 - 3\,xy + y^2 = 24.
\end{aligned}
$$

8. In the hyperbolas of problem 2, use the results of problem 6 to show that the directions of the asymptotes are given by the factors of the terms of the second degree.

8. The hyperbola as related to its asymptotes. — The equation of the hyperbola in simplest form,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

may also be written,

$$(bx - ay)(bx + ay) = a^2b^2,$$

whence,

$$\frac{bx - ay}{\sqrt{a^2 + b^2}} \cdot \frac{bx + ay}{\sqrt{a^2 + b^2}} = \frac{a^2b^2}{a^2 + b^2}.$$

Since $bx - ay = 0$ and $bx + ay = 0$ represent the asymptotes of this hyperbola, the final form states that the product of the perpendicular distances of any point on the hyperbola from the two asymptotes is constant. The converse proposition is also true, viz., if a point moves so that the product of its distances from two intersecting lines is a constant, the point moves on a hyperbola of which the two lines are the asymptotes. The proof of the converse is simply that the bisectors of the angles between the two given lines could be selected as axes of coördinates and, in consequence, the two lines would have as equations, expressions of the form $y - mx = 0$ and $y + mx = 0$. Any point which moves so that the product of its distances from these two lines is a constant would satisfy the equation

$$\frac{y - mx}{\sqrt{1 + m^2}} \cdot \frac{y + mx}{\sqrt{1 + m^2}} = k;$$

but this equation represents a hyperbola, and consequently the given locus is a hyperbola.

It follows from the above argument that the equation of any hyperbola differs by a constant from the product of the first-degree expressions which, put equal to zero, represent its asymptotes. The terms of the second degree in the hyperbola can be factored always into real linear factors in x and y (not necessarily rational so far as the coefficients are concerned) which as lines have the slopes of the asymptotes. (See problem 8 of the preceding list, and compare article 6.) A particularly simple type of hyperbola equation occurs quite frequently in practical problems and this type will be taken to illustrate the method which is, however, general.

Illustrative example. — Plot the curve

$$y = \frac{x}{1 - x}.$$

This equation may be written

$$y(1 - x) = x.$$

The only term of the second degree is xy. Placing the factors equal to zero, we have $x = 0$ and $y = 0$. The asymptotes are parallel to our coordinate axes. The equation can be written in the form

$$(x - h)(y - k) = c.$$

By inspection we note that the equation may be written

$$(y + 1)(x - 1) = - 1.$$

The asymptotes are given by $y + 1 = 0$ and $x - 1 = 0$.

The intersection point is the center of the given curve; further points should be plotted by substitution of values in the original equation.

This equation in i and d, $i = \frac{d}{1 - d}$, represents the relation between a given rate of discount for any interval and the corresponding rate of interest. If a bank in lending money takes out interest in advance, giving to the individual not the face of the loan but that amount less the interest upon that amount for the given interval for which the note is to run, the bank is said to discount the note. The rate of interest which the individual pays is obviously greater than the rate d which is used as the discount rate; the relation is

$$i = \frac{d}{1-d}.$$

In plotting the graph of this curve you would be interested only in values of i and d between .01 and .10, and you would confine your attention to the first quadrant, taking $\frac{1}{2}$ inch to represent .01 on both axes.

PROBLEMS

1. Plot the curve $p \cdot v = 1000$; show that it represents a hyperbola having the axes as asymptotes. This equation represents the relation between the pressure and volume of a quantity of gas which at a pressure of 1 atmosphere has a volume of 1000 cubic units, the temperature being kept constant.

2. Discuss the nature of the following curves, without making any transformation of axes; in the hyperbolas give the slopes of the asymptotes, and in the parabolas the slope of the axis.

a. $4x^2 - y^2 - 8y = 0.$

b. $4x^2 - 8y - 10 = 0.$

c. $4x^2 - 4xy - y^2 - 100 = 0.$

d. $4x^2 - 4xy + y^2 = 100y.$

e. $4x^2 - 4xy + y^2 = 100.$

f. $4x^2 - 4xy + 4y^2 = 100.$

g. $4x^2 - 4xy - 10x = 25.$

h. $4xy - 7x + 10y - 5 = 0.$

i. $xy = 15.$

j. $4x^2 + 4y^2 = 81.$

k. $3x^2 - 12x - 2y^2 - 10y - 15 = 0.$

l. $3x^2 - 12x + 2y^2 - 10y - 15 = 0.$

3. Transform to new axes so as to simplify the following equations to plot; select the appropriate method of substitution adapted to each equation.

a. $x^2 + 12\,xy + 4\,y^2 - 4\,x - 24\,y - 10 = 0.$

b. $x^2 + 3\,xy - 3\,y^2 - 10\,x - 15\,y + 24 = 0.$

c. $4\,x^2 - 4\,xy + 7\,y^2 - 10\,y + 4\,x - 25 = 0.$

d. $4\,x^2 - 12\,xy + 9\,y^2 - 6\,x - 10 = 0.$

e. $2\,x^2 + xy + 2\,y^2 - 6\,x + 6\,y - 15 = 0.$

f. $x^2 + 4\,xy - 2\,y^2 - 8\,x + 20\,y - 30 = 0.$

g. $x^2 - 3\,xy - 7\,x = 0.$

h. $2\,x^2 - 6\,x + 5\,y^2 - 20\,y - 10 = 0.$

4. Given that an aëroplane covers a distance of one hundred miles in t hours, its velocity in miles per hour is $\frac{100}{t}$, *i.e.* $v = \frac{100}{t}$; given that on different occasions the aëroplane covers 100 miles in 48 minutes (.8 hours), 1 hour, 1 hour 6 minutes, $1\frac{1}{2}$ hours, 1 hour and 24 minutes, 100 minutes, and 2 hours, respectively, find the velocities and plot a curve giving the relation between v and t. Choose units so that you can read from the curve between the extreme values the velocity within 2 miles per hour when the time of flight for 100 miles is given. Note that the curve is a hyperbola.

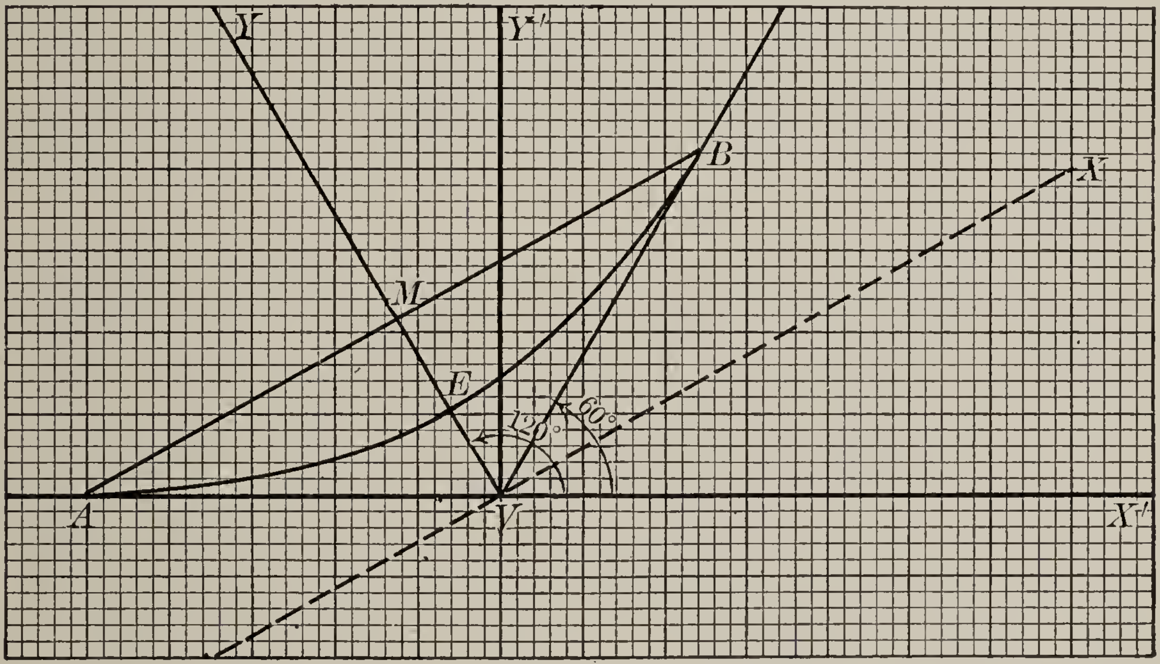
5. Given that an aëroplane covers on one trial 100 miles in 48 minutes, on another trial 125 miles in 61 minutes, and 156 miles in 71 minutes on a third trial, how could you compare graphically the corresponding velocities?

6. The air in an organ pipe vibrates in a manner somewhat similar to the motion of a pendulum; the number of such vibrations of the air in one second depends upon the length of the pipe and upon the velocity of sound in air; the formula $n = \frac{v}{2\,l}$, v in feet per sec. and l in feet, gives quite closely the number of vibrations. Plot the curve $n = \frac{1090}{2\,l}$, for values of l from 1 to 20 feet, choosing appropriate units. The curve gives the corresponding number of vibrations for pipes of different lengths. (See section 3, chapter 26.)

7. Discuss fully the equation

$$x^2 - 2\,xy + y^2 - 10\,y = 0.$$

8. The curve of transition on a railroad track in passing from one straight track to another is sometimes taken as parabolic, because of the fact that the slope changes uniformly with uniform increases of the horizontal length taken parallel to the tangent at the vertex of the parabola. Assuming that the track AV changes its direction by 60° to VB and that the transition points A and B from the straight line to the parabolas are taken on each track 500 feet from the point of intersection of the two directions, find the equation of the parabola. Note that the axis VY is inclined at an angle of 120° to the extension of AV; note that E, the vertex of the



Parabolic transition curve on a railroad track

The parabolic arc is used for vertical as well as for horizontal transition curves.

parabola, is midway between V and the point where the chord AB cuts the axis, since the tangent to a parabola cuts off from the vertex on the axis a distance equal to the distance cut off from the vertex on the axis by the perpendicular to the axis from the point of tangency. Find the equation of the curve with respect to the axis of the parabola as y-axis and the line through V at an angle of 30° with AV as x-axis; then transform to AV as x'-axis and a perpendicular to AV at V as y'-axis by turning through an angle of $-30°$, using the fundamental formulas for rotation of axes.

9. Assuming that a railroad track changes its direction by 40°, 30°, 20°, and 10° respectively, find the equations of the parabolic transition curves with transition points (A and B, as in figure) 500 feet from the intersection point of the two straight tracks.

10. In going over a hill the form of curve to which the track bed is rounded is often made parabolic. When the grade is the same on both sides of the highest point, the problem is precisely that of finding a parabolic arch. Assuming that in a horizontal distance of 5000 feet the hill rises 100 feet, find the equation of the parabola having the vertex at the highest point and passing through the point 100 feet lower at a horizontal distance of 5000 feet; find the four intermediate ordinates at distances 1000 feet apart.

11. An iron wire of diameter .2 cm. and length l cm., subjected to a tension T caused by a weight W grams, when caused to vibrate through its whole length has the number of vibrations determined by the equation

$$n = \frac{1}{2l}\sqrt{\frac{980\,W}{.077\,\pi}}; \qquad T = 980\,W.$$

When the weight is fixed and the length is variable, this gives a hyperbolic relation between n and l. For $W = 500$ grams the equation is approximately $n = \frac{1030}{l}$. Plot and discuss.

12. In the preceding problem suppose that l is fixed at 100 centimeters and that W varies between 100 grams and 2000 grams. What is the type of relationship? What would be the curve obtained by plotting to w- and n-axes?

13. The deck of any large vessel slopes from both bow and stern downwards towards amidships. The vertical section of the deck from bow to stern consists of two parabolas, having a common vertex at the middle of the ship. Plot the parabolas which are used for a vessel 400 feet long, having the highest point at the bow 8 feet above the vertex, and at the stern the deck 4 feet above the vertex. Use a different scale for y than for x, — at least twice as large.

14. Name the following curves, giving such facts as you can by inspection:

a. $3x + 2y - 5 = 0.$

b. $3x^2 + 2y - 5 = 0.$

c. $3x^2 + 2y^2 - 5 = 0.$

d. $3x^2 - 5x = 0.$

e. $3x^2 - 5xy - 5x = 10.$

f. $3x^2 + 3y^2 = 25.$

g. $3x^2 - 6xy + 3y^2 - 5x = 0.$

h. $3x^2 - 6xy + 3y^2 - 5 = 0.$

i. $3x^2 + 2y^2 + 5 = 0.$

j. $(3x + 2y - 5)(x - 3) = 0.$

k. $(3x + 2y - 5)(x - 3) = 10.$

l. $3x^2 + 3y^2 = 0.$

m. $xy - 7x + 6y - 18 = 0.$

n. $p(3 - v) = 6.$

o. $(1 + i)(1 - d) = 1.$

p. $\frac{1}{x} + \frac{1}{y} = 5.$

q. $V = \sqrt{\frac{T}{7 \cdot 7}}.$

r. $V = 331.7\sqrt{1 + \frac{T}{273}}.$

15. The highway over the Michigan Central R.R. tracks and over the Huron River, on the Whitmore Lake road near Ann Arbor, is rounded off (in profile) to a parabolic arc, rising 2.40 feet in a span of 240 feet. Show that the grade leading up to the arc should be a 4 % grade. Draw the arc to scale.

CHAPTER XXV

SOLUTION OF NUMERICAL ALGEBRAIC EQUATIONS

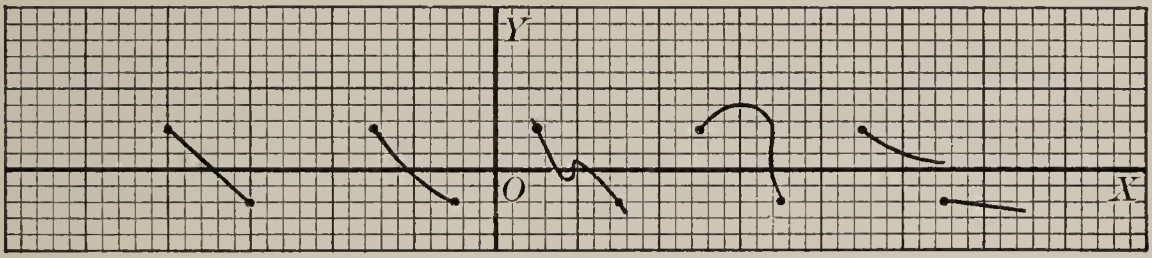
1. Solutions of algebraic equations. — By a solution of an equation of the type $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots a_{n-1}x + a_n = 0$, wherein n is a positive integer and a_0, a_1, $a_2 \cdots$ are constants, we understand a value which, substituted for x, reduces the left-hand member to zero. That such a solution always exists is proved by methods of higher mathematics. The theorem that every such rational integral algebraic equation has a root is called the *fundamental theorem of algebra;* it was first proved about a century ago by Gauss. The solution may be a real number or a complex number, and any constant coefficient may be real or complex; the latter involves the square root of a negative quantity and so is not representable as the abscissa of any point on our axis of positive and negative real numbers.

Certain types of algebraic equations are solvable in terms of the general constants which enter as coefficients. Thus $ax + b = 0$ is solvable in terms of a and b, and $ax^2 + bx + c = 0$ is solvable in terms of a, b, and c. It has been shown that the general cubic in one variable and the general biquadratic, or fourth degree equation, are solvable in this way, but the general equations of higher degree than the fourth are not solvable in this sense.

The approximate numerical solution of the real roots of rational integral equations with numerical coefficients is readily obtained and we have indicated in Chapter II and again in the preceding chapter, section 3, problems 8–10, the general method by which such solutions are obtained by substitution.

Simplifications for purposes of computation will be explained in this chapter.

2. Continuity. — The height of an individual is a continuous function of the age of the individual; by this we mean that in passing from one height to another the individual passes through every intermediate height. A graph representing age as abscissas and heights as ordinates will be a continuously connected curve. Upon this curve corresponding to any selected age, a_1, a period of time, there will be one and only one corresponding height, h_1, and corresponding to any second age, a_2, a second ordinate, h_2, representing height. The curve joining the two points (a_1, h_1) to (a_2, h_2) will be continuous and every intermediate height between the two given will be found to be represented by the ordinate corresponding to some age intermediate between the two given ages.



Four continuous graphs. One discontinuous

Continuity in passing from a positive to a negative value.

The rational integral function of x,

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$

in which n is any positive integer, is continuous between any two values of x, and will be represented by a continuous curve. This has been assumed in drawing the graph of $y = x^3 - 2x^2 - 18x + 24$, and in other graphs. The proof involves discussion somewhat too detailed and mathematically refined for an elementary course.

The symbol $f(x)$ will be used throughout the remainder of this chapter to represent a rational integral function of x of the type mentioned above.

3. Graph of $y = f(x)$ by location of points.

Give to x the appropriate values, find the corresponding values of y, and plot the points, connecting by a smooth curve. (See pages 70–71.)

Apply the remainder theorem, and employ synthetic division to determine values of the function corresponding to given values of x.

4. Remainder theorem and synthetic division. (See page 25.)

When $f(x)$ is divided by $x - a$, the remainder obtained by continuing the division until the remainder does not contain x is equal to the original expression with a put for x.

To divide $f(x)$ by $x - a$, employing synthetic division,

a. Arrange $f(x)$ in descending powers of x and write the coefficients horizontally, including zero coefficients for missing powers below the highest power which occurs.

b. Write $+a$ under $x - a$, the divisor, placed at the left. Under the coefficients of $f(x)$ as written leave space for a second horizontal row and draw a horizontal line.

c. Under the coefficient of the highest power of x, below the horizontal line drawn, place this coefficient again. Multiply by $+a$ and add to the following coefficient to the right. Place the sum below the line, vertically under the second coefficient; use this number below the line as multiplier of $+a$, and add the product to the third coefficient and continue this process until you have placed numbers under every coefficient (and the constant term) of the upper row. The final number which appears is the remainder and should be cut off by a vertical separator; the numbers under the horizontal line are coefficients in order from left to right of the quotient when $f(x)$ is divided by $x - a$.

Throughout this discussion a may be either positive or negative.

Illustrative Problem. — Divide $x^3 - 2x^2 - 18x + 24$ by $x - 3$, and use the remainder theorem to determine the value of this function of x when $x = 3$.

$$\begin{array}{rlll} & x^3 - 2x^2 & - 18x & + 24 \\ x - 3) & 1 - 2 & - 18 & + 24 \\ + 3) & + 3 & + 3 & - 45 \\ \hline & 1 + 1 & - 15 & (- 21 \end{array}$$

$x^2 + x - 15$ is the quotient and -21 is remainder; -21 is the value of $x^3 - 2x^2 - 18x + 24$ when 3 is substituted for x. Since

$$x^3 - 2x^2 - 18x + 24 \equiv (x^2 + x - 15)(x - 3) - 21,$$

we have, substituting 3,

$$\begin{aligned} 3^3 - 2 \cdot 3^2 - 18 \cdot 3 + 24 &\equiv (3^2 + 3 - 15)(3 - 3) - 21 \\ &\equiv 0 - 21. \end{aligned}$$

PROBLEMS

1. Locate ten points upon the graph of $y = 2x^3 + 3x^2 - 9x - 7$. Take the ten points between $x = -4$ and $x = +4$, including $\frac{1}{2}$ and $-\frac{1}{2}$; use the synthetic division method of finding the value of y except for $x = 0$, $x = 1$, and $x = -1$. Plot the points and draw a smooth curve connecting them; choose the y scale so as to keep the points on the paper. Locate the zeros of the function on the graph.

2. Plot the graph of the function $2x^3 + 3x^2 - 7$ between -3 and $+3$.

3. Plot the graph of the function $2x^3 - 9x - 7$. Note where the graph crosses the axis of x, thus locating the roots of the equation $2x^3 - 9x - 7 = 0$. Factor $2x^3 - 9x - 7$, dividing by the factor corresponding to the rational root which you have found; solve the resulting quadratic, and compare with the values found by the graph.

4. Plot the graph of the function $x^4 - 2x^3 + 3x^2 - 18x + 21$; select the appropriate interval to give the points of intersection with the x-axis.

5. Plot the graph of $y = x^4 - 3x^2 - 21$; locate the zeros of the function on the graph. Solve as an equation in quadratic form $x^4 - 3x^2 - 21 = 0$ and compare the solutions obtained with the roots located graphically.

6. Plot the graph of the function $4x^3 - 3x + .5$ in the interval from -1 to $+1$; substitute for x the values -1, $-.8$, $-.5$, $-.3$, $-.1$, 0, .1, .2, .3, .4, .5, .6, .7, .8, .9, and 1, finding the values in general, by the division method applying the remainder theorem. The roots of this equation represent the values of sin 10°, sin 50°, and sin − 70°. (See section 10, below.)

7. Locate one root between 0 and .1 of the equation

$$4x^3 - 3x + .05234 = 0,$$

by substituting for x the values 0, .01, .02, .03, up to .1. The value .05234 is the sine of three degrees which we obtained in problem 5, page 245. One root of this equation gives the sine of 1°.

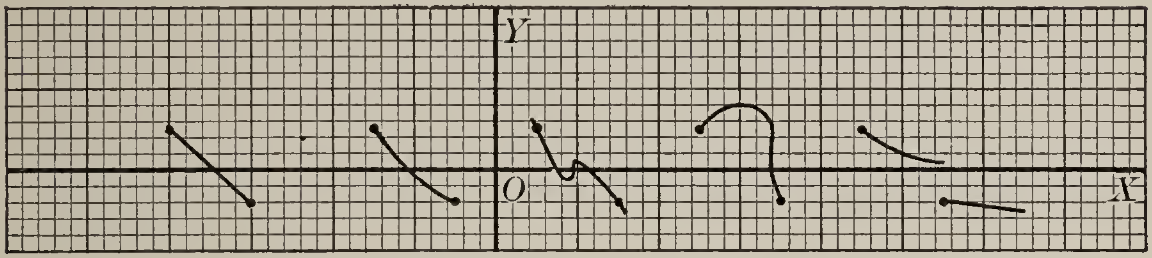
5. Number of roots. — A value of a_1, for which $f(a_1) = 0$, is a root of $f(x) = 0$. The remainder theorem applies and consequently $(x - a_1)$ is a factor of $f(x)$ since the remainder when $f(x)$ is divided by $x - a_1$ will be zero. Nothing in our argument requires that a_1 be a real number. Hence, dividing $f(x)$ by $(x - a_1)$, a new equation of degree one less will be obtained. This equation, by the fundamental theorem of algebra, also has a root, a_2, giving a quotient of degree $n - 1$. The number of such factors corresponds to the degree of the equation, n.

Every rational integral equation of the nth degree has n roots, and no more. For no further value of x could make the product, $a(x - a_1)(x - a_2)(x - a_3) \cdots (x - a_n)$, equal zero without making one of the factors zero and thus coinciding with one of the roots given.

6. Graphical location of real roots. — Any real root of a rational integral function of x equated to zero is a value of x which makes the ordinate in $y = f(x)$ equal to zero. The points in which the graph of the function of x crosses, or touches, the x-axis correspond to real roots of the equation, $f(x) = 0$, or zeros of the function.

Our assumption of continuity enables us to formulate the following theorem:

Between any two values $x = a$ and $x = b$, for which the two corresponding values of $f(x)$ are opposite in sign, there lies at least one real root of the equation $f(x) = 0$.



Four graphs passing continuously from $y = \frac{1}{2}$ to $y = -\frac{1}{3}$; one graph with a discontinuity

Thus to change continuously from $+\frac{1}{2}$ to $-\frac{1}{3}$, or from any positive value to any negative value, the function must pass through all values intermediate, including 0. At this point where the function of x is 0, the graph of $y = f(x)$ crosses the axis.

Illustrative problem. — Locate the roots of

$$x^3 - 2x^2 - 18x + 24 = 0.$$

Plot the graph of $y = x^3 - 2x^2 - 18x + 24$ by location of points. Give to x values from -5 to $+5$, find the corresponding values of y, and plot the points, connecting by a smooth curve. (See page 71.) Between $x = 1$ and $x = 2$, $f(x)$ changes from $+5$ to -12; there is a root between $x = 1$ and $x = 2$; between $x = 4$ and $x = 5$ there is a root, as $f(4)$ is -16 and $f(5)$ is $+9$; at $x = -4$ there is a root, as $f(-4)$ is 0.

7. Slope of $y = f(x)$. — The (h, k) method of finding the tangent at a point (x_1, y_1) on a curve applies, as we have stated in Chapter 18, section 11, to the graph of a rational integral function of x.

Thus in $y = x^3 - 2x^2 - 18x + 24$, let (x_1, y_1) be any point on the curve and $(x_1 + h, y_1 + k)$ a second point. It is desired to find the slope of the graph at (x_1, y_1)

$$y_1 = x_1^3 - 2x_1^2 - 18x_1 + 24,$$

and $y_1 + k = (x_1 + h)^3 - 2(x_1 + h)^2 - 18(x_1 + h) + 24$, since (x_1, y_1)

and $(x_1 + h,\ y_1 + k)$ are on the curve. Subtracting the upper from the lower equation, member for member, we have,

$$k = h(3\,x_1^2 - 4\,x_1 - 18) + h^2(3\,x_1 - 2) + h^3,$$

$$\frac{k}{h} = 3\,x_1^2 - 4\,x_1 - 18 + h(3\,x_1 - 2) + h^2.$$

Let h approach zero; the terms on the right containing h and h^2 will also approach zero, as the coefficients are constants.

$$\text{Limit}\ \frac{k}{h} = 3\,x_1^2 - 4\,x_1 - 18, \text{ as } h \doteq 0.$$

When $x_1 = 1$, the slope of the curve is -19; when $x_1 = 3$, the slope is -3; when $x_1 = 4$, the slope is $+14$.

A double root of any equation corresponds to a point at which the function is zero and the slope of the curve, obtained by the (h, k) method, is zero.

8. Slope and maximum and minimum points.— When the slope is zero, the curve is for the instant parallel to the x-axis. This is a necessary condition for a maximum or minimum point, *i.e.* a point at which the value of the function attains a greatest or a least value in some interval which includes the point.

This may be accepted by the student as graphically evident. A formal proof depends on the methods of the calculus, and rests essentially on the method used in finding the slope.

PROBLEMS

See the preceding list of problems.

1. Find the slope at any point (x_1, y_1) of each of the following curves and locate the maximum and minimum points on the curve by setting the slope equal to 0 and solving for x_1:

a. $y = 2\,x^3 + 3\,x^2 - 9\,x - 7.$

b. $y = 2\,x^3 + 3\,x^2 - 7.$

c. $y = x^4 - 3\,x^2 - 21.$

d. $y = 2\,x^3 - 9\,x - 7.$

e. $y = 4\,x^3 - 3\,x + .5.$

f. $y = 4\,x^3 - 3\,x + .05234.$

2. Find the slope at any point (x_1, y_1) on

$$y = x^4 - 2x^3 + 3x^2 - 18x + 21.$$

This gives the slope m as $m = 4x_1^3 - 6x_1^2 + 6x_1 - 18$. Plot the graph of $y = 4x^3 - 6x^2 + 6x - 18$, and note that the zeros of this function are the values of x_1 for which the slope of the curve $y = x^4 - 2x^3 + 3x^2 - 18x + 21$ is 0. These are values of x for which the original function has maximum and minimum values.

9. Historical note. — The solution early in the sixteenth century of the cubic and biquadratic was the undisputed achievement of a group of Italian mathematicians. Fiori, Tartaglia, and Cardan were involved in the solution of the cubic, while Ferrari, pupil of Cardan, solved the quartic. Not until the beginning of the nineteenth century was it shown that the general equations of higher degree are not solvable, this being the work of a brilliant young Norwegian named Abel.

10. The cubic applied to angle trisection. — By higher mathematics it has been demonstrated that geometrical problems which can be solved by ruler and compass correspond algebraically to problems whose solution can be effected by linear and quadratic equations and equations reducible to quadratics, *i.e.* by equations of which the roots will involve only quadratic irrationalities (square roots, and square roots of expressions involving only rational quantities and square roots). The trisection of an angle is a type of geometrical problem whose solution cannot be effected with ruler and compass; it is possible to reduce the trisection of an angle to an algebraical problem, the solution of the cubic equation.

Let the given angle which is to be trisected be denoted, for convenience, by 3α. Since this angle is given, the value of its sine is known. If the angle is given by a geometrical drawing, the ratio of the perpendicular h dropped from a point at a distance r from the vertex on one side to the second side to r, *i.e.* $\frac{h}{r}$, gives the sine of the angle. Let the value of the sine of the given angle be k.

Given $\sin 3\alpha = k$, find $\sin \alpha$.

$$\sin 3\alpha = \sin(2\alpha + \alpha) = \sin 2\alpha \cos \alpha + \cos 2\alpha \sin \alpha$$
$$= 2 \sin \alpha \cos^2 \alpha + (\cos^2 \alpha - \sin^2 \alpha) \sin \alpha$$

$$\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha.$$
$$k = 3 \sin \alpha - 4 \sin^3 \alpha.$$

This equation is a cubic in the unknown $\sin \alpha$; for convenience it may be written $k = 3x - 4x^3$, substituting x for $\sin \alpha$.

There are, in fact, three solutions of the cubic and these three solutions correspond to the fact that k is the sine not only of 3α, but also of $180° - 3\alpha$, and $n \cdot 360° + 3\alpha$, and $(2n + 1) 180° - 3\alpha$.

Thus the cubic which would give the sine of 10°, trisecting the angle of 30°, is $.5 = 3x - 4x^3$, or $4x^3 - 3x + .5 = 0$. The same cubic would be obtained if it were desired to trisect the angle of 150°, or of 390°, or 750°, …. There are an infinite number of angles which have this same sine, .5, but there will be only three different values involved when the sine of the third part of each of these angles is found. In the equation $4x^3 - 3x + .5 = 0$, the roots represent sin 10°, sin 50°, and sin 250°. (See problem 6, page 396.)

11. Closer approximation to located roots. — The method will be shown by a numerical illustration.

The equation

(1) $$x^3 - 2x^2 - 18x + 24 = 0,$$

of which the graph is given on page 71, evidently has a root between 4 and 5. To form the new equation whose roots are 4 less than the given equation, substitute $x' + 4$ for x, giving

(2) $$(x' + 4)^3 - 2(x' + 4)^2 - 18(x' + 4) + 24 = 0.$$

Assume that this gives

(3) $x'^3 + Bx'^2 + Cx' + D = 0$, in which B, C, and D can be obtained by expanding and combining terms in (2). The left-

hand members of equation (3) and equation (2) are then identical. Evidently, if $x-4$ is substituted for x' in (2), it will give the original equation, and consequently, if $x-4$ is substituted in (3), it will give the original equation. Substituting, we have

(4) $(x-4)^3 + B(x-4)^2 + C(x-4) + D = 0$, which is identical with the original equation.

If the left-hand member of this equation, *i.e.* the original, is divided by $x-4$, the remainder is D and the quotient is $(x-4)^2 + B(x-4) + C$; if this quotient is divided by $(x-4)$, the remainder is C; if the quotient of the preceding division is divided by $x-4$, the remainder is B.

$$\begin{array}{llllll}
x-4)\ x^3 & -\ 2x^2 & -18x & +24 & & \\
+4) & +\ 4 & +\ 8 & -40 & & \\ \cline{1-4}
1 & +\ 2 & -10 & (-16 & & D \\
 & +\ 4 & +24 & & & \\ \cline{1-3}
1 & +\ 6 & (+14 & & & C \\
 & +\ 4 & & & & \\ \cline{1-2}
1 & (+10 & & & & B
\end{array}$$

The continued division by $x-4$ is effected by the synthetic process, explained in section 4, above.

(5) $x'^3 + 10x'^2 + 14x' - 16 = 0$ is then the equation whose roots are 4 less than the roots of the original equation. This should be verified by substitution and expansion. The original equation has a root between 4 and 5. Hence this equation has a root between 0 and 1. By trial of tenths, .1, .2, ⋯ .9, this equation is found to have a root between .7 and .8. Hence the original equation has a root between 4.7 and 4.8.

Form the new equation whose roots are .7 less than the roots of (5).

$$\begin{array}{lllll}
x'-.7)1 & +10 & +14 & -16 & (.7 \\
+.7) & +\ \ .7 & +\ \ 7.49 & +15.043 & \\ \cline{1-4}
1 & +10.7 & +21.49 & (-\ \ .957 & \\
 & +\ \ .7 & +\ \ 7.98 & & \\ \cline{1-3}
1 & +11.4 & (+29.47 & & \\
 & +\ \ .7 & & & \\ \cline{1-2}
1 & (+12.1 & & &
\end{array}$$

(6) $$z^3 + 12.1z^2 + 29.47z - .957 = 0.$$

Equation (5) has a root between .7 and .8; hence equation (6) has a root between 0 and .1.

By trial of hundredths, trying .02, .03, .04 ··· it is found that this equation has a root $.03^+$, between .03 and .04 and evidently nearer .03.

Hence our original equation has a root 4.73^+.

In this way we can compute any real numerical root of a rational integral algebraical equation to any desired number of significant figures.

Illustrative Problems. — **1.** Find the cube root of 1,624,276 to four significant figures.

$x^3 - 1{,}624{,}276 = 0$. By trial, substituting 100, 200, ··· for x, this is found to have a root between 100 and 200.

$$\begin{array}{rrrrl}
x-100) & 1 & +\;\;0 & +\;\;\;\;0 & -\;1{,}624{,}276 \\
100) & & +\;100 & +\;10000 & -\;1{,}000{,}000 \\ \cline{2-5}
 & 1 & +\;100 & +\;10000\;(- & \;\;\;624{,}276 \\
 & & +\;100 & +\;20000 & \\ \cline{2-4}
 & 1 & +\;200\;(& +\;30000 & \\
 & & +\;100 & &
\end{array}$$

$$\begin{array}{rrrrl}
x'-10) & 1 & +\;300 & +\;30000 & -\;\;\;624{,}276 \\
10) & & +\;\;\;10 & +\;\;\;3100 & -\;\;\;331{,}000 \\ \cline{2-5}
 & 1 & +\;310 & +\;33100\;(- & \;\;\;293{,}276 \\
 & & +\;\;\;10 & +\;\;\;3200 & \\ \cline{2-4}
 & 1 & +\;320\;(& +\;36300 & \\
 & & +\;\;\;10 & &
\end{array}$$

By derivation, the roots are 100 less than the roots of (1); hence a root between 0 and 100. By trial, substituting, root between 10 and 20.

$$\begin{array}{rrrrl}
x''-7) & 1 & +\;330 & +\;36300 & -\;\;\;293{,}276 \\
7) & & +\;\;\;\;7 & +\;\;\;2359 & +\;\;\;270{,}613 \\ \cline{2-5}
 & 1 & +\;337 & +\;38659\;(- & \;\;\;\;\;22{,}663 \\
 & & +\;\;\;\;7 & +\;\;\;2408 & \\ \cline{2-4}
 & 1 & +\;344\;(& +\;41067 & \\
 & & +\;\;\;\;7 & & \\ \cline{2-5}
 & 1 & +\;351 & +\;41067 & -\;\;\;\;\;22{,}663
\end{array}$$

By derivation, has a root between 0 and 10. By trial, a root between 7 and 8.

By derivation, root between 0 and 1. By trial, between .5 and .6, and nearer to .5.

Hence the root of the original is 117.5, which may be partially checked by four-place logarithms.

2. Compute one negative root of

$$2x^4 + 10x^3 - 8x^2 - 11x + 19 = 0.$$

Negative root between -1 and -2.

$$\begin{array}{lrrrrr}
x+2) & 2 & +10 & -8 & -11 & +19 \\
-2) & & -4 & -12 & +40 & -58 \\
\hline
 & 2 & +6 & -20 & +29 & (-39 \\
 & & -4 & -4 & +48 & \\
\hline
 & 2 & +2 & -24 & (+77 & \\
 & & -4 & +4 & & \\
\hline
 & 2 & -2 & (-20 & & \\
 & & -4 & & & \\
\hline
x-.6) & 2 & -6 & -20 & +77 & -39 \\
+.6) & & +1.2 & -2.88 & -13.728 & +37.9632 \\
\hline
 & 2 & +4.8 & -22.88 & +63.272 & -(1.0368 \\
 & & +1.2 & -2.16 & -15.024 & \\
\hline
 & 2 & -3.6 & -25.04 & (+48.248 & \\
 & & +1.2 & -1.44 & & \\
\hline
 & 2 & -2.4 & (-26.48 & & \\
 & & +1.2 & & & \\
\hline
 & 2 & -1.2 & -26.48 & +48.248 & -1.0368
\end{array}$$

Root between 0 and 1. By trial, between .6 and .7.

By derivation has a root betwen 0 and .1. By trial, between .02 and .03.

The original equation has a root $-2 + .62^+$, or -1.38^-, *i.e.* between -1.38 and -1.37.

PROBLEMS

See the two preceding sets of problems and use the results obtained.

1. Compute to three significant figures the largest positive root of the following equations,

a. $2x^3 + 3x^2 - 9x - 7 = 0.$

b. $2x^3 + 3x^2 - 7 = 0.$

c. $2x^3 - 9x - 7 = 0.$

d. $x^4 - 2x^3 + 3x^2 - 18x + 21 = 0.$

2. Compute by the process indicated the positive root of $x^2 - 3x - 21 = 0$ to three decimal places; compute the same by solving as a quadratic, and compare as to efficiency the two methods.

3. Solve the equation $4x^3 - 3x + .5 = 0$, computing the smallest positive root to four decimal places. This is the value of sin 10°; check by your table of sines.

4. In problem 5, page 245, you have computed the sine of 3° to four decimal places. Write the cubic which will give the sine of 1°; compute the smallest positive root, and discuss to what decimal place it could be carried with propriety when the sine of three degrees is given to four decimal places.

5. Plot the graphs of the two equations $\begin{cases} xy = 1, \text{ and} \\ y = \frac{1}{4}(x^2 - 16). \end{cases}$ Note that the points of intersection give the solutions of the two equations regarded as simultaneous; but solving the two equations as simultaneous equations, we are led by substitution to the cubic $x \cdot \frac{1}{4}(x^2 - 16) = 1$, or $x^3 - 16x - 4 = 0$. Solve the cubic and compare with the solutions obtained graphically.

6. *Historical problem.* The great Archimedes proposed the problem to cut a sphere by a plane in such a way that the two segments of the sphere should have to each other a given ratio. Archimedes showed that the solution could be obtained as the intersection of a hyperbola and a parabola. If the diameter of the sphere is taken as 10 and k as the ratio of the larger to the smaller segment, this problem leads to the cubic

$$x^3 - 300x + 2000\frac{(k-1)}{k+1} = 0,$$

in which x represents the distance of the plane from the center of the sphere. Solve to two decimal places when $k = 2$. The plane at a distance x from the center then trisects the sphere.

7. In the preceding problem show that the solution may be obtained as the intersection of a hyperbola and a parabola.

8. A famous problem of antiquity is the problem to duplicate a given cube, *i.e.* to solve geometrically $x^3 = 2a^3$, a being the side of the given cube. Long before analytical geometry was invented it was known that the solution could be given as the intersection of the parabola $x^2 = ay$ with the parabola

$y^2 = 2\,ax$. Construct the graphical solution when a is taken as 10.

The problem may also be solved by the intersection of either of the two given parabolas with the hyperbola $xy = 2\,a^2$. Verify.

If two means, x and y, are inserted between a, and $2\,a$, *i.e.* $\frac{a}{x} = \frac{x}{y} = \frac{y}{2\,a}$, then x is the solution of the equation $x^3 = 2\,a^3$. This method reduced the problem of the duplication of the cube to the problem of inserting between two given numbers, or lines, two geometric means.

9. The volume of a spherical segment, greater than a hemisphere, of height $x + r$, is given by the expression

$$V = \frac{\pi}{3}(2\,r^3 + 3\,r^2x - x^3);$$

the volume of a sphere is $\frac{4}{3}\pi r^3$. Find the segment of a sphere of water of radius 10 which will be equal in weight to a sphere of wood, radius 10, which wood is only .6 as heavy as water. This leads to the cubic equation

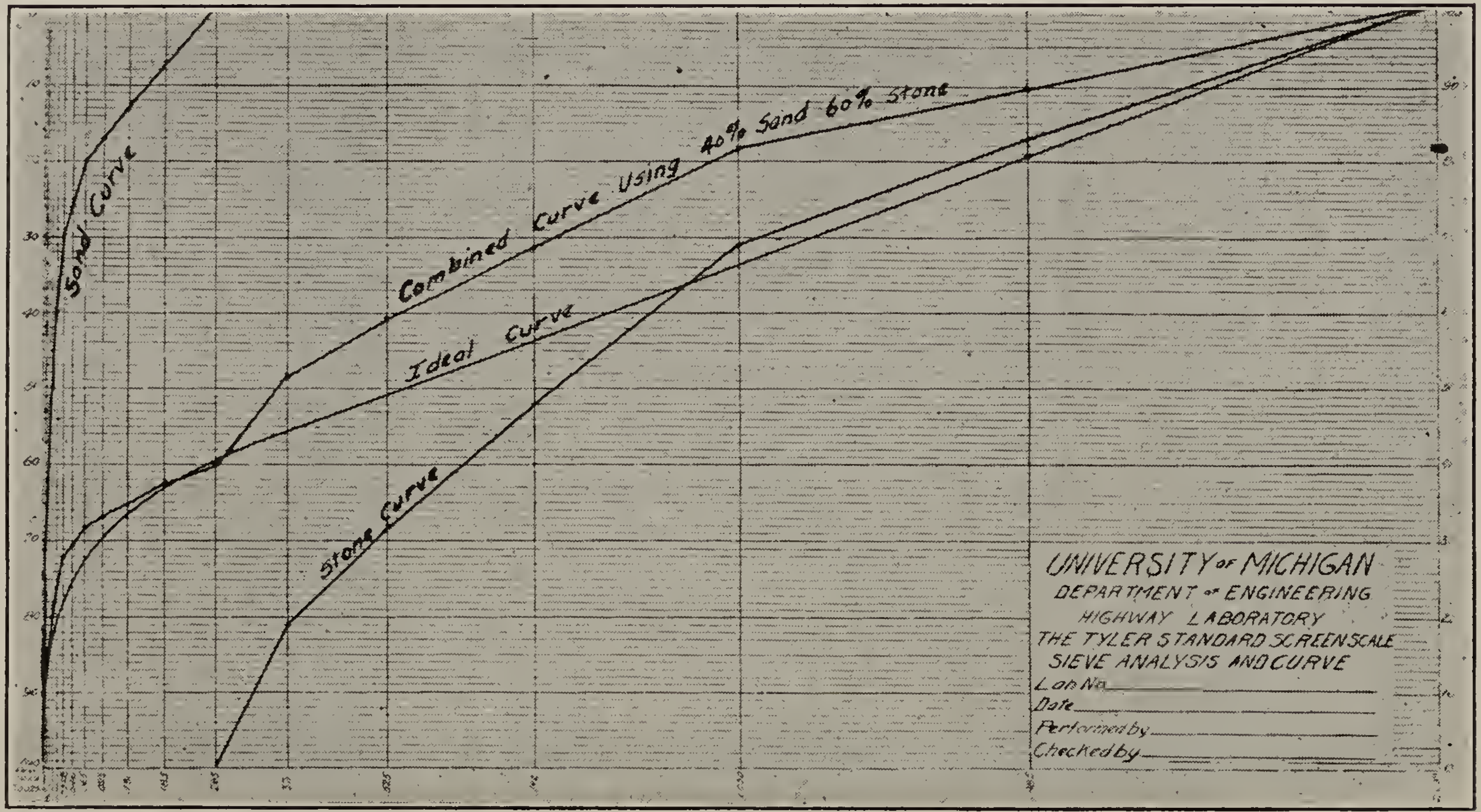
$$.6(\tfrac{4}{3}\,\pi r^3) = \frac{\pi}{3}(2\,r^3 + 3\,r^2x - x^3),$$

or

$$x^3 - 3\,r^2x + .4\,r^3 = 0,$$

and $r + x$ is the depth to which the sphere of wood will sink when it is placed in water. Compute this depth when $r = 10$.

10. Ice is only .92 as heavy as water. Use the equations of the preceding problem, substituting .92 for .6, to find the depth to which a spherical iceberg of radius 100 feet, if one were possible, would sink in water.



Ellipse and tangent to the ellipse used as an "ideal curve" to determine proper amounts of two or more given and analyzed types of gravel, sand, and stone to make the correct proportions for cement work

One of many uses of the conic sections in engineering work.

CHAPTER XXVI

WAVE MOTION

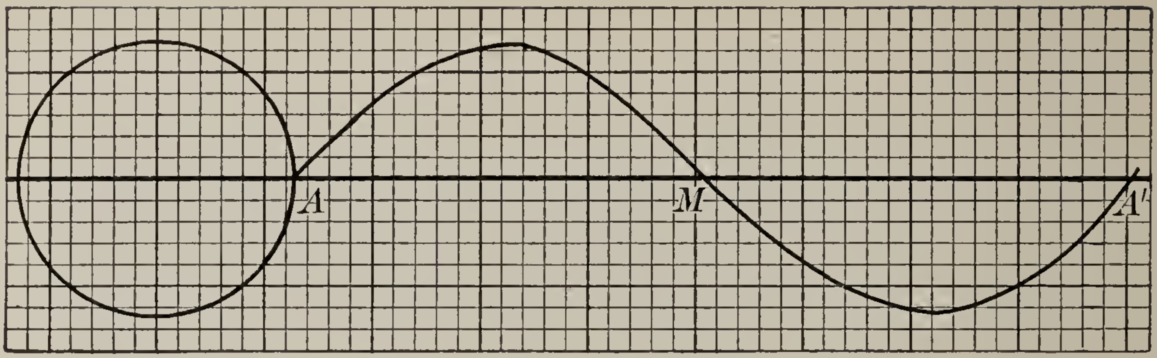
1. General. — In nature there are two types of recurrent motion, somewhat closely connected mathematically, in which repetition of motion occurs at regular intervals.

One type of this motion, in cycles as we may say, repeats the motion in one place, and is in a sense stationary. The tuning fork in motion moves through the same space again and again; a similar movement is the motion of a vibrating string. Of this stationary type may be mentioned the heartbeats, the pulse, the respiration, the tides, and the rotation of a wheel about its axis.

The second type of recurrent motion transmits or carries the vibratory impulse over an extent of space as well as time. The waves of the sea are of this character. Sound waves, electrical vibrations or waves, and radiant energy vibrations are transmitted by a process similar to that by which the waves of the sea are carried.

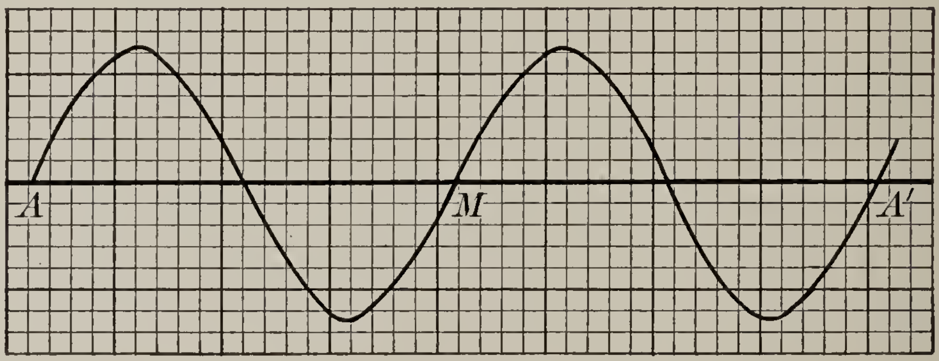
Both of these types of motion are representable mathematically by equations involving a sequence of trigonometric functions. To the fundamental and basic function involved, $y = \sin x$, we will direct our attention in the next section and to simple applications in other sections of this chapter.

2. The sine curve. — As a radius vector of unit length rotates in a plane with uniform velocity about a center, the sine of the angle θ fluctuates between 1 and -1. The variation of sine θ may be represented by the movement on the y-axis of the projection of the vector, and this movement of the



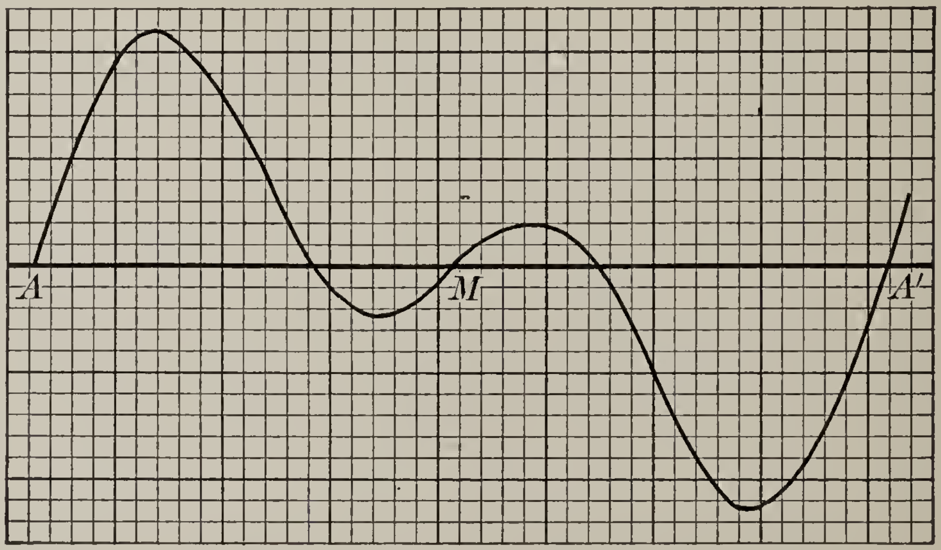
Graph of $y = \sin \theta$; a pure sinusoid

The length AA' equals the circumference of the circle; the amplitude, vertical distance between highest and lowest points, equals the diameter of the circle.



Graph of $y = \sin 2\,\theta$; a sinusoidal curve

The frequency is double that represented in the preceding graph.



$y = \sin \theta + \sin 2\,\theta$; obtained by addition of corresponding ordinates in the two preceding curves

This type of curve is obtained from a tuning fork having an octave overtone.

projection is termed simple harmonic motion, frequently abbreviated S. H. M. Precisely the same type of movement is given by the projection of the moving vector on the x-axis, $x = \cos\theta$, or on any line in the plane of the motion,

$$z = \cos(\theta - \epsilon),$$

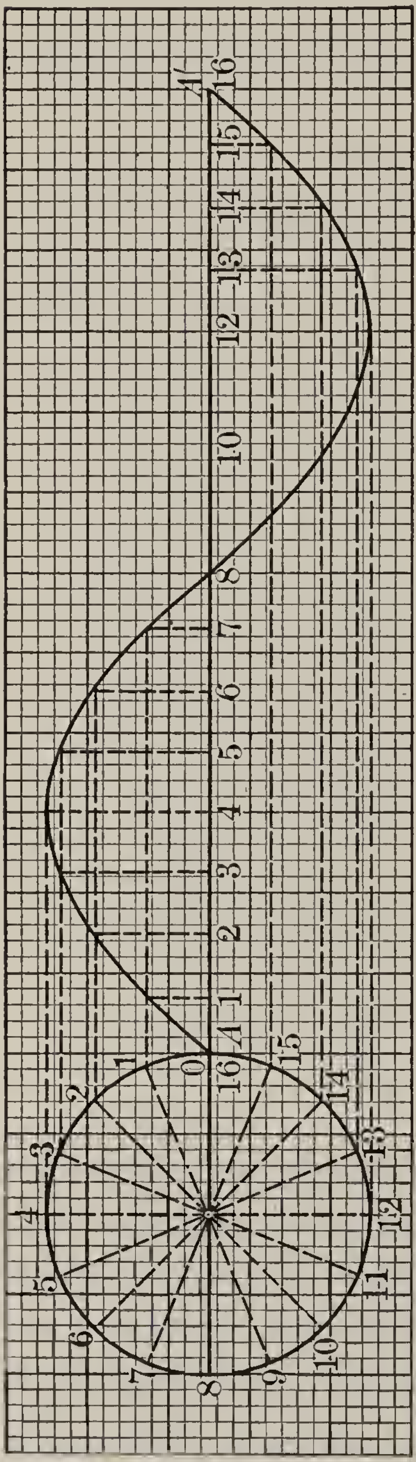
wherein ϵ is the slope angle of the line and z is the projection of the radius.

If the vector completes one revolution, $2\pi^r$, or 360°, in 1 second, the period of the motion is called 1 second, and the frequency, or the number of repetitions of the complete movement or cycle in a second, is 1 per second. If the complete revolution is effected in $\frac{1}{2}$ second, the period is $\frac{1}{2}$ second and the frequency 2 per second. The graphs of $y = \sin\theta$ and of $y = \sin 2\theta$ represent under these conditions the progress of the ordinate for uniform changes in θ, *i.e.* for uniform changes in the time, since the rotation is with constant angular velocity. For convenience the angle is conceived as measured in radians and the radius is taken on the x-axis as the unit to represent one radian; the abscissa then corresponds either to the angle measured in radians or to the length of arc traversed by the end of the moving vector. In plotting $y = \sin\theta$, values of θ from 0 to 360° or from 0 to $2\pi^r$ are plotted on the horizontal or θ-axis. Note particularly the points for which $\theta = 0$, 30°, 45°, 60°, 90°, ⋯ 180°, ⋯ 360°; or

$$0, \frac{\pi^r}{6}, \frac{\pi^r}{4}, \frac{\pi^r}{3}, \frac{\pi^r}{2}, \pi^r \cdots 2\pi^r.$$

Note that AA' on our diagram represents one complete cycle or period. For many purposes it is desirable to take t, the time (in seconds, usually), as the variable. The same graph then represents $y = \sin 2\pi t$, wherein AA' is taken equal to 1 and the horizontal axis is the t-axis. The same curve represents $y = \sin 20\pi t$, if AA' is taken as $\frac{1}{10}$ of 1 unit of time. The upper curve in our diagram is a pure sinusoid, the distance AA' representing the circumference of the circle of which the maximum ordinate is the radius.

The two curves plotted should be carefully studied; the lower curve has double the frequency of the upper and one half the period. The swing, amplitude as it is termed, is the same; the amplitude is the algebraic difference between the maximum and minimum values of the function.



Graphical method of locating points on a sinusoid

Any curve representing

$$y = a \sin \theta \quad \text{or} \quad a \sin k\theta$$

$$\text{or } a \sin (k\theta + \epsilon)$$

is called a sinusoid. We shall find that the graphs of $y = a \cos \theta$, $y = a \cos k\theta$, and $y = a \cos (k\theta + \epsilon)$ differ from the preceding only in position.

For most purposes it is convenient to plot time in complete units on the ordinary coördinate paper, the unit depending on the period of time in question. For a complete rotation in one minute ten seconds might be taken as one unit on the horizontal axis with the radius as vertical unit, and the curve would differ very slightly from our curve. The highest and lowest points would fall then at 15 and 45 respectively; 0, 5, 7.5, 10, 15, 20, and 30 seconds correspond then to 0, 30°, 45°, 60°, 90°, 120°, and 180° respectively.

Physicists and engineers commonly draw the sinusoidal curves which are of frequent occurrence entirely from graphical considerations. The circle with the desired amplitude is drawn; the angle between the axes is bisected and rebisected (as often as desired); an appropriate length for a complete cycle is taken on the horizontal axis, and this is divided into just as many parts, usually 16, as the circumference of the circle is divided by the axes and the bisecting lines which were drawn. At each point on the horizontal axis an ordinate is drawn and from each corresponding point on the circle a horizontal line is drawn to intersect the corresponding ordinate. Corresponding points have the same numbers if on the circle intersection points are numbered from the right-hand intersection with the horizontal axis counter-clockwise and numbered on the line from the left-hand end of the horizontal length taken to represent the time of one cycle, as indicated on our diagram. The two upper figures, page 408, were drawn by this method. The student is urged to make both the graphical construction and the construction by using the numerical values of the sines from the tables. Compare also the work under Section 11, Chapter VII.

PROBLEMS

1. Plot the curves $y = \sin \theta°$ and $y = \sin 3\,\theta°$ on the same sheet of coördinate paper; take 1 inch as radius and on the horizontal axis take 1 inch to represent 60°. For a pure sinusoid, $y = \sin x$, one unit on x should be the length of the radius; then 3.14^+ radians represents 180°, the second point in which the curve $y = \sin x$ cuts the axis of abscissas.

2. Plot $y = \sin 2\,\pi t$; note that $t = \frac{1}{10}, \frac{2}{10}, \cdots$ corresponds to 36° and multiples; take one unit for t as 6 times the radius chosen.

3. How could you interpret the curve of the preceding exercise as $y = \sin 4\,\pi t$?

4. Plot 10 points of $y = \sin(\theta - 30°)$. This curve is similar to the preceding; it is 30° behind, we may say, the regular sine curve; the "lag" is 30°; the two curves $y = \sin \theta$ and

$y = \sin(\theta - 30°)$ are said to be out of phase, the phase angle of the second being $-30°$. The "phase angle" is of particular importance in the theory and practice of alternating currents.

5. Plot the curve $E = 110 \sin \theta$. Note that if the horizontal scale be taken so that 1 inch represents 60° and the vertical scale such that 1 inch represents 110, the curve is precisely the first curve of problem 1. This curve represents the variable electromotive force (e. m. f.) developed by a generator which generates a maximum e. m. f. of 110 volts. To plot the curve no knowledge of electricity is necessary, but complete interpretation requires technical knowledge.

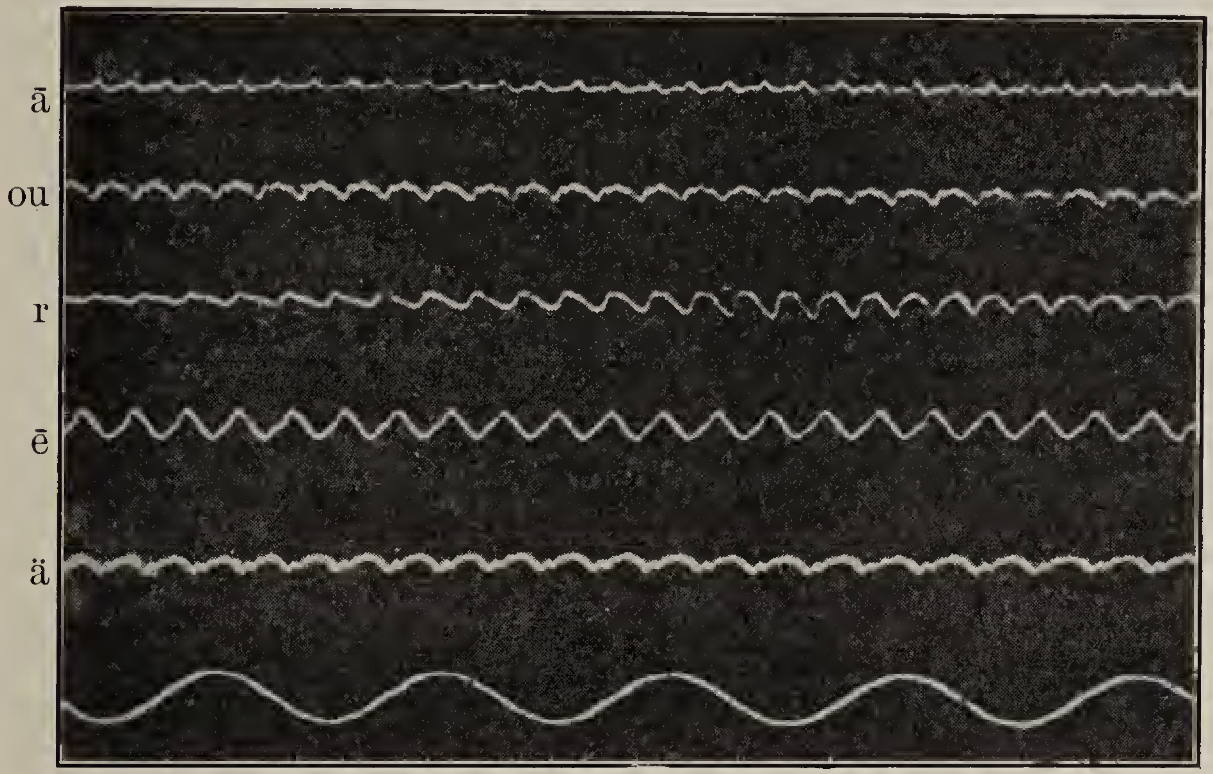
3. Sound waves. — If a tuning fork for note lower C is set to vibrating, the free bar makes 129 complete, back-and-forth, vibrations in one second. By attaching a fine point to the end of the bar and moving under this bar at a uniform rate, as it vibrates, a smoke-blackened paper, a sinusoidal curve is traced on the paper. Our curve is traced by a bar vibrating 50 times in 1 second.

The curve $y = \sin(50 \times 2\pi t)$

Tuning fork vibrations recorded on smoked paper.

In 1 second 50 complete vibrations are made; the vertical distance between the top and the bottom of the arcs represents the distance moved by any point on the moving bar; the motion is simple harmonic (S. H. M.). The period is $\frac{1}{50}$ second; the frequency is 50; the amplitude is about $\frac{1}{32}$ inch. If the smoked paper were moved with uniform velocity under the vibrating bar in such a way as to cover 50 times the circumference of a circle with radius $\frac{1}{64}$ of an inch, or $50 \times 2\pi \times \frac{1}{64}$ inch per second, the curve traced would be almost a perfect sinusoid of the type $y = \sin \theta$. The points move of course on arcs of curves, but the variation from a straight line is extremely slight.

Corresponding to each movement of the vibrating rod there is a movement of the air. As the bar moves to the right it compresses the layer of air to its right and that *compression* is immediately communicated to the layer of air to the right; as the bar moves back and to the left, the pressure on the adjacent air is released and a *rarefaction* takes place. In $\frac{1}{50}$ of 1 second you have the air adjacent to the rod *compressed,* back to normal, and *rarefied;* during this time the neighboring air is affected and the compression is communicated a distance which is the *wave length* of this given sound wave. In 1 second this disturbance is transmitted 1100 feet at 44° Fahrenheit. The wave length for this sound wave then is $\frac{1100}{50} = 22$ feet.



Vibration records produced by the voice

"ā" as in "āte"; "ou" as in "about"; "r" in "relay"; "ē" in "bē"; and "ä" in "fäther." The tuning fork record, frequency 50 per second, gives the vibration frequencies.

The wave length is commonly designated by λ. If v is the velocity, and t the time of one vibration, $\lambda = vt$.

The notes of the key of C on the natural scale have the following vibration frequencies:

c	d	e	f	g	a	b	c′
256	288	320	341.3	384	426.7	480	512

The intensity or loudness corresponds in the rod to the length of swing of the vibrating rod; as this amplitude decreases, the intensity of the sound decreases. For small amplitudes the vibratory motion gives a convenient way of measuring small intervals of time.

Thus on the above diagram if the tone of the note lower C, $y = \sin 256\,\pi t$, were represented, each complete wave would represent $\frac{1}{128}$ of 1 second; each half or each arch would represent $\frac{1}{256}$ of 1 second. Tuning bars, with periods $\frac{1}{50}$ and $\frac{1}{100}$ of 1 second are run electrically for timing purposes.

The curve $y = \sin\theta + \sin 2\,\theta$ represents the combination in sound of two tones which differ by an octave. Precisely the type of curve which is represented by our diagram can be produced mechanically by the record of a vibrating tuning fork[1] which sounds not only the principal note but also the octave overtone, due to the fact that the bar vibrates about the middle point at the same time that it vibrates about the end. Vibrating strings also have multiple vibration, overtones and other tones. Harmony is the result, in general, when the vibrating instrument gives vibrations which are connected with the fundamental vibration by simple numerical relations, like that of the overtone.

Thus the notes of the major chord, key of C, c, e, g, c, on the piano, have the vibration frequencies in the ratios 4 to 5 to 6 to 8.

4. Helical spring. — Similar to the vibrations of the air are those of a spiral wire spring which oscillates back and forth when a weight is suspended by the spring; the successive compressions and elongations of the wire correspond quite

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[1] See Miller, *The Science of Musical Sounds*, p. 188, for photograph.

closely to the condensations and rarefactions of the air. The position of the weight at any instant can be given by an equation entirely similar to the equation above of note C. Thus if the time of one complete vibration is $\frac{1}{2}$ second, and the maximum displacement is 4 inches, the equation is

$$y = 4 \sin 4 \pi t^r.$$

This gives the elevation above and below the point at which the weight comes to rest. Perfect elasticity of the spring is assumed.

5. Light waves. — Light waves have a much higher velocity than sound waves, 300×10^6 meters per second. The different wave lengths correspond to different colors, just as different wave lengths in sound waves correspond to different tones. The wave length of the light from burning sodium (D_2 of the spectrum) is 0.5890×10^{-6} meters per second, and for other colors varies for the visible spectrum between .39 and $.75 \times 10^{-6}$ meters. The vibration frequency of the sodium light is the number of these waves which occur in one second of time, hence since these waves cover 300×10^6 meters in one second the frequency n is such that

$$n \cdot \lambda = v, \text{ or } n \cdot 0.589 \times 10^{-6} = 300 \times 10^6,$$

whence $n = \dfrac{300 \times 10^6}{.589 \times 10^{-6}} = 509 \times 10^{12}$ vibrations per second.

Radiant energy is of the same general nature with longer waves. Light waves differ from the sound waves in having transverse vibrations, not longitudinal.

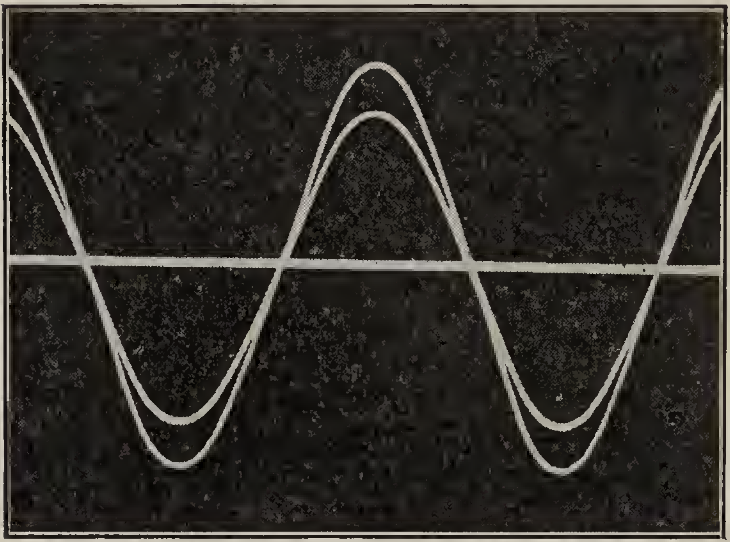
6. Electricity. — In electricity, particularly in the discussion of alternating currents, the sine curve plays a prominent rôle.

The equations $e = 156 \sin \theta,$

$$i = 4 \sin \theta,$$

and
$$\begin{aligned} p = ei = 624 \sin^2 \theta &= 624(\tfrac{1}{2} - \tfrac{1}{2} \cos 2\,\theta) \\ &= 312 - 312 \cos 2\,\theta, \end{aligned}$$

represent respectively the electromotive force, e, measured in volts, and the current, i, measured in amperes, and the power, p, measured in watts of an ordinary electric current.



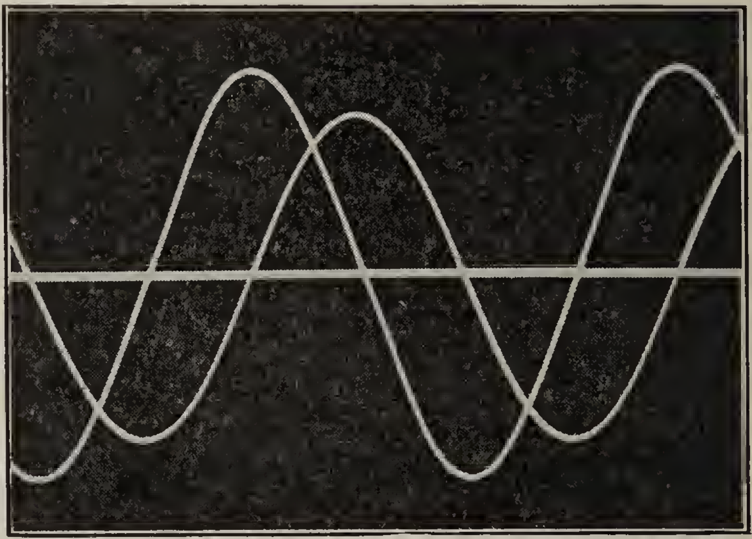
Sinusoids traced by electrical means

Oscillogram of an alternating current in which current and e. m. f. are "in phase."

In general, current and electromotive force are "out of phase"; the equations when the current lags 30° behind the electromotive force are,

$$e = 156 \sin \theta,$$
$$i = 4 \sin (\theta - 30°).$$

On the two diagrams we have represented by a photographic process the magnitude of the current and electromotive force of an alternating current. The current is represented by the curve with the smaller amplitude. In the first illustration current and e. m. f. are "in phase," and under these conditions a maximum of power is developed; in the second illustration current and e. m. f. are "out of phase," the current lagging behind the e. m. f.



Oscillogram showing current curve (lower) lagging 90° behind e. m. f. curve

The power at any instant delivered by an alternating current is given by the product of the current and the e.m. f. at that instant. Employing the formulas,

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta,$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta, \text{ whence}$$

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin\alpha\sin\beta,$$

show that $p = i\,e \sin\theta \sin(\theta - 30°)$ may be reduced to

$$\frac{i\,e}{2}[\cos 30° - \cos 2(\theta - 15°)].$$

Plot the curve showing the power at any instant, when

$$e = 156\cos\theta \quad \text{and} \quad i = 4\cos(\theta - 30°).$$

Note that this power curve is also a sinusoidal curve but placed with reference to a horizontal line which runs 270 units above the x-axis.

PROBLEMS

1. Plot the curves $y = \sin 256\,\pi\,t^r$ and $y = \sin 512\,\pi\,t^r$, using $\frac{1}{2}$ inch for 1 on the vertical axis and 6 half-inches for $\frac{1}{128}$ of 1 second on the t or horizontal axis. Treat the equations as $y = \sin 2\,\pi t^r$, and $y = \sin 4\,\pi t^r$, respectively, substituting for t, 0, .1, .2, .3, ···, .9, and 1 instead of $\frac{1}{10}$ of $\frac{1}{128}$, $\frac{2}{10}$ of $\frac{1}{128}$, ···.

Note that the unit $\frac{1}{128}$ taken as 1 on the horizontal axis, disposes of the difficulty of the awkward fractions.

2. What is the frequency of the vibrations in the curves of the preceding example? What are the corresponding wave lengths?

3. How would $y = \cos 2\,\pi t^r$ differ from the curve for $y = \sin 2\,\pi t^r$? Write 10 values of $y = \cos 2\,\pi t^r$ for $t = 0$, $\frac{1}{12}$, $\frac{1}{8}$, $\frac{1}{6}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{5}{12}$, $\frac{1}{2}$, ··· 1.

Note that these angles correspond to 0°, 30°, 45°, 60°, ··· respectively.

4. Use the equation $\cos\theta = \sin(90° + \theta)$ to show that $y = \sin\theta$ lags 90° behind $y = \cos\theta$.

5. Draw the graphs of $y = \cos\theta$ and $y = \cos 2\,\theta$; divide the arc of the circle into 24 equal parts and take the distance representing $2\,\pi^r$ as divisible by 24.

6. Draw the graph of $y = \sin\left(\frac{\pi}{4} + \theta\right)$ and compare with $y = \sin\theta$. Discuss the corresponding motion of the moving point on the vertical axis.

7. The limits of hearing are for vibrations of 16 per second and 40,000 per second. What are the corresponding wave lengths?

8. Plot on the same diagram the two curves,

$$e = 156 \sin\theta,$$
$$i = 4 \sin(\theta - 30°).$$

9. In problem 8 find the value of e for each 30° to 360°. This completes a "cycle" of values. The time of this movement in a 60-cycle system is $\frac{1}{60}$ of 1 second. What is the value of t for the angles given, and also for $\theta = 45°, 135°, 225°$, and 315°?

10. On the curve, on the same axes as the preceding, $i = 4 \sin(\theta - 30°)$, read the values of θ for the angles 30°, 45°, 60°, 90°, ⋯ to 360°. These may represent current in the circuit of problems 5 and 6; the current lags 30° behind the e. m. f. What interval of time is represented by the 30° lag?

11. Plot to the same axes the curves, $i \doteq 4 \sin(\theta + 40°)$,
$e = 156 \sin\theta$.

The curve of i here leads the curve of e by 40°.

In the case of i, what are convenient values of θ to plot without using tables?

12. Assuming that it takes $\frac{1}{60}$ of 1 second for one complete cycle of i or e in problem 8, find the time difference represented by the 40° angular difference. Find angles approximately corresponding to $\frac{1}{120}$, $\frac{1}{240}$, $\frac{1}{180}$, $\frac{1}{100}$, and $\frac{1}{200}$ of 1 second.

7. **Sine curve; circle; ellipse; cylinder.** — If a circular cylinder, such as the one in our diagram, is cut by any plane, the

intersection is an ellipse. Thus, the plane through *AOB* in our diagram, inclined at an angle of 45°, cuts the cylinder in an ellipse whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{2\,a^2}=1.$$

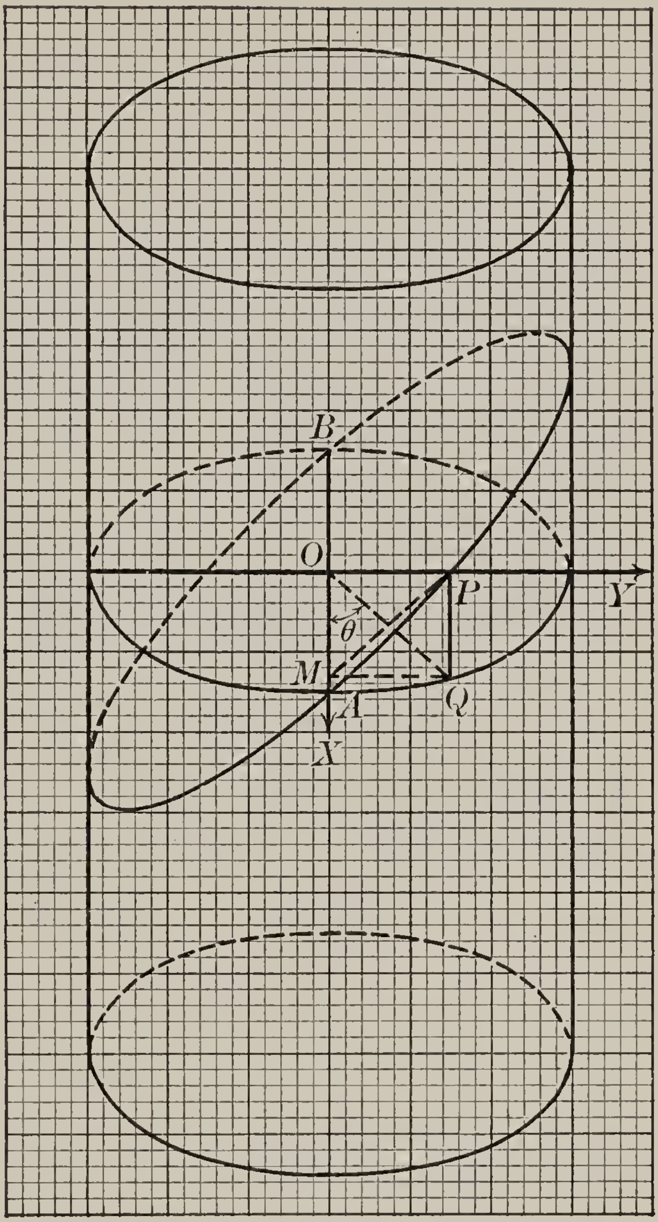
The circular base and any parallel section has the equation

$$\frac{x}{a^2}+\frac{y^2}{a^2}=1.$$

Taking the portion of the cylindrical surface between the elliptical curve and the circular curve through the same center and unrolling it gives a pure sinusoid,

$$y=a\sin\theta.$$

In the figure $PQ=QM$, since the cylinder is cut at an angle of 45°. But QM, the ordinate on the circle, equals $a\sin\theta$ and the arc AQ, which will be the abscissa, is $a\theta$.

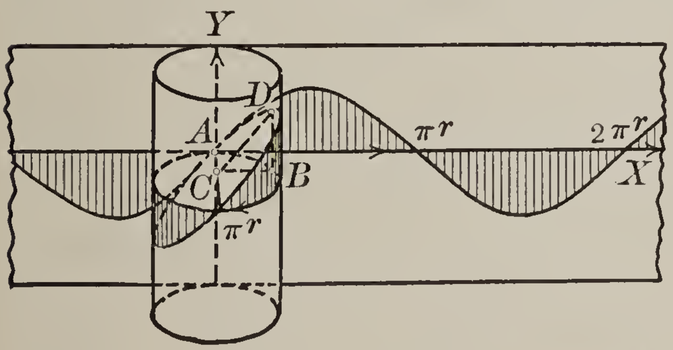


A plane intersecting all the elements of a circular cylinder cuts the surface in an ellipse

Hence the curve, when rolled out, is

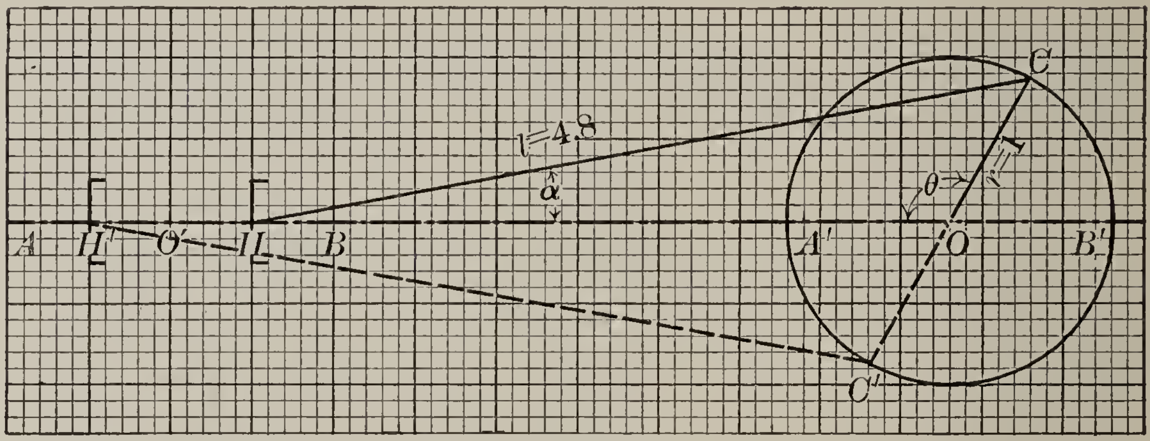
$$y=a\sin\theta$$

and will give a complete arch of the sinusoid if upper and lower portions of the surface are given.



Sinusoid developed by means of a circular cylinder

Experimentally the student can develop this curve by rolling a sheet of paper about a cylinder and cutting out with a sharp knife the required portions.



Piston-rod diagram

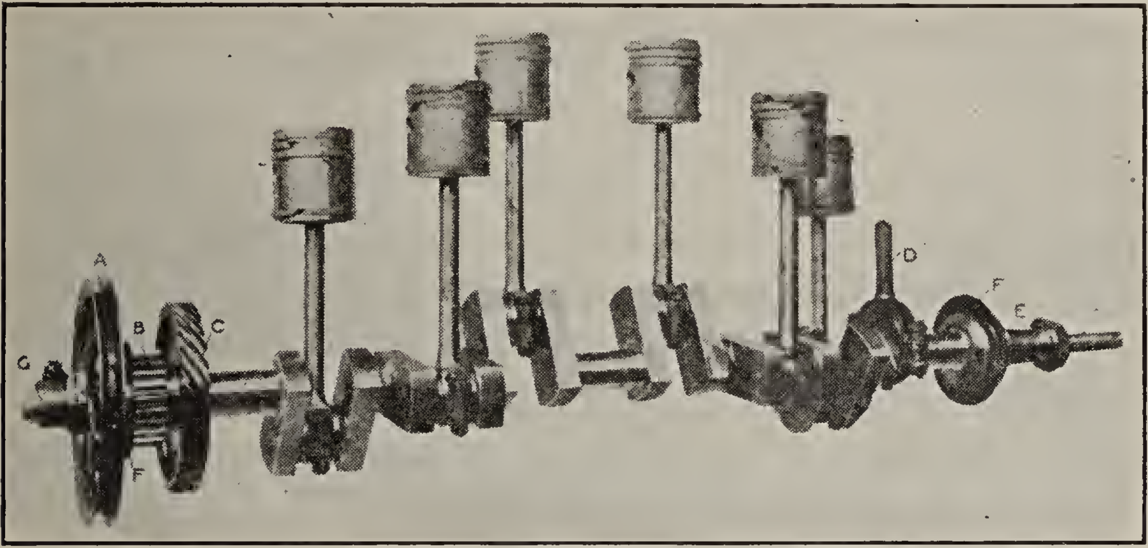
AB, the stroke ; *HC*, connecting rod ; *OC*, crank arm.

8. Piston-rod motion. — The common piston-rod motion of engines furnishes abundant trigonometric material, much of which is of sufficiently elementary character so that by the application of simple formulas problems of interest to the engineer can be solved.

The essential features for our purposes are the piston head H, the connecting rod HC of length l, the crank arm OC of length r, and the stroke AB, which is the distance through which the piston head H moves. Were the connecting rod infinite in extent, the motion of H would be simple harmonic motion when C is rotating with uniform velocity about O.

In modern engines the ratio of l to r varies from 3 to 1, low, to 4.8 to 1, which is approximately that of a Ford engine. It is desired to find for each position of the piston head the angle α of the connecting rod, the angle θ of the crank shaft, and also the effective pressure, called the tangential component, of the connecting rod to turn the crank shaft.

In the first place, when $l:r=4.8:1$, the angle α never exceeds arc sin $\frac{1}{4.8}\cdot$ Determine this angle in degrees. As the pressure P at H is horizontal, only a portion, $P\cos\alpha$, of this pressure is communicated to the connecting rod. Discuss the variation in pressure due to the inclination of the connecting rod and note that it is relatively small. Of course the pressure of the gas in the cylinder chamber is not uniform and this variation is much more serious than the variation due to the angle of a connecting rod.



Connecting rods and crank arms in six-cylinder automobile engine: ratio $l : r = 3 : 1$

Find also the maximum vertical pressure $P \sin \alpha$ on the cross-head support.

Show that the length of the stroke AB is equal to the diameter of the crank circle.

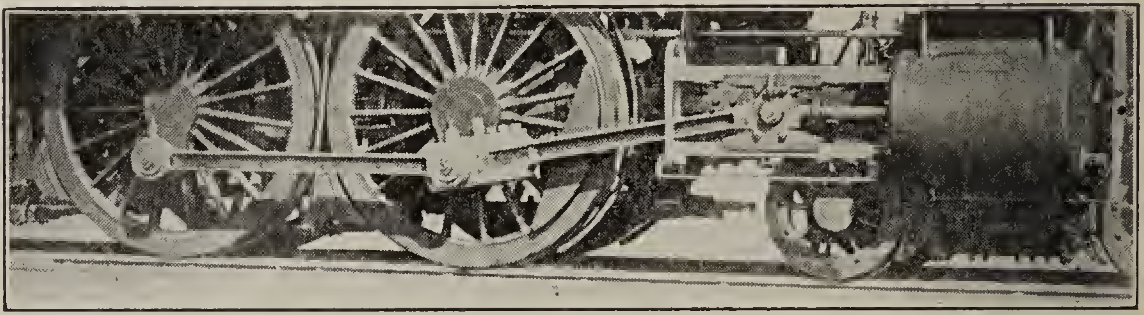
The position of the piston head is indicated in decimal parts of the total stroke $2r$, as measured from A. In our diagram the piston head is at .75 of the stroke. To determine α and θ when the position of the piston head is given we solve a triangle in which the three sides are known. Thus on our diagram $OO' = 4.8$; $HO = 4.3$; $HC = 4.8$; and $OC = 1$. Solve this triangle and determine the angles α and θ. Commonly a diagram is drawn in which the angles θ of the crank arm are plotted as abscissas and the piston displacements are plotted as ordinates.

PROBLEMS

1. Given $\theta = 0$, 10°, 20°, 30°, 60°, 90°, 120°, 150°, and 180°, find the corresponding values of α and the positions of H as decimal parts of the stroke. Plot angles as abscissas and piston displacements as ordinates. How could you complete this to 360°? Could you go on beyond 360°? Take $l : r = 4.8 : 1$.

2. Assuming that the piston rod transmits a uniform pressure of 200 lb., find the effective turning pressure when $\theta = 30°$, 60°, and 90°. Resolve the force at C into two components, one normal to the crank shaft and the other along the crank shaft. The component normal to the crank shaft, the tangential component, is effective, *i.e.* does the work. The radial pressure is also computed and is used to determine friction loss. Find the values of the radial pressure corresponding to above tangential pressure.

Compute in this case for the given angles the pressure on the cross-head support by the connecting rod due to the component, $P \sin \alpha$.



Connecting rod, crank arm, and cylinder on A. T. & S. F. locomotive 1444

The stroke is 30 inches and connecting rod 60 inches.

3. Draw the graph illustrating relative positions of the piston head and the connecting rod when the crank-pin is at the lowest point, in the locomotive illustrated above.

4. Find the number of strokes of the piston per minute when the train moves 60 miles per hour, given that the driving wheels are 57 inches in diameter.

CHAPTER XXVII

LAWS OF GROWTH

1. Compound interest function. — The function $S = P(1 + i)^n$ is of fundamental importance in other fields than in finance. Thus the growth of timber of a large forest tract may be expressed as a function of this kind, the assumption being that in a large tract the rate of growth may be taken as uniform from year to year. In the case of bacteria growing under ideal conditions in a culture, *i.e.* with unlimited food supplied, the increase in the number of bacteria per second is proportional to the number of bacteria present at the beginning of that second. Any function in which the rate of change or rate of growth at any instant t is directly proportional to the value of the function at the instant t obeys what has been termed the "law of organic growth," and may be expressed by the equation,

$$y = ce^{kt},$$

wherein c and k are constants determined by the physical facts involved, and e is a constant of nature analogous to π. The constant k is the proportionality constant and is negative when the quantity in question decreases; c is commonly positive;

$$e = 2.178 \cdots.$$

The values of the function of x, ce^{kx}, *increase according to the terms of a geometrical progression as the variable x increases in arithmetical progression.*

2. π and e. — A function can be found by methods of the calculus which is such that the rate of growth of the function at any instant t, or x, exactly equals the value of the function at that instant. This function is given by the equation,

$$y = e^t \text{ or } y = e^x.$$

The constant e is represented by the series

$$e = 1 + \tfrac{1}{1} + \tfrac{1}{\lfloor\underline{2}} + \tfrac{1}{\lfloor\underline{3}} + \tfrac{1}{\lfloor\underline{4}} \cdots,$$

wherein $\lfloor\underline{3}$, called factorial 3, represents $3 \times 2 \times 1$, and $\lfloor\underline{4} = 4 \times 3 \times 2 \times 1$, and, in general, $\lfloor\underline{n}$, n being a positive integer, represents the total product of all the integers from n down to 1. The sum of this series to 5 decimal places is 2.71828; to 10 places, = 2.7182818285.

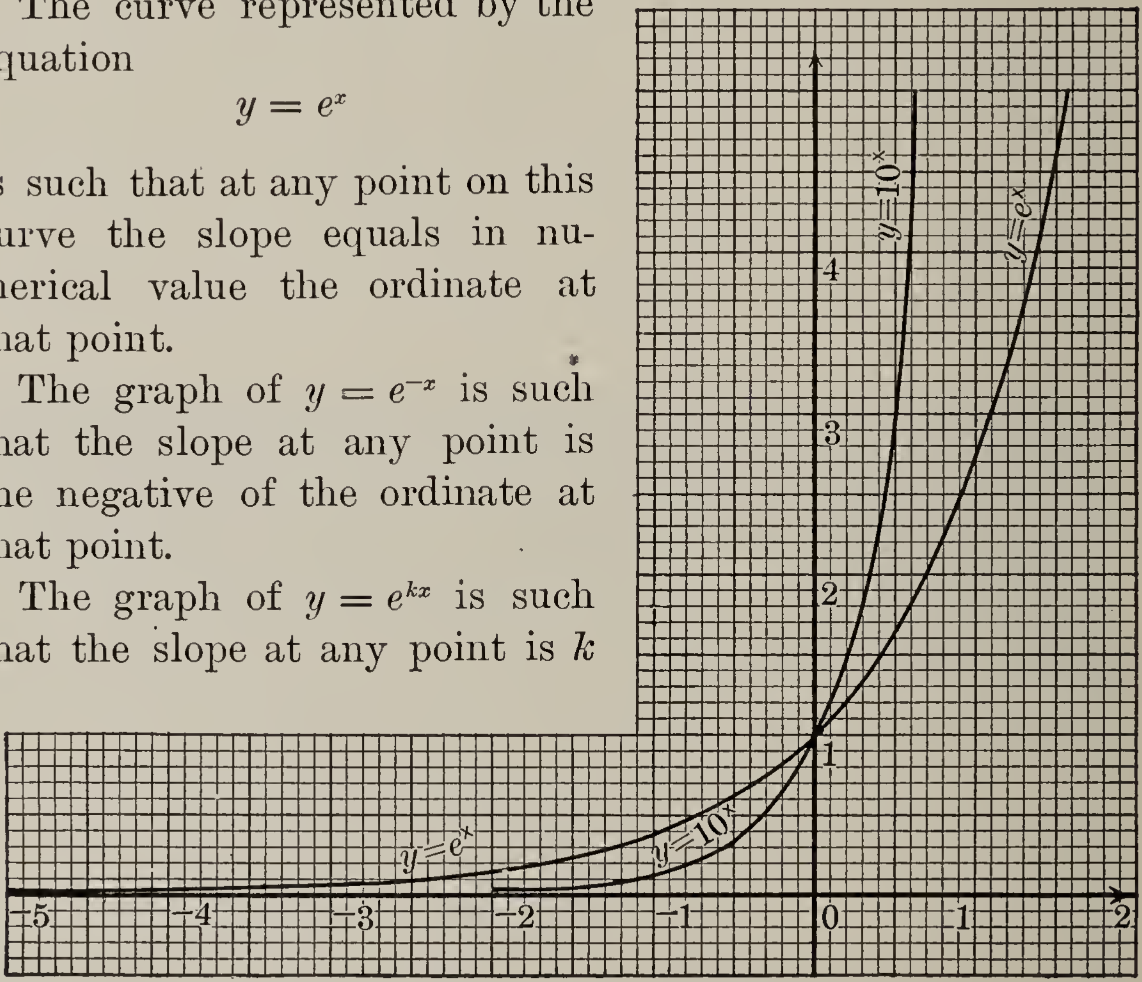
The curve represented by the equation

$$y = e^x$$

is such that at any point on this curve the slope equals in numerical value the ordinate at that point.

The graph of $y = e^{-x}$ is such that the slope at any point is the negative of the ordinate at that point.

The graph of $y = e^{kx}$ is such that the slope at any point is k times the corresponding value of the function at that point.



Graphs of $y = e^x$ and $y = 10^x$

Values of the function $y = e^x$ may be determined by logarithms.

Thus to find the points on $y = e^x$, for which

$x = -2, -1, 0, .1, .2, .5, .8, 1, 2,$ and 3 respectively,

we take first $\log y = x \log e$; since $\log e = .4343$

$x = -2,$	$\log y = -.8686,$ $= 9.1314 - 10$	$y = .135$
$x = -1,$	$\log y = -.4343,$ $= 9.5657 - 10$	$y = .368$
$x = 0,$	$\log y = 0,$	$y = 1$
$x = .1,$	$\log y = .0424,$	$y = 1.103$
$x = .2,$	$\log y = .0869,$	$y = 1.222$
$x = .5,$	$\log y = .2171,$	$y = 1.649$
$x = .8,$	$\log y = .3474,$	$y = 2.225$
$x = 1,$	$\log y = .4343,$	$y = 2.718$
$x = 2,$	$\log y = .8686,$	$y = 7.390$
$x = 3,$	$\log y = 1.3029,$	$y = 20.09$

Similarly, if $y = ce^{kx}$, $\log y = \log c + kx \log e$, and these values are obtained by logarithms.

The limit of the expression

$$\left(1 + \frac{1}{n}\right)^n$$

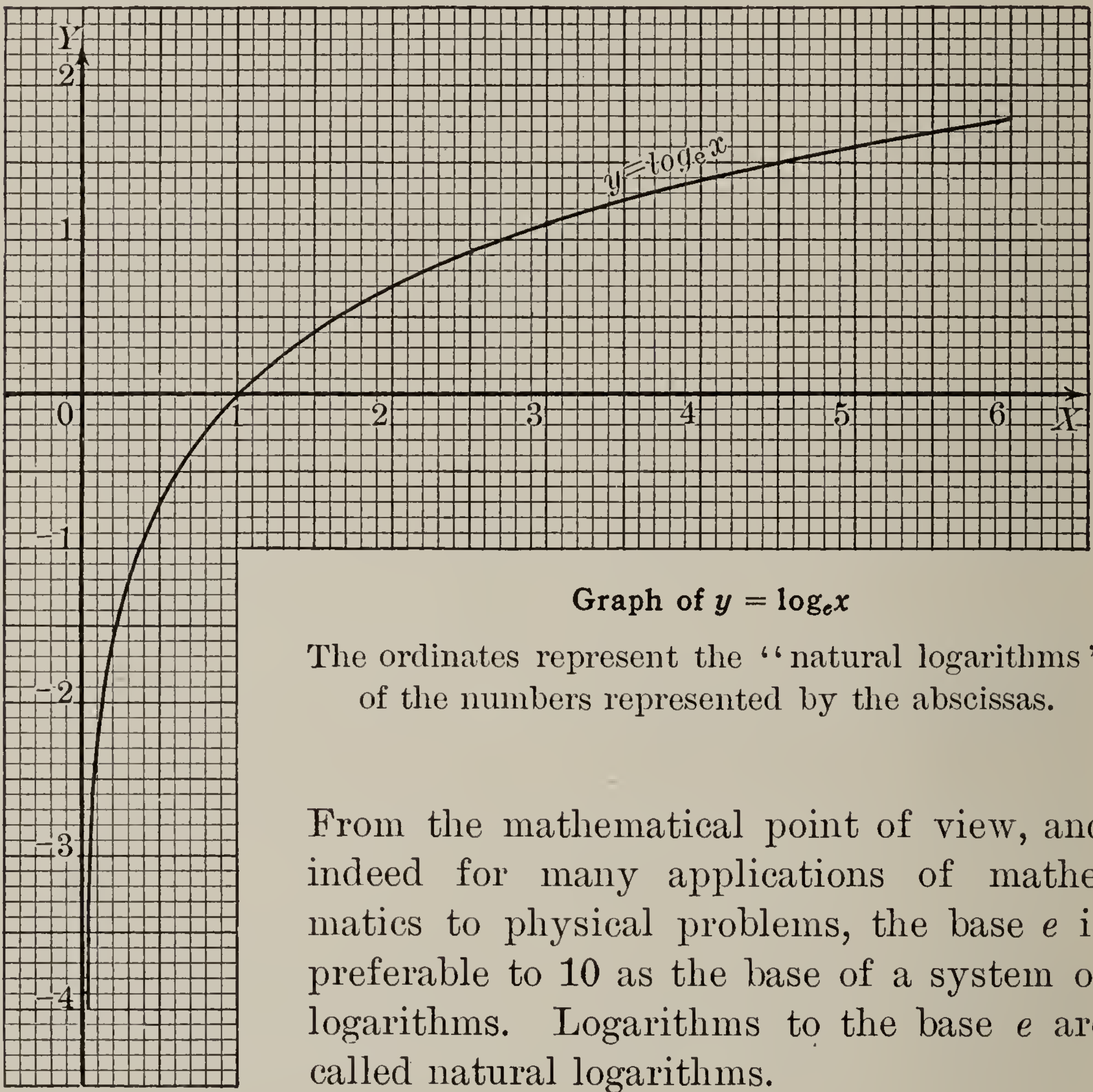
as n approaches infinity gives the value, e. When n is taken as a large positive integer, it can readily be shown that this expression

$$\left(1 + \frac{1}{n}\right)^n$$

has a value differing but slightly from e.

e and π may be called fundamental constants of nature; in mathematical work as applied to statistics and to physical problems of varied kinds these constants often appear.

3. Natural logarithms. — The first logarithms as computed by Napier were not calculated to any base, but were founded upon the comparison between an arithmetical and a corresponding geometrical progression. However, a base of the Napierian logarithms can be established, and it is approximately $\frac{1}{e}$.



Graph of $y = \log_e x$

The ordinates represent the "natural logarithms" of the numbers represented by the abscissas.

From the mathematical point of view, and indeed for many applications of mathematics to physical problems, the base e is preferable to 10 as the base of a system of logarithms. Logarithms to the base e are called natural logarithms.

4. Application. — The most immediate application of a function in which the growth is proportional to the function itself is to the air. The decrease in the pressure of the air at the distance h above the earth's surface is proportional to h.

The expression $P = 760\, e^{-\frac{h}{7990}}$ gives the numerical value of the pressure in millimeters of mercury for h measured in meters. The negative exponent indicates that the pressure decreases as h increases. In inches as units of length of the mercury column, h in feet,

$$P = 29.92\ e^{-\frac{h}{26200}}.$$

This is known as Halley's Law.

The growth of bean plants within limited intervals and the growth of children, again between quite restricted limits, follow approximately the law of organic growth. Radium in decomposing follows the same law, the rate of decrease at any instant being proportional to the quantity. In the case of vibrating bodies, like a pendulum, the rate of decrease of the amplitude follows this law; similarly in the case of a noise dying down and in certain electrical phenomena, the rate of decrease is proportional at any instant to the value of the function at the instant.

PROBLEMS

1. By experiment it has been found that 1000 of the so-called "hay bacteria" double their number, under favorable conditions, in 20 minutes. Find the rate of growth per minute. Take

$$n = 1000\ e^{kt},$$

and determine k by substituting $n = 2000$, $t = 20$, and solving for k. Determine the number that would grow from 1000 bacteria in 1 hour; in 1 day.

2. The cholera bacteria have been found, under favorable conditions, to double their number in 30 minutes. Determine the rate of growth per minute, and the number that would grow in one day from 1000.

NOTE. The favorable conditions cannot be continued for such a period.

3. Assuming that

$$P = 760\ e^{-\frac{h}{8000}}$$

find the value of h which will reduce the pressure of the air by 1 mm. Take the logarithm of both sides and note that

$-\frac{h}{8000}\log_{10} e$ must reduce log 760 to log 759. Find at what height the mercury column is reduced 1 cm. At what height would the pressure be reduced to 660? Are these heights ever attained?

4. Find the barometric pressure when

$h = 1000$, 5000, 10,000, and 15,000 feet, assuming
$P = 29.92$, when $h = 0$, in which P is the numerical value

of the pressure in inches of mercury and the height is h feet. Find the height for which the pressure decreases $\frac{1}{10}$ of 1 inch.

5. Show that if the height of the elevation is measured in miles, the pressure in inches is given approximately by the formula

$$P = 29.92\, e^{-\frac{h}{5}}.$$

Note that 26,000 is nearly 5 × 5280. The constant 26,200, above, is taken for simplicity instead of 26,240.

6. To what change in height does the maximum variation of the barometer recorded on the photograph, page 60, correspond?

7. Compute by the progressive method of Section 4, Chapter XII, the value of e, from the series,

$$= 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \frac{1}{\underline{|5}} + \frac{1}{\underline{|6}} + \frac{1}{\underline{|7}} + \frac{1}{\underline{|8}} + \cdots$$

by summing 8 terms to 8 decimal places.

8. Plot on the same diagram and compare the two graphs, of

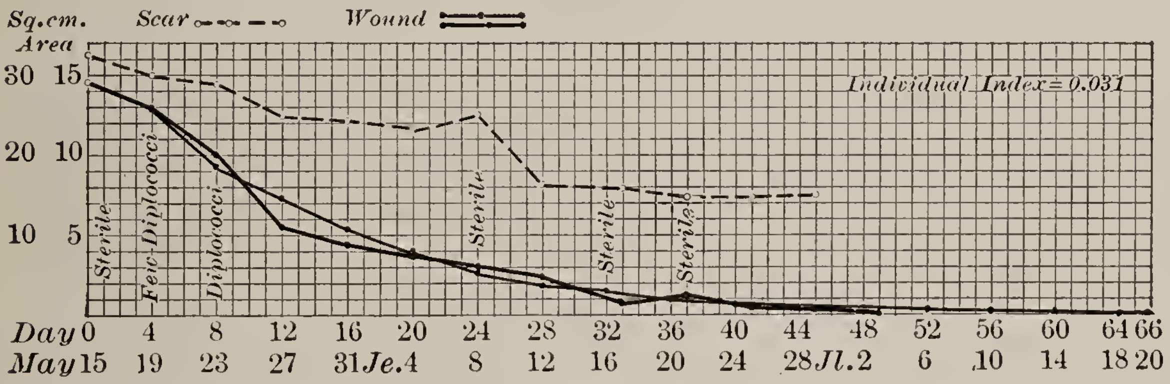
$$y = e^x \text{ and } y = 10^x.$$

9. Plot the curve $y = e^{-x^2}$, taking $\frac{1}{4}$ inch as .1 on the horizontal and on the vertical axes. This curve represents what is termed the normal distribution curve, which is of fundamental importance in all statistical work. In general, large groups of individuals may be distributed as to ability in any given quality over the area under such a curve; the middle abscissa at $x = 0$ represents average ability, and deviation to one side or the other represents, on one side, ability above the aver-

age and, on the other, ability below the average. The total number of all individuals considered is represented by the area between the curve and the x-axis.

An interesting graphical illustration of a normal distribution curve is the crowd at a football game when a great bleacher is not filled. The central aisles are all filled to a height representing the middle ordinate and from this out in either direction, the ordinates drop off, frequently in strikingly symmetrical manner, and corresponding quite closely to the normal distribution curve.

5. The curve of healing of a wound. — Closely allied to the formulas expressing the law of organic growth, $y = e^{kt}$, and the law of "organic decay," $y = e^{-kt}$, is a recently discovered law which connects algebraically by an equation and graphically by a curve, the surface-area of a wound, with time

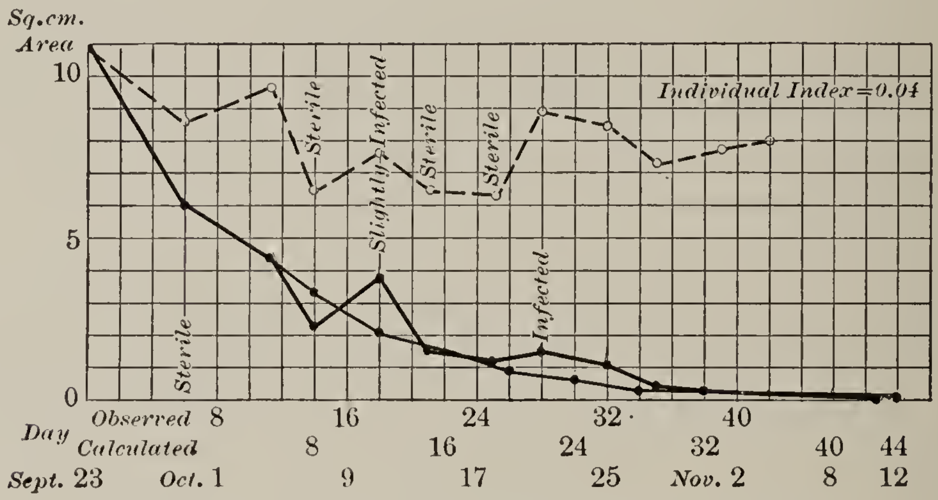


Progress of healing of a surface wound of the right leg, patient's age 31 years

The observed curve oscillates about the smoother, calculated curve.

expressed in days, measured from the time when the wound is aseptic or sterile. When this aseptic condition is reached, by washing and flushing continually with antiseptic solutions, two observations at an interval commonly of four days give the "index of the individual," and this index, and the two measurements of area of the wound-surface, enable the physician-scientist to determine the normal progress of the wound-surface, the expected decrease in area, for this wound-surface of this individual. The area of the wound is traced carefully on transparent paper, and then computed by using a mathematical machine, called a planimeter, which measures areas.

The areas of the wound are plotted as ordinates with the respective times of observation measured in days as abscissas. After each observation and computation of area the point so obtained is plotted to the same axes as the graph which gives the ideal or prophetic curve of healing. Two such ideal curves and also the actual observed curves are represented in our diagrams.



Progress of a surface wound of the right knee

Two infections in the course of healing are indicated.

When the observed area is found markedly greater than that determined by the ideal curve, the indication is that there is still infection in the wound. This is the case depicted, as will be noted, in the smaller diagram. A rather surprising and unexplained situation occurs frequently when the wound-surface heals more rapidly than the ideal curve would indicate; in this event secondary ulcers develop which bring the curve back to normal. This is the type which is represented by our larger diagram.

This application of mathematics to medicine is largely due to Dr. Alexis Carrel of the Rockefeller Institute of Medical Research. He noted that the larger the wound-surface, the more rapidly it healed, and that the rate of healing seemed to be proportional to the area. This proportionality constant is not the same for all values of the surface or we would have an equation of the form,

$$S = S_1 e^{-kt},$$

in which S_1 is the area at the time that the wound is rendered sterile and observations to be plotted really begin.

The actual formulas, as developed by Dr. P. Lecomte du Noüy of Base Hospital 21, Compiègne, France, are

$$i = \frac{S_1 - S_2}{6 S_1}, \tag{1}$$

giving the characteristic constant of the wound.

S_1 is the measure of the area, first observation; S_2 is a second measurement taken after 4 days.

$$S_n = S_{n-1}[1 - i(4 + \sqrt{4n})], \tag{2}$$

wherein S_n is the area after $4n$ days; similarly, S_{n-1} is the area after $4(n-1)$ days, etc.; each ordinate is obtained from the preceding; i is the constant as determined above.

Recent experiments by Dr. du Noüy show that there is a *normal* value of i dependent upon the age of the individual and the size of the wound, and that the individual index as determined by two observations will doubtless reveal facts concerning the general health of the individual.

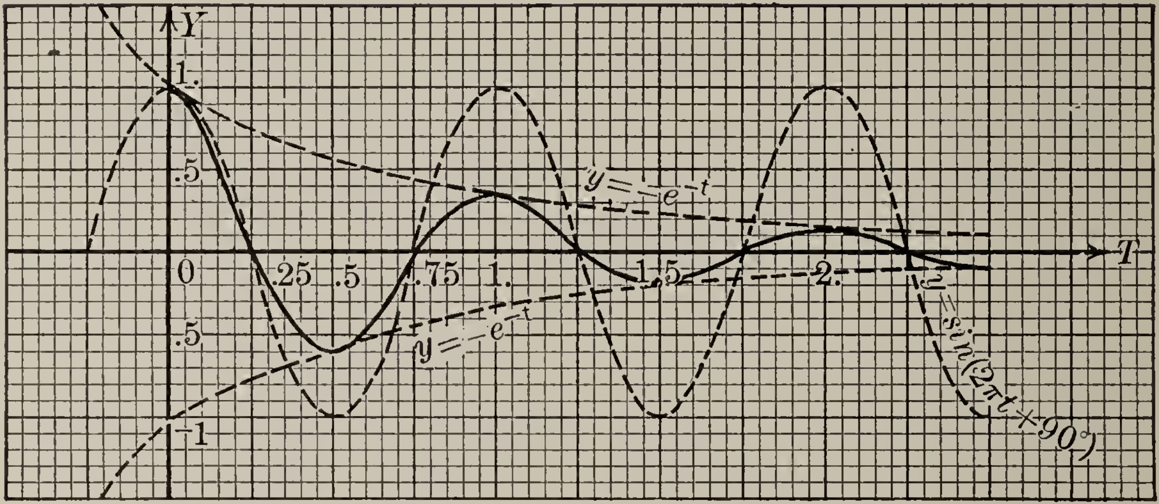
The data given are taken from the *Journal of Experimental Medicine*, reprints kindly furnished by Major George A. Stewart of the Rockefeller Institute. The diagrams are reproduced from the issue of Feb. 1, 1918, pp. 171 and 172, article by Dr. T. Tuffier and R. Desmarres, Auxiliary Hospital 75, Paris.

6. Damped vibrations. — The combination by multiplication of ordinates in the two functions, $y = e^{-k_1 t}$ and $y = \sin k_2 t$, which we have seen to be fundamental in the mathematical interpretation of many phenomena of nature, gives a formula which also has wide application.

The formula

$$y = e^{-k_1 t} \sin k_2 t,$$

expresses the law by which the decrease in intensity of the vibrations defined by $y = \sin k_2 t$ may be determined, under certain conditions.



Damped vibration curve by multiplication of ordinates

Thus a pendulum swinging in the air, when the friction is proportional to the velocity, has this form of equation as the equation of motion. We have indicated on our diagram the curves

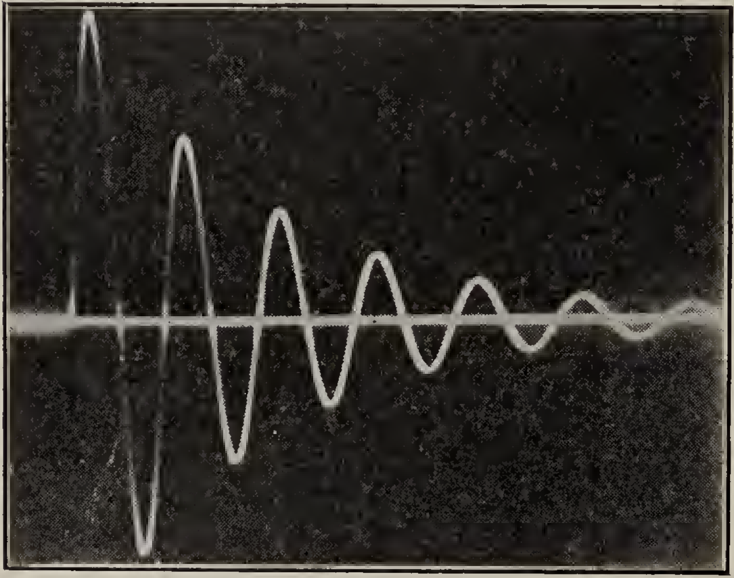
$$y = \sin\left(\frac{\pi}{2} + 2\,\pi t\right),$$

$$y = e^{-t},$$

and $y = -\,e^{-t}$ and the *damped vibration* curve

$$y = e^{-t} \sin 2\,\pi t.$$

The student should check the values, redrawing all the curves on double the scale of the illustration in the text.



Damped vibration produced electrically by the discharge of a condenser

A beautiful damped vibration curve is obtained by the discharge of an electrical condenser. In our illustration the equation

$$y = e^{-\frac{1}{2}t} \sin\left(2\,\pi t + \frac{\pi}{2}\right)$$

represents quite closely the curve, using the maximum ordinate as unity, *i.e.* when $t=0$, and on the horizontal axis the measure of the time in seconds of one complete vibration is taken as unity; in the electrical occurrence represented by this photograph, and the corresponding light phenomenon which produced the photograph, the action took place in about $\frac{1}{40}$ of one second, and $\frac{1}{250}$ of one second is approximately the time of one vibration on this curve.

PROBLEMS

1. Plot the following curves, in the order given:

$$y = \sin 2\,\pi t^r,$$

taking two half-inches to represent $t=1$ on the horizontal axis and 6.2 half-inches to represent unity on the vertical axis; use the graphical method.

$$y = e^{-\frac{t}{2}},$$

taking 6.2 half-inches to represent unity on the vertical axis, and two half-inches to represent one second on the horizontal axis.

$$y = e^{-\frac{t}{2}} \sin 2\pi t,$$

by multiplication of ordinates.

Note that by taking five half-inches to represent unity on the vertical axis each half-inch represents .2 and each twentieth of an inch represents .02. These facts are to be used when you multiply ordinates to obtain the third of these curves. For the values of the powers of e consult the table at the back of the book.

2. Given that an automobile wheel which is revolving freely at the rate of 400 revolutions per minute is allowed to come to rest by the action of the friction and air resistance; assuming that the subsequent velocities per minute are given at the end of t minutes by the equation

$$v = 400\,e^{-\frac{t}{10}},$$

to determine these velocities, plot the graph of the function. At what time will the number of revolutions be reduced to approximately 200 per minute? to 100? to 50? to 3? It may be assumed that at 3 revolutions per minute the law no longer holds and that the wheel will stop at about that time.

3. Given that the horizontal displacement of a second pendulum is 4 inches, and that the horizontal displacement is given by the equation

$$x = 4 \cos 2 \pi t,$$

and that the amplitudes are decreased according to the "law of organic decay," the position being given at any instant by the equation

$$x = 4 e^{-\frac{t}{100}} \cos 2 \pi t.$$

Find the displacement of the pendulum after 10 seconds; after 100 seconds; after one hour. When does the pendulum have a displacement of only 1 inch? of $\frac{1}{2}$ inch? of $\frac{1}{10}$ inch?

This type of retardation is found when the friction is proportional to the velocity.

4. Given that a fly-wheel revolving freely with a velocity of 500 revolutions per minute is allowed to come to rest. If the velocity at the end of t seconds is given by the equation

$$v = 500 e^{-\frac{t}{10}},$$

find the velocity at the end of 10 seconds; at the end of 100 seconds; at the end of 30 seconds. When will the velocity be reduced to 1 revolution per minute?

With a heavy oil as lubricator heavy fly-wheels follow approximately this law.

CHAPTER XXVIII

POLAR COÖRDINATES

(See Section 3, Chapter VII)

1. Uses. — For many purposes the representation of functions by the system of polar coördinates is desirable. Thus, effective pressure on the crank head by the piston head varies for every angle. It is convenient to give this pressure-diagram in polar coördinates; on every radius is plotted a length representing graphically the effective turning pressure on the crank for that angle.

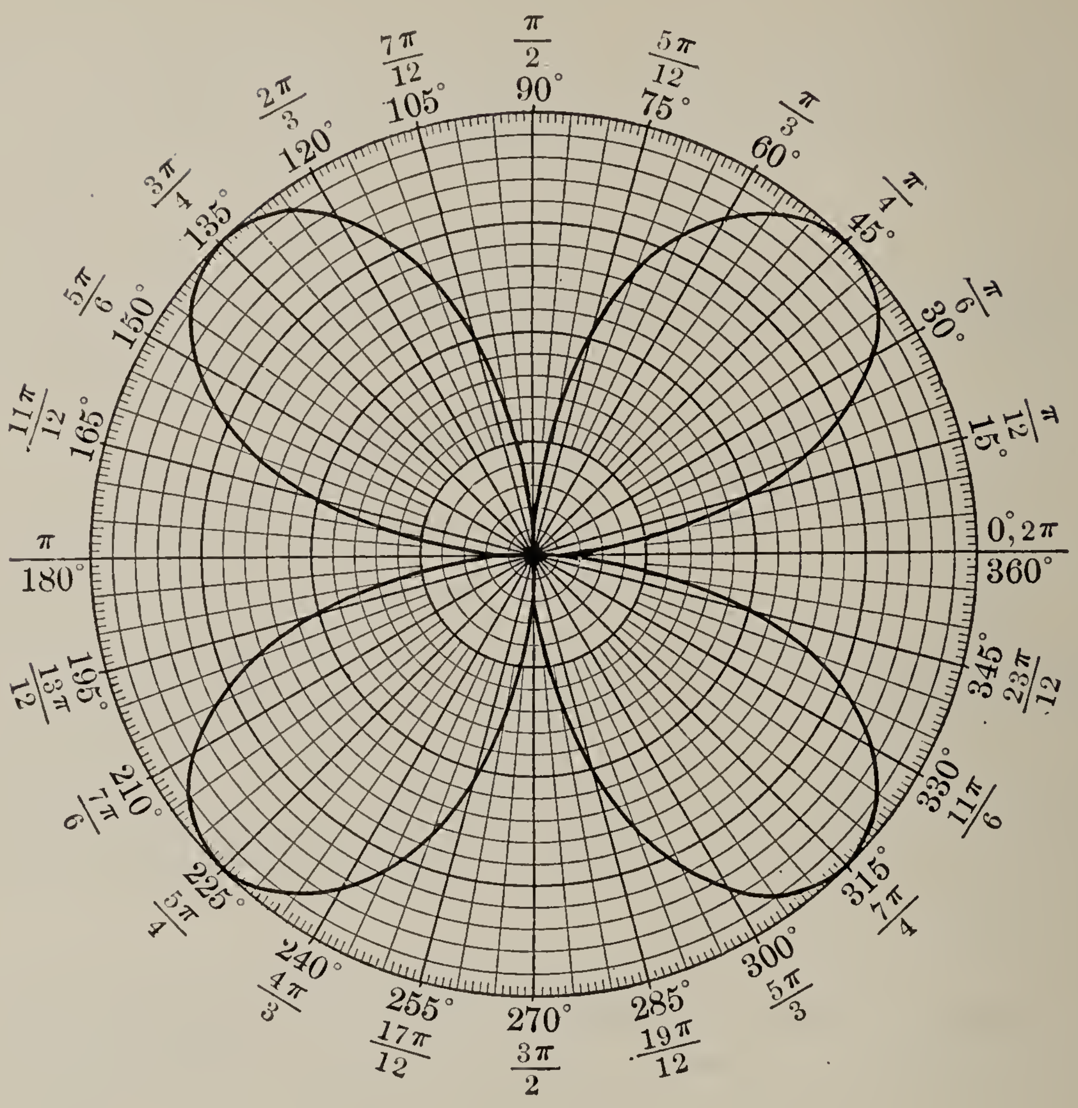
2. Plotting in polar coördinates. — The coördinates of points which satisfy an equation given in polar coördinates are obtained precisely as in rectangular coördinates. An equation in polar coördinates involves r and θ, radius vector and vectorial angle; by substituting in the given equation particular values of one of the variables and solving for the corresponding values of the other points on the curve are obtained.

θ	r
0°	0
10°	3.42
15°	5.00
20°	6.43
25°	7.66
30°	8.66
35°	9.40
40°	9.85
45°	10
50°	9.85

Illustrative problem. — Plot the curve

$$r = 10 \sin 2\,\theta.$$

Note that when $\theta = 10°$, $r = 10 \sin 20° = 3.42$, which length is plotted on the 10° line. Complete the work, showing how the second loop and other loops are obtained, by giving to θ values increasing by 5° intervals up to 360°. Note that no further computation is needed. Follow the progress of the curve on the diagram given on the next page.

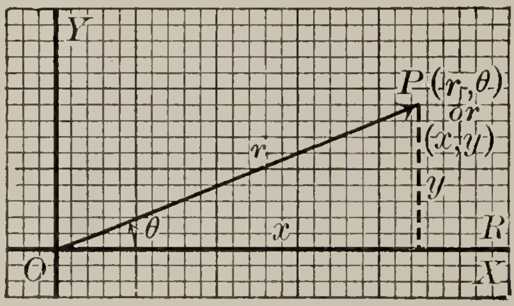


The formulas for transformation from rectangular coördinates to polar coördinates should be noted:

$$x = r \cos \theta,$$

$$y = r \sin \theta.$$

The formulas for transformation from polar to rectangular coördinates are:



$$r = \sqrt{x^2 + y^2},$$

$$\theta = \arctan \frac{y}{x},$$

and $$\sin \theta = \frac{y}{\sqrt{x^2 + y^2}},$$

or $$\cos \theta = \frac{x}{\sqrt{x^2 + y^2}}.$$

PROBLEMS

1. Plot the curve $r = 10 \sin \theta$. Prove that this is a circle. Take any point on the circle, as (r, θ), and show that the coördinates satisfy the equation.

2. Plot the following curves:

(*a*) $r \sin \theta = 5$.
(*b*) $r \cos \theta = 10$.
(*c*) $r \sin (\theta - 30°) = 10$.
(*d*) $r = 10 \sin 3\,\theta$.
(*e*) $r = 10 - 10 \cos \theta$.
(*f*) $r = 5$.

3. Plot the curve $r = 2\,a \tan \theta \sec \theta$; transform to rectangular coördinates. This curve is a "cissoid" and can be used in the "duplication of the cube" problem.

4. Plot the curve $r = 10 \sec \theta + 5$. This is a "conchoid of Nicomedes," and can be used to effect the solution of the problem to trisect any angle.

5. Plot the polar diagram of effective pressures on the crank for different angles of θ; use the data of problem 2 in the problems given under piston-rod motion.

6. Plot the curve $r = 10 - 10 \cos \theta$. This curve is called a "cardioid" because of its shape.

7. Plot the curve $r = 10 - 5 \cos \theta$. This is called a "limaçon of Pascal."

8. Plot $r = 10 - 20 \cos \theta$, another type of limaçon.

9. Show that the polar equation of any conic is

$$r = \frac{2\,m}{1 - e \cos \theta},$$

wherein $2\,m$ is one half of the right focal chord.

10. Plot the parabola

$$r = \frac{10}{1 - \cos \theta}.$$

11. Plot the hyperbola

$$r = \frac{10}{1 - 2\cos\theta}.$$

For what values of θ is r infinite in value? What directions do these values give? Are these lines from the origin then the asymptotes?

12. Plot the spiral of Archimedes given by

$$r = 10\,\theta.$$

13. Plot the hyperbolic spiral given by

$$r = \frac{10}{\theta}.$$

14. Plot $$r = \sin 2\theta.$$

15. Plot $r = \sin\theta + \sin 2\,\theta$, and compare the polar with the Cartesian (x, y) representation.

CHAPTER XXIX

COMPLEX NUMBERS

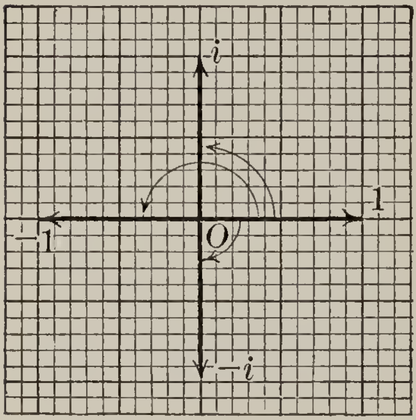
1. Object. — In the study of the number field, indicated in our first chapter, we found that in the extraction of square roots we were limited to positive numbers. Again in solving quadratic equations, and in the discussion of the roots of algebraic equations, we found that no number of the kind we had considered could occur as the even root of a negative quantity. We can extend the number field, removing the limitation that square roots and even roots must be taken of positive quantities only, by creating another class of numbers, complex numbers. These numbers, after the fundamental operations with them have been properly defined, apply to our algebraic equations, x and the constants being complex numbers. In the extended number field it is possible to prove that every rational integral equation has a root and that such an equation of the nth degree has n roots.

2. Complex numbers. — We define $\sqrt{-1}$, designated by i, as a number which, multiplied by itself, equals -1; this requires, then, an extension of the meaning of multiplication and a reëxamination of the fundamental processes as applied to the old numbers with this newly found number and other new numbers which follow directly from it. This discussion is given graphically in the next section. The square root of any other negative number, $-a$, is regarded as $\sqrt{a}\sqrt{-1}$, or $\sqrt{a}\cdot i$. Such a number, *e.g.* $\sqrt{-7}$, is called a *pure imaginary*. To add a pure imaginary to a real number both must be written and the combination is called a complex number.

Thus, $x+yi$ and $a+bi$, or $-3+2\sqrt{-1}$ and $\sqrt{5}-\sqrt{3}\,i$, are complex numbers.

Addition and multiplication are explained graphically in sections 6 and 7 below.

3. Graphical representation. — Our real numbers can all be conceived graphically, as well as analytically, as derived from the unit 1. Thus, integers are obtained by the repetition of the unit; fractions are obtained by the subdivision of the unit; and negative numbers are obtained from the negative unit, which in turn is obtained by reversing the direction of the positive unit. Analytically the imaginary unit repeated as a factor gives -1; graphically then we would desire an operation which repeated gives a reversal of direction. How is the reversing from $+1$ to -1 effected? Evidently by turning the positive unit through an angle of 180° or $-180°$. The $\sqrt{-1}$, or i, can be regarded then as represented by the middle position of this rotating unit, and the upper position is regarded as $+i$ and the lower as $-i$. This vertical line is taken as the axis of pure imaginaries. Thus, $\sqrt{-4}$, or $2\,i$, is represented two units up on this axis and $-\sqrt{-2}$ is represented $\sqrt{2}$ units down on this axis.



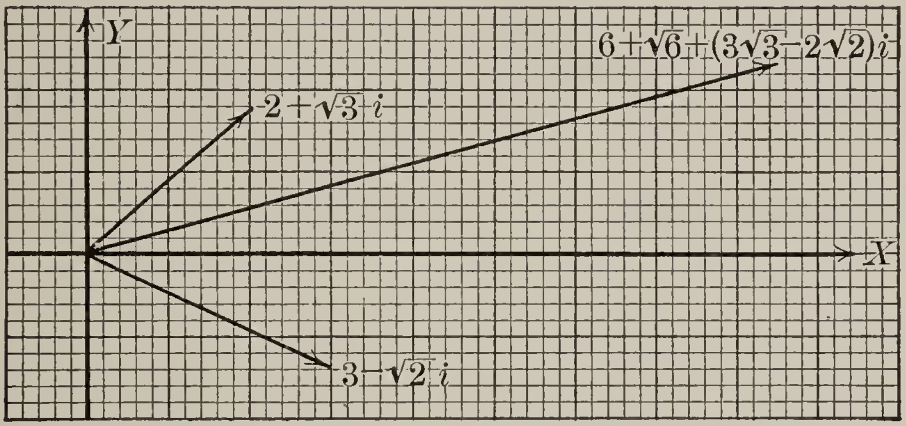
The imaginary unit obtained graphically

A complex number, $x+yi$, may now be uniquely represented by the point (x, y) in the complex plane, in which the y-axis coincides with the vertical axis of pure imaginaries.

The fundamentally important facts concerning these numbers are:

1. *Complex numbers are combined according to the laws of the real numbers* (which we have discussed in the first chapter), *noting that* $i^2=-1$.

2. *The combination of any two or more complex numbers, by the operations of addition, subtraction, multiplication, division* (except by zero), *involution, and evolution* (with certain exceptions), ALWAYS *produces a complex number.*



Representation of three complex numbers

Two complex numbers of the form $x + yi$ and $x - yi$, symmetrically placed with respect to the axis of reals, are called *conjugate* complex numbers. Their sum and their product are real numbers.

4. **Complex roots in pairs.** — In any rational integral algebraic equation with real coefficients, if $a + bi$ is a root of the equation, then $a - bi$ is also a root of the equation. The proof depends upon the fact that when $a + bi$ is substituted in

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots a_n,$$

the resulting expression is of the form $P + Qi$, in which P and Q being real numbers, P involves powers of a and the even powers of bi, and Q is obtained from expressions involving odd powers of bi. Now if

$$P + Qi = 0,$$

then $P = 0$ and $Q = 0$; otherwise you have a real number equal to a pure imaginary. Substituting $a - bi$ for x in

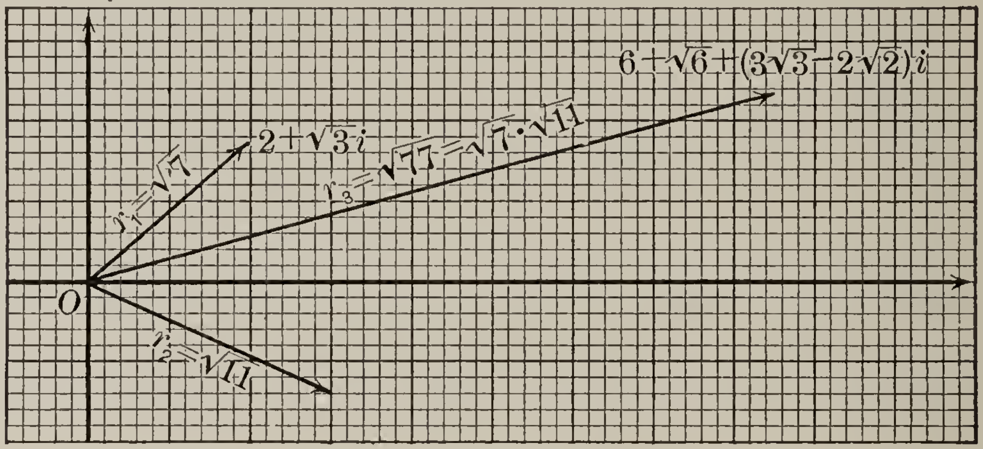
$$a_0x^n + a_1x^{n-1} + \cdots a_n$$

changes the signs of the terms involving the odd powers of i, and does not change the sign of the even powers. Hence $a - bi$ substituted gives

$$P - Qi;$$

but $P = 0$ and $Q = 0$, hence $P - Qi = 0$. Therefore $a - bi$ is also a root of the equation if $a + bi$ is a root. Complex roots go in pairs.

Illustrative problem. — 1. Find the product of $2 + \sqrt{-3}$ by $3 - \sqrt{-2}$ and put the product in the $x + yi$ form.



Graphical representation of the product of $2 + \sqrt{3}\, i$ by $3 - \sqrt{2}\, i$

$$(2 + \sqrt{3}\, i)(3 - \sqrt{2}\, i) = 6 + 3\sqrt{3}\, i - 2\sqrt{2}\, i - \sqrt{6}\, i^2,$$

but $i^2 = -1$, giving as product,

$$6 + \sqrt{6} + (3\sqrt{3} - 2\sqrt{2})i. \quad \textit{Ans.}$$

$$6 + \sqrt{6} = a; \ 3\sqrt{3} - 2\sqrt{2} = b.$$

2. Divide $3 + \sqrt{-3}$ by $5 - 2\sqrt{-3}$ and express the quotient in $x + yi$ form.

$$\frac{3 + \sqrt{-3}}{5 - 2\sqrt{-3}} = \frac{3 + \sqrt{3}\, i}{5 - 2\sqrt{3}\, i} = \frac{(3 + \sqrt{3}\, i)(5 + 2\sqrt{3}\, i)}{(5 - 2\sqrt{3}\, i)(5 + 2\sqrt{3}\, i)} = \frac{15 + 6\, i^2 + 11\sqrt{3}\, i}{25 - 12\, i^2}$$

$$= \frac{9 + 11\sqrt{3}\, i}{37} = \frac{9}{37} + \frac{11\sqrt{3}\, i}{37}.$$

3. Factor $x^2 + y^2$ into complex factors, linear in x and y.

$$x^2 + y^2 = (x + iy)(x - iy).$$

PROBLEMS

1. Write the conjugate complex numbers:

a. $3+\sqrt{-2}$.

b. $-4-2i$.

c. $-3-\sqrt{2}-\sqrt{-2}$.

d. $1+i$.

e. $-\sqrt{-7}$.

f. 3.

2. Rationalize the denominator in the following expressions by using the conjugate complex number as multiplier, reducing the quotient obtained in this way to the form $a+bi$.

a. $\dfrac{5-\sqrt{-2}}{3+\sqrt{-2}}$.

b. $\dfrac{5}{-4-2i}$.

c. $\dfrac{3+\sqrt{2}+2\sqrt{-2}}{-3-\sqrt{2}-\sqrt{-2}}$.

d. $\dfrac{2}{1+i}$.

e. $\dfrac{\sqrt{5}}{-\sqrt{-7}}$.

f. $\dfrac{5-\sqrt{-3}}{3}$.

3. Write the following expressions in the form $a+bi$:

a. $i+i^2+i^3+i^4$.

b. $i^5+3i^6+3i^7+4i^8$.

c. $i^{10}+i^{20}$.

d. $2i+3i^2+4i^3$.

e. $\dfrac{2}{3+2i}+\dfrac{2}{3-2i}$.

f. $\dfrac{2}{3+2i}-\dfrac{2}{3-2i}$.

4. Locate the points represented by the complex numbers in problem 1.

5. Square $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$; square $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$. These are roots of $x^3-1=0$. Multiply $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ by $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$. What is the cube of $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$?

6. Square $\dfrac{1-i}{\sqrt{2}}$ and $\dfrac{1+i}{\sqrt{2}}$. Give an equation with real coefficients which these numbers satisfy. What are the square roots of i?

5. Vectors. Polar representation of complex numbers. — The complex number $x+yi$ may be regarded either as determined by the point $P(x, y)$ or by the vector OP, which by its length and direction determines the position of P. The angle which OP as a ray makes with the x-axis is called θ, the *amplitude* or *angle*, and the length OP is called r, the *modulus* of the complex number. In other words, the polar coördinates of P are (r, θ);

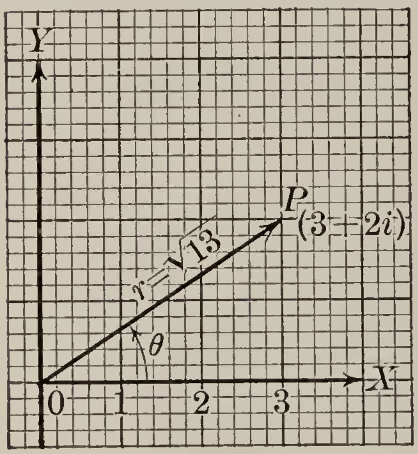
$$r=\sqrt{x^2+y^2},$$

$$\cos\theta=\frac{x}{r}, \text{ and } \sin\theta=\frac{y}{r}.$$

The complex number may be written in the form

$$r(\cos\theta+i\sin\theta),$$

which is termed the *polar form.*



Modulus and amplitude of a complex number

The *modulus*, r or $\sqrt{x^2+y^2}$, is a positive number representing the length of the vector or the distance of the point $x+iy$ from the origin. This modulus is sometimes called the *stretching factor* or the *tensor;* see section 7.

6. Addition of vectors. — When the complex number is represented by a vector, the sum of two complex numbers will be represented by the diagonal of the parallelogram formed by the two vectors; see Chapter IX, section 2. The student should verify the fact by a diagram.

7. Product of complex numbers. — Given two complex numbers, either in polar form or in rectangular form, the product of the two numbers is also a complex number; further *the modulus of the product is the product of the moduli,* and *the amplitude or angle of the product is the sum of the amplitudes of the factors.*

Let

$r_1(\cos\theta_1 + i\sin\theta_1)$; $x_1 + y_1 i$, mod $\sqrt{x_1^2 + y_1^2}$, ampl $\theta_1 = \tan^{-1}\frac{y_1}{x_1}$

and

$r_2(\cos\theta_2 + i\sin\theta_2)$; $x_2 + y_2 i$, mod $\sqrt{x_2^2 + y_2^2}$, ampl $\theta_2 = \tan^{-1}\frac{y_2}{x_2}$

be two complex numbers. Their product is

$r_1r_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$

$x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$, mod $\sqrt{x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2}$

ampl $\theta_3 = \tan^{-1}\frac{x_1y_2 + x_2y_1}{x_1x_2 - y_1y_2}$.

The polar product may be written

$$r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)],$$

showing that the product of the moduli r_1 and r_2 is the modulus r_1r_2 of the product and the amplitude is $\theta_1 + \theta_2$, the sum of the amplitudes. It is left as an exercise for the student to show that the analytical expressions for modulus and amplitude establish the same facts.

When any complex number is used as a multiplier, the modulus of the product is the modulus of the multiplicand stretched in the ratio of the modulus of the multiplier to unity. For this reason the modulus is sometimes termed the *stretching factor*.

8. De Moivre's theorem. — Evidently, if θ_1 and θ_2 are set equal to θ, our product formula may be written:

(1) $$[r(\cos\theta + i\sin\theta)]^2 = r^2(\cos 2\,\theta + i\sin 2\,\theta).$$

Evidently by *mathematical induction*, by simple introduction of one further factor $r(\cos\theta + i\sin\theta)$ at a time, it can be shown that

(2) $$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).$$

This theorem, which holds for all values of n, is called De Moivre's theorem. We have proved it only for n an integer. Taking the nth root of each member of equation (2), we have

$$(r^n)^{\frac{1}{n}}(\cos n\theta + i\sin n\theta)^{\frac{1}{n}} = r(\cos\theta + i\sin\theta).$$

Let $r^n = k$ and $n\theta = \theta'$, which, as no limitation was imposed on θ, imposes no limitation as to value on θ', and we have

$$[k(\cos\theta' + i\sin\theta')]^{\frac{1}{n}} = k^{\frac{1}{n}}\left(\cos\frac{\theta'}{n} + i\sin\frac{\theta'}{n}\right),$$

i.e. our formula holds for a fractional exponent of the form $\frac{1}{n}$. By raising to the mth power both sides, it can be shown to hold for any fractional exponent.

For $n = -1$,
$$[r(\cos\theta + i\sin\theta)]^{-1} = \frac{1}{r(\cos\theta + i\sin\theta)} = \frac{\cos\theta - i\sin\theta}{r(\cos^2\theta + \sin^2\theta)} = r^{-1}(\cos\theta - i\sin\theta),$$

whence

$$[r(\cos\theta + i\sin\theta)]^{-1} = r^{-1}[\cos(-\theta) + i\sin(-\theta)],$$

which establishes the formula when $n = -1$. By raising both sides to the nth power, n any rational number, the theorem is established for all rational exponents.

The theorem can be established also for irrational values of n.

PROBLEMS

1. Write the following complex numbers and their conjugates in polar form, giving modulus and amplitude:

a. $\frac{-1}{2} + \frac{\sqrt{3}}{2}i$.

b. $-4 - 2i$.

c. $-3 - \sqrt{2} - \sqrt{-2}$.

d. $1 + i$.

e. $3 + \sqrt{-2}$.

f. $-\sqrt{-7}$.

g. i.

h. 23.

i. $+\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

2. Show that $(\cos 30° + i\sin 30°)^2 = \cos 60° + i\sin 60°$, by multiplication.

3. Show that $(\cos 30° + i\sin 30°)^3 = i$, *i.e.* $\cos 90° + i\sin 90°$.

4. Show that $\frac{1}{\cos 30° + i\sin 30°} = \cos(-30°) + i\sin(-30)°$.

5. Show that $(\cos 60° + i \sin 60°)^{\frac{1}{2}} = \cos 30° + i \sin 30°$.

6. What is the value of $(\cos 30° + i \sin 30°)^{\frac{1}{2}}$ by De Moivre's theorem?

7. Plot, using 2 inches as 1 unit, $\cos 15° + i \sin 15°$, $\cos 30° + i \sin 30°$, $\cos 45° + i \sin 45°$.

8. Plot the point B, $\cos \frac{360°}{7} + i \sin \frac{360°}{7}$; connect by a chord with the point A, $\cos 0° + i \sin 0°$; take this length as a chord, successively seven times on the unit circle about O. This chord is the side of a regular inscribed polygon of seven sides.

9. Roots of unity. — Plotting the solutions of the following equations on the complex number diagram,

(See Section 3, above),

$x - 1 = 0$ gives one point, 1;

$x^2 - 1 = 0$ gives two points, 1 and -1;

$x^3 - 1 = 0$ gives three points, 1, $-\frac{1}{2} + \frac{\sqrt{3}}{2} i$, $-\frac{1}{2} - \frac{\sqrt{3}}{2} i$;

$x^4 - 1 = 0$ gives four points, 1, -1, i and $-i$;

$x^6 - 1 = 0$ gives six points,

1, -1, $-\frac{1}{2} + \frac{\sqrt{3}}{2} i$, $-\frac{1}{2} - \frac{\sqrt{3}}{2} i$, $+\frac{1}{2} + \frac{\sqrt{3}}{2} i$ and $+\frac{1}{2} - \frac{\sqrt{3}}{2} i$,

$x^8 - 1 = 0$, or $(x^4 - 1)(x^4 + 1) = 0$, gives eight points, which may be obtained by methods of quadratic equations. For

$$x^4 + 1 \equiv x^4 + 2x^2 + 1 - 2x^2 \equiv (x^2 + 1)^2 - (\sqrt{2}\, x)^2$$
$$\equiv (x^2 + 1 - \sqrt{2}\, x)(x^2 + 1 + \sqrt{2}\, x),$$

whence, $x^2 - \sqrt{2}\, x + 1 = 0$, $x = \frac{\sqrt{2} \pm \sqrt{-2}}{2} = \frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2} i$

and $x^2 + \sqrt{2}\, x + 1 = 0$, $x = \frac{-\sqrt{2} \pm \sqrt{-2}}{2} = -\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2} i$;

$x^8-1=0$ gives eight points, $1, -1, i, -i, \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i, \frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}$

$$-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i, \text{ and } -\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i.$$

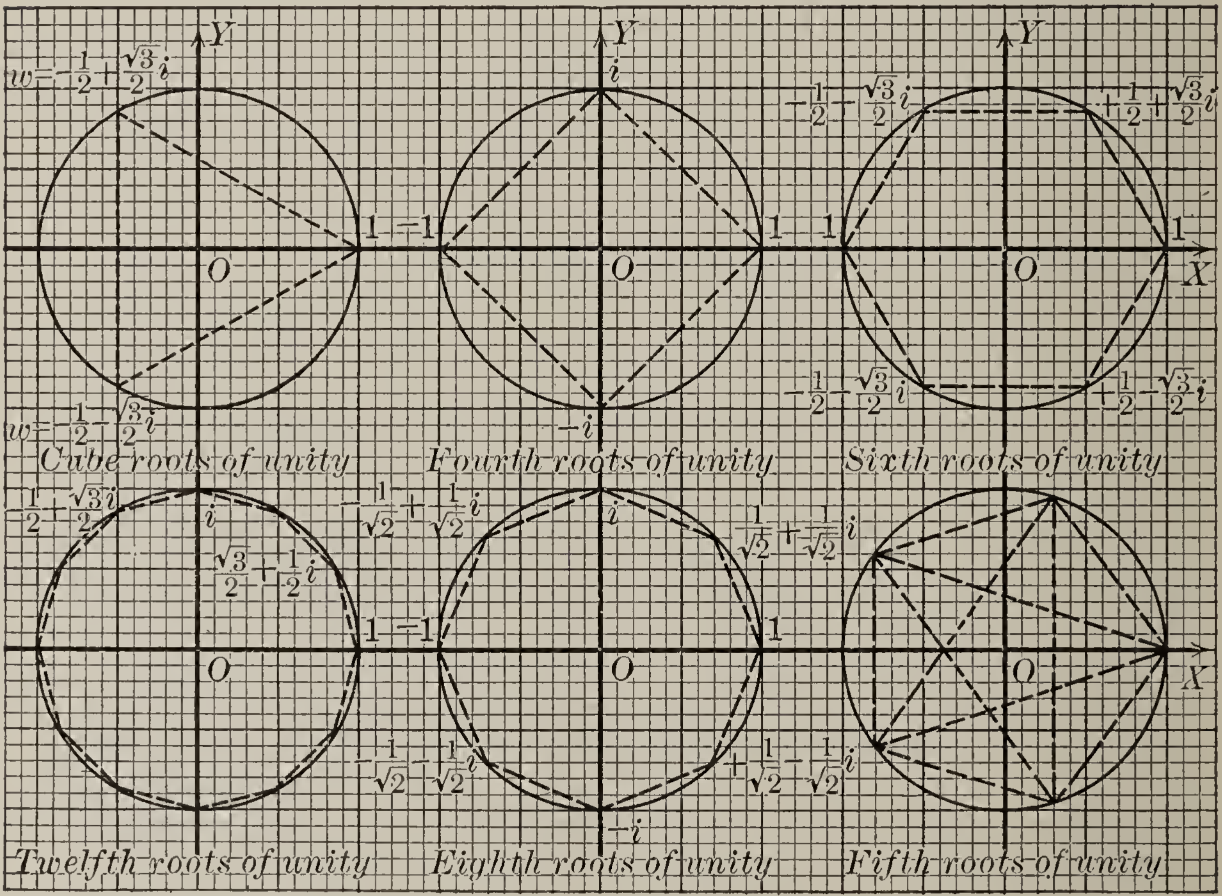
$x^5-1=0$ gives five points; see the solution obtained in problem 4, page 97. The solutions are

$$1, -\frac{\sqrt{5}+1}{4}-\frac{\sqrt{10-2\sqrt{5}}}{4}i, -\frac{\sqrt{5}+1}{4}+\frac{\sqrt{10-2\sqrt{5}}}{4}i,$$

$$\frac{-1+\sqrt{5}}{4}+\frac{\sqrt{10+2\sqrt{5}}}{4}i, \text{ and } \frac{-1+\sqrt{5}}{4}-\frac{\sqrt{10+2\sqrt{5}}}{4}i.$$

Plotting these points on the complex diagram gives the vertices of a regular pentagon.

Graphically representing these points we have the following diagrams:



Graphical solutions of $x^3-1=0$, $x^4-1=0$, $x^6-1=0$, $x^{12}-1=0$, $x^8-1=0$, and $x^5-1=0$

The roots of any equation of the form

$$x^n - 1 = 0, \; n \text{ integral},$$

are, aside from $+1$ or -1, complex numbers. Since any real number less than 1 when multiplied by itself gives a number less than 1 and since any real number greater than 1 multiplied by itself gives a number greater than 1, it follows that the roots other than 1 or -1 are complex. Further, the modulus of each of these complex roots is a real number which, taken n times as a factor, produces 1; hence the modulus of any nth root of unity is 1. We need then to know only the real part of any root of unity to plot it, since the root itself, having a modulus 1, lies on the unit circle, $x^2 + y^2 = 1$.

The nth roots of unity can be obtained graphically by finding the angles which repeated n times give 360° or integral multiples of 360°.

Thus for the twelfth roots of unity these angles are 0°, 30°, 60°, 90°, 120°, 150°, 180°, 210°, 240°, 270°, 300°, and 330°. Writing the corresponding complex numbers in polar form we have the twelve twelfth roots of unity. If we went farther, taking 360°, 390°, 420°, ···, we would simply repeat values already obtained. The twelfth roots of unity are, then,

$$1, \frac{\sqrt{3}}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{\sqrt{3}}{2}i, i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{\sqrt{3}}{2} + \frac{1}{2}i, -1,$$

$$-\frac{\sqrt{3}}{2} - \frac{1}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, -i, +\frac{1}{2} - \frac{\sqrt{3}}{2}i, \text{ and } +\frac{\sqrt{3}}{2} - \frac{1}{2}i,$$

as on the figure.

10. Historical note. — Just as negative numbers were generally accepted only after a graphical scheme of representation of these numbers was introduced by Descartes, so imaginary numbers were neglected and even rejected by mathematicians until a graphical system of representation was found.

In 1797 a Norwegian, Caspar Wessel, presented the scheme of representation of complex numbers to the Danish Academy, but public recognition of his work is only recent. A Frenchman, J. R. Argand, discovered the same system independently in 1806 and it appears that the German J. C. F. Gauss in 1831 again independently rediscovered this graphical method. The latter made extensive use of the diagram and since then these numbers play a vital rôle in the development of algebra. Quite recently the practical importance of these numbers has been more generally recognized by physicists and engineers. Applications have been made to problems in electricity, to the steam turbine by Steinmetz, and to numerous other vector problems.

The term *imaginary* is a misnomer, as our development shows. So far as actuality is concerned, $3+\sqrt{-3}$ exists as a number quite as much as 3 or $\sqrt{3}$; all numbers are the product of intelligence reacting on the experiences of life, and in this sense all numbers are imaginary, the product of the imagination.

11. Mathematical unity. — The complex numbers are fittingly chosen to conclude our treatment of plane analytic geometry, elementary algebra, and elementary trigonometry since, as the observant student will have noticed, we have here involved the fundamental principles of these subjects as well as theorems of plane geometry.

We might note that while regular polygons of seven and nine sides cannot be constructed with ruler and compass, since the solutions of these equations lead to cubics which cannot be solved in terms of quadratic irrationalities, there are other polygons having a prime number of sides which can be so constructed. Gauss, when only 19 years old, showed that the polygon of 17 sides, and, in general, the polygon of sides $2^{2^n}+1$ in number, when this number is prime, can be constructed with ruler and compass. The corresponding algebraic fact is that $x^m-1=0$, when m is a prime number equal to $2^{2^n}+1$, is solvable by the methods of quadratics, and the roots can be expressed in functions involving only square roots.

PROBLEMS

1. Solve $x^3 - 1 = 0$ and plot the points on the polar diagram.

2. Solve similarly $x^3 + 1 = 0$ and plot.

3. Solve $x^5 + 1 = 0$. Using the results given for $x^5 - 1 = 0$, write all the solutions of $x^{10} - 1 = 0$.

4. Derive the formulas for $\cos 2\theta$ and $\sin 2\theta$, using De Moivre's theorem.

HINT. $\cos 2\theta + i \sin 2\theta = (\cos\theta + i \sin\theta)^2$. Square the right-hand member and then equate $\cos 2\theta$ to the real part and put $\sin 2\theta$ equal to the coefficient of i.

5. Derive formulas for $\cos 3\theta$ and $\sin 3\theta$ by the process of problem 4.

6. Show on the diagram how to obtain the twelfth roots of unity. What equation do these numbers satisfy?

7. Show geometrically and algebraically how you can obtain the solutions of the equation

$$x^{24} - 1 = 0$$

from the complex diagram. Use also the formulas for $\sin \frac{1}{2}\theta$ and $\cos \frac{1}{2}\theta$ to obtain $\sin 15°$ and $\cos 15°$ from $\sin 30°$ and $\cos 30°$.

8. Solve $x^8 - 1 = 0$, and plot the points on the complex diagram.

9. Solve $x^{16} - 1 = 0$. What angles are involved?

CHAPTER XXX

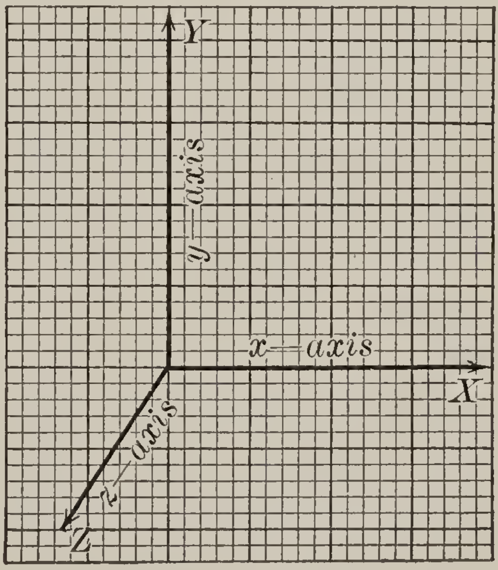
SOLID ANALYTIC GEOMETRY: POINTS AND LINES

1. The third and fourth dimensions. — We have found that on a line the position of any point may be given by a single number, x, which locates the point with reference to one fixed point on the line. The single number, commonly x, represents distance, in terms of some unit of length, from the point of reference, and direction by means of a + or − sign. In a plane the position of any point may be given by a pair of numbers which locate the point with reference to two fixed lines in the plane. The two numbers, x and y commonly, represent the distances in determined order, in terms of some unit of length, from each of the two given lines of reference, and direction as before. By analogy, continuing with the proper changes, it is obvious that in space the position of any point may be given by a set of three numbers which locate the point with reference to three fixed planes in space. The three numbers, x, y, and z commonly, represent the distances in determined order, in terms of some unit of length, from the three given planes of reference, and the direction in each case is determined by the algebraic sign of the number. If the analogy could be continued we could state that the position of any point in a four-dimensional space would probably be given by a set of four numbers which locate the point with reference to four given "three-dimensional" spaces. The four numbers, x, y, z, and w commonly, would then represent the "distances" in determined order, in terms of some unit of length, from each of the spaces of reference. Without a precise definition of what we mean by "distance" of a point in "four-dimensional space" from a "three-dimensional space" these analogies must be regarded as purely fanciful, and devoid of physical significance.

Location of a Point

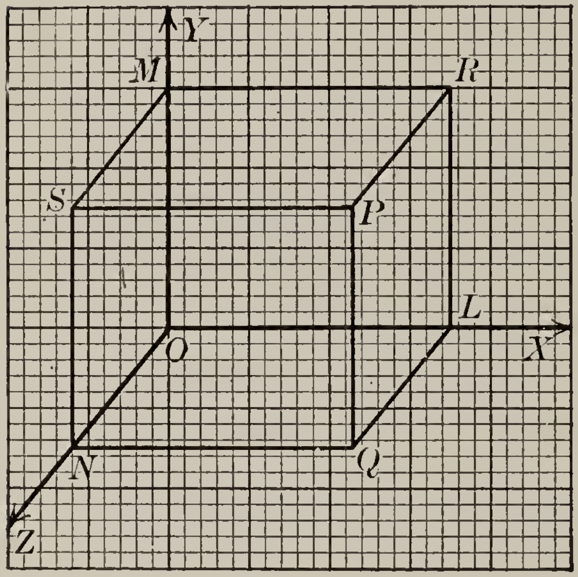
Upon or in a	*With reference to*	*By means of*
line, one dimensional.	one point, zero dimensional.	one variable, x.
plane, two dimensional	two lines, one dimensional.	two variables, (x, y).
space (ordinary), three dimensional.	three planes, two dimensional.	three variables, (x, y, z).
hyperspace, four dimensional.	four three-spaces, three dimensional.	four variables, (x, y, z, w).
. . . .	. . . .	. . . .
n-space, n dimensional.	n $(n-1)$-spaces, $(n-1)$ dimensional.	n variables, $(x_1, x_2, x_3, \cdots x_n)$.

2. Space coördinates. — The position of a point in ordinary space is determined by location with respect to three intersecting planes, called the *coördinate planes.* Just as our lines of reference were chosen perpendicular to each other, for convenience, in plane analytics, so here the planes of reference are taken mutually perpendicular, like the three sides of a box or like the front wall, the floor, and the left-hand wall of a room. The three lines of intersection of these planes with each other in pairs are called the *axes of coördinates,* designated as *x-axis, y-axis,* and *z-axis;* the three planes are named *xy-, xz-,* and *yz-planes* respectively; the point common to the three planes and to the axes is called the *origin.*



Axes in space

The numbers x, y, and z represent respectively distances from the yz-, xz-, and xy-planes; direction is determined as indicated by the arrowheads upon the diagram. No general agreement has been reached as to which axis to use as the vertical axis. The system shown is called a *right-handed system*, since the 90° rotation of the positive ray of the x-axis into the positive y ray advances a right-handed screw along the z-axis, and similarly with the other axes, by cyclical interchange in x, y, z order. The positive directions of these three axes can be represented by the thumb, first finger, and second finger of the right hand.

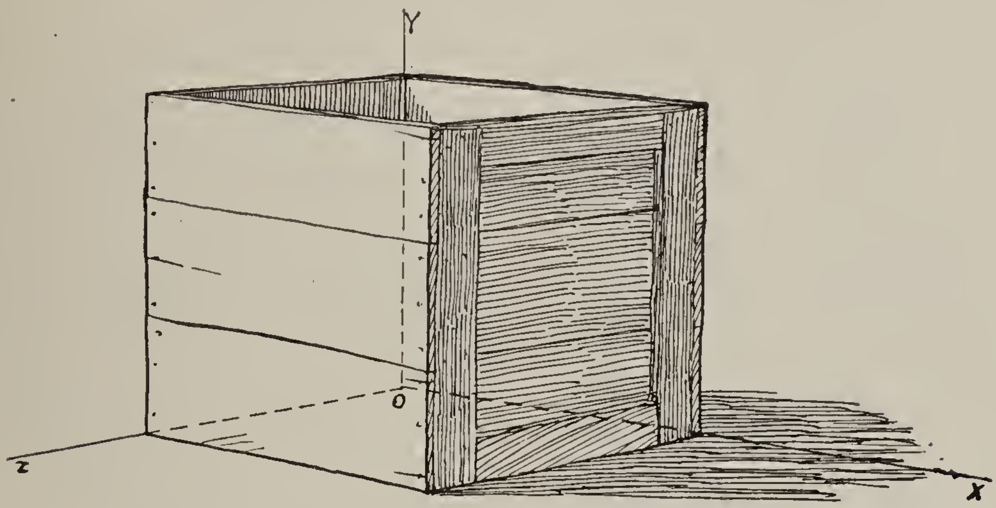


$O(0, 0, 0)$; $L(3.5, 0, 0)$; $M(0, 3, 0)$; $N(0, 0, 2)$; $Q(3.5, 0, 2)$; $R(3.5, 8, 0)$; $S(0, 3, 2)$; $P(3.5, 3, 2)$

To determine the coördinates of any point P in space, planes are drawn or conceived through the point parallel to the coördinate planes; the distances OL, OM, and ON cut off on the axes are given with proper sign as the coördinates (x, y, z) of the point P.

Space is divided by the coördinate planes into eight divisions, called octants. The signs of the coördinates of any point within an octant are given in xyz order to distinguish the octants. Thus the $+ - -$ octant is at the right, below, and back.

To every point in space corresponds one set of coördinates and only one, and, conversely, to every set of three numbers corresponds one and only one point in space. When a point is given by its coördinates, the position is determined on the diagram by passing a plane through the x-axis at the x of the point, parallel to the yz-plane; on the intersection of this plane with the xy-plane indicate the y coördinate. The third coördinate must be represented in perspective, and the direction and length of these units in perspective are taken parallel and equal to the units represented on the third axis.



Drawing by L. Makielski.

An artist's conception of a rectangular solid in space

3. Fundamental propositions of solid geometry. — The following propositions of solid geometry have constant application in our further work. The student would do well to review these propositions in any elementary work on solid geometry and further to verify the reasonableness of these propositions on our figures.

a. Two planes intersect in a straight line.

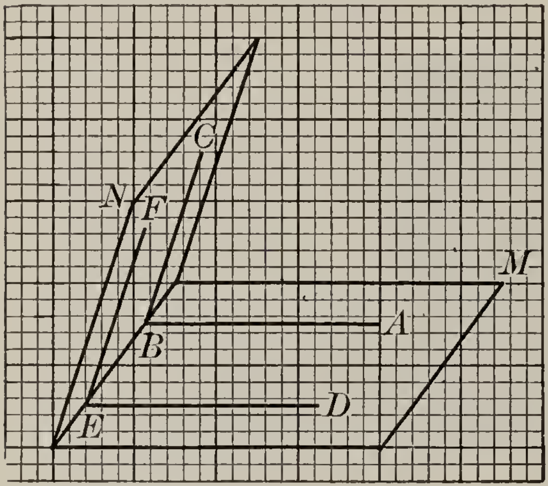
b. If a line is perpendicular to each of two intersecting lines, it is perpendicular to the plane of the two lines, *i.e.* it is perpendicular to every line in the plane of the two given lines.

Thus, PQ on the diagram below is perpendicular to QL, to QM, to QD, and to every line in the xz-plane which passes through Q. A line in the xz-plane which does not pass through Q does not intersect PQ, but the angle which it makes with PQ is defined as the angle which any parallel to it which does intersect PQ makes with PQ. This will then be a right angle.

c. The angles between two pairs of parallel lines are equal or supplementary.

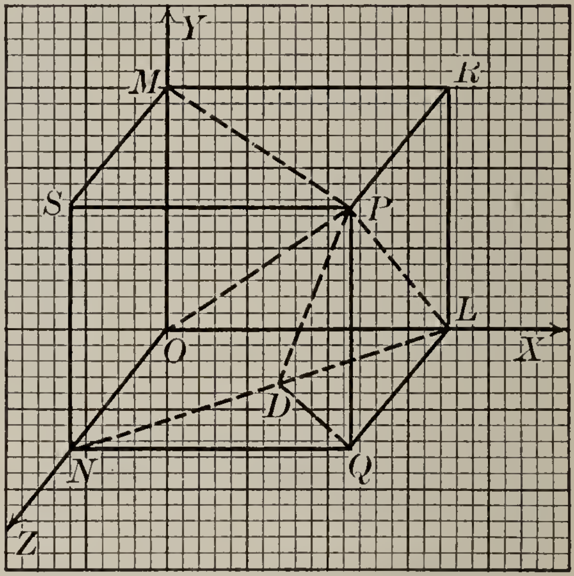
d. If two planes are perpendicular to a third, their intersection line is perpendicular to the third.

e. The dihedral angle between two planes is measured by the plane angle formed by two lines, one in each plane, both perpendicular to the edge of the dihedral angle.

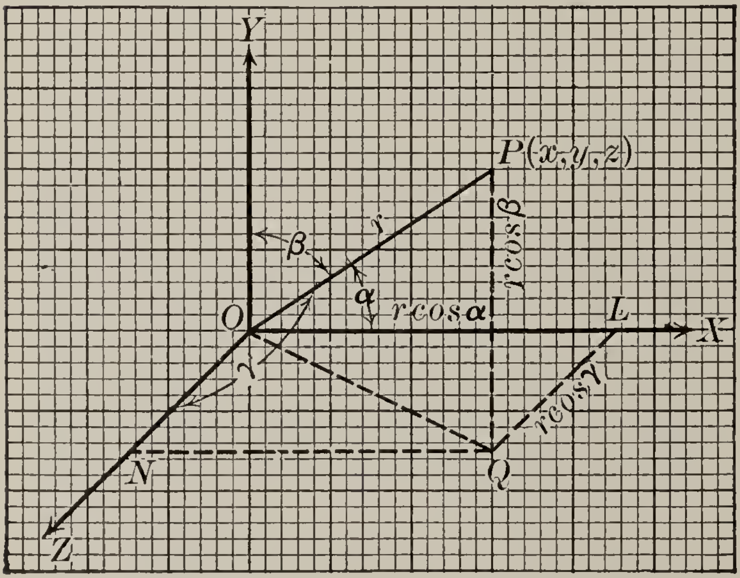


The dihedral angle, between two planes, is measured by the angle between two lines

f. If a line (PQ) is perpendicular to a plane (xz) and from the foot of the perpendicular a second perpendicular (QL or QD) is drawn to any line (OX or NL) in the plane (zx), then the line connecting any point (P) on the first perpendicular to the intersection point (L or D) is perpendicular to the line (OX or NL) in the plane.



NOTE. — We will refer to these propositions as 3 *a*, 3 *b*, 3 *c*, 3 *d*, 3 *e*, and 3 *f*.



Point $P(x, y, z)$ or $P(r, \alpha, \beta, \gamma)$

4. Vectorial representation. — For some purposes it is convenient to think of three numbers (x, y, z) as representing the vector from the origin to the point. The length OP of the vector is called r, and the vectorial angles which this vector makes with the x-, y-, and z-axes are termed *direction angles* and are represented by the letters, α, β, and γ, respectively.

By proposition 3 *f*, the triangles PLO, PQO, and PNO are right triangles. Hence

$$x = r \cos \alpha, \quad y = r \cos \beta, \quad z = r \cos \gamma.$$

Further, the square on the diagonal $OP(= r)$ of our rectangular prism is the sum of $\overline{OQ}^2$ and $\overline{QP}^2$, and $\overline{OQ}^2 = \overline{ON}^2 + \overline{NQ}^2$, whence

$$r^2 = r^2 \cos^2 \alpha + r^2 \cos^2 \beta + r^2 \cos^2 \gamma,$$

or

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

Evidently, also,

$$r^2 = x^2 + y^2 + z^2.$$

5. Parametric equations of a line. — The equations

$$x = r \cos \alpha, \quad y = r \cos \beta, \quad z = r \cos \gamma,$$

when α, β, and γ are fixed and r is a variable parameter, serve as the equations of the straight line OP. Cos α, cos β, and cos γ are called *direction cosines* of this line.

Evidently any point $E(a, b, c)$ on the line OP produced in either direction satisfies this relationship, if r is taken as the distance from O to E. If E is on the other side of the origin from P, then the direction angles of the vector OE are supplementary to α, β, and γ and have cosines opposite in sign to cos α, cos β, cos γ. In this case r is taken as negative and it is evident that with this interpretation the coördinates of the point E satisfy the given equations.

A line which does not pass through O has the same direction cosines as the line parallel to it through O, positive directions on both being the same. If such a line passes through $P(x_1, y_1, z_1)$ in space, the corresponding parametric equations are

$$x - x_1 = r \cos \alpha, \; y - y_1 = r \cos \beta, \; z - z_1 = r \cos \gamma.$$

In plane analytics the corresponding formulas for straight lines are

$$x = r\cos\alpha,\ y = r\cos\beta, \text{ where } \cos\beta = \sin\alpha$$

and

$$x - x_1 = r\cos\alpha,\ y - y_1 = r\cos\beta.$$

The student should check these formulas and show their relations to the ordinary equations used.

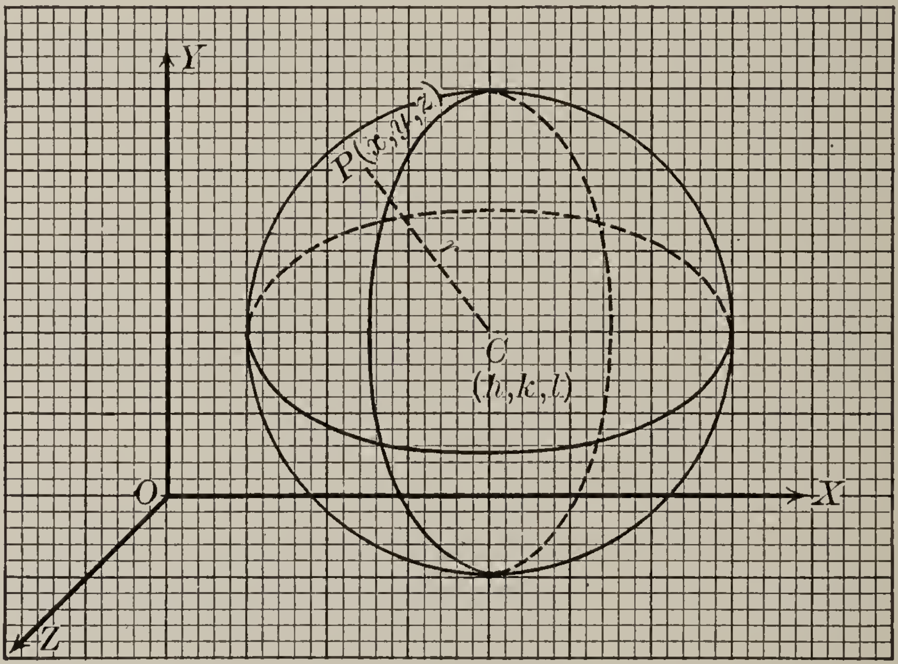
6. Distance formulas and spheres. —

$$r^2 = x^2 + y^2 + z^2$$

is the square of the distance from (x, y, z) to $(0, 0, 0)$.

$$x^2 + y^2 + z^2 = r^2$$

is an equation which is satisfied by every point on a sphere.



The sphere: $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

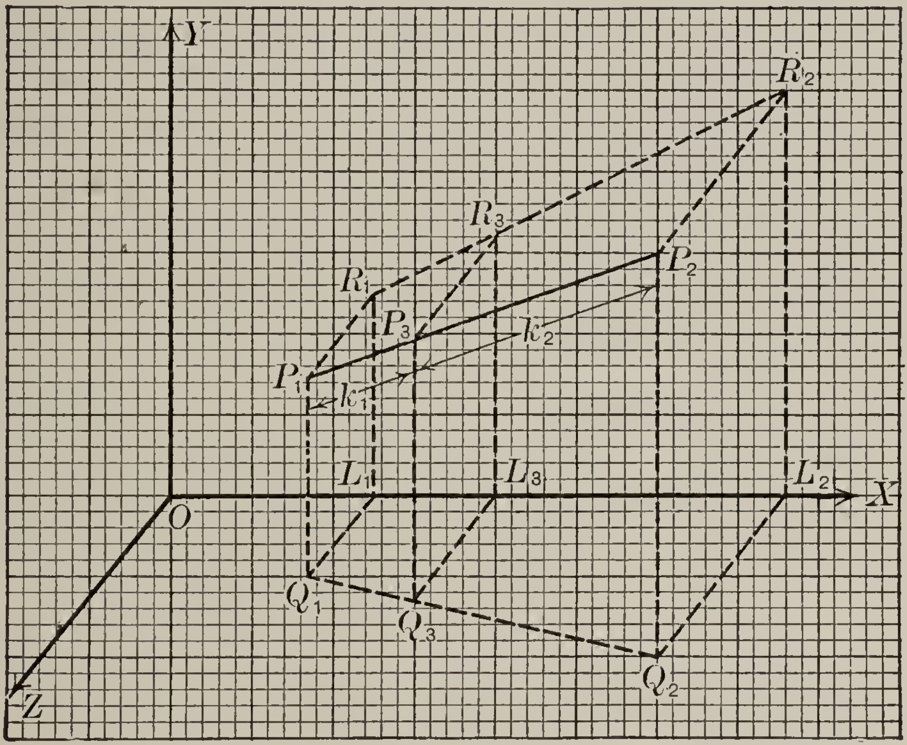
is the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2).

$$(x - h)^2 + (y - k)^2 + (z - l)^2$$

represents the square of the distance from (x, y, z) to (h, k, l) and hence, $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$ is the equation of a sphere whose center is (h, k, l) and whose radius is r.

7. Point of division.

$$x = \frac{k_2x_1 + k_1x_2}{k_1 + k_2},\quad y = \frac{k_2y_1 + k_1y_2}{k_1 + k_2},\quad z = \frac{k_2z_1 + k_1z_2}{k_1 + k_2}.$$



$$\frac{P_1P_3}{P_3P_2} = \frac{Q_1Q_3}{Q_3Q_2} = \frac{L_1L_3}{L_3L_2} = \frac{x_3 - x_1}{x_2 - x_3} = \frac{y_3 - y_1}{y_2 - y_3} = \frac{z_3 - z_1}{z_2 - z_3} = \frac{k_1}{k_2}$$

Step for step, and letter for letter, the proof follows that given for the point of division in a plane; the only change is that the z-term is added.

Thus, since $\frac{P_1P_3}{P_3P_2} = \frac{k_1}{k_2}$, it follows that $\frac{L_1L_3}{L_3L_2} = \frac{k_1}{k_2}$, etc.

PROBLEMS

1. What does the equation $x = 3$ or $x - 3 = 0$ represent on a line, the x-axis, *i.e.* when you are considering points on a line? What does this equation represent in plane analytics? What does this equation represent in space analytics?

2. What does the equation $x^2 = 9$ represent in one dimensional analytics? How are these two points located with reference to the origin? What does the equation $x^2 + y^2 = 9$

represent in the xy-plane? How are the points, which lie on this locus, located with reference to the origin? What does the equation $x^2 + y^2 + z^2 = 9$ probably represent in xyz-coördinates?

3. Where do all lines lie which make an angle of $+30°$ with the x-axis? Where do all lines lie which make an angle of $+70°$ with the y-axis? Where does a line lie which has the angle α equal to 30° and the angle β equal to 70°? Determine the angle γ, using the relation $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. Given that $\alpha = 30°$ and $\beta = 60°$, what is the value of γ? If $\alpha = 30°$, $\beta = 45°$, what is the value of γ?

4. Write the equations, in parametric form, of a line through the origin which has the direction angles $\alpha = 30°$, $\beta = 70°$, and the third angle as determined in the preceding problem. Write the equations when these angles are 30°, 60°, and 90°.

5. Write the equations of lines parallel to the two lines of the preceding problem and passing through the point $(-2, 3, -7)$.

6. Write the equation of the sphere whose radius is 10 and whose center is the point $(2, -3, 4)$. Find three other points on this sphere.

7. Find the coördinates of the points of trisection of the line joining $A(-2, 3, -7)$ to $B(2, -3, -4)$. If the line AB is extended through B by its own length, what are the coördinates of the point so determined?

8. Given that the parametric equations of a line are

$$x = 3r,\ y = -2r,\ z = -5r,$$

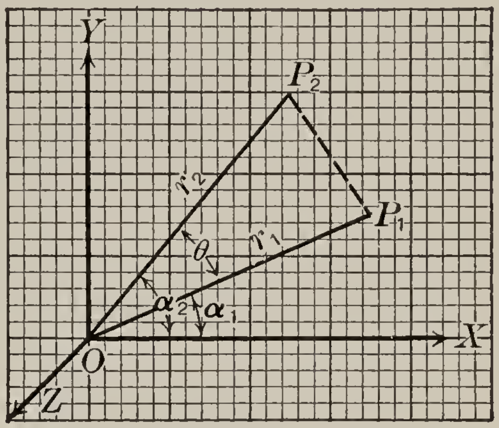
find 10 points upon the line by giving to r values from -4 to $+5$. Determine the direction cosines of this line. Determine from your trigonometric tables the angles α, β, and γ.

9. Given the parametric equations of a line

$$x - 3 = 3r,\ y + 5 = -2r, \text{ and } z - 7 = -5r,$$

find 10 points on the locus; determine the direction cosines and the angles α, β, and γ.

8. Angle between two lines. — We have had occasion to note that the angle between two non-intersecting, or *skew*, lines in space is defined as the angle between two intersecting lines which are respectively parallel to these given lines. For convenience these parallels may be taken through O.



The angle between two lines in space

The angle θ between two lines having direction cosines $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ respectively is obtained as follows:

Take any two points P_1 and P_2, one on each line, having vector distances r_1 and r_2. Evidently

$$\overline{P_1P_2}^2 = r_1^2 + r_2^2 - 2\, r_1 r_2 \cos\theta.$$

But

$$\overline{P_1P_2}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

Equating the two values and canceling, noting that

$$r_1^2 = x_1^2 + y_1^2 + z_1^2, \text{ and } r_2^2 = x_2^2 + y_2^2 + z_2^2,$$

this gives

$$\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{r_1r_2}.$$

Now

$$x_1 = r_1 \cos\alpha_1,\ y_1 = r_1\cos\beta_1,\ z_1 = r_1\cos\gamma_1,$$

and

$$x_2 = r_2\cos\alpha_2,\ y_2 = r_2\cos\beta_2,\ z_2 = r_2\cos\gamma_2.$$

Substituting,

$$\cos\theta = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2,$$

a relation which is independent, as it must be, of the particular points P_1 and P_2 chosen.

The corresponding formula in the plane for the angle between two lines is

$$\cos\theta = \cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2.$$

The adaptation and the proof are left to the student as an exercise. Compare with formula on page 247.

Frequently the direction cosines of a line in space are represented by l, m, n or l_1, m_1, n_1, etc. The condition that two lines be perpendicular is evidently $l_1l_2 + m_1m_2 + n_1n_2 = 0$, and the condition for parallel lines is that $l_1 = l_2$, $m_1 = m_2$, and $n_1 = n_2$. It can be shown by using vectors that the relation $l_1l_2 + m_1m_2 + n_1n_2 = 1$, combined with $l_1^2 + m_1^2 + n_1^2 = 1$ and $l_2^2 + m_2^2 + n_2^2 = 1$ reduces to $l_1 = l_2$, $m_1 = m_2$, and $n_1 = n_2$.

9. First-degree equation. — We now show that the equation

$$(a) \qquad Ax + By + Cz + D = 0$$

represents a plane. A plane is, by definition, a surface which is such that the straight line joining any two points in the surface lies wholly on the surface, *i.e.* any other point on the line is also on the surface.

Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be any two points which satisfy equation (a).

$$(b) \qquad \therefore Ax_1 + By_1 + Cz_1 + D = 0.$$

$$(c) \qquad Ax_2 + By_2 + Cz_2 + D = 0.$$

Let $P_3(x_3, y_3, z_3)$ be any other point on the line joining $P(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ and let this point divide P_1P_2 into segments such that $\frac{P_1P_3}{P_3P_2} = \frac{k_1}{k_2}$.

Then, x_3, y_3, and z_3 may be written

$$x_3 = \frac{k_2x_1 + k_1x_2}{k_1 + k_2}, \quad y_3 = \frac{k_2y_1 + k_1y_2}{k_1 + k_2}, \quad z_3 = \frac{k_2z_1 + k_1z_2}{k_1 + k_2}.$$

Substituting these values in the left-hand member of equation (a), we have

$$A\frac{k_2x_1 + k_1x_2}{k_1 + k_2} + B\frac{k_2y_1 + k_1y_2}{k_1 + k_2} + C\frac{k_2z_1 + k_1z_2}{k_1 + k_2} + D,$$

which may be written, by rearrangement of terms,

$$\frac{k_2}{k_1 + k_2}(Ax_1 + By_1 + Cz_1 + D) + \frac{k_1}{k_1 + k_2}(Ax_2 + By_2 + Cz_2 + D);$$

but this expression, by (*b*) and (*c*) above, is zero; hence the point $P_3(x_3, y_3, z_3)$, any point on the straight line joining P_1 to P_2, which points are on the locus of (*a*), is also on this locus.

$$\therefore Ax + By + Cz + D = 0 \text{ represents a plane.}$$

The converse proposition is demonstrated in section 5 of the next chapter.

A plane to pass through three given points is determined by substitution of the three points in equation (*a*) and solving for three of the constants, *e.g.* A, B, and C, in terms of the fourth. If the fourth constant chosen happens to be zero, another selection must be made.

Illustrative problem. — Find the plane through (4, 4, 4), (3, 0, − 5), and (0, − 3, − 7).

Let $Ax + By + Cz + D = 0$ represents the equation of the plane. Substituting,

$$\begin{aligned} \times 3\ (1)\quad & 4A + 4B + 4C + D = 0. \\ (2)\quad & 3A \phantom{{}+4B} - 5C + D = 0. \\ \times 4\ (3)\quad & - 3B - 7C + D = 0. \end{aligned}$$

Since it happens that only A, C, and D occur in (2), eliminate B between (1) and (3) by multiplying (1) by 3 and (3) by 4 and adding, obtaining

$$\begin{aligned} (4)\quad & 12A - 16C + 7D = 0. \\ -4\ (2)\quad & 3A - 5C + D = 0. \end{aligned}$$

Eliminate A between (4) and (2) by multiplying (3) by − 4 and adding to (4). This gives

$$(5)\quad 4C + 3D = 0;\quad C = -\tfrac{3}{4}D.$$

Substituting in (2) the value of C found gives

$$3A + \tfrac{15}{4}D + D = 0,\ A = -\tfrac{19}{12}D.$$

Substituting value of C in (3) gives

$$-3B + \tfrac{21}{4}D + D = 0;\quad B = \tfrac{25}{12}D.$$

The equation of the plane may be written,

$$-\tfrac{19}{12}Dx + \tfrac{25}{12}Dy - \tfrac{3}{4}Dz + D = 0, \text{ or}$$

$$-19x + 25y - 9z + 12 = 0. \quad \textit{Ans.}$$

PROBLEMS

1. Find the angle between the two lines of problem 5 of the preceding list of problems.

2. Find the direction cosines proportional to 3, -2, and -5; find those proportional to 2, 3, and -4; find the angle between two lines having these direction cosines that you have found.

3. Find the equation of a plane whose intercepts on the axes are, respectively, 3, -5, and 7.

4. Find the equation of a plane through the points (3, 0, 5), (-2, 11, 7), and (0, 11, 7).

5. The parametric equations of any line through (3, 0, 5) can be written

$$x - 3 = r \cos \alpha,\ y - 0 = r \cos \beta,\ z - 5 = r \cos \gamma.$$

If this line is to pass through (-2, 11, 7), these coördinates must satisfy these equations. Make the substitutions, respectively; square and add the corresponding numbers and thus obtain the value of r, giving the distance of the point (3, 0, 5) from (-2, 11, 7). Find then the values of $\cos \alpha$, $\cos \beta$, and $\cos \gamma$.

6. If a rectangular box with sides parallel to the coördinate planes has the line joining (3, 0, 5) to (-2, 11, 7) as a principal diagonal, find the lengths of the sides, the length of the diagonal, and so find the direction cosines of the line joining the two points.

7. Discuss the loci of the following equations and find three points on each locus:

a. $3x + 11 = 0.$
b. $x^2 - 9 = 0.$
c. $z - 5 = 0.$
d. $x - y - 5 = 0.$
e. $z - 2y + 10 = 0.$
f. $x + 2y + 3z - 8 = 0.$

8. Where do points lie which are common to the loci of the two following equations: $x = 3$, and $y = 5$?
$x - 3 = 0$ and $z - 2y + 10 = 0$?

CHAPTER XXXI

SOLID ANALYTICS; FIRST-DEGREE EQUATIONS AND EQUATIONS IN TWO VARIABLES

1. Locus of an equation in three variables. — Any equation involving three variables has for its locus a surface which may, in special forms of the equation, reduce to one or more lines or points. We obtain points on such a surface by giving values to two coördinates, *e.g.*, x and y, and solving for the third, *e.g.*, z. Thus we have found that any first-degree equation represents a plane.

NOTE. $x^2 + y^2 + z^2 = 0$ represents only a point (0, 0, 0), or a point sphere.

$x^2 + y^2 = 0$ represents the z-axis since everywhere on this axis $x = 0$ and $y = 0$.

2. Intersections of loci. — (See Chapter V, Section 2.) Any point which satisfies two equations involving three variables lies, in general, upon a curve which is common to the two surfaces represented.

When three equations are regarded as simultaneous, points of intersection of the three surfaces are obtained. Under special relations between the three given equations, these points may lie upon a line, but, in general, three simultaneous rational integral algebraic equations determine a finite number of points of intersection.

Just as a family of lines through the intersection of two given lines is obtained in Chapter V, Section 4, in the form $l_1 + kl_2 = 0$, so the equation $f_1(x, y, z) + kf_2(x, y, z) = 0$

represents a family of surfaces which pass through the intersection curves of the two given surfaces.

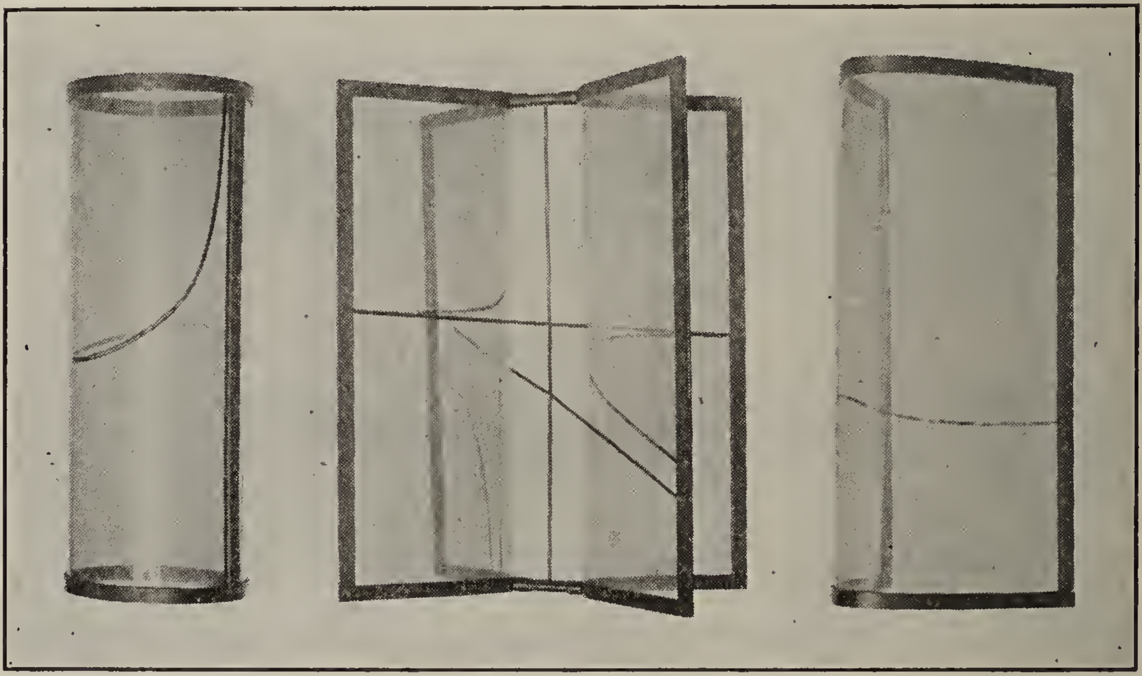
Thus, $x^2 + y^2 + z^2 = 25$ represents a sphere of radius 5; $x^2 = 9$ represents two planes parallel to the yz-plane; the equation

$$x^2 + y^2 + z^2 - 25 - k(x^2 - 9) = 0$$

represents for all values of k a surface through the intersections of the sphere and the plane. For $k = -1$, this surface reduces to a cylinder,

$$y^2 + z^2 - 16 = 0.$$

3. Cylindrical surfaces. — Any equation in two variables, as x and y, represents in space a cylinder whose axis is parallel to the axis designated by the third variable.



Cylindrical surfaces:

ELLIPTIC HYPERBOLIC PARABOLIC

The curves indicated on these surfaces are cubic space curves.

If an equation $f(x, y) = 0$ is given in x and y, any point (x_1, y_1) which satisfies the equation will lie upon the curve in the xy-plane given by $f(x, y) = 0$. Considered as a point in space, the point $(x_1, y_1, 0)$ satisfies the equation, and further it is evident that (x_1, y_1, z), irrespective of the value of z, will also satisfy $f(x, y) = 0$, since the z-coördinate does not enter at all.

Thus, (3, 4, 0) satisfies the equation

$$x^2 + y^2 - 25 = 0$$

and also the points (3, 4, 1) or (3, 4, – 10) or (3, 4, 8,) will satisfy the equation

$$x^2 + y^2 - 25 = 0.$$

But all points (x_1, y_1, z), for varying values of z only, lie on a parallel to the z-axis through (x_1, y_1) and hence all points on the surface generated by a straight line moving parallel to the z-axis and touching the curve $f(x, y) = 0$, in the xy-plane, lie upon a cylinder. The curve $f(x, y) = 0$ is called the *directrix* of the cylinder and the moving line is called the *generator* or *element* of the cylindrical surface. Similarly, when an equation is given in x and z or in y and z, a cylinder is represented.

A plane given by a first-degree equation in two variables, or one variable, is a special case of the preceding.

4. Straight line as the intersection of two planes. — Just as the equation $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$ represents in the plane the straight line joining $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$, so the three equations,

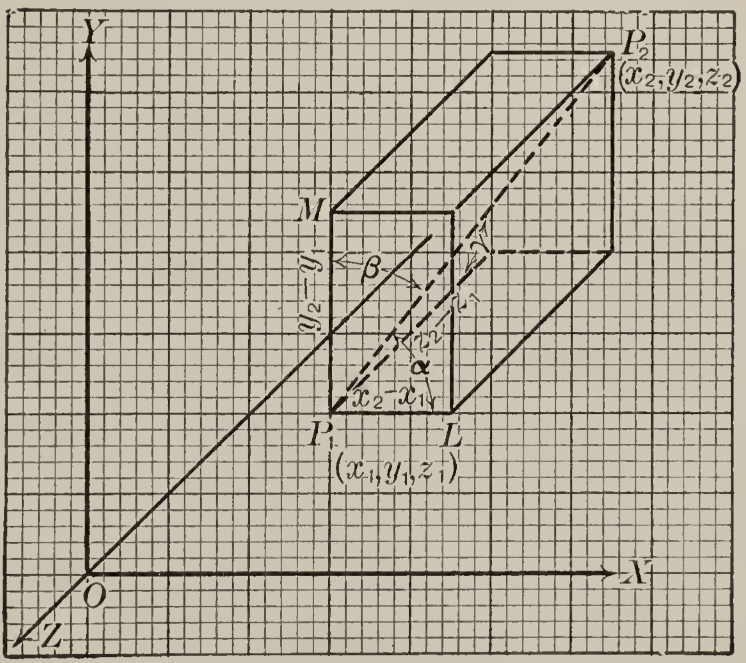
$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1},$$

represent in space the straight line joining

$$P_1(x_1, y_1, z_1)$$

to $P_2(x_2, y_2, z_2)$.

There are three equalities which are obtained by leaving out in turn each of the fractions, but there are only two independent equations, as



Line joining P_1 to P_2 in space

the third equality would follow always from the first two which were given.

These formulas can be obtained directly from the properties of similar triangles, or from the parametric equations of section 5 of the preceding chapter. The latter method brings out the important fact that the values $x_2 - x_1$, $y_2 - y_1$, and $z_2 - z_1$ are proportional to the direction cosines of the given line, and the values themselves of these cosines can be obtained, using the fact that the sum of the three squares is equal to unity. The derivation of the theorems mentioned is left as an exercise to the student.

The parametric forms of the equations of a straight line may be written,

$$\frac{x - x_1}{\cos \alpha} = \frac{y - y_1}{\cos \beta} = \frac{z - z_1}{\cos \gamma} = r,$$

or

$$\frac{x - x_1}{k \cos \alpha} = \frac{y - y_1}{k \cos \beta} = \frac{z - z_1}{k \cos \gamma} = \frac{r}{k}.$$

Further, any equations which can be put in one of the two forms above represent a straight line, and the denominators of the fractions are proportional to the direction cosines of the line.

The equations of the straight line in the form

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c},$$

wherein a, b, and c are necessarily proportional to the direction cosines of the line, are called the *standard* or *symmetrical* equations of the line.

In general, any curve in space is given as the intersection of two surfaces by the equations of the two surfaces. In particular, the straight line is given by the equations of any two planes which pass through the line. Of the infinite number of planes, the pencil of planes, which pass through a given line, the three planes, called *projecting planes* of the line, which are parallel to the coördinate axes are of particular importance. These equations will evidently be first-degree equations in two variables. In the standard form the equality of any two members gives one of the projecting planes through the given line.

Illustrative problem. — Find the direction cosines of the straight line determined by the two planes

(a) $$x + y - 3z - 5 = 0.$$

(b) $$3x - y - 5z - 11 = 0.$$

Find the projecting planes parallel to the coördinate axes (or perpendicular to the coördinate planes); find the points where this line pierces the coördinate planes.

Any plane through the line of intersection is given by

(c) $$x + y - 3z - 5 + k(3x - y - 5z - 11) = 0.$$

Giving to k the value $-\frac{3}{5}$, which is equivalent to multiplying (a) by 5, and (b) by -3 and adding, and simplifying you have,

$$5x + 5y - 15z - 25 - 9x + 3y + 15z + 33 = 0, \text{ or}$$

(d) $-4x + 8y + 8 = 0$, as the *plane of projection* on the xy-plane.

Eliminating y, $k = 1$, gives,

(e) $4x - 8z - 16 = 0$ or $x - 2z - 4 = 0$, the *plane of projection* on the xz-plane.

Eliminating x, $k = -\frac{1}{3}$, gives,

(f) $-4y + 4z + 4 = 0$, which might have been obtained from (d) and (e), the *plane of projection* on the yz-plane.

Solving for x, in (d) and (e),

$$x = 2(y + 1) \text{ and}$$
$$x = 2(z + 2).$$
$$x = 2(y + 1) = 2(z + 2).$$
$$\frac{x - 0}{2} = \frac{y + 1}{1} = \frac{z + 2}{1}.$$

The denominators 2, 1, and 1 are proportional to the direction cosines of this line. Hence

$$\cos\alpha = 2m,\ \cos\beta = 1m \text{ and } \cos\gamma = 1m,$$

giving,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 4m^2 + m^2 + m^2 = 1;\ 6m^2 = 1;\ m = \pm\frac{1}{\sqrt{6}}.$$

Either sign may be taken, but for convenience, make $\cos\alpha$ positive.

$$\cos\alpha = \frac{2}{\sqrt{6}},\ \cos\beta = \frac{1}{\sqrt{6}},\ \cos\gamma = \frac{1}{\sqrt{6}}, \text{ the } \textit{direction cosines.}$$

To find where this line pierces any coördinate plane, as $z = 0$, solve the equation of the coördinate plane as simultaneous with the two given planes which determine the line.

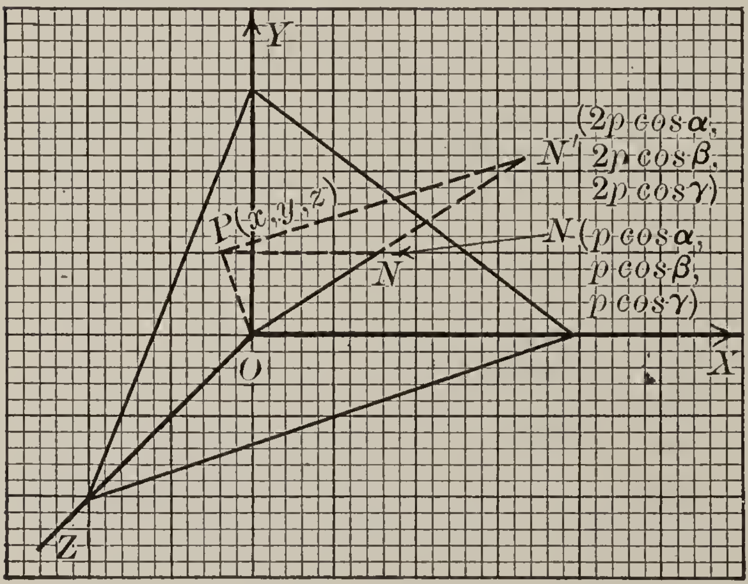
This gives here $(4, 1, 0)$ as the *piercing point* with the xy-plane. Similarly we find the intersection with any plane.

A parallel line to our given line through a given point would be determined by two planes through the given point parallel to the two given planes which determine the line. Why? Determine the parallel line through $(1, -5, 6)$.

5. Normal form of the equation of a plane. — (See Section 3, Chapter IX.) In the plane, the equation $x \cos \alpha + y \sin \alpha - p = 0$, which may be written $x \cos \alpha + y \cos \beta - p = 0$, represents the equation of a straight line in normal form, which line is such that the perpendicular from the origin upon it has the length p and makes the angles α and β with x-axis and y-axis. Similarly, in space, the equation

$$x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0$$

represents a plane which is such that the perpendicular from the origin upon it has the length p and makes the angles α, β, and γ with the x-axis, y-axis, and z-axis respectively.



ON of length p; ON' of length $2p$
Direction angles of ONN': α, β, γ

Evidently, if a plane is given and a perpendicular ON of length p, having direction cosines α, β, and γ, is dropt from the origin to this plane, the point N is $(p \cos \alpha, p \cos \beta, p \cos \gamma)$ and the extension of the perpendicular by the length p gives the point N' $(2p \cos \alpha, 2p \cos \beta, 2p \cos \gamma)$. Any point $P(x, y, z)$ which is equidistant from $O(0, 0, 0)$ and $N'(2p \cos \alpha, 2p \cos \beta, 2p \cos \gamma)$

lies on our plane. Writing and equating these distances, we have,

$$x^2 + y^2 + z^2 = (x - 2\,p \cos \alpha)^2 + (y - 2\,p \cos \beta)^2 + (z - 2\,p \cos \beta)^2.$$

Whence,

$$(4\,p \cos \alpha)x + (4\,p \cos \beta)y + (4\,p \cos \gamma)z = 4\,p^2(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma),$$

giving finally

$$x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0$$

as the equation.

In the plane the distance from any point (x_1, y_1) to a line is obtained by writing the equation of the line in normal form and substituting therein for x and y, x_1 and y_1. In space the distance of a point (x_1, y_1, z_1) from a plane is obtained by writing the equation of the plane in normal form and substituting therein these coördinates of the point for x, y, and z, respectively.

To reduce a linear equation to normal form, you divide the equation through, after transposing all terms to the left-hand member, by the square root of the sum of the squares of the coefficients of x, y, and z, choosing the sign opposite to the sign of the constant term. The proof is not similar to the proof of the corresponding theorem in plane analytics.

Parallel planes are represented by linear equations having the corresponding coefficients, of x, y, and z, equal or proportional.

PROBLEMS

1. Put the following equations in normal form and determine the distance of each plane from the origin:

a. $2\,x - 3\,y + 4\,z - 11 = 0.$

b. $x + y + z - 5 = 0.$

c. $2\,x - 3\,y - 11 = 0.$ *d.* $z - 7 = 0.$

Determine the direction cosines and the direction angles of the normals to each of the above planes.

2. Find the equations in standard form of a line from (1, 2, 5) perpendicular to the first plane in problem 1; through (0, 0, 0) perpendicular to the second plane in problem 1; through $(-2, -3, 4)$ perpendicular to the third plane in problem 1. Determine in each of these three problems the intersection of the perpendicular with the plane.

3. Find the piercing points with the coördinate planes of the following lines:

a. $2x - 3y + 4z - 11 = 0$ and $x - y + z - 5 = 0$.
b. $2x - 3y + 4z - 11 = 0$ and $z - 7 = 0$.
c. $2x - 3y - 11 = 0$ and $z - 7 = 0$.

4. Put the three lines of problem 3 in standard form. Note that in the second and third cases, since the given line lies in a plane parallel to the xy-plane, the line makes an angle of 90° with the z-axis, *i.e.* $\cos\gamma = 0$. The equations of the second and third lines in standard form would have a zero denominator, and so it is better to put these equations in the form given in the third of these problems. The values of $\cos\alpha$ and $\cos\beta$ are determined here from the equation $2x - 3y - 11 = 0$, giving $\cos\alpha = \frac{2}{\sqrt{13}}$ and $\cos\beta = \frac{3}{\sqrt{13}}$.

5. Find the equations of the straight lines through the two points:

a. $(3, 5, -2)$ and $(0, 0, 7)$.
b. $(3, 5, -2)$ and $(0, 0, 0)$.
c. $(3, 5, -2)$ and $(-3, 5, +2)$.

6. Find the angle between the lines

$$\frac{x+4}{2} = \frac{y}{-1} = \frac{z-3}{-3} \text{ and } \frac{x-2}{-2} = \frac{y}{5} = \frac{z-1}{7}.$$

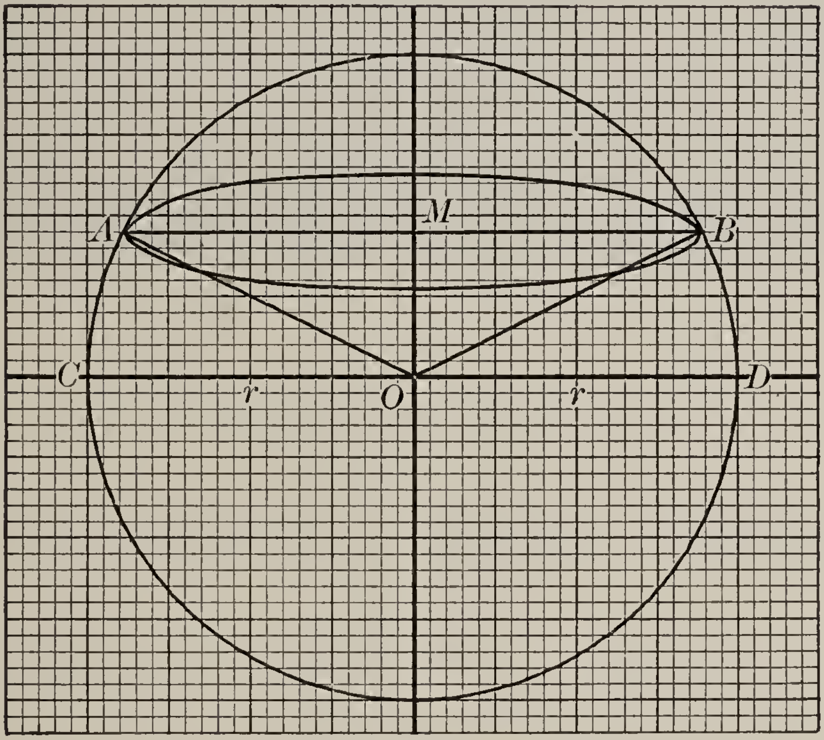
7. Do the two lines in problem 6 intersect? How can you determine whether any two given lines intersect? Note that the problem is entirely analogous to the problem in plane analytics as to whether three given lines intersect, and is solved in the same manner. Write the equations of two intersecting lines.

8. Determine the curve in which the sphere

$$x^2 + y^2 + z^2 - 400 = 0$$

is intersected by the plane $y - 9 = 0$. Note that substituting $y = 9$ is equivalent to writing

$$x^2 + y^2 + z^2 - 25 - (y + 9)(y - 9) = 0,$$



Sphere cut by a plane

which gives, of course, a new surface passing through the intersection curve of the first two surfaces.

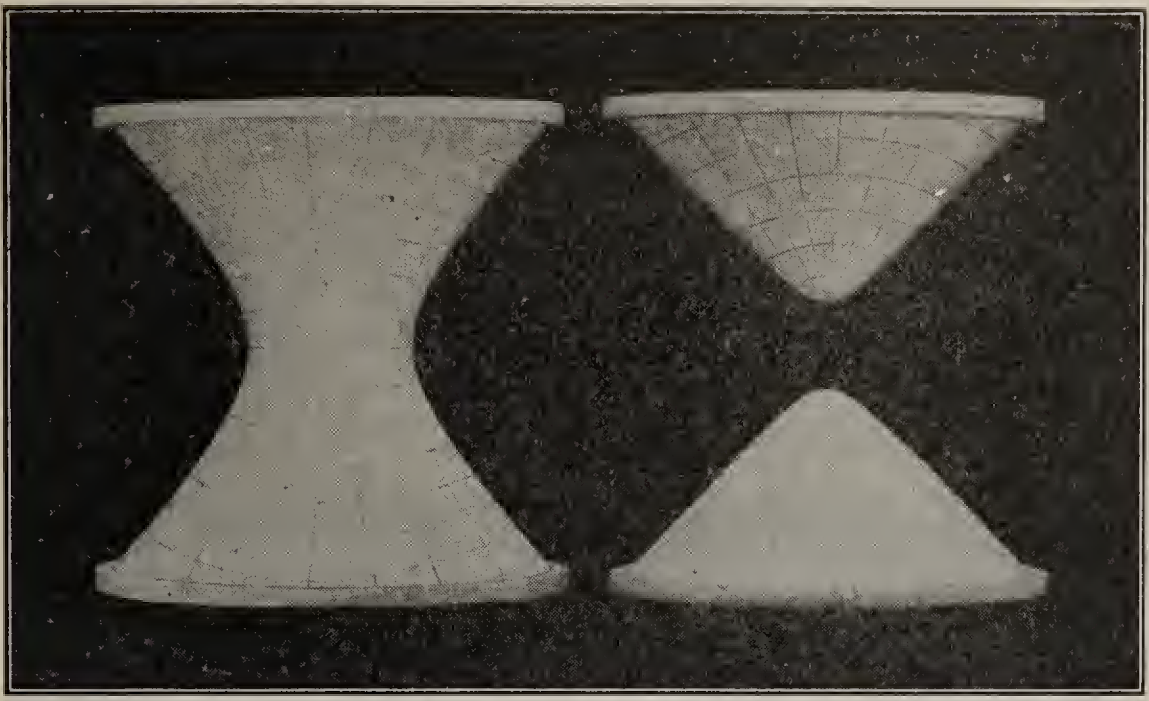
9. Find the intersection of the sphere $x^2 + y^2 + z^2 - 100 = 0$ and the cylinder $x^2 + y^2 - 36 = 0$.

10. Upon what cylinder, parallel to one of the coördinate axes, does the intersection of the plane $x = 5$ with the surface $x^2 + 4y^2 = 25z$ lie?

11. Find the intersection of the line

$$x - 1 = 2r,\ y - 2 = 3r,\ z + 3 = -5r$$

with the sphere $x^2 + y^2 + z^2 - 100 = 0$ by substituting these values in the equation of the sphere and solving for r. Note that since the right-hand coefficients, 2, 3, and -5, are not the direction cosines of this line, but only proportional to them, the values of r obtained are not the distances from $(1, 2, -3)$ to the points of intersection with the sphere, but are proportional to these distances. The points of intersection are obtained by substituting the values of r found back in the equations of the line and solving for (x, y, z).



Hyperboloid of one sheet   Hyperboloid of two sheets

CHAPTER XXXII

SOLID ANALYTICS: QUADRIC SURFACES

1. General equation. — In plane coördinates, any equation of the second degree represents a conic section. Similarly, in space coördinates, any equation of the second degree represents a quadric surface. The types of quadric surfaces, limited in number, are closely allied to the types of conic sections. In plane analytics, it is shown that the general equation of the second degree, containing the "cross-term" xy, introduces no new curves, only the same curves, represented by the different types of equations in which no xy-term appears. It is likewise true in space that the general equation containing any or all of the "cross-terms," yz, xz, and xy, presents no surfaces different from those which may be represented by the general equation containing no cross-term.

Methods of transformation of coördinates quite similar to those discussed in Chapter XXIV apply to space coördinates, but the limitations of a first course preclude any discussion of the methods and results.

Any surface given by an equation of the second degree is cut by any plane in some form (including, of course, limiting forms) of conic section. The coördinate planes very evidently cut any quadric surface in a conic, since the curve of intersection in the coördinate plane is given by an equation of the second degree in the two variables of that plane. The transformations mentioned above are desirable for the general proof, but another method is indicated below.

Ellipsoids:

SPHERE PROLATE SPHEROID OBLATE SPHEROID GENERAL ELLIPSOID

2. Ellipsoids. — The equation of a sphere has been given as

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2.$$

An ellipsoid is given by the equation

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}+\frac{(z-l)^2}{c^2}=1.$$

This surface is related to the three spheres,

$$(x-h)^2+(y-k)^2+(z-l)^2=a^2,$$
$$(x-h)^2+(y-k)^2+(z-l)^2=b^2,$$
$$(x-h)^2+(y-k)^2+(z-l)^2=c^2,$$

very much as the ellipse is related to its auxiliary circles.

The parametric equations of the above ellipsoid are

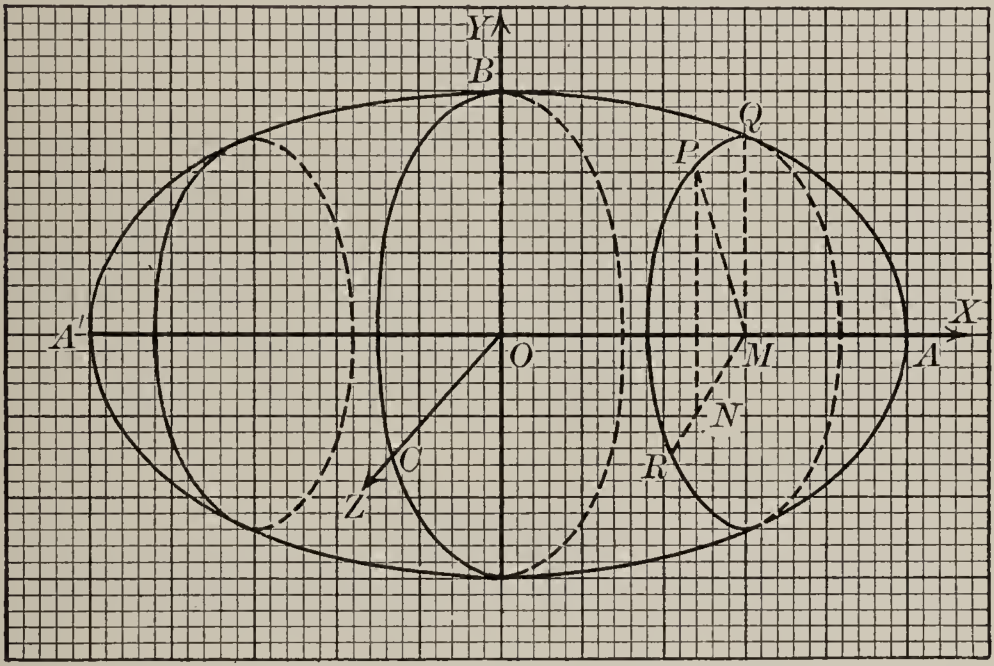
$$x-h=a\cos\alpha,$$
$$y-k=b\cos\beta,$$
$$z-l=c\cos\gamma.$$

The elimination of α, β, and γ, employing

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1,$$

gives the equation of the ellipsoid in the standard form above.

The quantities a, b, and c represent the semi-axes of the ellipsoid. If two of these denominators are equal to each other, the ellipsoid is an ellipsoid of revolution about an axis parallel to the axis corresponding to the term with the odd denominator.



Ellipsoid of revolution, with x-axis as axis of revolution

Thus,
$$\frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{16} = 1$$
is an ellipsoid of revolution, obtained by revolving the curve $\frac{x^2}{25} + \frac{y^2}{16} = 1$ about the x-axis.

The derivation of the formula of the ellipsoid of revolution is as follows, $PN^2 + NM^2 = PM^2$, but $QM = PM$, radii of the circle QPR about M, with lettering as indicated on diagram given above. Now for all points on this circle the x-coördinate is the same,

$$y^2 + z^2 = PM^2 = QM^2 = 16\left(1 - \frac{x^2}{25}\right),$$

which is a relation true for every point on the circle obtained by rotating the point Q on the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ about its axis. But Q is any point on the ellipse, and hence P may be any point on the surface obtained by revolving the ellipse about its axis.

Hence, for every point on this surface,

$$y^2+z^2=16\left(1-\frac{x^2}{25}\right),$$

or

$$\frac{x^2}{25}+\frac{y^2}{16}+\frac{z^2}{16}=1.$$

Any ellipsoid of revolution obtained by revolving an ellipse about its major axis is called a *prolate spheroid*, and is shaped like a football; an *oblate spheroid* is obtained by rotating an ellipse about its minor axis, and is shaped like a circular cushion or the earth.

PROBLEMS

1. Find the equation of the sphere having the center at the origin and passing through the point $(-2, 5, 6)$. Give the seven points which lie on this sphere and are symmetrically situated to the given point with respect to the coördinate planes.

2. Find the equation of the preceding sphere if the center is at $(3, -2, 12)$. Find by using conditions of symmetry with respect to planes through the center parallel to the coördinate planes seven further points on this sphere.

3. Write the equation of the ellipsoid having the center at the origin and semi-axes equal to 2, 3, and 5 respectively (x, y, and z order). Find three points on this ellipsoid. Write the equations of three circles which lie on this surface. Write the equations of the traces on the coördinate planes, *i.e.* the intersections with these planes. Draw the graph.

4. If a football is 10 inches long with a diameter of 8 inches, write the equation of the surface, assuming it to be an ellipsoid. Draw the graph to scale.

5. Assuming that an air cushion 18 inches in diameter and 6 inches high is an ellipsoid, write the equation of the surface. Draw the graph to scale.

6. Find the six principal foci of the ellipsoid in problem 3. These are the foci of the traces on the coördinate planes.

3. Hyperboloids. — By rotating the hyperbola

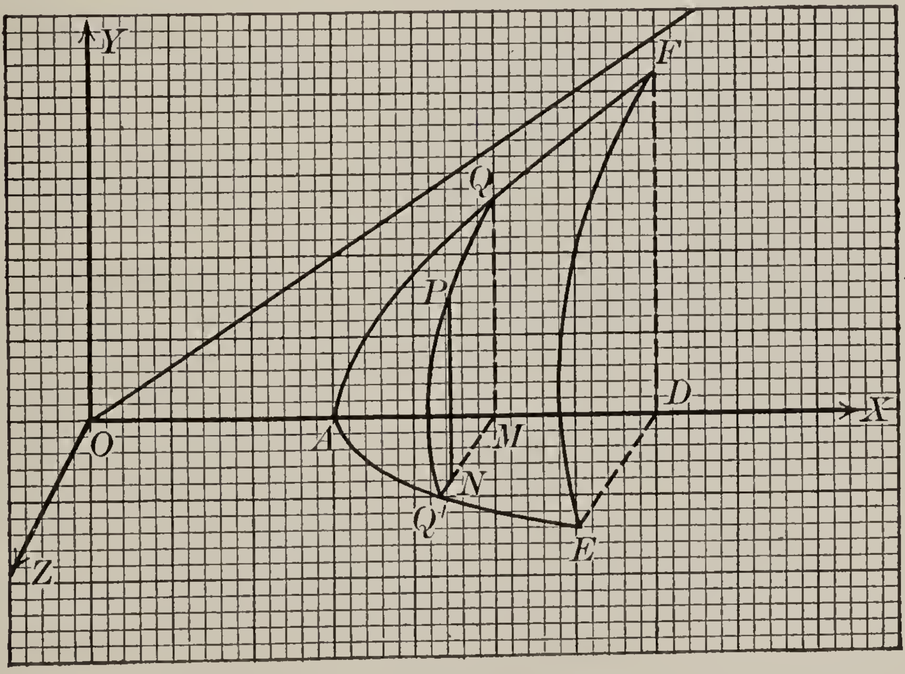
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

about either axis, a hyperboloid of revolution is obtained. Rotation about the principal axis, the x-axis here, gives a surface of two separated parts, called a *hyperboloid of revolution of two sheets.* The equation is,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{b^2}=1.$$

The method of derivation, which we outline, is general, and being applied to the surface obtained by revolving any curve, $y=f(x)$ about the x-axis, will give the equation of the surface in the form $y^2+z^2=[f(x)]^2$.

Given $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, revolved about the x-axis.

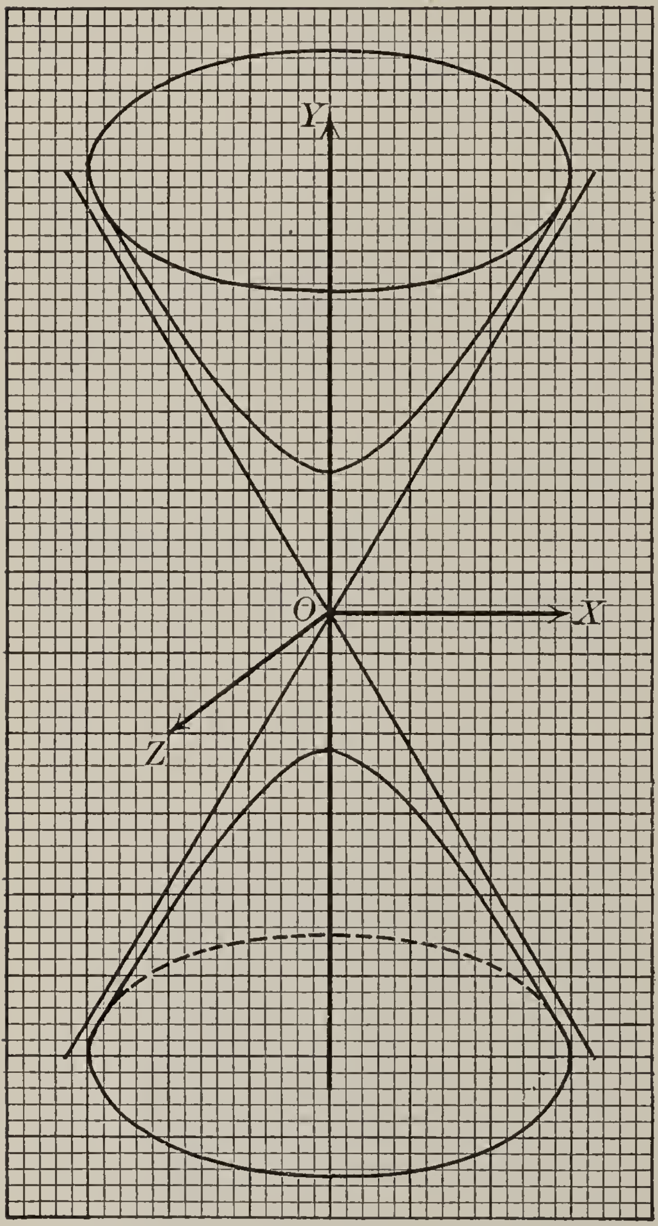


Hyperboloid of two sheets, of revolution

The curves are slightly distorted.

Any point $P(x, y, z)$ on this surface is obtained by the rotation of a point $Q(x, y, 0)$ about the x-axis.

The point Q generates a circle in a plane parallel to the yz-plane, in which x has everywhere the value given by OM.



Hyperboloid of two sheets

y-axis as principal axis.

The equation of this circle is

$$y^2 + z^2 = r^2 = \overline{PM}^2 = \overline{QM}^2.$$

This radius r is evidently a function of x, being defined by the original equation given; hence,

$$r^2 = (\text{ordinate on the hyperbola})^2 = b^2\left(\frac{x^2}{a^2} - 1\right).$$

Hence the point $P(x, y, z)$ satisfies the equation

$$y^2 + z^2 = b^2\left(\frac{x^2}{a^2} - 1\right),$$

or, by rearrangement of terms,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{b^2} = 1,$$

the *hyperboloid of revolution of two sheets.*

Note that precisely this surface would have been obtained by revolving $\frac{x^2}{a^2} - \frac{z}{b^2} = 1$ in the xz-plane about the x-axis.

Similarly, the *hyperboloid of revolution of one sheet* is obtained by revolving a hyperbola about its conjugate axis. The preceding hyperbola revolved about the y-axis gives

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{a^2}=1.$$

The student will note that the axis of rotation in each case is given by the odd term.

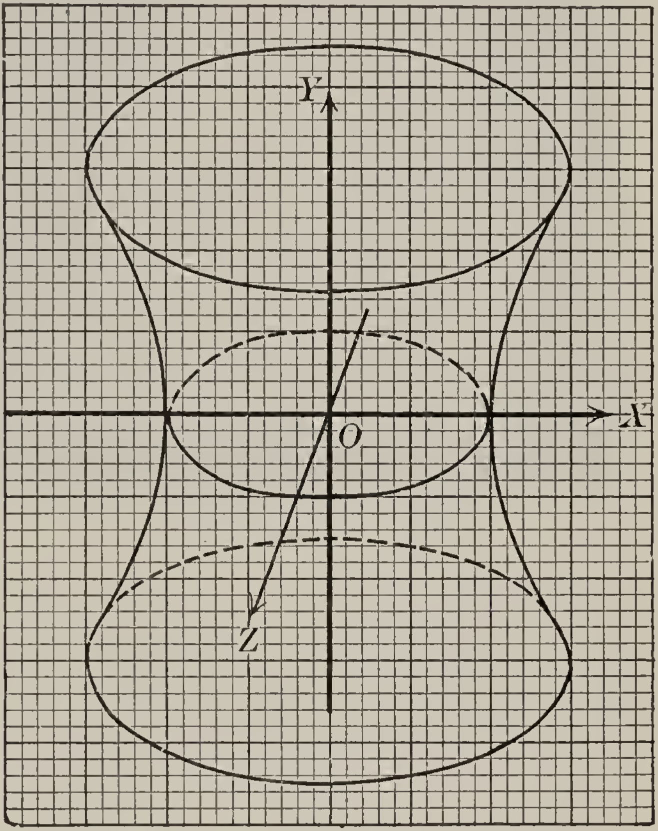
Corresponding to these surfaces of revolution are the general hyperboloids,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

hyperboloid of two sheets, and

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

hyperboloid of one sheet, which represent in each case a surface having a principal axis parallel to the axis of the odd term, *e.g.* the first has the x-axis as principal axis and the second has the y-axis as principal axis. Changing the principal axis to another of the coördinate axes interchanges two of the algebraic signs in the equation.



Hyperboloid of revolution, one sheet
y-axis as principal axis.

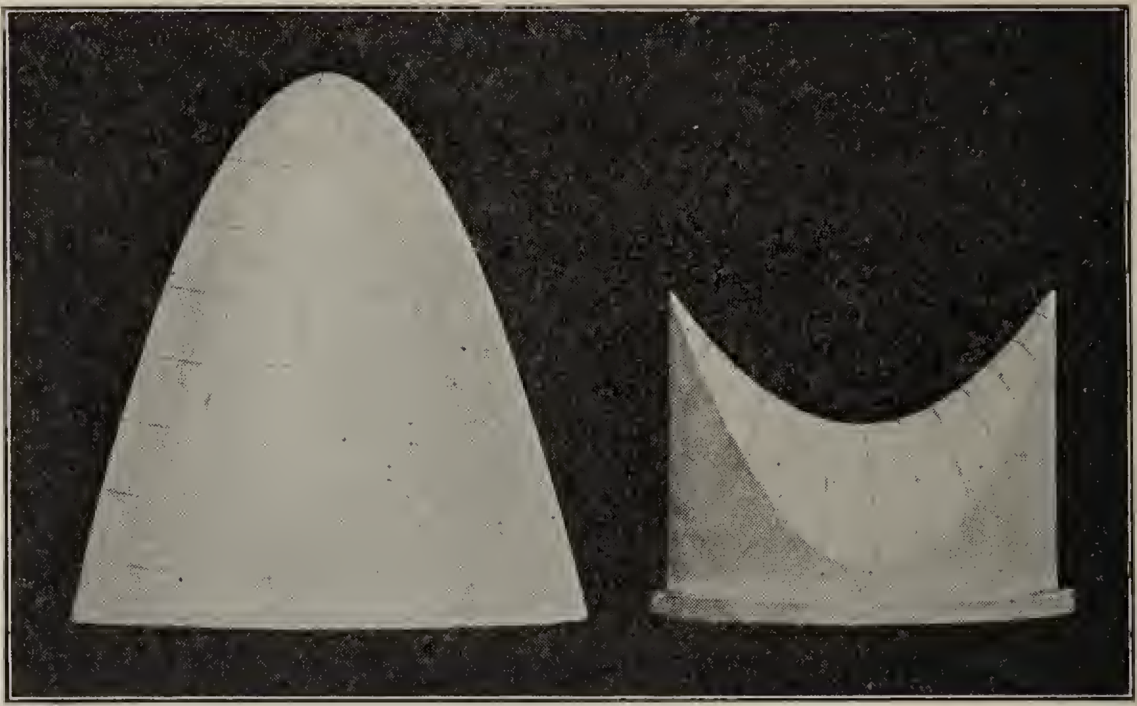
4. Paraboloids. — By revolving the parabola $y^2=4\,ax$ about its axis the surface $y^2+z^2=4\,ax$ is obtained.

This is called a *paraboloid of revolution*, or a *circular paraboloid*, and is the type of surface which is fundamental in theater and auditorium construction. The derivation of the equation is left as an exercise for the student.

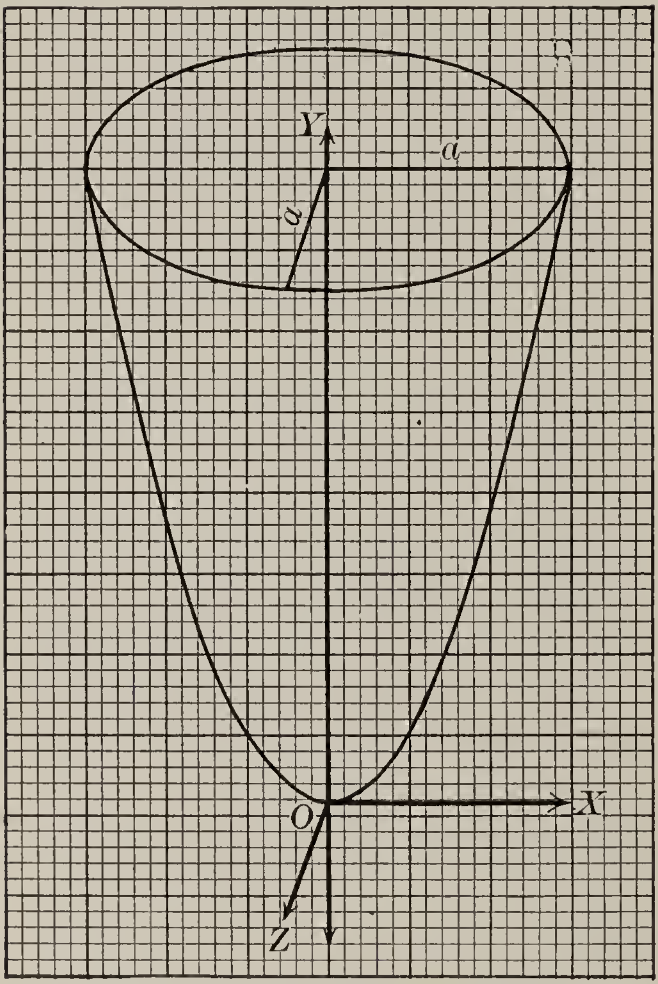
The *elliptic paraboloid* is given by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=4\,z,$$

and sections parallel to the xy-plane are ellipses.



Elliptic paraboloid **Hyperbolic paraboloid**



Elliptic paraboloid of revolution

The corresponding standard forms with y-axis and with x-axis, respectively, as principal axis are,

$$\frac{x^2}{a^2}+\frac{z^2}{c^2}=4\,y,$$

and $$\frac{y^2}{b^2}+\frac{z^2}{c^2}=4\,x.$$

The equation

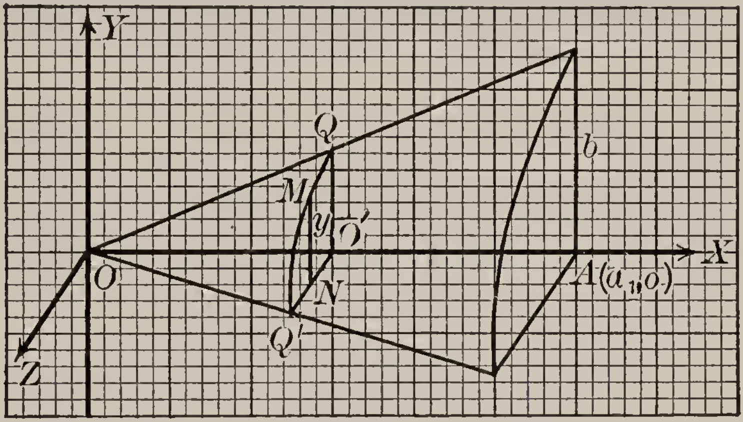
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=4\,z$$

gives the most complicated of the quadratic surfaces, the *hyperbolic paraboloid,* a saddle-shaped surface which is here represented by a photograph of a model of such a surface.

The hyperbolic paraboloid may be generated by the motion of a given parabola, $x^2 - 4a^2z = 0$, moving parallel to the xz-plane and having its vertex moving on the parabola

$$y^2 = -4b^2z.$$

5. Cones. — If any straight line is revolved about another straight line in its plane as an axis, a cone of revolution is generated. Limiting forms are the cylinder, when the revolving line is parallel to the axis, a plane when the revolving line is perpendicular to the axis, and a straight line when the revolving line coincides with its axis.



Cone generated by a straight line rotating about an axis in the same plane

Let $y = \frac{b}{a}x$ revolve about OX. The cone of revolution generated has the equation

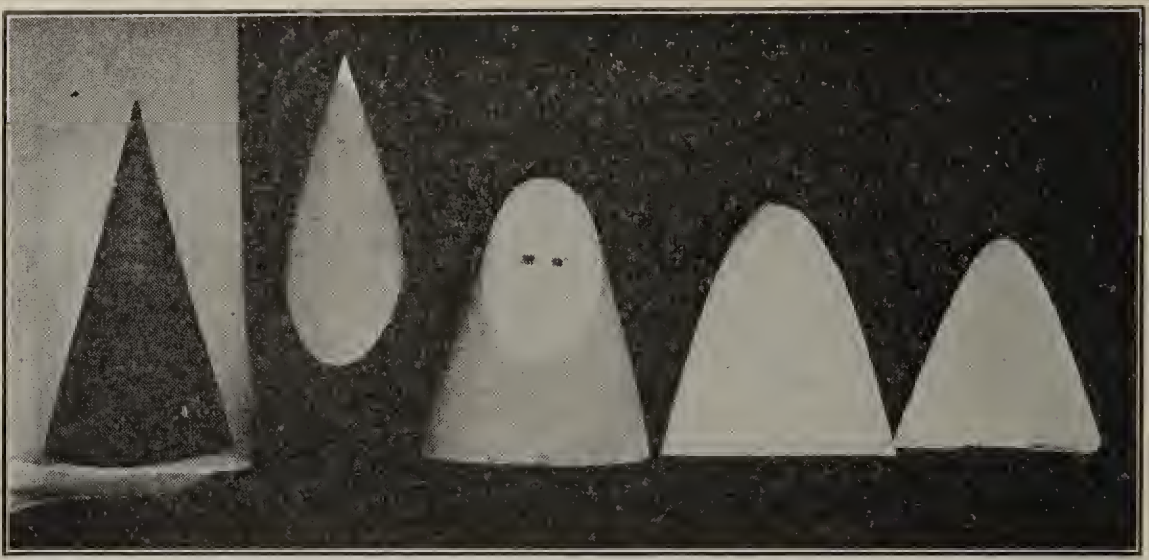
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{b^2} = 0.$$

The cone is itself a limiting form of the hyperboloids, as will be noted below.

The general equation of the cone, whose axis is the y-axis and whose vertex is the origin, is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0.$$

6. Conic sections. — The method which we will here outline to prove that every plane section of a cone may be given by

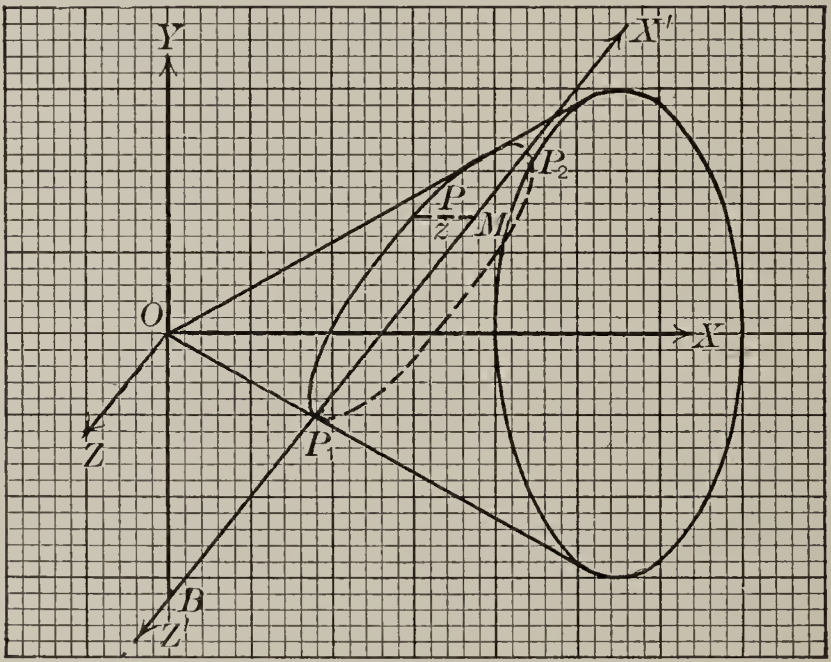


Conic sections: ELLIPSE PARABOLA HYPERBOLA

an equation of the second degree applies to a cone having an ellipse, parabola, or hyperbola as a base as well as to the circular base, which is taken for convenience.

Given the cone $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{b^2}=0.$

Evidently, any one of the planes given by $x=k$ cuts this



Cone cut in an ellipse by a plane

cone in a circular section, with a point as limiting case when $x=0$.

The planes $y=k$ and $z=k$ cut this cone in hyperbolas. When $y=0$ or $z=0$, the hyperbola "degenerates" into two straight lines intersecting at the vertex.

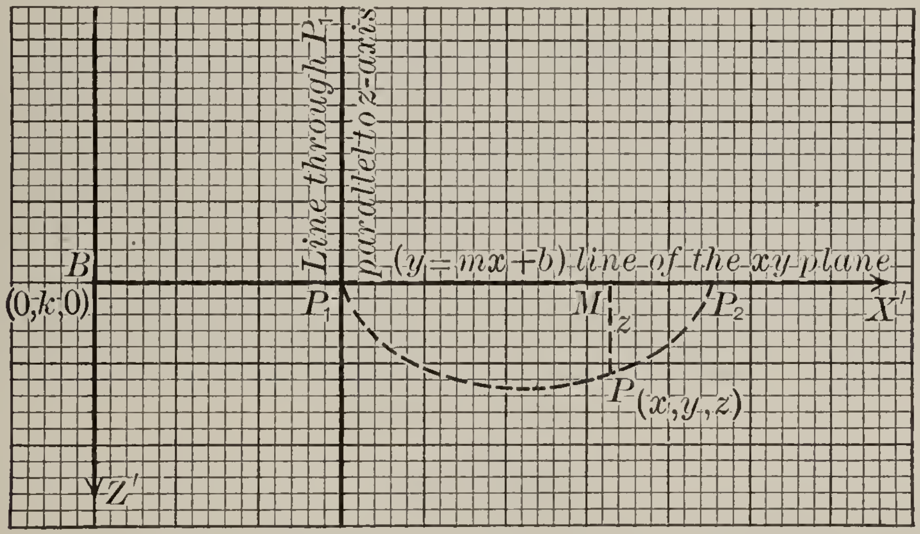
The plane $y=mx+k$ cuts the cone in a curve, which we will refer in this plane to the line $y=mx+k$, intersection of the xy-plane and the cutting plane, and the line $y=k$, the intersection of the yz-plane and the cutting plane, as axes (x' and z') of coördinates. From any point $P(x, y, z)$ on the curve of intersection, drop a perpendicular PM to the xy-plane. The intersection curve satisfies the equation

$$\frac{x^2}{a^2}-\frac{(mx+k)^2}{b^2}-\frac{z^2}{b^2}=0,$$

or

$$b^2x^2-a^2(mx+k)^2-a^2z^2=0.$$

Evidently, $PM=z=z'$, since PM is drawn in one of two perpendicular planes, and perpendicular to the other.



The elliptic section depicted in its own plane

Further, $BM=x'=x\sqrt{1+m^2}$, whence $x=\dfrac{x'}{\sqrt{1+m^2}}$. Substituting, we have,

$$\frac{b^2x'^2}{1+m^2}-\frac{a^2m^2x'^2}{1+m^2}-\frac{2\,a^2km}{\sqrt{1+m^2}}x-a^2k^2-a^2z'^2=0,$$

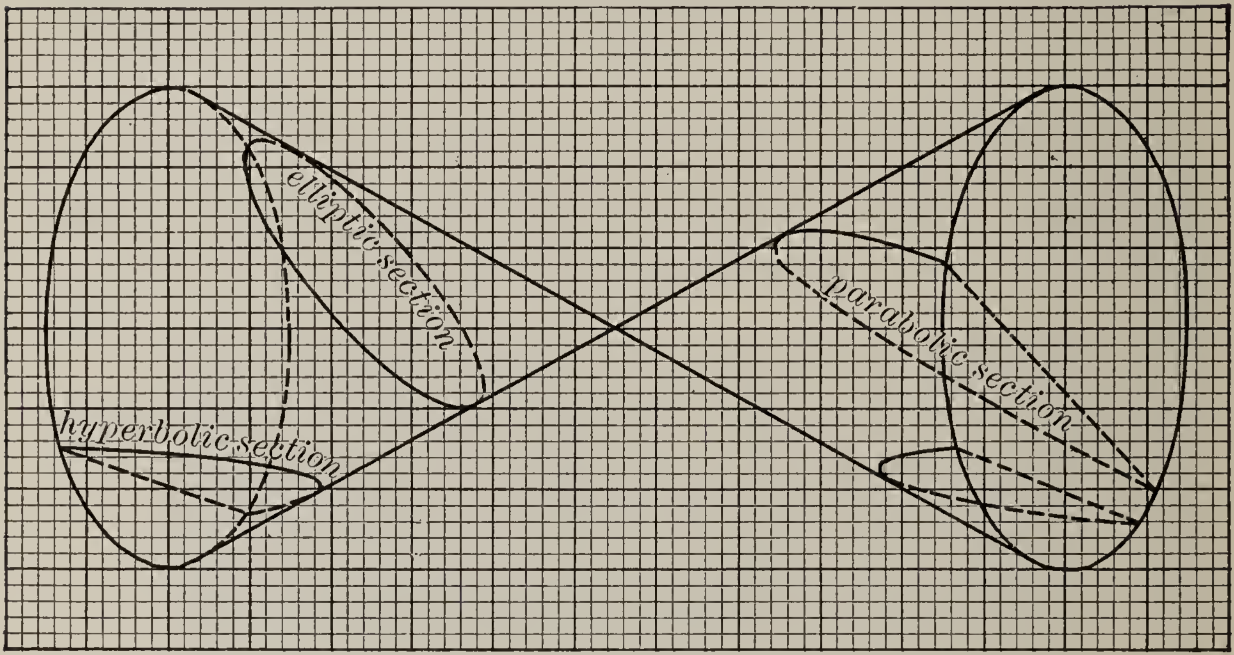
or

$$(b^2-a^2m^2)x'^2-a^2z'^2(1+m^2)-2\,a^2km\sqrt{1+m^2}x-a^2k^2(1+m^2)=0.$$

But this is an equation of the second degree in x' and z'. Further, the coefficient of x'^2 is $(b^2 - a^2m^2)$ and of z'^2 is $-a^2(1+m^2)$; hence the curve is an ellipse if $m^2 > \frac{b^2}{a^2}$, a parabola if $m^2 = \frac{b^2}{a^2}$, and a hyperbola if $m^2 < \frac{b^2}{a^2}$.

When the cutting plane has the form $y = mx + nz + k$, the proof is more complicated but not essentially different.

Every section of a cone may be represented by an equation of the second degree in two variables.



Ellipse, parabola, hyperbola, and two straight lines as intersections of a cone by a plane

PROBLEMS

1. Name and discuss the following surfaces:

a. $x^2 + y^2 + z^2 - 100 = 0.$

b. $x^2 + 2y^2 + 3z^2 - 100 = 0.$

c. $x^2 + 2y^2 - 4z^2 - 100 = 0.$

d. $x^2 + 2y^2 - 100 = 0.$

e. $x^2 + 2y^2 - 4z^2 = 0.$

f. $x^2 - 2y^2 - 4z^2 - 100 = 0.$

g. $x^2 - 100 = 0.$

h. $x^2 - 2y^2 - 100z = 0.$

i. $x^2 - 2y^2 = 0.$

j. $x^2 + 2y^2 = 0.$

k. $x^2 + 2y^2 + 4z^2 = 0.$

l. $x^2 + 2y^2 + 4z^2 + 100 = 0.$

2. How would the addition of a term, $10x$, affect the locus of each of the preceding twelve expressions? Discuss the change in each locus produced by changing the sign of x^2 in each expression from + to −.

3. Find the equation of the cone obtained by revolving the line in the xy-plane $y = 4x - 10$ about the x-axis; about the y-axis; about the z-axis.

4. Find the equation of the paraboloid obtained by revolving the parabola $z^2 = 8x$ about the x-axis; find the equation of the surface obtained by revolving this surface about the z-axis. Why has the latter surface not received particular discussion?

5. Find the equation of the cone obtained by revolving the line $y = 4x - 10$ about the line $y = 6$. Note that this differs from the problems which we have considered in the text only by a change of origin, or a transformation of the type

$$x = x' + h, \quad y = y' + k, \quad z = z' + l.$$

6. Find the locus of a point which is equidistant from the point (4, 0, 0) and from the plane $x + 4 = 0$. What is the surface?

7. Find the locus of a point the sum of whose distances from the points (4, 0, 0) and (− 4, 0, 0) is constant and equal to 10. What is the surface?

8. Find the locus of a point the difference of whose distances from two points (4, 0, 0) and (− 4, 0, 0) is constant and equal to 6.

9. How would you find in space coördinates the distance from a point to a line? Apply to finding the distance from (1, 3, − 5) to the line $\frac{x-2}{3} = \frac{y-3}{-1} = \frac{z-8}{4}$.

7. Limiting forms. — The limiting forms corresponding to the ellipsoid are given by equations of the type

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0,$$

or
$$\frac{(x-h)^2}{a^2}+\frac{(y--k)^2}{b^2}+\frac{(z-l)^2}{c^2}=0,$$

the first of which represents the point O (0, 0, 0) and the second the point (h, k, l).

The method of approaching this limit is best indicated by writing the equation as

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=k.$$

As k approaches 0, the semi-axes $\sqrt{ka^2}$, $\sqrt{kb^2}$, and $\sqrt{kc^2}$ approach 0 as a limit.

In a similar way the hyperboloid equations approach, as limits, the equations representing cones asymptotic to the given hyperboloids.

The limiting forms of the paraboloids are equations in two variables, and reduce to two planes, or to cylinders.

The limiting forms of equations in two variables representing cylinders correspond, with proper and more or less evident changes, to the limiting forms of the corresponding equations in plane analytics.

Thus, any equation of the second degree $f(x, y, z)=0$, whether in one or two or three variables, of which the left-hand member can be factored into two real linear factors in the variables, represents two planes which constitute also a type of quadric surface.

8. Applications. — The applications of the conic sections which have been given in plane analytics are strictly applications of surfaces, or solids having these surfaces as boundaries.

Thus, a bridge having a parabolic arch uses a solid having a parabolic cylinder as bounding surface.

The equation of the paraboloid in the Hill Auditorium, with the foot as unit of length, is $y^2 + z^2 = 70.02\,x$; the skylight in the ceiling of the Hill Auditorium is bounded by an elliptical cylinder,

$$\frac{x^2}{76^2} + \frac{y^2}{50^2} = 1, \text{ dimensions in feet.}$$

Any of the automobile reflectors are paraboloids, in general, of revolution. Hyperboloids are used as revolving cones in the manufacture of iron pipes; these pipes are passed between two revolving cones whose axes are inclined at 90° to straighten the pipes.

9. Circular sections. — Given the ellipsoid represented by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

the question arises as to what planes cut this surface in circular sections.

The method which we have given above, under section 7, for determining the nature of the curve cut out of the cone by the plane $y = mx + k$ applies to this problem. In the ellipsoid above planes $y = k$ cut the surface in ellipses. The development as given shows that any plane $y = mx$ cuts this surface in a curve given by the intersection, also, of the surface,

$$\frac{x^2}{a^2} + \frac{m^2x^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and the given plane, or a curve,

$$\frac{x'^2}{a^2(1+m^2)} + \frac{m^2x'^2}{b^2(1+m^2)} + \frac{z'^2}{c^2} = 1,$$

referred to the line $\begin{cases} y = mx \\ z = 0 \end{cases}$ and the line $\begin{cases} y = 0 \\ x = 0 \end{cases}$ as axes of reference, both lying in the plane of the section. This curve can be written,

$$c^2(b^2 - a^2m^2)x'^2 + a^2b^2(1+m^2)z'^2 = a^2b^2c^2(1+m^2).$$

Equating the coefficients of x'^2 and z'^2 gives

$$m^2 = \frac{b^2(a^2 - c^2)}{a^2(c^2 - b^2)},$$

$$m = \pm \frac{b\sqrt{a^2 - c^2}}{a\sqrt{c^2 - b^2}}.$$

The two planes, $y = \pm \frac{b\sqrt{a^2 - c^2}}{a\sqrt{c^2 - b^2}} x$, and all planes parallel to them, cut this surface in circular sections. For these to be real planes c must be intermediate in value between a and b. If c is not intermediate between a and b, then planes either of the form $y = mz$ or $y = nx$ will make real circular sections.

The method applies to elliptic cylinders, to elliptic cones, and to hyperboloids, as well as to the ellipsoid.

A simpler method, assuming c as the intermediate value, is to find in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ a diameter of length $2\,c$; this diameter with the z-axis determines the plane of a circular section.

10. Tangent planes and tangent lines. — The formula

$$Ax_1x + By_1y + G(x + x_1) + F(y + y_1) + C = 0,$$

which gives the tangent to

$$Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0,$$

at the point $P_1(x_1, y_1)$ on the curve applies in space analytics, with the addition of the corresponding z^2 and z terms, to give the tangent plane to the quadric surface at a point $P_1(x_1, y_1, z_1)$ on the surface.

The tangent plane at $P_1(x_1, y_1, z_1)$ to the surface

$$Ax^2 + By^2 + Cz^2 + 2\,Gx + 2\,Fy + 2\,Ez + K = 0$$

is given by the equation,

$$Ax_1x + By_1y + Cz_1z + G(x + x_1) + F(y + y_1) + E(z + z_1) + K = 0.$$

When the point $P_1(x_1, y_1, z_1)$ is not on the quadric surface, this equation represents not the tangent plane but the polar plane of the point (x_1, y_1, z_1) with respect to the surface. For any point outside of the surface, tangent planes to the surface have their points of tangency situated upon a plane, the polar plane of the point $P_1(x_1, y_1, z_1)$. A more complete discussion of the polar plane would reveal many other points of similarity between the polar plane as related to its quadric surface and the polar line as related to its conic.

The intersection of a tangent plane at a point $P_1(x_1, y_1, z_1)$ on the surface with any other plane through P_1 gives a tangent line to the surface at P_1.

11. Ruled surfaces. Generating lines. — Any surface which can be generated by the motion of a straight line moving according to some law is termed a ruled surface. Evidently, by its method of generation, such a surface has straight-line elements, called rectilinear generators, which lie wholly upon the surface.

Certain of the quadric surfaces are ruled surfaces. Evidently all the cylinders, the cones, and the pairs of planes belong in this class. The ellipsoid, being confined to a finite portion of space, does not have right-line elements lying wholly upon the surface; nor do the elliptic paraboloid and the hyperboloid of two sheets have right-line elements.

The hyperbolic paraboloid and the hyperboloid of one sheet do have rectilinear generators. We will find the equation of the families of lines which lie wholly upon one of the surfaces in question; the method will apply to the other ruled quadric surfaces.

Any point upon the hyperboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

very evidently satisfies the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 - \frac{z^2}{c^2},$$

which may be written,

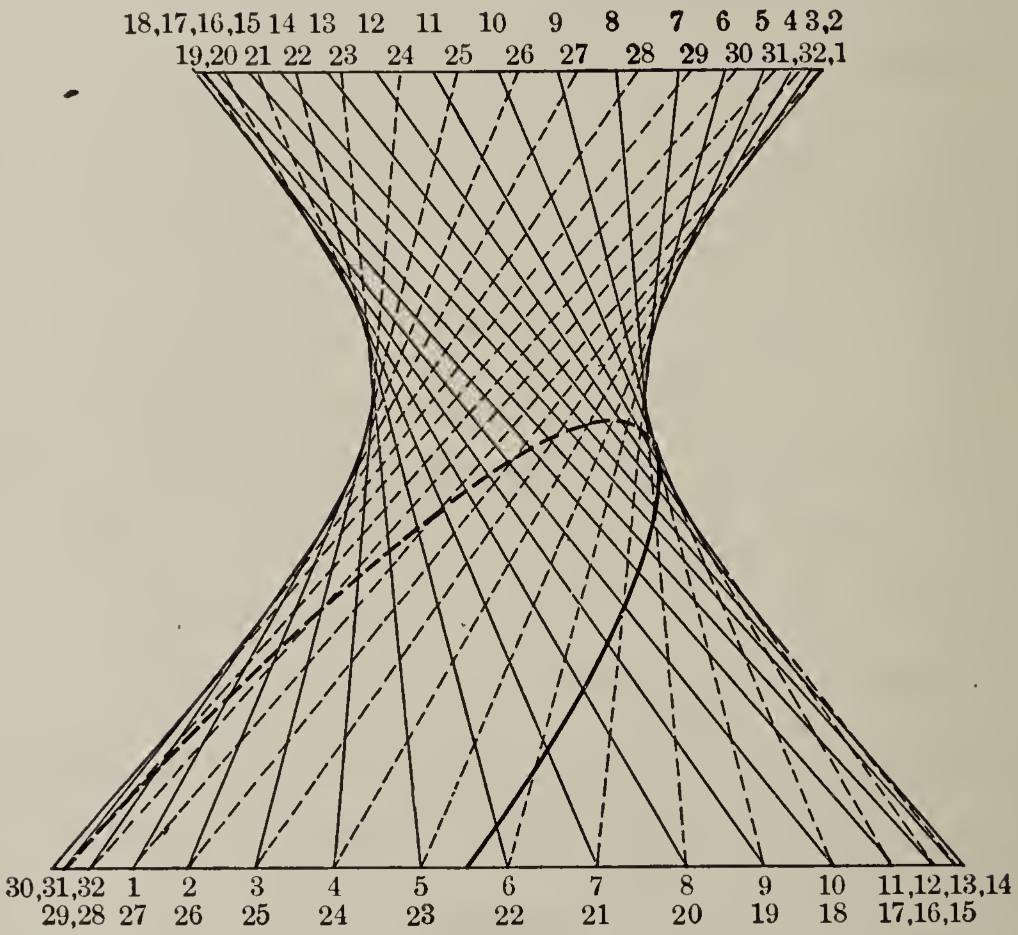
$$\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{a}+\frac{y}{b}\right)=\left(1-\frac{z}{c}\right)\left(1+\frac{z}{c}\right).$$

This indicates that any point which satisfies the pair of linear equations

$$\frac{x}{a}-\frac{y}{b}=k\left(1-\frac{z}{c}\right),$$

$$\frac{x}{a}+\frac{y}{b}=\frac{1}{k}\left(1+\frac{z}{c}\right)$$

will satisfy the equation of our surface since it will make the product represented by the left-hand member of our equation,



The right-line generators on this hyperboloid of revolution are formed by connecting corresponding points on two circular sections

An elliptic section is also indicated.

in the second form above, equal to the product of the factors, representing the right-hand member. But every point which satisfies the pair of equations for any given value of k lies upon a straight line, the intersection of the two planes given by the linear equations. Hence, every point upon this line lies upon the given surface for any value of k.

It can be shown that no two lines of this family of lines, *i.e.* no two lines given by two values of k, intersect.

Another family of lines also lies upon this surface. The equations of this second family of lines, with the parameter k, are as follows:

$$\frac{x}{a}-\frac{y}{b}=k\left(1+\frac{z}{c}\right),$$

$$\frac{x}{a}+\frac{y}{b}=\frac{1}{k}\left(1-\frac{z}{c}\right).$$

Every member of this family of lines can be shown to intersect every member of the preceding family and no member of its own family.

PROBLEMS

1. Find the equations of the rectilinear generators of the following surfaces:

a. $x^2-y^2-z^2=0.$

b. $x^2+y^2-z^2=16.$

c. $x^2-y^2-4z=0.$

d. $x^2-y^2=0.$

2. Find the circular sections of the following surfaces:

a. $\dfrac{x^2}{25}-\dfrac{y^2}{16}-\dfrac{z^2}{9}=1.$

b. $\dfrac{x^2}{25}-\dfrac{y^2}{16}-\dfrac{z^2}{9}=0.$

c. $x^2+4y^2=9z.$

d. $x^2+4y^2=9.$

e. $\dfrac{x^2}{100}+\dfrac{y^2}{36}-\dfrac{z^2}{16}=1.$

3. Write the equations of the tangent planes to each of the surfaces in the preceding problem at the point (x_1, y_1, z_1) in each case upon the given surface.

4. In problem 1 *a*, above, take $k_1 = 1$ and $k_2 = 2$ and show that these two lines of the same family of rectilinear generators do not intersect. Write the second family of rectilinear generators of the same surface and show that, taking $k = 1$ (any other value would do), this line does intersect a given line $(k_1 = 1)$ of the first set. How could you make this proof general?

TABLES

Constants with their logarithms.

	NUMBER	LOGARITHM
Base of natural logarithms	$e = 2.71828183$	0.4342945
Modulus of common logarithms	$u = 0.43429448$	9.6377843–10
Circumference of a circle in degrees	= 360	2.5563025
Circumference of a circle in minutes	= 21600	4.3344538
Circumference of a circle in seconds	= 1296000	6.1126050
Radian expressed in degrees	= 57.29578	1.7581226
Radian expressed in minutes	= 3437.7468	3.5362739
Radian expressed in seconds	= 206264.806	5.3144251
Ratio of a circumference to diameter	$\pi = 3.14159265$	0.4971499

$\pi = 3.14159265358979323846264338328$

VOLUMES AND WEIGHTS

Cubic inches in 1 gallon (U S.)	= 231	2.3636
Gallons in 1 cubic foot	= 7.48	.8739
Cubic inches in 1 bushel	= 2150.4	3.3325
Pounds per cubic foot of water (4° C.)	= 62.43	1.7954
Pounds per cubic foot of air (0° C.)	= 0.0807	8.9069–10
Cubic feet in 1 cubic meter	= 35.32	1.5480
Cubic meters in 1 cubic yard	= 0.76	9.8808–10
Cubic inches in 1 liter	= 61.03	1.7855
Liters in 1 gallon (U. S.)	= 3.786	.5783
Pounds in 1 kilogram	= 2.2 or 2.205	.3434
Metric ton in pounds	= 2205	3.3434
Volume of sphere, $\frac{4}{3}\pi r^3$	$= 4.1888\ r^3$	
	4.1888	.6221

LENGTHS AND AREAS

Inches in 1 meter (by Act of Congress)	= 39.37	1.5952
Feet in 1 rod, 16.5 ; yards in 1 rod	= 5.5	
Square feet in 1 acre	= 43560	4.6391

160 square rods = 1 acre ; 640 acres = 1 square mile ;
3.281 feet = 1.094 yards = 1 meter.

Squares and cubes of integers, 1 to 100.

Square roots and cube roots of 1 to 100. Reciprocals of 1 to 100.

n	Square n^2	Cube n^3	Square Root $\sqrt{n}$	Cube Root $\sqrt[3]{n}$	Reciprocal $\frac{1}{n}$	n	Square n^2	Cube n^3	Square Root $\sqrt{n}$	Cube Root $\sqrt[3]{n}$	Reciprocal $\frac{1}{n}$
1	1	1	1.000	1.000	1.0000	**51**	2,601	132,651	7.141	3.708	.0196
2	4	8	1.414	1.260	.5000	**52**	2,704	140,608	7.211	3.733	.0192
3	9	27	1.732	1.442	.3333	**53**	2,809	148,877	7.280	3.756	.0189
4	16	64	2.000	1.587	.2500	**54**	2,916	157,464	7.348	3.780	.0185
5	25	125	2.236	1.710	.2000	**55**	3,025	166,375	7.416	3.803	.0182
6	36	216	2.449	1.817	.1667	**56**	3,136	175,616	7.483	3.826	.0179
7	49	343	2.646	1.913	.1429	**57**	3,249	185,193	7.550	3.849	.0175
8	64	512	2.828	2.000	.1250	**58**	3,364	195,112	7.616	3.871	.0172
9	81	729	3.000	2.080	.1111	**59**	3,481	205,379	7.681	3.893	.0170
10	100	1,000	3.162	2.154	.1000	**60**	3,600	216,000	7.746	3.915	.0167
11	121	1,331	3.317	2.224	.0909	**61**	3,721	226,981	7.810	3.936	.0164
12	144	1,728	3.464	2.289	.0833	**62**	3,844	238,328	7.874	3.958	.0161
13	169	2,197	3.606	2.351	.0769	**63**	3,969	250,047	7.937	3.979	.0159
14	196	2,744	3.742	2.410	.0714	**64**	4,096	262,144	8.000	4.000	.0156
15	225	3,375	3.873	2.466	.0667	**65**	4,225	274,625	8.062	4.021	.0154
16	256	4,096	4.000	2.520	.0625	**66**	4,356	287,496	8.124	4.041	.0152
17	289	4,913	4.123	2.571	.0588	**67**	4,489	300,763	8.185	4.062	.0149
18	324	5,832	4.243	2.621	.0556	**68**	4,624	314,432	8.246	4.082	.0147
19	361	6,859	4,359	2.668	.0526	**69**	4,761	328,509	8.307	4.102	.0145
20	400	8,000	4.472	2.714	.0500	**70**	4,900	343,000	8.367	4.121	.0143
21	441	9,261	4.583	2.759	.0476	**71**	5,041	357,911	8.426	4.141	.0141
22	484	10,648	4,690	2.802	.0455	**72**	5,184	373,248	8.485	4.160	.0139
23	529	12,167	4.796	2.844	.0436	**73**	5,329	389,017	8.544	4.179	.0137
24	576	13,824	4.899	2.884	.0417	**74**	5,476	405,224	8.602	4.198	.0135
25	625	15,625	5.000	2.924	.0400	**75**	5,625	421,875	8.660	4.217	.0133
26	676	17,576	5.099	2.962	.0385	**76**	5,776	438,976	8.718	4.236	.0132
27	729	19,683	5.196	3.000	.0370	**77**	5,929	456,533	8.775	4.254	.0130
28	784	21,952	5.292	3.037	.0357	**78**	6,084	474,552	8.832	4.273	.0128
29	841	24,389	5.385	3.072	.0345	**79**	6,241	493,039	8.888	4.291	.0127
30	900	27,000	5.477	3.107	.0333	**80**	6,400	512,000	8.944	4.309	.0125
31	961	29,791	5.568	3.141	.0323	**81**	6,561	531,441	9.000	4.327	.0123
32	1,024	32,768	5.657	3.175	.0313	**82**	6,724	551,368	9.055	4.344	.0122
33	1,089	35,937	5.745	3.208	.0303	**83**	6,889	571,787	9.110	4.362	.0120
34	1,156	39,304	5.831	3.240	.0294	**84**	7,056	592,704	9.165	4.380	.0119
35	1,225	42,875	5.916	3.271	.0286	**85**	7,225	614,125	9.220	4.397	.0118
36	1,296	46,656	6.000	3.302	.0278	**86**	7,396	636,056	9.274	4.414	.0116
37	1,369	50,653	6.083	3.332	.0270	**87**	7,569	658,503	9.327	4.431	.0115
38	1,444	54,872	6.164	3.362	.0263	**88**	7,744	681,472	9.381	4.448	.0114
39	1,521	59,319	6.245	3.391	.0256	**89**	7,921	704,969	9.434	4.465	.0112
40	1,600	64,000	6.325	3.420	.0250	**90**	8,100	729,000	9.487	4.481	.0111
41	1,681	68,921	6.403	3.448	.0244	**91**	8,281	753,571	9.539	4.498	.0110
42	1,764	74,088	6.481	3.476	.0238	**92**	8,464	778,688	9.592	4.514	.0109
43	1,849	79,507	6.557	3.503	.0233	**93**	8,649	804,357	9.644	4.531	.0108
44	1,936	85,184	6.633	3.530	.0227	**94**	8,836	830,584	9.695	4.547	.0106
45	2,025	91,125	6.708	3.557	.0222	**95**	9,025	857,375	9.747	4.563	.0105
46	2,116	97,336	6.782	3.583	.0217	**96**	9,216	884,736	9.798	4.579	.0104
47	2,209	103,823	6.856	3.609	.0213	**97**	9,409	912,673	9.849	4.595	.0103
48	2,304	110,592	6.928	3.634	.0208	**98**	9,604	941,192	9.899	4.610	.0102
49	2,401	117,649	7.000	3.659	.0204	**99**	9,801	970,299	9.950	4.626	.0101
50	2,500	125,000	7.071	3.684	.0200	**100**	10,000	1,000,000	10.000	4.642	.0100
n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$\frac{1}{n}$	n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$\frac{1}{n}$

Logarithms of numbers from 100 to 549.

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

To avoid interpolation in the first ten lines, use tables on the following pages.

	22
1	2.2
2	4.4
3	6.6
4	8.8
5	11.0
6	13.2
7	15.4
8	17.6
9	19.8

	21	**20**	**19**
1	2.1	2.0	1.9
2	4.2	4.0	3.8
3	6.3	6.0	5.7
4	8.4	8.0	7.6
5	10.5	10.0	9.5
6	12.6	12.0	11.4
7	14.7	14.0	13.3
8	16.8	16.0	15.2
9	18.9	18.0	17.1

	18	**17**	**16**
1	1.8	1.7	1.6
2	3.6	3.4	3.2
3	5.4	5.1	4.8
4	7.2	6.8	6.4
5	9.0	8.5	8.0
6	10.8	10.2	9.6
7	12.6	11.9	11.2
8	14.4	13.6	12.8
9	16.2	15.3	14.4

	15	**14**	**13**
1	1.5	1.4	1.3
2	3.0	2.8	2.6
3	4.5	4.2	3.9
4	6.0	5.6	5.2
5	7.5	7.0	6.5
6	9.0	8.4	7.8
7	10.5	9.8	9.1
8	12.0	11.2	10.4
9	13.5	12.6	11.7

	11		**12**
1	1.1	1	1.2
2	2.2	2	2.4
3	3.3	3	3.6
4	4.4	4	4.8
5	5.5	5	6.0
6	6.6	6	7.2
7	7.7	7	8.4
8	8.8	8	9.6
9	9.9	9	10.8

Interpolate mentally, using the multiplication table.

Logarithms of numbers from 550 to 999.

	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025

Interpolate mentally, using the multiplication table.

	0	1	2	3	4	5	6	7	8	9
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
	0	1	2	3	4	5	6	7	8	9

Interpolate mentally, using the multiplication table.

Logarithms of numbers between 1000 and 1499.

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
100	0000	0004	0009	0013	0017	0022	0026	0030	0035	0039
101	0043	0048	0052	0056	0060	0065	0069	0073	0077	0082
102	0086	0090	0095	0099	0103	0107	0111	0116	0120	0124
103	0128	0133	0137	0141	0145	0149	0154	0158	0162	0166
104	0170	0175	0179	0183	0187	0191	0195	0199	0204	0208
105	0212	0216	0220	0224	0228	0233	0237	0241	0245	0249
106	0253	0257	0261	0265	0269	0273	0278	0282	0286	0290
107	0294	0298	0302	0306	0310	0314	0318	0322	0326	0330
108	0334	0338	0342	0346	0350	0354	0358	0362	0366	0370
109	0374	0378	0382	0386	0390	0394	0398	0402	0406	0410
110	0414	0418	0422	0426	0430	0434	0438	0441	0445	0449
111	0453	0457	0461	0465	0469	0473	0477	0481	0484	0488
112	0492	0496	0500	0504	0508	0512	0515	0519	0523	0527
113	0531	0535	0538	0542	0546	0550	0554	0558	0561	0565
114	0569	0573	0577	0580	0584	0588	0592	0596	0599	0603
115	0607	0611	0615	0618	0622	0626	0630	0633	0637	0641
116	0645	0648	0652	0656	0660	0663	0667	0671	0674	0678
117	0682	0686	0689	0693	0697	0700	0704	0708	0711	0715
118	0719	0722	0726	0730	0734	0737	0741	0745	0748	0752
119	0755	0759	0763	0766	0770	0774	0777	0781	0785	0788
120	0792	0795	0799	0803	0806	0810	0813	0817	0821	0824
121	0828	0831	0835	0839	0842	0846	0849	0853	0856	0860
122	0864	0867	0871	0874	0878	0881	0885	0888	0892	0896
123	0899	0903	0906	0910	0913	0917	0920	0924	0927	0931
124	0934	0938	0941	0945	0948	0952	0955	0959	0962	0966
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
125	0969	0973	0976	0980	0983	0986	0990	0993	0997	1000
126	1004	1007	1011	1014	1017	1021	1024	1028	1031	1035
127	1038	1041	1045	1048	1052	1055	1059	1062	1065	1069
128	1072	1075	1079	1082	1086	1089	1092	1096	1099	1103
129	1106	1109	1113	1116	1119	1123	1126	1129	1133	1136
130	1139	1143	1146	1149	1153	1156	1159	1163	1166	1169
131	1173	1176	1179	1183	1186	1189	1193	1196	1199	1202
132	1206	1209	1212	1216	1219	1222	1225	1229	1232	1235
133	1239	1242	1245	1248	1252	1255	1258	1261	1265	1268
134	1271	1274	1278	1281	1284	1287	1290	1294	1297	1300
135	1303	1307	1310	1313	1316	1319	1323	1326	1329	1332
136	1335	1339	1342	1345	1348	1351	1355	1358	1361	1364
137	1367	1370	1374	1377	1380	1383	1386	1389	1392	1396
138	1399	1402	1405	1408	1411	1414	1418	1421	1424	1427
139	1430	1433	1436	1440	1443	1446	1449	1452	1455	1458
140	1461	1464	1467	1471	1474	1477	1480	1483	1486	1489
141	1492	1495	1498	1501	1504	1508	1511	1514	1517	1520
142	1523	1526	1529	1532	1535	1538	1541	1544	1547	1550
143	1553	1556	1559	1562	1565	1569	1572	1575	1578	1581
144	1584	1587	1590	1593	1596	1599	1602	1605	1608	1611
145	1614	1617	1620	1623	1626	1629	1632	1635	1638	1641
146	1644	1647	1649	1652	1655	1658	1661	1664	1667	1670
147	1673	1676	1679	1682	1685	1688	1691	1694	1697	1700
148	1703	1706	1708	1711	1714	1717	1720	1723	1726	1729
149	1732	1735	1738	1741	1744	1746	1749	1752	1755	1758
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

Logarithms of numbers between 1500 and 1999.

	0	1	2	3	4	5	6	7	8	9
150	1761	1764	1767	1770	1772	1775	1778	1781	1784	1787
151	1790	1793	1796	1798	1801	1804	1807	1810	1813	1816
152	1818	1821	1824	1827	1830	1833	1836	1838	1841	1844
153	1847	1850	1853	1855	1858	1861	1864	1867	1870	1872
154	1875	1878	1881	1884	1886	1889	1892	1895	1898	1901
155	1903	1906	1909	1912	1915	1917	1920	1923	1926	1928
156	1931	1934	1937	1940	1942	1945	1948	1951	1953	1956
157	1959	1962	1965	1967	1970	1973	1976	1978	1981	1984
158	1987	1989	1992	1995	1998	2000	2003	2006	2009	2011
159	2014	2017	2019	2022	2025	2028	2030	2033	2036	2038
160	2041	2044	2047	2049	2052	2055	2057	2060	2063	2066
161	2068	2071	2074	2076	2079	2082	2084	2087	2090	2092
162	2095	2098	2101	2103	2106	2109	2111	2114	2117	2119
163	2122	2125	2127	2130	2133	2135	2138	2140	2143	2146
164	2148	2151	2154	2156	2159	2162	2164	2167	2170	2172
165	2175	2177	2180	2183	2185	2188	2191	2193	2196	2198
166	2201	2204	2206	2209	2212	2214	2217	2219	2222	2225
167	2227	2230	2232	2235	2238	2240	2243	2245	2248	2251
168	2253	2256	2258	2261	2263	2266	2269	2271	2274	2276
169	2279	2281	2284	2287	2289	2292	2294	2297	2299	2302
170	2304	2307	2310	2312	2315	2317	2320	2322	2325	2327
171	2330	2333	2335	2338	2340	2343	2345	2348	2350	2353
172	2355	2358	2360	2363	2365	2368	2370	2373	2375	2378
173	2380	2383	2385	2388	2390	2393	2395	2398	2400	2403
174	2405	2408	2410	2413	2415	2418	2420	2423	2425	2428
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
175	2430	2433	2435	2438	2440	2443	2445	2448	2450	2453
176	2455	2458	2460	2463	2465	2467	2470	2472	2475	2477
177	2480	2482	2485	2487	2490	2492	2494	2497	2499	2502
178	2504	2507	2509	2512	2514	2516	2519	2521	2524	2526
179	2529	2531	2533	2536	2538	2541	2543	2545	2548	2550
180	2553	2555	2558	2560	2562	2565	2567	2570	2572	2574
181	2577	2579	2582	2584	2586	2589	2591	2594	2596	2598
182	2601	2603	2605	2608	2610	2613	2615	2617	2620	2622
183	2625	2627	2629	2632	2634	2636	2639	2641	2643	2646
184	2648	2651	2653	2655	2658	2660	2662	2665	2667	2669
185	2672	2674	2676	2679	2681	2683	2686	2688	2690	2693
186	2695	2697	2700	2702	2704	2707	2709	2711	2714	2716
187	2718	2721	2723	2725	2728	2730	2732	2735	2737	2739
188	2742	2744	2746	2749	2751	2753	2755	2758	2760	2762
189	2765	2767	2769	2772	2774	2776	2778	2781	2783	2785
190	2788	2790	2792	2794	2797	2799	2801	2804	2806	2808
191	2810	2813	2815	2817	2819	2822	2824	2826	2828	2831
192	2833	2835	2838	2840	2842	2844	2847	2849	2851	2853
193	2856	2858	2860	2862	2865	2867	2869	2871	2874	2876
194	2878	2880	2882	2885	2887	2889	2891	2894	2896	2898
195	2900	2903	2905	2907	2909	2911	2914	2916	2918	2920
196	2923	2925	2927	2929	2931	2934	2936	2938	2940	2942
197	2945	2947	2949	2951	2953	2956	2958	2960	2962	2964
198	2967	2969	2971	2973	2975	2978	2980	2982	2984	2986
199	2989	2991	2993	2995	2997	2999	3002	3004	3006	3008
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

Log sin A° from 0° to 45°.

A°	0′	10′	20′	30′	40′	50′	60′	A°	d.
0		7.4637	7648	9408	*0658	*1627	*2419	**89°**	
1	8.2419	3088	3668	4179	4637	5050	5428	**88°**	
2	5428	5776	6097	6397	6677	6940	7188	**87°**	
3	7188	7423	7645	7857	8059	8251	8436	**86°**	
4	8436	8613	8783	8946	9104	9256	9403	**85°**	
5	9403	9545	9682	9816	9945	*0070	*0192	**84°**	
6	9.0192	0311	0426	0539	0648	0755	0859	**83°**	
7	0859	0961	1060	1157	1252	1345	1436	**82°**	91
8	1436	1525	1612	1697	1781	1863	1943	**81°**	80
9	1943	2022	2100	2176	2251	2324	2397	**80°**	73
10	9.2397	2468	2538	2606	2674	2740	2806	**79°**	68
11	2806	2870	2934	2997	3058	3119	3179	**78°**	62
12	3179	3238	3296	3353	3410	3466	3521	**77°**	57
13	3521	3575	3629	3682	3734	3786	3837	**76°**	53
14	3837	3887	3937	3986	4035	4083	4130	**75°**	49
15	4130	4177	4223	4269	4314	4359	4403	**74°**	46
16	4403	4447	4491	4533	4576	4618	4659	**73°**	43
17	4659	4700	4741	4781	4821	4861	4900	**72°**	40
18	4900	4939	4977	5015	5052	5090	5126	**71°**	38
19	5126	5163	5199	5235	5270	5306	5341	**70°**	36
20	9.5341	5375	5409	5443	5477	5510	5543	**69°**	34
21	5543	5576	5609	5641	5673	5704	5736	**68°**	32
22	5736	5767	5798	5828	5859	5889	5919	**67°**	31
23	5919	5948	5978	6007	6036	6065	6093	**66°**	29
24	6093	6121	6149	6177	6205	6232	6259	**65°**	28
				30′					
25	6259	6286	6313	6340	6366	6392	6418	**64°**	27
26	6418	6444	6470	6495	6521	6546	6570	**63°**	25
27	6570	6595	6620	6644	6668	6692	6716	**62°**	24
28	6716	6740	6763	6787	6810	6833	6856	**61°**	23
29	6856	6878	6901	6923	6946	6968	6990	**60°**	22
30	9.6990	7012	7033	7055	7076	7097	7118	**59°**	21
31	7118	7139	7160	7181	7201	7222	7242	**58°**	21
32	7242	7262	7282	7302	7322	7342	7361	**57°**	20
33	7361	7380	7400	7419	7438	7457	7476	**56°**	19
34	7476	7494	7513	7531	7550	7568	7586	**55°**	18
35	7586	7604	7622	7640	7657	7675	7692	**54°**	18
36	7692	7710	7727	7744	7761	7778	7795	**53°**	17
37	7795	7811	7828	7844	7861	7877	7893	**52°**	16
38	7893	7910	7926	7941	7957	7973	7989	**51°**	16
39	7989	8004	8020	8035	8050	8066	8081	**50°**	15
40	9.8081	8096	8111	8125	8140	8155	8169	**49°**	15
41	8169	8184	8198	8213	8227	8241	8255	**48°**	14
42	8255	8269	8283	8297	8311	8324	8338	**47°**	14
43	8338	8351	8365	8378	8391	8405	8418	**46°**	13
44	8418	8431	8444	8457	8469	8482	8495	**45°**	13
A°	60′	50′	40′	30′	20′	10′	0′	A°	d.

Log cos A° from 45° to 90°.

Do not interpolate, but use the special table which gives these values by minutes.

P. P.

	92	**90**	**88**	**86**	**84**
1	9.2	9.0	8.8	8.6	8.4
2	18.4	18.0	17.6	17.2	16.8
3	27.6	27.0	26.4	25.8	25.2
4	36.8	36.0	35.2	34.4	33.6
5	46.0	45.0	44.0	43.0	42.0
6	55.2	54.0	52.8	51.6	50.4
7	64.4	63.0	61.6	60.2	58.8
8	73.6	72.0	70.4	68.8	67.2
9	82.8	81.0	79.2	77.4	75.6

	82	**80**	**78**	**76**	**74**
1	8.2	8.0	7.8	7.6	7.4
2	16.4	16.0	15.6	15.2	14.8
3	24.6	24.0	23.4	22.8	22.2
4	32.8	32.0	31.2	30.4	29.6
5	41.0	40.0	39.0	38.0	37.0
6	49.2	48.0	46.8	45.6	44.4
7	57.4	56.0	54.6	53.2	51.8
8	65.6	64.0	62.4	60.8	59.2
9	73.8	72.0	70.2	68.4	66.6

	72	**70**	**68**	**66**	**64**
1	7.2	7.0	6.8	6.6	6.4
2	14.4	14.0	13.6	13.2	12.8
3	21.6	21.0	20.4	19.8	19.2
4	28.8	28.0	27.2	26.4	25.6
5	36.0	35.0	34.0	33.0	32.0
6	43.2	42.0	40.8	39.6	38.4
7	50.4	49.0	47.6	46.2	44.8
8	57.6	56.0	54.4	52.8	51.2
9	64.8	63.0	61.2	59.4	57.6

	62	**60**	**58**	**56**	**54**
1	6.2	6.0	5.8	5.6	5.4
2	12.4	12.0	11.6	11.2	10.8
3	18.6	18.0	17.4	16.8	16.2
4	24.8	24.0	23.2	22.4	21.6
5	31.0	30.0	29.0	28.0	27.0
6	37.2	36.0	34.8	33.6	32.4
7	43.4	42.0	40.6	39.2	37.8
8	49.6	48.0	46.4	44.8	43.2
9	55.8	54.0	52.2	50.4	48.6

	52	**50**	**48**	**46**	**44**
1	5.2	5.0	4.8	4.6	4.4
2	10.4	10.0	9.6	9.2	8.8
3	15.6	15.0	14.4	13.8	13.2
4	20.8	20.0	19.2	18.4	17.6
5	26.0	25.0	24.0	23.0	22.0
6	31.2	30.0	28.8	27.6	26.4
7	36.4	35.0	33.6	32.2	30.8
8	41.6	40.0	38.4	36.8	35.2
9	46.8	45.0	43.2	41.4	39.6

P. P.

Log sin A° from 45° to 90°.

A°	0′	10′	20′	30′	40′	50′	60′	A°	d.
45	8495	8507	8520	8532	8545	8557	8569	**44**	12
46	8569	8582	8594	8606	8618	8629	8641	**43**	12
47	8641	8653	8665	8676	8688	8699	8711	**42**	12
48	8711	8722	8733	8745	8756	8767	8778	**41**	11
49	8778	8789	8800	8810	8821	8832	8843	**40**	11
50	9.8843	8853	8864	8874	8884	8895	8905	**39**	10
51	8905	8915	8925	8935	8945	8955	8965	**38**	10
52	8965	8975	8985	8995	9004	9014	9023	**37**	9
53	9023	9033	9042	9052	9061	9070	9080	**36**	10
54	9080	9089	9098	9107	9116	9125	9134	**35**	9
55	9134	9142	9151	9160	9169	9177	9186	**34**	9
56	9186	9194	9203	9211	9219	9228	9236	**33**	8
57	9236	9244	9252	9260	9268	9276	9284	**32**	8
58	9284	9292	9300	9308	9315	9323	9331	**31**	8
59	9331	9338	9346	9353	9361	9368	9375	**30**	7
60	9.9375	8393	9390	9397	9404	9411	9418	**29**	7
61	9418	9425	9432	9439	9446	9453	9459	**28**	6
62	9459	9466	9473	9479	9486	9492	9499	**27**	7
63	9499	9505	9512	9518	9524	9530	9537	**26**	7
64	9537	9543	9549	9555	9561	9567	9573	**25**	6
65	9573	9579	9584	9590	9596	9602	9607	**24**	5
66	9607	9613	9618	9624	9629	9635	9640	**23**	5
67	9640	9646	9651	9656	9661	9667	9672	**22**	5
68	9672	9677	9682	9687	9692	9697	9702	**21**	5
69	9702	9706	9711	9716	9721	9725	9730	**20**	5
				30′					
70	9.9730	9734	9739	9743	9748	9752	9757	**19**	5
71	9757	9761	9765	9770	9774	9778	9782	**18**	4
72	9782	9786	9790	9794	9798	9802	9806	**17**	4
73	9806	9810	9814	9817	9821	9825	9828	**16**	3
74	9828	9832	9836	9839	9843	9846	9849	**15**	3
75	9849	9853	9856	9859	9863	9866	9869	**14**	3
76	9869	9872	9875	9878	9881	9884	9887	**13**	3
77	9887	9890	9893	9896	9899	9901	9904	**12**	3
78	9904	9907	9909	9912	9914	9917	9919	**11**	2
79	9919	9922	9924	9927	9929	9931	9934	**10**	3
80	9.9934	9936	9938	9940	9942	9944	9946	**9**	2
81	9946	9948	9950	9952	9954	9956	9958	**8**	2
82	9958	9959	9961	9963	9964	9966	9968	**7**	2
83	9968	9969	9971	9972	9973	9975	9976	**6**	1
84	9976	9977	9979	9980	9981	9982	9983	**5**	1
85	9983	9985	9986	9987	9988	9989	9989	**4**	0
86	9989	9990	9991	9992	9993	9993	9994	**3**	1
87	9994	9995	9995	9996	9996	9997	9997	**2**	0
88	9997	9998	9998	9999	9999	9999	9999	**1**	0
89	9999	*0000	*0000	*0000	*0000	*0000	*0000	**0**	0
A°	**60′**	**50′**	**40′**	**30′**	**20′**	**10′**	**0′**	**A°**	**d.**

P. P.

	42	**40**	**38**	**36**
1	4.2	4.0	3.8	3.6
2	8 4	8.0	7.6	7.2
3	12.6	12.0	11.4	10.8
4	16.8	16.0	15.2	14.4
5	21.0	20.0	19.0	18.0
6	25.2	24.0	22.8	21.6
7	29.4	28.0	26.6	25.2
8	33.6	32.0	30.4	28.8
9	37.8	36.0	34.2	32.4

	34	**32**	**30**	**28**
1	3.4	3.2	3.0	2.8
2	6.8	6.4	6.0	5.6
3	10.2	9.6	9.0	8.4
4	13.6	12.8	12.0	11.2
5	17.0	16.0	15.0	14.0
6	20.4	19.2	18.0	16.8
7	23.8	22.4	21.0	19.6
8	27.2	25.6	24.0	22.4
9	30.6	28.8	27.0	25.2

	26	**24**	**22**	**20**
1	2.6	2.4	2.2	2.0
2	5.2	4.8	4.4	4.0
3	7.8	7.2	6.6	6.0
4	10.4	9.6	8.8	8.0
5	13.0	12.0	11.0	10.0
6	15.6	14.4	13.2	12.0
7	18.2	16.8	15.4	14.0
8	20.8	19.2	17.6	16.0
9	23.4	21.6	19.8	18.0

	19	**18**	**17**	**16**
1	1.9	1.8	1.7	1.6
2	3.8	3.6	3.4	3.2
3	5.7	5.4	5.1	4.8
4	7.6	7.2	6.8	6.4
5	9.5	9.0	8.5	8.0
6	11.4	10.8	10.2	9.6
7	13.3	12.6	11.9	11.2
8	15.2	14.4	13.6	12.8
9	17.1	16.2	15.3	14.4

	15	**14**	**13**	**12**
1	1.5	1.4	1.3	1.2
2	3.0	2.8	2.6	2.4
3	4.5	4.2	3.9	3.6
4	6.0	5.6	5.2	4.8
5	7.5	7.0	6.5	6.0
6	9.0	8.4	7.8	7.2
7	10.5	9.8	9.1	8.4
8	12.0	11.2	10.4	9.6
9	13.5	12.6	11.7	10.8

P. P.

	12
1	1.2
2	2.4
3	3.6
4	4.8
5	6.0
6	7.2
7	8.4
8	9.6
9	10.8

Interpolate mentally, using the multiplication table.

Log cos A° from 0° to 45°.

Logarithms of tangents and cotangents, 0° to 23°.

A°		0′	10′	20′	30′	40′	50′	60′		A°	d.
0′	tan		7.4637	7648	9409	*0658	*1627.	*2419	cot	89°	
log	cot		2.5363	2352	0591	*9342	*8373	*7581	tan	log	
1°	tan	8.2419	3089	3669	4181	4638	5053	5431	cot	88°	
log	cot	1.7581	6911	6331	5819	5362	4947	4569	tan	log	
2°	tan	8.5431	5779	6101	6401	6682	6945	7194	cot	87°	
log	cot	1.4569	4221	3899	3599	3318	3055	2806	tan	log	
3°	tan	8.7194	7429	7652	7865	8067	8261	8446	cot	86°	
log	cot	1.2806	2571	2348	2135	1933	1739	1554	tan	log	
4°	tan	8.8446	8624	8795	8960	9118	9272	9420	cot	85°	
log	cot	1.1554	1376	1205	1040	0882	0728	0580	tan	log	
5°	tan	8.9420	9563	9701	9836	9966	*0093	*0216	cot	84°	
log	cot	1.0580	0437	0299	0164	0034	*9907	*9784	tan	log	
6°	tan	9.0216	0336	0453	0567	0678	0786	0891	cot	83°	
log	cot	0.9784	9664	9547	9433	9322	9214	9109	tan	log	
7°	tan	9.0891	0995	1096	1194	1291	1385	1478	cot	82°	
log	cot	0.9109	9005	8904	8806	8709	8615	8522	tan	log	
8°	tan	9.1478	1569	1658	1745	1831	1915	1997	cot	81°	87
log	cot	0.8522	8431	8342	8255	8169	8085	8003	tan	log	87
9°	tan	9.1997	2078	2158	2236	2313	2389	2463	cot	80°	78
log	cot	0.8003	7922	7842	7764	7687	7611	7537	tan	log	78
					30′						
10°	tan	9.2463	2536	2609	2680	2750	2819	2887	cot	79°	71
log	cot	0.7537	7464	7391	7320	7250	7181	7113	tan	log	71
11°	tan	9.2887	2953	3020	3085	3149	3212	3275	cot	78°	65
log	cot	0.7113	7047	6980	6915	6851	6788	6725	tan	log	65
12°	tan	9.3275	3336	3397	3458	3517	3576	3634	cot	77°	60
log	cot	0.6725	6664	6603	6542	6483	6424	6366	tan	log	60
13°	tan	9.3634	3691	3748	3804	3859	3914	3968	cot	76°	56
log	cot	0.6366	6309	6252	6196	6141	6086	6032	tan	log	56
14°	tan	9.3968	4021	4074	4127	4178	4230	4281	cot	75	52
log	cot	0.6032	5979	5926	5873	5822	5770	5719	tan	log	52
15°	tan	9.4281	4331	4381	4430	4479	4527	4575	cot	74°	49
log	cot	0.5719	5669	5619	5570	5521	5473	5425	tan	log	49
16°	tan	9.4575	4622	4669	4716	4762	4808	4853	cot	73°	46
log	cot	0.5425	5378	5331	5284	5238	5192	5147	tan	log	46
17°	tan	9.4853	4898	4943	4987	5031	5075	5118	cot	72°	44
log	cot	0.5147	5102	5057	5013	4969	4925	4882	tan	log	44
18°	tan	9.5118	5161	5203	5245	5287	5329	5370	cot	71°	42
log	cot	0.4882	4839	4797	4755	4713	4671	4630	tan	log	42
19°	tan	9.5370	5411	5451	5491	5531	5571	5611	cot	70°	40
log	cot	0.4630	4589	4549	4509	4469	4429	4389	tan	log	40
20°	tan	9.5611	5650	5689	5727	5766	5804	5842	cot	69°	39
log	cot	0.4389	4350	4311	4273	4234	4196	4158	tan	log	39
21°	tan	9.5842	5879	5917	5954	5991	6028	6064	cot	68°	37
log	cot	0.4158	4121	4083	4046	4009	3972	3936	tan	log	37
22°	tan	9.6064	6100	6136	6172	6208	6243	6279	cot	67°	36
log	cot	0.3936	3900	3864	3828	3792	3757	3721	tan	log	36
A°		60′	50′	40′	30′	20′	10′	0′		A°	d.

Do not interpolate, but use the special table for log tan from 0° to 9°, and log cot from 81° to 90°.

P. P.

	82	80	78	76	74
1	8.2	8.0	7.8	7.6	7.4
2	16.4	16.0	15.6	15.2	14.8
3	24.6	24.0	23.4	22.8	22.2
4	32.8	32.0	31.2	30.4	29.6
5	41.0	40.0	39.0	38.0	37.0
6	49.2	48.0	46.8	45.6	44.4
7	57.4	56.0	54.6	53.2	51.8
8	65.6	64.0	62.4	60.8	59.2
9	73.8	72.0	70.2	68.4	66.6

	72	70	68	66	64
1	7.2	7.0	6.8	6.6	6.4
2	14.4	14.0	13.6	13.2	12.8
3	21.6	21.0	20.4	19.8	19.2
4	28.8	28.0	27.2	26.4	25.6
5	36.0	35.0	34.0	33.0	32.0
6	43.2	42.0	40.8	39.6	38.4
7	50.4	49.0	47.6	46.2	44.8
8	57.6	56.0	54.4	52.8	51.2
9	64.8	63.0	61.2	59.4	57.6

	62	60	58	56	54
1	6.2	6.0	5.8	5.6	5.4
2	12.4	12.0	11.6	11.2	10.8
3	18.6	18.0	17.4	16.8	16.2
4	24.8	24.0	23.2	22.4	21.6
5	31.0	30.0	29.0	28.0	27.0
6	37.2	36.0	34.8	33.6	32.4
7	43.4	42.0	40.6	39.2	37.8
8	49.6	48.0	46.4	44.8	43.2
9	55.8	54.0	52.2	50.4	48.6

	53	52	51	50	49
1	5.3	5.2	5.1	5.0	4.9
2	10.6	10.4	10.2	10.0	9.8
3	15.9	15.6	15.3	15.0	14.7
4	21.2	20.8	20.4	20.0	19.6
5	26.5	26.0	25.5	25.0	24.5
6	31.8	31.2	30.6	30.0	29.4
7	37.1	36.4	35.7	35.0	34.3
8	42.4	41.6	40.8	40.0	39.2
9	47.7	46.8	45.9	45.0	44.1

	48	47	46	45	44
1	4.8	4.7	4.6	4.5	4.4
2	9.6	9.4	9.2	9.0	8.8
3	14.4	14.1	13.8	13.5	13.2
4	19.2	18.8	18.4	18.0	17.6
5	24.0	23.5	23.0	22.5	22.0
6	28.8	28.2	27.6	27.0	26.4
7	33.6	32.9	32.2	31.5	30.8
8	38.4	37.6	36.8	36.0	35.2
9	43.2	42.3	41.4	40.5	39.6

P. P.

Logarithms of tangents and cotangents, 67° to 90°.

Logarithms of tangents and cotangents, 23° to 46°.

P. P.

	43	**42**
1	4.3	4.2
2	8.6	8.4
3	12.9	12.6
4	17.2	16.8
5	21.5	21.0
6	25.8	25.2
7	30.1	29.4
8	34.4	33.6
9	38.7	37.8

	41	**40**
1	4.1	4.0
2	8.2	8.0
3	12.3	12.0
4	16.4	16.0
5	20.5	20.0
6	24.6	24.0
7	28.7	28.0
8	32.8	32.0
9	36.9	36.0

	39	**38**
1	3.9	3.8
2	7.8	7.6
3	11.7	11.4
4	15.6	11.4
5	19.5	19.0
6	23.4	22.8
7	27.3	26.6
8	31.2	30.4
9	35.1	34.2

	37	**36**
1	3.7	3.6
2	7.4	7.2
3	11.1	10.8
4	14.8	14.4
5	18.5	18.0
6	22.2	21.6
7	25.9	25.2
8	29.6	28.8
9	33.3	32.4

	35	**34**
1	3.5	3.4
2	7.0	6.8
3	10.5	10.2
4	14.0	13.6
6	17.5	17.0
6	21.0	20.4
7	24.5	23.8
8	28.0	27.2
9	31.5	30.6

P. P.

A°	**0′**	**10′**	**20′**	**30′**	**40′**	**50′**	**60′**	**A°**	**d.**
23° tan	9.6279	6314	6348	6383	6417	6452	6486	cot **66°**	34
log cot	0.3721	3686	3652	3617	3583	3548	3514	tan log	34
24° tan	9.6486	6520	6553	6587	6620	6654	6687	cot **65°**	33
log cot	0.3514	3480	3447	3413	3380	3346	3313	tan log	33
25° tan	9.6687	6720	6752	6785	6817	6850	6882	cot **64°**	33
log cot	0.3313	3280	3248	3215	3183	3150	3118	tan log	33
26° tan	9.6882	6914	6946	6977	7009	7040	7072	cot **63°**	32
log cot	0.3118	3086	3054	3023	2991	2960	2928	tan log	32
27° tan	9.7072	7103	7134	7165	7196	7226	7257	cot **62°**	31
log cot	0.2928	2897	2866	2835	2804	2774	2743	tan log	31
28° tan	9.7257	7287	7317	7348	7378	7408	7438	cot **61°**	30
log cot	0.2743	2713	2683	2652	2622	2592	2562	tan log	30
29° tan	9.7438	7467	7497	7526	7556	7585	7614	cot **60°**	29
log cot	0.2562	2533	2503	2474	2444	2415	2386	tan log	29
30° tan	9.7614	7644	7673	7701	7730	7759	7788	cot **59°**	29
log cot	0.2386	2356	2327	2299	2270	2241	2212	tan log	29
31° tan	9.7788	7816	7845	7873	7902	7930	7958	cot **58°**	28
log cot	0.2212	2184	2155	2127	2098	2070	2042	tan log	28
32° tan	9.7958	7986	8014	8042	8070	8097	8125	cot **57°**	28
log cot	0.2042	2014	1986	1958	1930	1903	1875	tan log	28
33° tan	9.8125	8153	8180	8208	8235	8263	8290	cot **56°**	27
log cot	0.1875	1847	1820	1792	1765	1737	1710	tan log	27
34° tan	9.8290	8317	8344	8371	8398	8425	8452	cot **55°**	27
log cot	0.1710	1683	1656	1629	1602	1575	1548	tan log	27
				30′					
35° tan	9.8452	8479	8506	8533	8559	8586	8613	cot **54°**	27
log cot	0.1548	1521	1494	1467	1441	1414	1387	tan log	27
36° tan	9.8613	8639	8666	8692	8718	8745	8771	cot **53°**	26
log cot	0.1387	1361	1334	1308	1282	1255	1229	tan log	26
37° tan	9.8771	8797	8824	8850	8876	8902	8928	cot **52°**	26
log cot	0.1229	1203	1176	1150	1124	1098	1072	tan log	26
38° tan	9.8928	8954	8980	9006	9032	9058	9084	cot **51°**	26
log cot	0.1072	1046	1020	0994	0968	0942	0916	tan log	26
39° tan	9.9084	9110	9135	9161	9187	9212	9238	cot **50°**	26
log cot	0.0916	0890	0865	0839	0813	0788	0762	tan log	26
40° tan	9.9238	9264	9289	9315	9341	9366	9392	cot **49°**	26
log cot	0.0762	0736	0711	0685	0659	0634	0608	tan log	26
41° tan	9.9392	9417	9443	9468	9494	9519	9544	cot **48°**	25
log cot	0.0608	0583	0557	0532	0506	0481	0456	tan log	25
42° tan	9.9544	9570	9595	9621	9646	9671	9697	cot **47°**	25
log cot	0.0456	0430	0405	0379	0354	0329	0303	tan log	25
43° tan	9.9697	9722	9747	9772	9798	9823	9848	cot **46°**	25
log cot	0.0303	0278	0253	0228	0202	0177	0152	tan log	25
44° tan	9.9848	9874	9899	9924	9949	9975	0000	cot **45°**	25
log cot	0.0152	0126	0101	0076	0051	0025	0000	tan log	25
45° tan	0.0000	0025	0051	0076	0101	0126	0152	cot **44°**	25
log cot	0.0000	*9975	*9949	*9924	*9899	*9874	*9848	tan log	25
A°	**60′**	**50′**	**40′**	**30′**	**20′**	**10′**	**0′**	**A°**	**d.**

P. P.

	33	**32**
1	3.3	3.2
2	6.6	6.4
3	9.9	9.6
4	13.2	12.8
5	16.5	16.0
6	19.8	19.2
7	23.1	22.4
8	26.4	25.6
9	29.7	28.8

	31	**30**
1	3.1	3.0
2	6.2	6.0
3	9.3	9.0
4	12.4	12.0
5	15.5	15.0
6	18.6	18.0
7	21.7	21.0
8	24.8	24.0
9	27.9	27.0

	29	**28**
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	27	**26**
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	25	**24**
1	2.5	2.4
2	5.0	4.8
3	7.5	7.2
4	10.0	9.6
5	12.5	12.0
6	15.0	14.4
7	17.5	16.8
8	20.0	19.2
9	22.5	21.6

P. P.

Logarithms of tangents and cotangents, 44° to 67°.

Log sin by minutes from 0° to 9°.

A	′	0′	1′	2′	3′	4′	5′	6′	7′	8′	9′	10′		A
0	0	——	6.4637	7648	9408	*0658	*1627	*2419	*3088	*3668	*4180	*4637	50	
	10	7.4637	5051	5429	5777	6099	6398	6678	6942	7190	7425	7648	40	
	20	7648	7859	8061	8255	8439	8617	8787	8951	*109	*261	*408	30	
	30	7.9408	551	689	822	952	*078	*200	*319	*435	*548	*658	20	
	40	8.0658	765	870	972	*072	*169	*265	*358	*450	*539	*627	10	
	50	8.1627	713	797	880	961	*041	*119	*196	*271	*346	*419	0	**89**
1	0	8.2419	490	561	630	699	766	832	898	962	*025	*088	50	
	10	8.3088	150	210	270	329	388	445	502	558	613	668	40	
	20	668	722	775	828	880	931	982	*032	*082	*131	*179	30	
	30	8.4179	227	275	322	368	414	459	504	549	593	637	20	
	40	637	680	723	765	807	848	890	930	971	*011	*050	10	
	50	8.5050	090	129	167	206	243	281	318	355	392	428	0	**88**
2	0	8.5428	464	500	535	571	605	640	674	708	742	776	50	
	10	776	809	842	875	907	939	972	*003	*035	*066	*097	40	
	20	8.6097	128	159	189	220	250	279	309	339	368	397	30	
	30	397	426	454	483	511	539	567	595	622	650	677	20	
	40	677	704	731	758	784	810	837	863	889	914	940	10	
	50	940	965	991	*016	*041	*066	*090	*115	*140	*164	*188	0	**87**
3	0	8.7188	212	236	260	283	307	330	354	377	400	423	50	
	10	423	445	468	491	513	535	557	580	602	623	645	40	
	20	645	667	688	710	731	752	773	794	815	836	857	30	
	30	857	877	898	918	939	959	979	999	*019	*039	*059	20	
	40	8.8059	078	098	117	137	156	175	194	213	232	251	10	
	50	251	270	289	307	326	345	363	381	400	418	436	0	**86**
4	0	8.8436	454	472	490	508	525	543	560	578	595	613	50	
	10	613	630	647	665	682	699	716	733	749	766	783	40	
	20	783	799	816	833	849	865	882	898	914	930	946	30	
	30	946	962	978	994	*010	*026	*042	*057	*073	*089	*104	20	
	40	8.9104	119	135	150	166	181	196	211	226	241	256	10	
	50	256	271	286	301	315	330	345	359	374	389	403	0	**85**
5	0	8.9403	417	432	446	460	475	489	503	517	531	545	50	
	10	545	559	573	587	601	614	628	642	655	669	682	40	
	20	682	696	709	723	736	750	763	776	789	803	816	30	
	30	816	829	842	855	868	881	894	907	919	932	945	20	
	40	945	958	970	983	996	*008	*021	*033	*046	*058	*070	10	
	50	9.0070	083	095	107	120	132	144	156	168	180	192	0	**84**
6	0	9.0192	204	216	228	240	252	264	276	287	299	311	50	
	10	311	323	334	346	357	369	380	392	403	415	426	40	
	20	426	438	449	460	472	483	494	505	516	527	539	30	
	30	539	550	561	572	583	594	605	616	626	637	648	20	
	40	648	659	670	680	691	702	712	723	734	744	755	10	
	50	755	765	776	786	797	807	818	828	838	849	859	0	**83**
7	0	9.0859	869	879	890	900	910	920	930	940	951	961	50	
	10	961	971	981	991	*001	*011	*020	*030	*040	*050	*060	40	
	20	9.1060	070	080	089	099	109	118	128	138	147	157	30	
	30	157	167	176	186	195	205	214	224	233	242	252	20	
	40	252	261	271	280	289	299	308	317	326	336	345	10	
	50	345	354	363	372	381	390	399	409	418	427	436	0	**82**
8	0	9.1436	445	453	462	471	480	489	498	507	516	525	50	
	10	525	533	542	551	560	568	577	586	594	603	612	40	
	20	612	620	629	637	646	655	663	672	680	689	697	30	
	30	697	705	714	722	731	739	747	756	764	772	781	20	
	40	781	789	797	806	814	822	830	838	847	855	863	10	
	50	863	871	879	887	895	903	911	919	927	935	943	0	**81**
A		10′	9′	8′	7′	6′	5′	4′	3′	2′	1′	0′	′	A

Log cos by minutes from 81° to 90°.

Log tan by minutes from 0° to 9°.

°	′	0′	1′	2′	3′	4′	5′	6′	7′	8′	9′	10′		
0	0		6.4637	7648	9408	*0658	*1627	*2419	*3088	*3668	*4180	*4637	50	
	10	7.4637	5051	5429	5777	6099	6398	6678	6942	7190	7425	7648	40	
	20	7648	7860	8062	8255	8439	8617	8787	8951	*109	*261	*409	30	
	30	7.9409	551	689	823	952	*078	*200	*319	*435	*548	*658	20	
	40	8.0658	765	870	972	*072	*170	*265	*359	*450	*540	*627	10	
	50	8.1627	713	798	880	962	*041	*120	*196	*272	*346	*419	0	**89**
1	0	8.2419	491	562	631	700	767	833	899	963	*026	*089	50	
	10	8.3089	150	211	271	330	389	446	503	559	614	669	40	
	20	669	723	776	829	881	932	983	*033	*083	132	181	30	
	30	8.4181	229	276	323	370	416	461	506	551	595	638	20	
	40	638	682	725	767	809	851	892	933	973	*013	*053	10	
	50	8.5053	092	131	170	208	246	283	321	358	394	431	0	**88**
2	0	8.5431	467	503	538	573	608	643	677	711	745	779	50	
	10	779	812	845	878	911	943	975	*007	*038	*070	*101	40	
	20	8.6101	132	163	193	223	254	283	313	343	372	401	30	
	30	401	430	459	487	515	544	571	599	627	654	682	20	
	40	682	709	736	762	789	815	842	868	894	920	945	10	
	50	945	971	996	*021	*046	*071	*096	*121	*145	*170	*194	0	**87**
3	0	8.7194	218	242	266	290	313	337	360	383	406	429	50	
	10	429	452	475	497	520	542	565	587	609	631	652	40	
	20	652	674	696	717	739	760	781	802	823	844	865	30	
	30	865	886	906	927	947	967	988	*008	*028	*048	*067	20	
	40	8.8067	087	107	126	146	165	185	204	223	242	261	10	
	50	261	280	299	317	336	355	373	392	410	428	446	0	**86**
4	0	8.8446	465	483	501	518	536	554	572	589	607	624	50	
	10	624	642	659	676	694	711	728	745	762	778	795	40	
	20	795	812	829	845	862	878	895	911	927	944	960	30	
	30	960	976	992	*008	*024	*040	*056	*071	*087	*103	*118	20	
	40	8.9118	134	150	165	180	196	211	226	241	256	272	10	
	50	272	287	302	316	331	346	361	376	390	405	420	0	**85**
5	0	8.9420	434	449	463	477	492	506	520	534	549	563	50	
	10	563	577	591	605	619	633	646	660	674	688	701	40	
	20	701	715	729	742	756	769	782	796	809	823	836	30	
	30	836	849	862	875	888	901	915	927	940	953	966	20	
	40	966	979	992	*005	*017	*030	*043	*055	*068	*080	*093	10	
	50	9.0093	105	118	130	143	155	167	180	192	204	216	0	**84**
6	0	9.0216	228	240	253	265	277	289	300	312	324	336	50	
	10	336	348	360	371	383	395	407	418	430	441	453	40	
	20	453	464	476	487	499	510	521	533	544	555	567	30	
	30	567	578	589	600	611	622	633	645	656	667	678	20	
	40	678	688	699	710	721	732	743	754	764	775	786	10	
	50	786	796	807	818	828	839	849	860	871	881	891	0	**83**
7	0	9.0891	902	912	923	933	943	954	964	974	984	995	50	
	10	995	*005	*015	*025	*035	*045	*055	*066	*076	*086	*096	40	
	20	9.1096	106	116	125	135	145	155	165	175	185	194	30	
	30	194	204	214	223	233	243	252	262	272	281	291	20	
	40	291	300	310	319	329	338	348	357	367	376	385	10	
	50	385	395	404	413	423	432	441	450	460	469	478	0	**82**
8	0	9.1478	487	496	505	515	524	533	542	551	560	569	50	
	10	569	578	587	596	605	613	622	631	640	649	658	40	
	20	658	667	675	684	693	702	710	719	728	736	745	30	
	30	745	754	762	771	779	788	797	805	814	822	831	20	
	40	831	839	848	856	864	873	881	890	898	906	915	10	
	50	915	923	931	940	948	956	964	973	981	989	997	0	**81**
		10′	**9′**	**8′**	**7′**	**6′**	**5′**	**4′**	**3′**	**2′**	**1′**	**0′**	′	°

Log cot by minutes from 81° to 90°.

Numerical values of the sine function, 0° to 45°.

°	0′	10′	20′	30′	40′	50′	60′		d.
0	0.0000	0029	0058	0087	0116	0145	0175	**89**	29
1	0175	0204	0233	0262	0291	0320	0349	**88**	29
2	0349	0378	0407	0436	0465	0494	0523	**87**	29
3	0523	0552	0581	0610	0640	0669	0698	**86**	29
4	0698	0727	0756	0785	0814	0843	0872	**85**	29
5	0.0872	0901	0929	0958	0987	1016	1045	**84**	29
6	1045	1074	1103	1132	1161	1190	1219	**83**	29
7	1219	1248	1276	1305	1334	1363	1392	**82**	29
8	1392	1421	1449	1478	1507	1536	1564	**81**	29
9	1564	1593	1622	1650	1679	1708	1736	**80**	29
10	0.1736	1765	1794	1822	1851	1880	1908	**79**	29
11	1908	1937	1965	1994	2022	2051	2079	**78**	28
12	2079	2108	2136	2164	2193	2221	2250	**77**	28
13	2250	2278	2306	2334	2363	2391	2419	**76**	28
14	2419	2447	2476	2504	2532	2560	2588	**75**	28
15	0.2588	2616	2644	2672	2700	2728	2756.	**74**	28
16	2756	2784	2812	2840	2868	2896	2924	**73**	28
17	2924	2952	2979	3007	3035	3062	3090	**72**	28
18	3090	3118	3145	3173	3201	3228	3256	**71**	28
19	3256	3283	3311	3338	3365	3393	3420	**70**	27
20	0.3420	3448	3475	3502	3529	3557	3584	**69**	27
21	3584	3611	3638	3665	3692	3719	3746	**68**	27
22	3746	3773	3800	3827	3854	3881	3907	**67**	27
23	3907	3934	3961	3987	4014	4041	4067	**66**	27
24	4067	4094	4120	4147	4173	4200	4226	**65**	26
				30′					
25	0.4226	4253	4279	4305	4331	4358	4384	**64**	26
26	4384	4410	4436	4462	4488	4514	4540	**63**	26
27	4540	4566	4592	4617	4643	4669	4695	**62**	26
28	4695	4720	4746	4772	4797	4823	4848	**61**	26
29	4848	4874	4899	4924	4950	4975	5000	**60**	25
30	0.5000	5025	5050	5075	5100	5125	5150	**59**	25
31	5150	5175	5200	5225	5250	5275	5299	**58**	25
32	5299	5324	5348	5373	5398	5422	5446	**57**	24
33	5446	5471	5495	5519	5544	5568	5592	**56**	24
34	5592	5616	5640	5664	5688	5712	5736	**55**	24
35	0.5736	5760	5783	5807	5831	5854	5878	**54**	24
36	5878	5901	5925	5948	5972	5995	6018	**53**	23
37	6018	6041	6065	6088	6111	6134	6157	**52**	23
38	6157	6180	6202	6225	6248	6271	6293	**51**	23
39	6293	6316	6338	6361	6383	6406	6428	**50**	22
40	0.6428	6450	6472	6494	6517	6539	6561	**49**	22
41	6561	6583	6604	6626	6648	6670	6691	**48**	22
42	6691	6713	6734	6756	6777	6799	6820	**47**	22
43	6820	6841	6862	6884	6905	6926	6947	**46**	21
44	6947	6967	6988	7009	7030	7050	7071	**45**	21
	60′	**50′**	**40′**	**30′**	**20′**	**10′**	**0′**	°	**d.**

P. P.

	30	**29**
1	3.0	2.9
2	6.0	5.8
3	9.0	8.7
4	12.0	11.6
5	15.0	14.5
6	18.0	17.4
7	21.0	20.3
8	24.0	23.2
9	27.0	26.1

	28	**27**
1	2.8	2.7
2	5.6	5.4
3	8.4	8.1
4	11.2	10.8
5	14.0	13.5
6	16.8	16.2
7	19.6	18.9
8	22.4	21.3
9	25.2	24.3

	26	**25**
1	2.6	2.5
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	24	**23**
1	2.4	2.3
2	4.8	4.6
3	7.2	6.9
4	9.6	9.2
5	12.0	11.5
6	14.4	13.8
7	16.8	16.1
8	19.2	18.4
9	21.6	20.7

	22	**21**	**20**
1	2.2	2.1	2.0
2	4.4	4.2	4.0
3	6.6	6.3	6.0
4	8.8	8.4	8.0
5	11.0	10.5	10.0
6	13.2	12.6	12.0
7	15.4	14.7	14.0
8	17.6	16.8	16.0
9	19.8	18.9	18.0

P. P.

Numerical values of the cosine function, 45° to 90°.

Numerical values of the sine function, 45° to 90°.

°	0′	10′	20′	30′	40′	50′	60′		d.
45	0.7071	7092	7112	7133	7153	7173	7193	**44**	20
46	7193	7214	7234	7254	7274	7294	7314	**43**	20
47	7314	7333	7353	7373	7392	7412	7431	**42**	20
48	7431	7451	7470	7490	7509	7528	7547	**41**	19
49	7547	7566	7585	7604	7623	7642	7660	**40**	19
50	0.7660	7679	7698	7716	7735	7753	7771	**39**	18
51	7771	7790	7808	7826	7844	7862	7880	**38**	18
52	7880	7898	7916	7934	7951	7969	7986	**37**	18
53	7986	8004	8021	8039	8056	8073	8090	**36**	17
54	8090	8107	8124	8141	8158	8175	8192	**35**	17
55	0.8192	8208	8225	8241	8258	8274	8290	**34**	16
56	8290	8307	8323	8339	8355	8371	8387	**33**	16
57	8387	8403	8418	8434	8450	8465	8480	**32**	16
58	8480	8496	8511	8526	8542	8557	8572	**31**	15
59	8572	8587	8601	8616	8631	8646	8660	**30**	15
60	0.8660	8675	8689	8704	8718	8732	8746	**29**	14
61	8746	8760	8774	8783	8802	8816	8829	**28**	14
62	8829	8843	8857	8870	8884	8897	8910	**27**	14
63	8910	8923	8936	8949	8962	8975	8988	**26**	13
64	8988	9001	9013	9026	9038	9051	9063	**25**	12
65	0.9063	9075	9088	9100	9112	9124	9135	**24**	12
66	9135	9147	9159	9171	9182	9194	9205	**23**	12
67	9205	9216	9228	9239	9250	9261	9272	**22**	11
68	9272	9283	9293	9304	9315	9325	9336	**21**	11
69	9336	9346	9356	9367	9377	9387	9397	**20**	10
				30′					
70	0.9397	9407	9417	9426	9436	9446	9455	**19**	10
71	9455	9465	9474	9483	9492	9502	9511	**18**	9
72	9511	9520	9528	9537	9546	9555	9563	**17**	9
73	9563	9572	9580	9588	9596	9605	9613	**16**	8
74	9613	9621	9628	9636	9644	9652	9659	**15**	8
75	0.9659	9667	9674	9681	9689	9696	9703	**14**	7
76	9703	9710	9717	9724	9730	9737	9744	**13**	7
77	9744	9750	9757	9763	9769	9775	9781	**12**	6
78	9781	9787	9793	9799	9805	9811	9816	**11**	6
79	9816	9822	9827	9833	9838	9843	9848	**10**	5
80	0.9848	9853	9858	9863	9868	9872	9877	**9**	5
81	9877	9881	9886	9890	9894	9899	9903	**8**	4
82	9903	9907	9911	9914	9918	9922	9925	**7**	4
83	9925	9929	9932	9936	9939	9942	9945	**6**	3
84	9945	9948	9951	9954	9957	9959	9962	**5**	3
85	0.9962	9964	9967	9969	9971	9974	9976	**4**	2
86	9976	9978	9980	9981	9983	9985	9986	**3**	2
87	9986	9988	9989	9990	9992	9993	9994	**2**	1
88	9994	9995	9996	9997	9997	9998	9998	**1**	1
89	9998	9999	9999	*0000	*0000	*0000	*0000	**0**	0
	60′	**50′**	**40′**	**30′**	**20′**	**10′**	**0′**	°	**d.**

P. P.

	21	**20**	**19**	**18**
1	2.1	2.0	1.9	1.8
2	4.2	4.0	3.8	3.6
3	6.3	6.0	5.7	5.4
4	8.4	8.0	7.6	7.2
5	10.5	10.0	9.5	9.0
6	12.6	12.0	11.4	10.8
7	14.7	14.0	13.3	12.6
8	16.8	16.0	15.2	14.4
9	18.9	18.0	17.1	16.2

	17	**16**	**15**	**14**
1	1.7	1.6	1.5	1.4
2	3.4	3.2	3.0	2.8
3	5.1	4.7	4.5	4.2
4	6.8	6.4	6.0	5.6
5	8.5	8.0	7.5	7.0
6	10.2	9.6	9.0	8.4
7	11.9	11.2	10.5	9.8
8	13.6	12.8	12.0	11.2
9	15.3	14.4	13.5	12.6

	13	**12**	**11**	**10**
1	1.3	1.2	1.1	1.0
2	2.6	2.4	2.2	2.0
3	3.9	3.6	3.3	3.0
4	5.2	4.8	4.4	4.0
5	6.5	6.0	5.5	5.0
6	7.8	7.2	6.6	6.0
7	9.1	8.4	7.7	7.0
8	10.4	9.6	8.8	8.0
9	11.7	10.8	9.9	9.0

Interpolate mentally, using the multiplication table.

P. P.

Numerical values of the cosine function, 0° to 45°.

Numerical values of the tangent function, 0° to 45°.

°	0′	10′	20′	30′	40′	50′	60′		d.
0	0.0000	0029	0058	0087	0116	0145	0175	**89**	29
1	0175	0204	0233	0262	0291	0320	0349	**88**	29
2	0349	0378	0407	0437	0466	0495	0524	**87**	29
3	0524	0553	0582	0612	0641	0670	0699	**86**	29
4	0699	0729	0758	0787	0816	0846	0875	**85**	29
5	0.0875	0904	0934	0963	0992	1022	1051	**84**	29
6	1051	1080	1110	1139	1169	1198	1228	**83**	30
7	1228	1257	1287	1317	1346	1376	1405	**82**	30
8	1405	1435	1465	1495	1524	1554	1584	**81**	30
9	1584	1614	1644	1673	1703	1733	1763	**80**	30
10	0.1763	1793	1823	1853	1883	1914	1944	**79**	30
11	1944	1974	2004	2035	2065	2095	2126	**78**	30
12	2126	2156	2186	2217	2247	2278	2309	**77**	30
13	2309	2339	2370	2401	2432	2462	2493	**76**	31
14	2493	2524	2555	2586	2617	2648	2679	**75**	31
15	0.2679	2711	2742	2773	2805	2836	2867	**74**	31
16	2867	2899	2931	3962	2994	3026	3057	**73**	32
17	3057	3089	3121	3153	3185	3217	3249	**72**	32
18	3249	3281	3314	3346	3378	3411	3443	**71**	32
19	3443	3476	3508	3541	3574	3607	3640	**70**	33
20	0.3640	3673	3706	3739	3772	3805	3839	**69**	33
21	3839	3872	3906	3939	3973	4006	4040	**68**	34
22	4040	4074	4108	4142	4176	4210	4245	**67**	34
23	4245	4279	4314	4348	4383	4417	4452	**66**	34
24	4452	4487	4522	4557	4592	4628	4663	**65**	35
				30′					
25	0.4663	4699	4734	4770	4806	4841	4877	**64**	36
26	4877	4913	4950	4986	5022	5059	5095	**63**	36
27	5095	5132	5169	5206	5243	5280	5317	**62**	37
28	5317	5354	5392	5430	5467	5505	5543	**61**	38
29	5543	5581	5619	5658	5696	5735	5774	**60**	38
30	0.5774	5812	5851	5890	5930	5969	6009	**59**	39
31	6009	6048	6088	6128	6168	6208	6249	**58**	40
32	6249	6289	6330	6371	6412	6453	6494	**57**	41
33	6494	6536	6577	6619	6661	6703	6745	**56**	42
34	6745	6787	6830	6873	6916	6959	7002	**55**	43
35	0.7002	7046	7089	7133	7177	7221	7265	**54**	44
36	7265	7310	7355	7400	7445	7490	7536	**53**	45
37	7536	7581	7627	7673	7720	7766	7813	**52**	46
38	7813	7860	7907	7954	8002	8050	8098	**51**	48
39	8098	8146	8195	8243	8292	8342	8391	**50**	49
40	0.8391	8441	8491	8541	8591	8642	8693	**49**	50
41	8693	8744	8796	8847	8899	8952	9004	**48**	52
42	9004	9057	9110	9163	9217	9271	9325	**47**	54
43	9325	9380	9435	9490	9545	9601	9657	**46**	55
44	9657	9713	9770	9827	9884	9942	*0000	**45**	57
	60′	**50′**	**40′**	**30′**	**20′**	**10′**	**0′**	°	**d.**

P. P.

	29	**30**	**31**	**32**	**33**
1	2.9	3.0	3.1	3.2	3.3
2	5.8	6.0	6.2	6.4	6.6
3	8.7	9.0	9.3	9.6	9.9
4	11.6	12.0	12.4	12.8	13.2
5	14.5	15.0	15.5	16.0	16.5
6	17.4	18.0	18.6	19.2	19.8
7	20.3	21.0	21.7	22.4	23.1
8	23.2	24.0	24.8	25.6	26.4
9	26.1	27.0	27.9	28.8	29.7
	34	**35**	**36**	**37**	**38**
1	3.4	3.5	3.6	3.7	3.8
2	6.8	7.0	7.2	7.4	7.6
3	10.2	10.5	10.8	11.1	11.4
4	13.6	14.0	14.4	14.8	15.2
5	17.0	17.5	18.0	18.5	19.0
6	20.4	21.0	21.6	22.2	22.8
7	23.8	24.5	25.2	25.9	26.6
8	27.2	28.0	28.8	29.6	30.4
9	30.6	31.5	32.4	33.3	34.2
	39	**40**	**41**	**42**	**43**
1	3.9	4.0	4.1	4.2	4.3
2	7.8	8.0	8.2	8.4	8.6
3	11.7	12.0	12.3	12.6	12.9
4	15.6	16.0	16.4	16.8	17.2
5	19.5	20.0	20.5	21.0	21.5
6	23.4	24.0	24.6	25.2	25.8
7	27.3	28.0	28.7	29.4	30.1
8	31.2	32.0	32.8	33.6	34.4
9	35.1	36.0	36.9	37.8	38.7
	44	**45**	**46**	**47**	**48**
1	4.4	4.5	4.6	4.7	4.8
2	8.8	9.0	9.2	9.4	9.6
3	13.2	13.5	13.8	14.1	14.4
4	17.6	18.0	18.4	18.8	19.2
5	22.0	22.5	23.0	23.5	24.0
6	26.4	27.0	27.6	28.2	28.8
7	30.8	31.5	32.2	32.9	33.6
8	35.2	36.0	36.8	37.6	38.4
9	39.6	40.5	41.4	42.3	43.2
	49	**50**	**51**	**52**	**53**
1	4.9	5.0	5.1	5.2	5.3
2	9.8	10.0	10.2	10.4	10.6
3	14.7	15.0	15.3	15.6	15.9
4	19.6	20.0	20.4	20.8	21.2
5	24.5	25.0	25.5	26.0	26.5
6	29.4	30.0	30.6	31.2	31.8
7	34.3	35.0	35.7	36.4	37.1
8	39.2	40.0	40.8	41.6	42.4
9	44.1	45.0	45.9	46.8	47.7
	54	**55**	**56**	**57**	**58**
1	5.4	5.5	5.6	5.7	5.8
2	10.8	11.0	11.2	11.4	11.6
3	16.2	16.5	16.8	17.1	17.4
4	21.6	22.0	22.4	22.8	23.2
5	27.0	27.5	28.0	28.5	29.0
6	32.4	33.0	33.6	34.2	34.8
7	37.8	38.5	39.2	39.9	40.6
8	43.2	44.0	44.8	45.6	46.4
9	48.6	49.5	50.4	51.3	52.2

P. P.

Numerical values of the cotangent function, 45° to 90°.

Numerical values of the tangent function, 45° to 90°.

°	0′	10′	20′	30′	40′	50′	60′		d.
45	1.000	1.006	1.012	1.018	1.024	1.030	1.036	**44**	6
46	1.036	1.042	1.048	1.054	1.060	1.066	1.072	**43**	6
47	1.072	1.079	1.085	1.091	1.098	1.104	1.111	**42**	6
48	1.111	1.117	1.124	1.130	1.137	1.144	1.150	**41**	6
49	1.150	1.157	1.164	1.171	1.178	1.185	1.192	**40**	7
50	1.192	1.199	1.206	1.213	1.220	1.228	1.235	**39**	7
51	1.235	1.242	1.250	1.257	1.265	1.272	1.280	**38**	8
52	1.280	1.288	1.295	1.303	1.311	1.319	1.327	**37**	8
53	1.327	1.335	1.343	1.351	1.360	1.368	1.376	**36**	8
54	1.376	1.385	1.393	1.402	1.411	1.419	1.428	**35**	9
55	1.428	1.437	1.446	1.455	1.464	1.473	1.483	**34**	9
56	1.483	1.492	1.501	1.511	1.520	1.530	1.540	**33**	10
57	1.540	1.550	1.560	1.570	1.580	1.590	1.600	**32**	10
58	1.600	1.611	1.621	1.632	1.643	1.653	1.664	**31**	11
59	1.664	1.675	1.686	1.698	1.709	1.720	1.732	**30**	11
60	1.732	1.744	1.756	1.767	1.780	1.792	1.804	**29**	12
61	1.804	1.816	1.829	1.842	1.855	1.868	1.881	**28**	13
62	1.881	1.894	1.907	1.921	1.935	1.949	1.963	**27**	14
63	1.963	1.977	1.991	2.006	2.020	2.035	2.050	**26**	14
64	2.050	2.066	2.081	2.097	2.112	2.128	2.145	**25**	16
65	2.145	2.161	2.177	2.194	2.211	2.229	2.246	**24**	17
66	2.246	2.264	2.282	2.300	2.318	2.337	2.356	**23**	18
67	2.356	2.375	2.394	2.414	2.434	2.455	2.475	**22**	20
68	2.475	2.496	2.517	2.539	2.560	2.583	2.605	**21**	22
69	2.605	2.628	2.651	2.675	2.699	2.723	2.747	**20**	24
				30′					
70	2.747	2.773	2.798	2.824	2.850	2.877	2.904	**19**	26
71	2.904	2.932	2.960	2.989	3.018	3.047	3.078	**18**	29
72	3.078	3.108	3.140	3.172	3.204	3.237	3.271	**17**	32
73	3.271	3.305	3.340	3.376	3.412	3.450	3.487	**16**	36
74	3.487	3.526	3.566	3.606	3.647	3.689	3.732	**15**	41
75	3.732	3.776	3.821	3.867	3.914	3.962	4.011	**14**	46
76	4.011	4.061	4.113	4.165	4.219	4.275	4.331	**13**	53
77	4.331	4.390	4.449	4.511	4.574	4.638	4.705	**12**	62
78	4.705	4.773	4.843	4.915	4.989	5.066	5.145	**11**	73
79	5.145	5.226	5.309	5.396	5.485	5.576	5.671	**10**	88
80	5.671	5.769	5.871	5.976	6.084	6.197	6.314	**9**	
81	6.314	6.435	6.561	6.691	6.827	6.968	7.115	**8**	
82	7.115	7.269	7.429	7.596	7.770	7.953	8.144	**7**	
83	8.144	8.345	8.556	8.777	9.010	9.255	9.514	**6**	
84	9.514	9.788	10.078	10.385	10.712	11.059	11.430	**5**	
85	11.430	11.826	12.251	12.706	13.197	13.727	14.301	**4**	
86	14.301	14.924	15.605	16.350	17.169	18.075	19.081	**3**	
87	19.081	20.206	21.470	22.904	24.542	26.432	28.636	**2**	
88	28.636	31.242	34.368	38.188	42.964	49.104	57.290	**1**	
89	57.290	68.750	85.940	114.59	171.89	343.77	infinit.	**0**	
	60′	**50′**	**40′**	**30′**	**20′**	**10′**	**0′**	°	**d.**

P. P.

	6	**7**	**8**	**9**	**10**	**11**
1	0.6	0.7	0.8	0.9	1.0	1.1
2	1.2	1.4	1.6	1.8	2.0	2.2
3	1.8	2.1	2.4	2.7	3.0	3.3
4	2.4	2.8	3.2	3.6	4.0	4.4
5	3.0	3.5	4.0	4.5	5.0	5.5
6	3.6	4.2	4.8	5.4	6.0	6.6
7	4.2	4.9	5.6	6.3	7.0	7.7
8	4.8	5.6	6.4	7.2	8.0	8.8
9	5.4	6.3	7.2	8.1	9.0	9.9

	12	**13**	**14**	**15**	**16**
1	1.2	1.3	1.4	1.5	1.6
2	2.4	2.6	2.8	3.0	3.2
3	3.6	3.9	4.2	4.5	4.8
4	4.8	5.2	5.6	6.0	6.4
5	6.0	6.5	7.0	7.5	8.0
6	7.2	7.8	8.4	9.0	9.6
7	8.4	9.1	9.8	10.5	11.2
8	9.6	10.4	11.2	12.0	12.8
9	10.8	11.7	12.6	13.5	14.4

	17	**18**	**19**	**20**	**21**
1	1.7	1.8	1.9	2.0	2.1
2	3.4	3.6	3.8	4.0	4.2
3	5.1	5.4	5.7	6.0	6.3
4	6.8	7.2	7.6	8.0	8.4
5	8.5	9.0	9.5	10.0	10.5
6	10.2	10.8	11.4	12.0	12.6
7	11.9	12.6	13.3	14.0	14.7
8	13.6	14.4	15.2	16.0	16.8
9	15.3	16.2	17.1	18.0	18.9

	22	**23**	**24**	**25**	**26**
1	2.2	2.3	2.4	2.5	2.6
2	4.4	4.6	4.8	5.0	5.2
3	6.6	6.9	7.2	7.5	7.8
4	8.8	9.2	9.6	10.0	10.4
5	11.0	11.5	12.0	12.5	13.0
6	13.2	13.8	14.4	15.0	15.6
7	15.4	16.1	16.8	17.5	18.2
8	17.6	18.4	19.2	20.0	20.8
9	19.8	20.7	21.6	22.5	23.4

	27	**28**	**59**	**62**	**63**
1	2.7	2.8	5.9	6.2	6.3
2	5.4	5.6	11.8	12.4	12.6
3	8.1	8.4	17.7	18.6	18.9
4	10.8	11.2	23.6	24.8	25.2
5	13.5	14.0	29.5	31.0	31.5
6	16.2	16.8	35.4	37.2	37.8
7	18.9	19.6	41.3	43.4	44.1
8	21.6	22.4	47.2	49.6	50.4
9	24.3	25.2	53.1	55.8	56.7

	64	**66**	**68**	**70**	**72**
1	6.4	6.6	6.8	7.0	7.2
2	12.8	13.2	13.6	14.0	14.4
3	19.2	19.8	20.4	21.0	21.6
4	25.6	26.4	27.2	28.0	28.8
5	32.0	33.0	34.0	35.0	36.0
6	38.4	39.6	40.8	42.0	43.2
7	44.8	46.2	47.6	49.0	50.4
8	51.2	52.8	54.4	56.0	57.6
9	57.6	59.4	61.2	63.0	64.8

P. P.

Numerical values of the cotangent function, 0° to 45°.

Radian measure of angles, 0° to 180°
or
Length of arc in unit circle for angle 0° to 180° at center.

A°	RADIANS	A°	RADIANS	A°	RADIANS	A°	RADIANS
1°	0.017	46°	0.803	91°	1.588	136°	2.374
2°	0.035	47°	0.820	92°	1.606	137°	2.391
3°	0.052	48°	0.838	93°	1.623	138°	2.409
4°	0.070	49°	0.855	94°	1.641	139°	2.426
5°	0.087	50°	0.873	95°	1.658	140°	2.443
6°	0.105	51°	0.890	96°	1.676	141°	2.461
7°	0.122	52°	0.908	97°	1.693	142°	2.478
8°	0.140	53°	0.925	98°	1.710	143°	2.496
9°	0.157	54°	0.942	99°	1.728	144°	2.513
10°	0.175	55°	0.960	100°	1.745	145°	2.531
11°	0.192	56°	0.977	101°	1.763	146°	2.548
12°	0.209	57°	0.995	102°	1.780	147°	2.566
13°	0.227	58°	1.012	103°	1.798	148°	2.583
14°	0.244	59°	1.031	104°	1.815	149°	2.601
15°	0.262	60°	1.047	105°	1.833	150°	2.618
16°	0.279	61°	1.065	106°	1.850	151°	2.635
17°	0.297	62°	1.082	107°	1.868	152°	2.653
18°	0.314	63°	1.100	108°	1.885	153°	2.670
19°	0.332	64°	1.117	109°	1.902	154°	2.688
20°	0.349	65°	1.134	110°	1.920	155°	2.705
21°	0.367	66°	1.152	111°	1.937	156°	2.723
22°	0.384	67°	1.169	112°	1.955	157°	2.740
23°	0.401	68°	1.187	113°	1.972	158°	2.758
24°	0.419	69°	1.204	114°	1.990	159°	2.775
25°	0.436	70°	1.222	115°	2.007	160°	2.793
26°	0.454	71°	1.239	116°	2.025	161°	2.810
27°	0.471	72°	1.257	117°	2.042	162°	2.827
28°	0.489	73°	1.274	118°	2.059	163°	2.845
29°	0.506	74°	1.292	119°	2.077	164°	2.862
30°	0.524	75°	1.309	120°	2.094	165°	2.880
31°	0.541	76°	1.326	121°	2.112	166°	2.897
32°	0.559	77°	1.344	122°	2.129	167°	2.915
33°	0.576	78°	1.361	123°	2.147	168°	2.932
34°	0.593	79°	1.379	124°	2.164	169°	2.950
35°	0.611	80°	1.396	125°	2.182	170°	2.967
36°	0.628	81°	1.414	126°	2.199	171°	2.985
37°	0.646	82°	1.431	127°	2.217	172°	3.002
38°	0.663	83°	1.449	128°	2.234	173°	3.019
39°	0.681	84°	1.466	129°	2.251	174°	3.037
40°	0.698	85°	1.484	130°	2.269	175°	3.054
41°	0.716	86°	1.501	131°	2.286	176°	3.072
42°	0.733	87°	1.518	132°	2.304	177°	3.089
43°	0.750	88°	1.536	133°	2.321	178°	3.107
44°	0.768	89°	1.553	134°	2.339	179°	3.124
45°	0.785	90°	1.571	135°	2.356	180°	3.142

Minutes as Decimals of One Degree or Seconds as Decimals of One Minute

1	.017	31	.517
2	.033	32	.533
3	.050	33	.550
4	.067	34	.567
5	.083	35	.583
6	.100	36	.600
7	.117	37	.617
8	.133	38	.633
9	.150	39	.650
10	.167	40	.667
11	.183	41	.683
12	.200	42	.700
13	.217	43	.717
14	.233	44	.733
15	.250	45	.750
16	.267	46	.767
17	.283	47	.783
18	.300	48	.800
19	.317	49	.817
20	.333	50	.833
21	.350	51	.850
22	.367	52	.867
23	.383	53	.883
24	.400	54	.900
25	.417	55	.917
26	.433	56	.933
27	.450	57	.950
28	.467	58	.967
29	.483	59	.983
30	.500	60	1.000

Growth Function, e^x
Decay Function, e^{-x}

x t	$\log_e x$ $\log_e t$	e^x OR e^t		e^{-x} OR e^{-t}	
		Value	$\log_{10}$	Value	$\log_{10}$
0.0	$-\infty$	1.000	0.000	1.000	0.000
0.1	−2.303	1.105	0.043	0.905	9.957
0.2	−1.610	1.221	0.087	0.819	9.913
0.3	−1.204	1.350	0.130	0.741	9.870
0.4	−0.916	1.492	0.174	0.670	9.826
0.5	−0.693	1.649	0.217	0.607	9.783
0.6	−0.511	1.822	0.261	0.549	9.739
0.7	−0.357	2.014	0.304	0 497	9.696
0.8	−0.223	2.226	0.347	0.449	9.653
0.9	−0.105	2.460	0.391	0.407	9.609
1.0	0.000	2.718	0.434	0.368	9.566
1.1	0.095	3.004	0.478	0.333	9.522
1.2	0.182	3.320	0.521	0.301	9.479
1.3	0.262	3.769	0.565	0.273	9.435
1.4	0.336	4.055	0.608	0.247	9.392
1.5	0.405	4.482	0.651	0.223	9.349
1.6	0.470	4.953	0.695	0.202	9.305
1.7	0.531	5.474	0.738	0.183	9.262
1.8	0.588	6.050	0.782	0.165	9.218
1.9	0.642	6.686	0.825	0.150	9.175
2.0	0.693	7.389	0.869	0.135	9.131
2.1	0.742	8.166	0.912	0.122	9.088
2.2	0.788	9.025	0.955	0.111	9.045
2.3	0.833	9.974	0.999	0.100	0.001
2.4	0.875	11.02	1.023	0.091	8.958
2.5	0.916	12.18	1.086	0.082	8.914
2.6	0.956	13.46	1.129	0.074	8.871
2.7	0.993	14.88	1.173	0.067	8.827
2.8	1.030	16.44	1.216	0.061	8.784
2.9	1.065	18.17	1.259	0.055	8.741
3.0	1.099	20.09	1.303	0.050	8.697
3.1	1.132	22.20	1.346	0.045	8.654
3.2	1.163	24.53	1.390	0.041	8.610
3.3	1.193	27.11	1.433	0.037	8.567
3.4	1.224	29.96	1.477	0.033	8.523
3.5	1.253	33.12	1.520	0.030	8.480
4.0	1.386	54.60	1.737	0.018	8.263
4.5	1.504	90.02	1.954	0.0111	8.046
5.0	1.609	148.4	2.171	0.0067	7.829
6.0	1.792	403.4	2.606	0.0025	7.394
7.0	1.946	1096.6	3.040	0.0009	6.960
8.0	2.079	2981.0	3.474	0.0003	6.526
9.0	2.197	8103.1	3.909	0.0001	6.091
10.0	2.303	22026.	4.343	0.0000	5.657

The accumulation of 1 at the end of n years. $r^n = (1+i)^n$.

Years.	1½%.	2%.	2½%.	3%.	4%.	5%.	6%.	Years.
1	1.0150	1.0200	1.0250	1.0300	1.0400	1.0500	1.0600	**1**
2	1.0302	1.0404	1.0506	1.0609	1.0816	1.1025	1.1236	**2**
3	1.0457	1.0612	1.0769	1.0927	1.1249	1.1576	1.1910	**3**
4	1.0614	1.0824	1.1038	1.1255	1.1699	1.2155	1.2625	**4**
5	1.0773	1.1041	1.1314	1.1593	1.2167	1.2763	1.3382	**5**
6	1.0934	1.1262	1.1597	1.1941	1.2653	1.3401	1.4185	**6**
7	1.1098	1.1487	1.1887	1.2299	1.3159	1.4071	1.5036	**7**
8	1.1265	1.1717	1.2184	1.2668	1.3686	1.4775	1.5938	**8**
9	1.1434	1.1951	1.2489	1.3048	1.4233	1.5513	1.6895	**9**
10	1.1605	1.2190	1.2801	1.3439	1.4802	1.6289	1.7908	**10**
11	1.1779	1.2434	1.3121	1.3842	1.5395	1.7103	1.8983	**11**
12	1.1956	1.2682	1.3449	1.4258	1.6010	1.7959	2.0122	**12**
13	1.2136	1.2936	1.3785	1.4685	1.6651	1.8856	2.1329	**13**
14	1.2318	1.3195	1.4130	1.5126	1.7317	1.9799	2.2609	**14**
15	1.2502	1.3459	1.4483	1.5580	1.8009	2.0789	2.3966	**15**
16	1.2690	1.3728	1.4845	1.6047	1.8730	2.1829	2.5404	**16**
17	1.2880	1.4002	1.5216	1.6528	1.9473	2.2920	2.6928	**17**
18	1.3073	1.4282	1.5597	1.7024	2.0258	2.4066	2.8543	**18**
19	1.3270	1.4568	1.5987	1.7535	2.1068	2.5270	3.0256	**19**
20	1.3469	1.4859	1.6386	1.8061	2.1911	2.6533	3.2071	**20**
21	1.3671	1.5157	1.6796	1.8603	2.2788	2.7860	3.3996	**21**
22	1.3876	1.5460	1.7216	1.9161	2.3699	2.9253	3.6035	**22**
23	1.4084	1.5769	1.7646	1.9736	2.4647	3.0715	3.8197	**23**
24	1.4295	1.6084	1.8087	2.0328	2.5633	3.2251	4.0489	**24**
25	1.4509	1.6406	1.8539	2.0938	2.6658	3.3864	4.2919	**25**
26	1.4727	1.6734	1.9003	2.1566	2.7725	3.5557	4.5494	**26**
27	1.4948	1.7069	1.9478	2.2213	2.8834	3.7335	4.8223	**27**
28	1.5172	1.7410	1.9965	2.2879	2.9987	3.9201	5.1117	**28**
29	1.5400	1.7758	2.0464	2.3566	3.1187	4.1161	5.4184	**29**
30	1.5631	1.8114	2.0976	2.4273	3.2434	4.3219	5.7435	**30**
31	1.5865	1.8476	2.1500	2.5001	3.3731	4.5380	6.0881	**31**
32	1.6103	1.8845	2.2038	2.5751	3.5081	4.7649	6.4534	**32**
33	1.6345	1.9222	2.2589	2.6523	3.6484	5.0032	6.8406	**33**
34	1.6590	1.9607	2.3153	2.7319	3.7943	5.2533	7.2510	**34**
35	1.6839	1.9999	2.3732	2.8139	3.9461	5.5160	7.6861	**35**
36	1.7091	2.0399	2.4325	2.8983	4.1039	5.7918	8.1473	**36**
37	1.7348	2.0807	2.4933	2.9852	4.2681	6.0814	8.6361	**37**
38	1.7608	2.1223	2.5557	3.0748	4.4388	6.3855	9.1543	**38**
39	1.7872	2.1647	2.6196	3.1670	4.6164	6.7048	9.7035	**39**
40	1.8140	2.2080	2.6851	3.2620	4.8010	7.0400	10.2857	**40**
50	2.1052	2.6916	3.4371	4.3839	7.1067	11.4674	18.4202	**50**
60	2.4432	3.2810	4.3998	5.8916	10.5196	18.6792	32.9877	**60**
70	2.8355	3.9996	5.6321	7.9178	15.5716	30.4264	59.0759	**70**
80	3.2907	4.8754	7.2096	10.6409	23.0498	49.6514	105.7960	**80**
90	3.8190	5.9431	9.2289	14.3005	34.1193	80.7304	189.4645	**90**
100	4.4321	7.2447	11.8137	19.2186	50.5050	131.5013	339.3021	**100**
Years.	1½%.	2%.	2½%.	3%.	4%.	5%.	6%.	Years.

The present value of 1 due in n years. $v^n = (1 + i)^{-n}$.

Years.	1½%.	2%.	2½%.	3%.	4%.	5%.	6%.	Years.
1	0.9852	0.9804	0.9756	0.9709	0.9615	0.9524	0.9434	1
2	0.9707	0.9612	0.9518	0.9426	0.9246	0.9070	0.8900	2
3	0.9563	0.9423	0.9286	0.9151	0.8890	0.8638	0.8396	3
4	0.9422	0.9238	0.9060	0.8885	0.8548	0.8227	0.7921	4
5	0.9283	0.9057	0.8839	0.8626	0.8219	0.7835	0.7473	5
6	0.9145	0.8880	0.8623	0.8375	0.7903	0.7462	0.7050	6
7	0.9010	0.8706	0.8413	0.8131	0.7599	0.7107	0.6651	7
8	0.8877	0.8535	0.8207	0.7894	0.7307	0.6768	0.6274	8
9	0.8746	0.8368	0.8007	0.7664	0.7026	0.6446	0.5919	9
10	0.8617	0.8203	0.7812	0.7441	0.6756	0.6139	0.5584	10
11	0.8489	0.8043	0.7621	0.7224	0.6496	0.5847	0.5268	11
12	0.8364	0.7885	0.7436	0.7014	0.6246	0.5568	0.4970	12
13	0.8240	0.7730	0.7254	0.6810	0.6006	0.5303	0.4688	13
14	0.8118	0.7579	0.7077	0.6611	0.5775	0.5051	0.4423	14
15	0.7999	0.7430	0.6905	0.6419	0.5553	0.4810	0.4173	15
16	0.7880	0.7284	0.6736	0.6232	0.5339	0.4581	0.3936	16
17	0.7764	0.7142	0.6572	0.6050	0.5134	0.4363	0.3714	17
18	0.7649	0.7002	0.6412	0.5874	0.4936	0.4155	0.3503	18
19	0.7536	0.6864	0.6255	0.5703	0.4746	0.3957	0.3305	19
20	0.7425	0.6730	0.6103	0.5537	0.4564	0.3769	0.3118	20
21	0.7315	0.6598	0.5954	0.5375	0.4388	0.3589	0.2942	21
22	0.7207	0.6468	0.5809	0.5219	0.4220	0.3418	0.2775	22
23	0.7100	0.6342	0.5667	0.5067	0.4057	0.3256	0.2618	23
24	0.6995	0.6217	0.5529	0.4919	0.3901	0.3101	0.2470	24
25	0.6892	0.6095	0.5394	0.4776	0.3751	0.2953	0.2330	25
26	0.6790	0.5976	0.5262	0.4637	0.3607	0.2812	0.2198	26
27	0.6690	0.5859	0.5134	0.4502	0.3468	0.2678	0.2074	27
28	0.6591	0.5744	0.5009	0.4371	0.3335	0.2551	0.1956	28
29	0.6494	0.5631	0.4887	0.4243	0.3207	0.2429	0.1846	29
30	0.6398	0.5521	0.4767	0.4120	0.3083	0.2314	0.1741	30
31	0.6303	0.5412	0.4651	0.4000	0.2965	0.2204	0.1643	31
32	0.6210	0.5306	0.4538	0.3883	0.2851	0.2099	0.1550	32
33	0.6118	0.5202	0.4427	0.3770	0.2741	0.1999	0.1462	33
34	0.6028	0.5100	0.4319	0.3660	0.2636	0.1904	0.1379	34
35	0.5939	0.5000	0.4214	0.3554	0.2534	0.1813	0.1301	35
36	0.5851	0.4902	0.4111	0.3450	0.2437	0.1727	0.1227	36
37	0.5764	0.4806	0.4011	0.3350	0.2343	0.1644	0.1158	37
38	0.5679	0.4712	0.3913	0.3252	0.2253	0.1566	0.1092	38
39	0.5595	0.4620	0.3817	0.3158	0.2166	0.1491	0.1031	39
40	0.5513	0.4529	0.3724	0.3066	0.2083	0.1420	0.0972	40
50	0.4750	0.3715	0.2909	0.2281	0.1407	0.0872	0.0543	50
60	0.4093	0.3048	0.2273	0.1697	0.0951	0.0535	0.0303	60
70	0.3527	0.2500	0.1776	0.1263	0.0642	0.0329	0.0169	70
80	0.3039	0.2051	0.1387	0.0940	0.0434	0.0202	0.0095	80
90	0.2619	0.1683	0.1084	0.0699	0.0293	0.0124	0.0053	90
100	0.2256	0.1380	0.0846	0.0520	0.0198	0.0076	0.0029	100
Years.	1½%.	2%.	2½%.	3%.	4%.	5%.	6%.	Years.

The accumulation of an annuity of 1 per annum at the end of n years.

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}.$$

Years.	1½%.	2%.	2½%.	3%.	4%.	5%.	6%.	Years.
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	**1**
2	2.0150	2.0200	2.0250	2.0300	2.0400	2.0500	2.0600	**2**
3	3.0452	3.0604	3.0756	3.0909	3.1216	3.1525	3.1836	**3**
4	4.0909	4.1216	4.1525	4.1836	4.2465	4.3101	4.3746	**4**
5	5.1523	5.2040	5.2563	5.3091	5.4163	5.5256	5.6371	**5**
6	6.2296	6.3081	6.3877	6.4684	6.6330	6.8019	6.9753	**6**
7	7.3230	7.4343	7.5474	7.6625	7.8983	8.1420	8.3938	**7**
8	8.4328	8.5830	8.7361	8.8923	9.2142	9.5491	9.8975	**8**
9	9.5593	9.7546	9.9545	10.1591	10.5828	11.0266	11.4913	**9**
10	10.7027	10.9497	11.2034	11.4638	12.0061	12.5779	13.1808	**10**
11	11.8633	12.1687	12.4835	12.8078	13.4864	14.2068	14.9716	**11**
12	13.0412	13.4121	13.7956	14.1920	15.0258	15.9171	16.8699	**12**
13	14.2368	14.6803	15.1404	15.6178	16.6268	17.7130	18.8821	**13**
14	15.4504	15.9739	16.5190	17.0863	18.2919	19.5986	21.0151	**14**
15	16.6821	17.2934	17.9319	18.5989	20.0236	21.5786	23.2760	**15**
16	17.9324	18.6393	19.3802	20.1569	21.8245	23.6575	25.6725	**16**
17	19.2014	20.0121	20.8647	21.7616	23.6975	25.8404	28.2129	**17**
18	20.4894	21.4123	22.3863	23.4144	25.6454	28.1324	30.9057	**18**
19	21.7967	22.8406	23.9460	25.1169	27.6712	30.5390	33.7600	**19**
20	23.1237	24.2974	25.5447	26.8704	29.7781	33.0660	36.7856	**20**
21	24.4705	25.7833	27.1833	28.6765	31.9692	35.7193	39.9927	**21**
22	25.8376	27.2990	28.8629	30.5368	34.2480	38.5052	43.3923	**22**
23	27.2251	28.8450	30.5844	32.4529	36.6179	41.4305	46.9958	**23**
24	28.6335	30.4219	32.3490	34.4265	39.0826	44.5020	50.8156	**24**
25	30.0630	32.0303	34.1578	36.4593	41.6459	47.7271	54.8645	**25**
26	31.5140	33.6709	36.0117	38.5530	44.3117	51.1135	59.1564	**26**
27	32.9867	35.3443	37.9120	40.7096	47.0842	54.6691	63.7058	**27**
28	34.4815	37.0512	39.8598	42.9309	49.9676	58.4026	68.5281	**28**
29	35.9987	38.7922	41.8563	45.2189	52.9663	62.3227	73.6398	**29**
30	37.5387	40.5681	43.9027	47.5754	56.0849	66.4389	79.0582	**30**
31	39.1018	42.3794	46.0003	50.0027	59.3283	70.7608	84.8017	**31**
32	40.6883	44.2270	48.1503	52.5028	62.7015	75.2988	90.8898	**32**
33	42.2986	46.1116	50.3540	55.0778	66.2095	80.0638	97.3432	**33**
34	43.9331	48.0338	52.6129	57.7302	69.8579	85.0670	104.1838	**34**
35	45.5921	49.9945	54.9282	60.4620	73.6522	90.3203	111.4348	**35**
36	47.2760	51.9944	57.3014	63.2759	77.5983	95.8363	119.1209	**36**
37	48.9851	54.0343	59.7339	66.1742	81.7022	101.6281	127.2681	**37**
38	50.7199	56.1149	62.2273	69.1594	85.9703	107.7095	135.9042	**38**
39	52.4807	58.2372	64.7830	72.2342	90.4092	114.0950	145.0585	**39**
40	54.2679	60.4020	67.4026	75.4013	95.0255	120.7998	154.7620	**40**
50	73.6828	84.5794	97.4843	112.7969	152.6671	209.3480	290.3359	**50**
60	96.2147	114.0515	135.9916	163.0534	237.9907	353.5837	533.1282	**60**
70	122.3638	149.9779	185.2841	230.5941	364.2905	588.5285	967.9322	**70**
80	152.7109	193.7720	248.3827	321.3630	551.2450	971.2288	1746.5999	**80**
90	187.9299	247.1567	329.1543	443.3489	827.9833	1594.6073	3141.0752	**90**
100	228.8030	312.2323	432.5487	607.2877	1237.6237	2610.0252	5638.3681	**100**
Years.	1½%.	2%.	2½%.	3%	4%.	5%.	6%.	Years.

The present value of an annuity of 1 for n years,

$$a_{\overline{n}|} = \frac{1 - v^n}{i}.$$

Years.	1½%.	2%.	2½%.	3%.	4%.	5%.	6%.	Years.
1	0.9852	0.9804	0.9756	0.9709	0.9615	0.9524	0.9434	**1**
2	1.9559	1.9416	1.9274	1.9135	1.8861	1.8594	1.8334	**2**
3	2.9122	2.8839	2.8560	2.8286	2.7751	2.7232	2.6730	**3**
4	3.8544	3.8077	3.7620	3.7171	3.6299	3.5460	3.4651	**4**
5	4.7827	4.7135	4.6458	4.5797	4.4518	4.3295	4.2124	**5**
6	5.6972	5.6014	5.5081	5.4172	5.2421	5.0757	4.9173	**6**
7	6.5982	6.4720	6.3494	6.2303	6.0021	5.7864	5.5824	**7**
8	7.4859	7.3255	7.1701	7.0197	6.7327	6.4632	6.2098	**8**
9	8.3605	8.1622	7.9709	7.7861	7.4353	7.1078	6.8017	**9**
10	9.2222	8.9826	8.7521	8.5302	8.1109	7.7217	7.3601	**10**
11	10.0711	9.7868	9.5142	9.2526	8.7605	8.3064	7.8869	**11**
12	10.9075	10.5753	10.2578	9.9540	9.3851	8.8633	8.3838	**12**
13	11.7315	11.3484	10.9832	10.6350	9.9856	9.3936	8.8527	**13**
14	12.5434	12.1062	11.6909	11.2961	10.5631	9.8986	9.2950	**14**
15	13.3432	12.8493	12.3814	11.9379	11.1184	10.3797	9.7122	**15**
16	14.1313	13.5777	13.0550	12.5611	11.6523	10.8378	10.1059	**16**
17	14.9076	14.2919	13.7122	13.1661	12.1657	11.2741	10.4773	**17**
18	15.6726	14.9920	14.3534	13.7535	12.6593	11.6896	10.8276	**18**
19	16.4262	15.6785	14.9789	14.3238	13.1340	12.0853	11.1581	**19**
20	17.1686	16.3514	15.5892	14.8775	13.5903	12.4622	11.4699	**20**
21	17.9001	17.0112	16.1845	15.4150	14.0292	12.8212	11.7641	**21**
22	18.6208	17.6580	16.7654	15.9369	14.4511	13.1630	12.0416	**22**
23	19.3309	18.2922	17.3321	16.4436	14.8568	13.4886	12.3034	**23**
24	20.0304	18.9139	17.8850	16.9355	15.2470	13.7986	12.5504	**24**
25	20.7196	19.5235	18.4244	17.4131	15.6221	14.0940	12.7834	**25**
26	21.3986	20.1210	18.9506	17.8768	15.9828	14.3752	13.0032	**26**
27	22.0676	20.7069	19.4640	18.3270	16.3296	14.6430	13.2105	**27**
28	22.7267	21.2813	19.9649	18.7641	16.6631	14.8981	13.4062	**28**
29	23.3761	21.8444	20.4535	19.1885	16.9837	15.1411	13.5907	**29**
30	24.0158	22.3965	20.9303	19.6004	17.2920	15.3725	13.7648	**30**
31	24.6461	22.9377	21.3954	20.0004	17.5885	15.5928	13.9291	**31**
32	25.2671	23.4683	21.8492	20.3888	17.8736	15.8027	14.0840	**32**
33	25.8790	23.9886	22.2919	20.7658	18.1476	16.0025	14.2302	**33**
34	26.4817	24.4986	22.7238	21.1318	18.4112	16.1929	14.3681	**34**
35	27.0756	24.9986	23.1452	21.4872	18.6646	16.3742	14.4982	**35**
36	27.6607	25.4888	23.5563	21.8323	18.9083	16.5469	14.6210	**36**
37	28.2371	25.9695	23.9573	22.1672	19.1426	16.7113	14.7368	**37**
38	28.8051	26.4406	24.3486	22.4925	19.3679	16.8679	14.8460	**38**
39	29.3646	26.9026	24.7303	22.8082	19.5845	17.0170	14.9491	**39**
40	29.9158	27.3555	25.1028	23.1148	19.7928	17.1591	15.0463	**40**
50	34.9997	31.4236	28.3623	25.7298	21.4822	18.2559	15.7619	**50**
60	39.3803	34.7609	30.9087	27.6756	22.6235	18.9293	16.1614	**60**
70	43.1549	37.4987	32.8979	29.1234	23.3945	19.3427	16.3845	**70**
80	46.4073	39.7445	34.4518	30.2008	23.9154	19.5965	16.5091	**80**
90	49.2099	41.5869	35.6658	31.0024	24.2673	19.7523	16.5787	**90**
100	51.6247	43.0983	36.6141	31.5989	24.5050	19.8479	16.6175	**100**
Years.	1½%.	2%.	2½%.	3%.	4%.	5%.	6%.	Years.

The annual sinking fund which will accumulate to 1 at the end of n years.

$$\frac{1}{s_{\overline{n|}}} = \frac{i}{(1+i)^n - 1}. \quad \text{To obtain } \frac{1}{a_{\overline{n|}}} \text{ add } i, \text{ since } \frac{1}{a_{\overline{n|}}} = \frac{1}{s_{\overline{n|}}} + i.$$

Years.	1½%.	2%.	2½%.	3%.	4%.	5%.	6%.	Years.
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	**1**
2	0.4963	0.4950	0.4938	0.4926	0.4902	0.4878	0.4854	**2**
3	0.3284	0.3268	0.3251	0.3235	0.3203	0.3172	0.3141	**3**
4	0.2444	0.2426	0.2408	0.2390	0.2355	0.2320	0.2286	**4**
5	0.1941	0.1922	0.1902	0.1884	0.1846	0.1810	0.1774	**5**
6	0.1605	0.1585	0.1566	0.1546	0.1508	0.1470	0.1434	**6**
7	0.1366	0.1345	0.1325	0.1305	0.1266	0.1228	0.1191	**7**
8	0.1186	0.1165	0.1145	0.1125	0.1085	0.1047	0.1010	**8**
9	0.1046	0.1025	0.1005	0.0984	0.0945	0.0907	0.0870	**9**
10	0.0934	0.0913	0.0893	0.0872	0.0833	0.0795	0.0759	**10**
11	0.0843	0.0822	0.0801	0.0781	0.0741	0.0704	0.0668	**11**
12	0.0767	0.0746	0.0725	0.0705	0.0666	0.0628	0.0593	**12**
13	0.0702	0.0681	0.0660	0.0640	0.0601	0.0565	0.0530	**13**
14	0.0647	0.0626	0.0605	0.0585	0.0547	0.0510	0.0476	**14**
15	0.0599	0.0578	0.0558	0.0538	0.0499	0.0463	0.0430	**15**
16	0.0558	0.0537	0.0516	0.0496	0.0458	0.0423	0.0390	**16**
17	0.0521	0.0500	0.0479	0.0460	0.0422	0.0387	0.0354	**17**
18	0.0488	0.0467	0.0447	0.0427	0.0390	0.0355	0.0324	**18**
19	0.0459	0.0438	0.0418	0.0398	0.0361	0.0327	0.0296	**19**
20	0.0432	0.0412	0.0391	0.0372	0.0336	0.0302	0.0272	**20**
21	0.0409	0.0388	0.0368	0.0349	0.0313	0.0280	0.0250	**21**
22	0.0387	0.0366	0.0346	0.0327	0.0292	0.0260	0.0230	**22**
23	0.0367	0.0347	0.0327	0.0308	0.0273	0.0241	0.0213	**23**
24	0.0349	0.0329	0.0309	0.0290	0.0256	0.0225	0.0197	**24**
25	0.0333	0.0312	0.0293	0.0274	0.0240	0.0210	0.0182	**25**
26	0.0317	0.0297	0.0278	0.0259	0.0226	0.0196	0.0169	**26**
27	0.0303	0.0283	0.0264	0.0246	0.0212	0.0183	0.0157	**27**
28	0.0290	0.0270	0.0251	0.0233	0.0200	0.0171	0.0146	**28**
29	0.0278	0.0258	0.0239	0.0221	0.0189	0.0160	0.0136	**29**
30	0.0266	0.0246	0.0228	0.0210	0.0178	0.0151	0.0126	**30**
31	0.0256	0.0236	0.0217	0.0200	0.0169	0.0141	0.0118	**31**
32	0.0246	0.0226	0.0208	0.0190	0.0159	0.0133	0.0110	**32**
33	0.0236	0.0217	0.0199	0.0182	0.0151	0.0125	0.0103	**33**
34	0.0228	0.0208	0.0190	0.0173	0.0143	0.0118	0.0096	**34**
35	0.0219	0.0200	0.0182	0.0165	0.0136	0.0111	0.0090	**35**
36	0.0212	0.0192	0.0175	0.0158	0.0129	0.0104	0.0084	**36**
37	0.0204	0.0185	0.0167	0.0151	0.0122	0.0098	0.0079	**37**
38	0.0197	0.0178	0.0161	0.0145	0.0116	0.0093	0.0074	**38**
39	0.0191	0.0172	0.0154	0.0138	0.0111	0.0088	0.0069	**39**
40	0.0184	0.0166	0.0148	0.0133	0.0105	0.0083	0.0065	**40**
50	0.0136	0.0118	0.0103	0.0089	0.0066	0.0048	0.0034	**50**
60	0.0104	0.0088	0.0074	0.0061	0.0042	0.0028	0.0019	**60**
70	0.0182	0.0067	0.0054	0.0043	0.0027	0.0017	0.0010	**70**
80	0.0065	0.0052	0.0040	0.0031	0.0018	0.0010	0.0006	**80**
90	0.0053	0.0040	0.0030	0.0023	0.00121	0.00063	0.00032	**90**
100	0.0044	0.0032	0.0023	0.0016	0.00081	0.00038	0.00018	**100**
Years.	1½%.	2%.	2½%.	3%.	4%.	5%.	6%.	Years.

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