A NEW PERSPECTIVE ON MATHEMATICS EDUCATION COMING FROM HISTORY: THE EXAMPLE OF INTEGRAL CALCULUS

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Abstract

This research deals with a possible use of history of mathematics in mathematics education. In particular, history can be a fundamental element for the introduction of the concept of integral through a problem-centred and intuitive approach. Therefore, what follows is dedicated to the teaching of mathematics in the last years of secondary schools, where infinitesimal calculus is addressed. The thesis here proposed is that the resort to Archimedes' use of exhaustion method and to Newton's initial lemmas expounded in his Principia Mathematica *are useful means to reach a genetic comprehension of the concept of integral. Hence, two demonstrations by Archimedes and two lemmas by Newton are used to prove such thesis. A further idea here proposed is that history of mathematics can be of help for an interdisciplinary education.*
Keywords: *interdisciplinary education, mathematics education, science history, secondary schools*

Introduction

This research concerns the possible use of the history of mathematics within mathematical education in the last two years of the secondary schools because it regards the teaching of integral calculus.

History of mathematics can be a useful element in mathematics education because:

1) starting from history, the pupils can understand the conceptual origin of the mathematical concepts and to realize that they have been created to solve «concrete» mathematical problems. The formalization of the concepts is a successive step, when the numbers and the width of problems increased, and the necessity of precise definitions became necessary. However, if the teaching begins with the formal apparatus there is the risk that the learners do not guess its scope and utility and mistake the real meaning of the formalization.

2) A historical approach is useful for the students to realize the human aspect of mathematics. Mathematics exists thank to the hard work of the mathematicians. It is a human activity in which before reaching the final presentation of a concept or solution of a problem, a long history of failures, attempts and improvements has been needed. Mathematics, if appropriately presented is neither sterile nor mechanic, though, obviously, several technicalities have to be learned.

3) Nowadays «interdisciplinarity» is a common place, but to conceive an, at

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least partially, interdisciplinary education is not easy. History of mathematics can be of help. We will speak of Archimedes and Newton in reference to the birth of integral calculus. It might be possible to conceive an educational itinerary in which the teachers of history, physics, mathematics, and history of philosophy (where this discipline is taught) collaborate. For example, in the programme of history, it would be possible to stress the aspects connected to history of science and in the programme of physics the genetic problems which determined the introduction of certain concepts might be highlighted. In this way, an education based both on the connection of different subjects and on the understanding of the conceptual grounds of each specific discipline could be implemented.

Area of Curvilinear Figures and Integral Calculus

The example I am proposing concerns the calculation of curvilinear and mystilinear areas. Today the integral calculus and the concept of definite integral is used. The integral calculus is taught in the secondary schools after the formal introduction of the concepts of function, of limit and of the differential calculus. I am not claiming this educative methodology to be incorrect. Rather, a different one will be proposed for the teachers to have diverse options to follow in the difficult enterprise to make mathematics an interesting subject for most learners. The teachers themselves will be able to decide, according to pupils' level and propensities, the most appropriate approach.

Thus, before introducing the formal concepts of infinitesimal calculus, it would be possible to proceed like this: introduce only the concept of function. After that, analyse the way in which the problem of the calculation of curvilinear areas was faced by Archimedes (287-212 B.C.). This will offer the possibility:

a) to understand what the concept of limit is and to grasp that it is necessary while dealing with the areas of curvilinear figure.
b) To guess the profound meaning of the proves *ad absurdum* as well as the nuances and the subtleties behind such difficult problems as those proposed.
c) to comprehend the greatness and limits of Archimedes' approach as well as the necessity to overcome such an approach if you intend to achieve a general treatment of curvilinear areas. Therefore, Newton's concept of integral will be introduced.
d) Finally, after that the learners will have understood all the conceptual and computational bases of the infinitesimal calculus, a more formal approach will be proposed. But now the learners will fully grasp the conceptual meaning of such a formalization.

Archimedes and the Exhaustion Method: First Example

The procedures of Archimedes are extremely interesting because they seem an «anticipation» of the integral calculus, at least from a conceptual, if not operative, standpoint.

In fact, Archimedes invented two methods correlated to the calculations of curvilinear surfaces and volumes: the method of barycentres and the method of

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exhaustion. We will focus on two examples of the latter. The method of exhaustion was invented by Eudoxus of Cnidus (first half of the 4th century BC) and it is also used in the last books of *Euclid*'s Elements (Euclid 2008, Book XII, Propositions 2, 5, 10, 11, 12, 18, pp. 473-475, 480-481, 486-495, 503-504). However, Archimedes was the true master of this method.

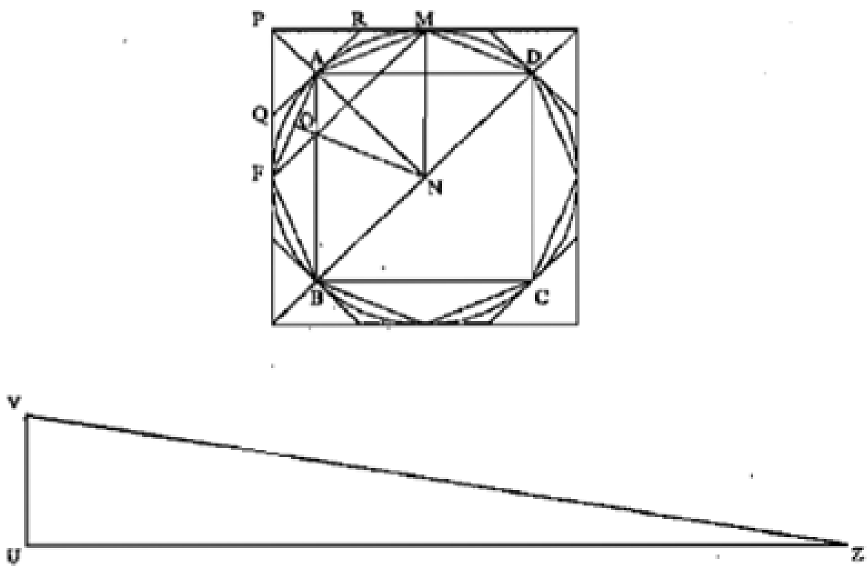
It works like this: you want to prove that two quantities *A* and *B* (areas or volume of two figures) are equal. Suppose they are not and be $A>B$. You construct a series of figures which approximate the figure *A* by defect and among them you find one whose difference from *A* is less than the supposed difference A-B. Then, you are able to prove that this figure should be, at the same time, greater and less than *B*, which is absurd.

Afterwards, you suppose $A<B$ and, in a similar manner, you show a figure which should be bigger and less than *B*. Absurd. Since *A* cannot be either greater of less than *B*, it is equal.

The area of the circle. From the first years of the secondary schools, the pupils know that, given a circle whose radius is *r*, its surface is πr^2. However, if you ask them to prove this statement, though in an intuitive manner, probably no one will answer the question, because the answer is not easy (try to do this «experiment»).

The first mathematician to solve such a problem was Archimedes in his work *Measurement of the circle* (Archimedes 2010, pp. 91-98). He developed the following brilliant reasoning to prove that «Each circle has the same surface as the right triangle *E* of which a leg (cathetus) is equal to the radius and the other leg to the circumference».

Figure 1
The Diagram Drawn by Archimedes to Determine the Circle's Area



Note: Adapted from Archimede 1974, p. 226. The editor uses Latin rather than Greek letters to indicate points

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Archimedes proceeds like this (Fig. 1): suppose the circle *ABCD* to have an area *S* greater than *E*. It is possible to inscribe the square *ABCD* of area *Q* in the circle. Suppose *Q*<*E* or, as Archimedes would say, the area of the four circular segments *AD, DC, CB, BA* is bigger than the difference by which the area of the circle exceeds the area of the triangle *E*. Bisect the four arcs of circumference *AD, DC, CB, BA*. Be *F* and *M* two of such points of bisection. Let us construct the regular octagon *FA, AM, MD, …*, whose area is *O.* Suppose *O*>*E*, or, as Archimedes wrote, that the sum of the areas of the eight circular segments *FA*, *AM*, *MD*, … is less than the difference by which the area of the circle exceeds the area of the triangle *E* (if you suppose *O*<*E*, it is possible to continue bisecting the arcs of the circumference until reaching a polygon of 2^n sides whose area is greater than *E*, but less than *S*). From the centre *N* of the circle, draw the perpendicular *NO* to the side *AF* of the octagon. Such a segment is less than the altitude of the triangle, which is equal to the radius. The octagon's perimeter is less of the rectified circumference.

Therefore, the octagon's area is less than *E*. But this is absurd because we have supposed *O*>*E.* Therefore, the area of the circle cannot be greater than that of the triangle.

Through a similar reasoning it is possible to prove that the area of the circle cannot be less than that of the triangle.

Commentary to use in an educational perspective. The just expounded demonstration gives rise to several comments which are very instructive. Some of them concern the internal logic of the proof; others are relative to the concepts used by Archimedes in comparison to modern notions. Let us begin with the first kind of comments:

a) In the proof there is a critical point: Archimedes seems to give for granted that the perimeter of the polygon inscribed in the circumference is less than the circumference itself. In fact, this problem does not exist because in the assumptions of his work *On the sphere and cylinder* Archimedes writes as first Assumption: "Of all lines which have the same extremities the straight line is the least" (Archimedes 2010, p. 3).

b) Further possible problem: it might happen that the difference between the area of the circle and that of the inscribed polygons diminishes but without tending to 0. Archimedes considered this problem, too. In propositions 3-6 of the first book of *On the sphere and cylinder,* he proved to be possible to inscribe in or circumscribe to a circle a series of regular polygons whose area has any ratio (or difference) with the area of the circle (Archimedes 2010, pp. 6-10).

Now let us come to commentaries which are useful not only to compare the modern concepts with those used by Archimedes, but also to introduce the learners to such concepts showing that their necessity derives from "concrete" mathematical problems as the one we are analysing. I am specifically referring to the notion of limit: it is usually introduced as the limit of a function. Given a function f:R→R and a point x_0 of its domain you say that $f(x) = l$, if for any real number $\varepsilon > 0$, a real number δ exists such that, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$. This definition is easily adaptable to the cases in which x tends to infinity or l is infinite.

Now, what is present in Archimedes' proof is the limit of a series of figure of which we consider the function-area, which can be reduced to a real function. To be more precise, Archimedes - to transcribe his idea in modern terms - considers as domain the perimeters of the regular polygons whose number of sides is of the form 2^n, also including the circumference – let us name *P* such a set – and, after that, the areas of each of such figures, also including the area of the circle – let us name S such a set – regarded as functions of the perimeters. The perimeter and the areas are real numbers; therefore, the function-area f can be represented as : $R \rightarrow R$. Be P_0 the length of the circumference and P the perimeter of any inscribed polygon, so that $f(P_0)$ is the area of the circle and $f(P)$ is the area of the regular polygon whose perimeter is P. Thus, in the reasoning of Archimedes it is implied that, for any real number $\varepsilon > 0$, a real number δ exists such that, if $|P - P_0| < \delta$, then $|f(P) - f(P_0)| < \varepsilon$. This is a simple transcription in a modern and relatively formalised language of what exists in Archimedes. There is no interpretative stretch. Hence, from an educative point of view, it is to highlight that this easy proof offered by Archimedes seems a natural way to introduce the concept of limit of a function; before in an intuitive manner – as the teacher might simply say that, when the difference between the perimeters of the inscribed polygons and the circumference becomes smaller and smaller, the same happens with the differences between the area of the polygons and that of the circle – and, afterwards, in a more formalised way.

On the other hand, it is appropriate to show the pupils a characteristic of Archimedes proof: the concept of limit is not used to develop a calculation through which to compute by means of formal steps, which might be repeatable, *mutatis mutandis*, to solve other problems. Rather, it is used to develop an *ad absurdum* reasoning which works because of the specific characteristics of the regular polygons inscribed in a circle. Therefore, while, in the genial Archimedes' procedure the general concept of limit is present, a general method to calculate the areas of, at least, a class of curvilinear figures is missing. As we will see, there are "ingredients" and ideas which can be used in other cases, but a general method of areas calculus lacks. Therefore, the students will perfectly understand the need to develop a method more general than Archimedes'.

An interesting and very instructive consideration concerns the way in which Archimedes used the ad *absurdum reasoning*. In this kind of proves, the assumption the theorem to be false (in particular, the assumption that the area of the triangle E is less than that of the circle) implies a proposition p and its negation not-p to be true. In our case the proposition p is: «the area of the octagon is bigger that the area of the triangle». The proposition not-p is not a proposition which is true because it is an axiom or a theorem previously demonstrated, as it happens usually in the *ad absurdum* proves. Rather, the propositions p and not-p are both deduced during the argumentation. In a sense, in this proof, and in general, in the proves for exhaustion, the hypothesis the theorem to be false destroys itself. There is not a contradiction with an already established truth. This is a particular of the exhaustion method worth being stressed to the pupils.

Archimedes and the Exhaustion Method: Second Example

It is appropriate to present a further example of the way in which the exhaustion method can be applied because the conceptual and didactical considerations which can be drawn hold a remarkable educative value.

The theorem whose prove will be proposed concerns one of the most famous results obtained by Archimedes: the surface of every sphere is the quadruple of its great circle. Namely, if r is the radius of the sphere, its surface is $4\pi r^2$. This statement is proved by Archimedes in *On the Sphere and Cylinder* as Proposition 33 of the first book (Archimedes 2010, pp. 39-41). Five other theorems proved by Archimedes in the same

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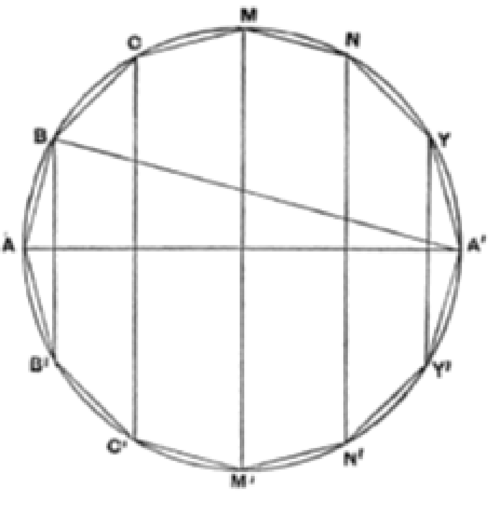
work are necessary along the demonstration. For convenience of the reader, I mention them:

1) Theorem I: Given two unequal magnitudes A and B, with $A>B$, it is possible to find two segments, whose lengths are resp. S and s such that $S: s < A: B$. (*On the Sphere and Cylinder*, I, 2, Archimedes 2010, p. 5).
2) Theorem II: Given two unequal magnitudes A and B, with $A>B$ and a circle, it is possible to inscribe a regular polygon of side l and to circumscribe a regular polygon of side L, so that $L: l < A: B$. (*Ibidem*, I,3, pp. 6-7).
3) Theorem III: Consider a regular polygon *ABCM*... whose number of sides is of the form $4n$, being n a natural number, inscribed in a great circle of a sphere. Rotate this polygon around the sphere's diameter *AA'*, then the area of the figure obtained by a complete rotation is less than four times the area of the sphere's great circle (*Ibidem*, I, 25, pp. 32-33).

Figure 2

The Diagram Used by Archimedes to Prove Proposition 25, book I, On the Sphere and Cylinder

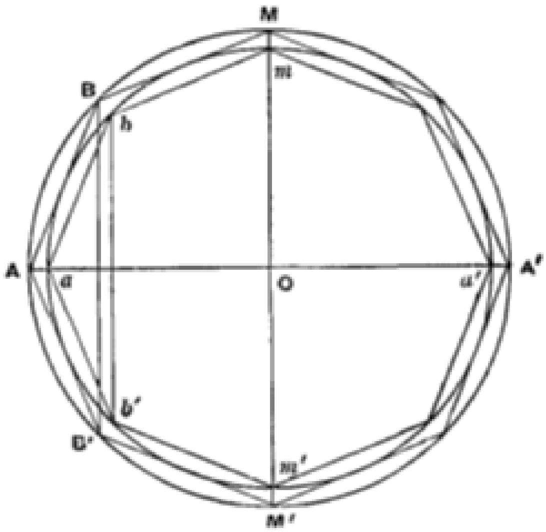


Note: Retrieved from Archimedes 2010, p. 32. The editor uses Latin rather than Greek letters to indicate points.

4) Theorem IV: If the regular polygon of $4n$ sides is circumscribed to a great circle of the sphere, and it rotates around a sphere's diameter in a manner similar as the polygon of the previous theorem, the area of the figure obtained by a complete rotation is bigger than the area of the sphere (*Ibidem*, I, 28, p. 35-36).
5) Theorem V: Consider the figure of rotation constructed in the two previous theorems: the areas of the circumscribed and of the inscribed figures are as the squares of the sides' ratio of the polygon circumscribed to the great circle of the sphere to that inscribed. The volumes of the two figures are as the cubes of the sides (*Ibidem*, I, 32, pp. 38-39)

Figure 3

The Diagram Used by Archimedes to Prove Proposition 32, Book I, On the Sphere and Cylinder



Note: Retrieved from Archimedes 2010, p. 38. The editor uses Latin rather than Greek letters to indicate points.

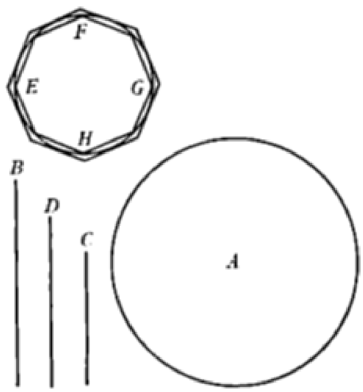
Let us not see Archimedes' proof of Proposition 33: be *S* a sphere and *A* a circle equivalent to four time the sphere's great circle. It is required to prove that the surfaces *s*(*S*) of *S* and *s*(*A*) of *A* are equal. Suppose, *ad absurdum*, *s*(*S*) > *s*(*A*). Under this hypothesis, it is possible to choose two segments *B* and *C* such that $\frac{B}{C} < \frac{s(S)}{s(A)}$ (Theorem I). Be *D* the mean proportional between *B* and *C*, namely *B*: *C* = *D*: *C*. Be the sphere cut by a plane through its centre along a great circle *EFGH*. Inscribe in and circumscribe to the sphere two similar polygons with a number 4*n* of sides, under the condition that between the side *P* of that circumscribed and the side *p* of that inscribed the relation *P*: *p* < *B*: *D* (Theorem II) holds. Consider the figures generated as in Theorem III and IV. Indicate the surface of that inscribed as *s*(*I*) and the surface of that circumscribed as *s*(*C*). Then *s*(*C*)*:* *s*(*I*) < *s*(*S*)*:* *s*(*A*), which can be deduced by the following series of steps:

I.	$s(C): s(I) = P^2: p^2$	(Theorem V).
II.	$P: p < B: D$	(Assumption).
III.	$B:C < s(S): s(A)$	(Assumption).
IV.	$B: D = D: C$	(Assumption).
V.	$B: C = B^2: D^2$	(Direct consequence of IV).
VI.	$s(S): s(A) > B^2: D^2$	(From III and V).
VII.	$B^2: D^2 > P^2: p^2$	(From II).
VIII.	$s(C): s(I) < s(S):s(A)$	(From I, VI and VII).

However, this is impossible because $s(C) > s(S)$ (Theorem IV) and $s(I) < s(A)$. This absurd conclusion implies that the area of the sphere cannot be bigger than four times the area of its great circle. Through an analogous reasoning, Archimedes proved that it cannot be even smaller. Thence, it is equal.

Figure 4

The Diagram Used by Archimedes to Prove Proposition 33, Book I, On the Sphere and Cylinder



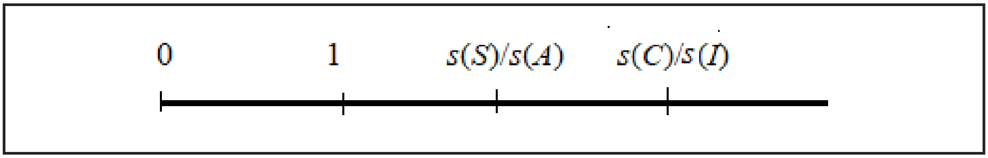
Note: Retrieved from Archimede 2004, p. 154. The editor uses Latin rather than Greek letters to indicate points

Commentary to use in an educational perspective. This proof offers many reasons of interest which can represent useful ideas to develop from an educative standpoint. They concern the internal logic of the proof and its level of generality. Let us begin with the first question.

Logic of the proof: in the demonstration concerning the area of the circle, Archimedes used the difference between the area of the circle itself and the area of the inscribed (circumscribed) regular polygons of 2^n sides. The area of those inscribed (the case we have analysed) approximate with an arbitrary exactness the area of the circle.

In this second proof, the idea is to resort to ratios rather than to differences. One might say that the space between the areas of the circumscribed and the inscribed figures include the area of the sphere the more precisely the more the number of the sides of the polygons inscribed and circumscribed to a great circle of the sphere increases. In particular, to prove *ab absurdum* that the ratio $\frac{s(S)}{s(A)}$ cannot be greater than 1, Archimedes used the fact that the ratio $\frac{s(C)}{s(I)} > 1$ and that $\frac{s(C)}{s(I)} > \frac{s(S)}{s(A)}$. Thus, his *ad absurdum* construction can be represented like this:

Figure 5
Diagram which Summarizes Archimedes' Reasoning. Adapted from Archimede 1974, p. 152



During the prove he demonstrated that, supposing $\frac{s(S)}{s(A)} > 1$, you reach a situation where $\frac{s(C)}{s(I)} < \frac{s(S)}{s(A)}$, which is impossible.

Where is the concept of limit here involved? It is involved in the construction which exploits theorem III because, if you indicate by 1 + ε the ratio *A/B*, being an arbitrary real number (but, of course, we are interested in the case in which ε is "very small), this theorem states that *L*:*l*<1 + ε, namely that it tends to 1, which, in its turn, implies that the ratio between the areas $\frac{s(C)}{s(I)}$ can differ from 1 by any arbitrary small value. This allows Archimedes to insert, *ad absurdum*, $\frac{s(C)}{s(I)}$ between 1 and $\frac{s(S)}{s(A)}$. Thus, the concept of limit is implied in this demonstration and is in every proof by exhaustion.

We have seen two different applications of the exhaustion method, one based on the difference and one based on the ratio between figures. The latter application is more complex than the former because in the former only two figures are used in the proof (the circle and the series of the inscribed polygons, in the part of the proof here proposed; the circle and the series of the circumscribed polygons in the other part), whereas in the latter three figures are used at the same time: the series of circumscribed figures, the sphere, and the series of inscribed figures.

The first form of the exhaustion method can be called *by approximation*, the second form *by compression based on a ratio*. There is a third form, namely the *compression based on a difference*, but it is not necessary to enter further details for the aims of this research (on this subject see the seminal, pp. 130-133).

These logical considerations here presented are rather subtle and can be extremely useful to develop the logical capabilities of the learners. Thus, it is advisable that the teachers develop them in front of the learners. This will be instructive. Let us move on some more general reflections.

General reflections on Archimedes' method. Archimedes proved several theorems by exhaustion. The two demonstrations we have seen are paradigmatic of some general features which connote all the proves by exhaustion:

a) The concept of limit is present and well conceived. Archimedes guessed that the only way to calculate, in general, the areas and the volumes of curvilinear or mystilinear figures was to approximate them with an infinite series of rectilinear figures, or, at least, of figures whose areas or volumes were already known.
b) What expounded in the item a) is sufficient to define that of Archimedes a method, in the sense that each prove by exhaustion is based on some leading ideas which are always the same. On the other hand, the exhaustion method is

not a systematic procedure which can be applied always in the same manner to calculate areas and volumes. For every figure Archimedes analysed, some specific theorems and constructions are necessary. They are not logically deducible from the theorems and constructions applied in different cases. Thus, the exhaustion method is not a mechanical procedure based on constant and invariable steps. Though a series of general ideas exist, each case requires some different specific reasonings.

c) Furthermore, Archimedes, as it is typical of the tradition of Greek mathematics, does not offer the numerical value of the required area or volume, but proves the identity of an unknown area with a known area. It is, however, to point out that in no works of the Greeks mathematics you find explicit formulas such as "area of a triangle is equal to basis by altitude". In Euclid's *Elements* there are propositions (See, for instance Euclid 2008, I, 36, pp. 37-38 for the area of a parallelogram and Euclid 2008, I, 37-I, 40 pp. 38-41 for the area of a triangle) from which it results clearly that the Greeks had reached many elementary formulas for areas and volume, but such formulas are not explicitly stated.

d) The exhaustion method is not a heuristic method; it is a demonstrative method, which does not allow you to calculate the value of a certain area or volume if you do not know it; rather, it permits to prove an area or volume to be equal to another area or volume if you have reached this result otherwise. This is understandable if you think that we are speaking of an *ab absurdum* procedure.

e) Finally, a brief historiographic note has to be added: many important historians of science thought that Archimedes' method of exhaustion contains most of the elements typical of the integral calculus. Heath, albeit with many reservations and specifications on the use of the infinite in Greek mathematics, basically thought that Archimedes' exhaustion method "anticipated" integral calculus (Heath in Archimedes 2010, pp. CXLII-CLIV). Many other scholars followed his opinion. In contrast to this view, far more recently, Napolitani and Saito have underlined the fundamental differences between Archimedes' procedure and infinitesimal calculus. According to their train of thoughts, there is no generality in Archimedes' exhaustion proves (see, i.e., Napolitani 2001, Saito 2013). My opinion, as also clarified in these pages, is in between the two: the concept of limit is present in Archimedes in almost modern form, but nothing as a calculation of limits, a necessary – though not yet sufficient - ingredient for the integral calculus, exists (see also Bussotti 2019). I point out that, in this item, I have given only a very slight idea (the aim of this research is not historical, but educative) of a complex and articulated historiographical debate, with a great number of protagonists.

Once described this fascinating picture, which can arouse the interest and the curiosity of the pupils, the reason to look for a general method which will permit to calculate directly the numerical value of curvilinear areas and volumes will be evident. The teachers, at this point, might ask the pupils which elements of the Archimedes' method they think to be necessary in order to develop a general procedure and which elements have, instead, to be modified. The result of this discussion should be that the notion of limit has to be conserved as well as the idea to approximate the curvilinear areas and volumes with known areas and volumes. It is clear that the next step will be to better clarify the techniques of approximation. The aim is to transform Archimedes' use of approximation, which is genial and correct, but which needs a new different idea for any application, into a systematic procedure. This process will bring to the concept of integral. In this manner, the learners will understand profoundly the problem behind mathematics, which will, hence, result to be a conceptually rich discipline and not merely a set of techniques to learn by heart. The best way to introduce this further step is Newton's work.

Newton and the Concept of Integral

Newton dedicated several of his works to the concept of integral and to the integral calculus. However, for our aims, it is sufficient to analyse the lemmas on the concept of integral included by Newton in the first section of his *Mathematical Principles of Natural Philosophy* (1687 first edition, 1713, second edition, 1726 third edition; see Newton 1999), which is entitled "method of first and last ratios". Here the concept of limit is used in an almost modern manner and more explicitly than in Archimedes, though its formalization in ε-δ terms was given only in the 1820s. Newton's text opens with the celebrate "Lemma I", which states:

> "Quantities, and also ratios of quantities, which in any finite time constantly tend to equality, and which before the end of that time approach so close to one another that their difference is less than any given quantity *d*, become ultimately equal".
> Proof: "If you deny this, let them become ultimately unequal, and let their ultimate difference be *D*. Then they cannot approach so close to equality that their difference is less than the given difference *d*, contrary to the hypothesis". (Newton 1999, p. 433).

Two considerations can be proposed in an educative context:

1) Newton considers two quantities or two ratios of quantities whose difference tends to become smaller than any given *d* and claims their limit to be 0. This is the meaning of the expression that the two quantities or ratio of quantities "become ultimately equal". What Newton indicates with *d* has the same meaning as ε in the formal definition of limit. However, Newton proceeds in a manner which is slightly different from ours: he considers the limit of two quantities which are both functions of the time. In practice, Newton is claiming that if, being $g(t)$ and $f(t)$ functions of the time t and if $\lim_{t \to t_0} g(t) - f(t) = 0$ (or equivalently if $|t - t_0| < \delta$, then $|g(t) - f(t)| < \varepsilon$ or d, as Newton writes) the values of the two functions can be considered equal in t_0, which is absolutely true if g and f are continuous in t_0, as Newton implicitly presupposes. In an educative perspective, it is important to remark that the reference to the concept of function is a way to grasp Newton's ideas and to transcribe them in a language the pupils know and which creates, hence, no problem. However, properly speaking, Newton did not have the concept of function. For, he spoke of "quantities" which is a very generic expression that cannot be identified with our formal notion of function. This notwithstanding,

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the concept of limit can subsist and correctly used also in a mathematics in which the notion of function is not exactly stated, though we refer always the limit to the functions. Only as a consequence to solve mathematical problems for which the concepts had to be defined in a more perspicuous manner, we reached the formal definition of basic notions such as function and limit of a function.

2) Newton sees the concept of limit in a dynamical, a physical manner. For, the independent value is time and, hence, the functions $g(t)$ and $f(t)$ represent, at least in principle, physical quantities. It is intuitive that they might represent spaces expressed in function of time, or velocities, or accelerations varying in the time. It will be instructive to show that, at the dawn of modern science, the concept of limit was presented not in terms of a generic and abstract independent variable, but in terms of the variable Newton assumed as the ground of his physics: time. In spite of the fact that time is a specific variable, Newton's procedure has a general value, whatever the independent variable be.

Let us now see how Newton introduced the concept of integral.

Lemma II: «If in any figure *AacE*, comprehended by the straight lines *Aa* and *AE* and the curve *acE*, any number of parallelograms *Ab, EC, Cd*, ... are inscribed upon equal bases *AB, BC, CD*, ... and have sides *Bb, Cc, Dd*, . . . parallel to the side *Aa* of the figure; and if the parallelograms *aKbl, bLwcm, cMdn*, ... are completed; if then the width of these parallelograms is diminished and their number increased indefinitely, I say that the ultimate ratios which the inscribed figure *AKbLcMdqD*, the circumscribed figure *AalbmcndoE*, and the curvilinear figure *AabcdE* have to one another are ratios of equality».

Proof: «For the difference of the inscribed and circumscribed figures is the sum of the parallelograms *Kl, Lm, Mn*, and *Do*, that is (because they all have equal bases), the rectangle having as base *Kb* (the base of one of them) and as altitude *Aa* (the sum of the altitudes), that is, the rectangle *ABla.* But this rectangle, because its width *AB* is diminished indefinitely, becomes less than any given rectangle. Therefore (by Lemma 1) the inscribed figure and the circumscribed figure and, all the more, the intermediate curvilinear figure become ultimately equal».

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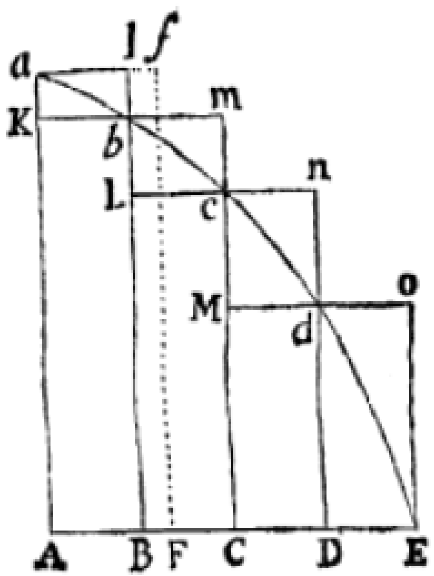
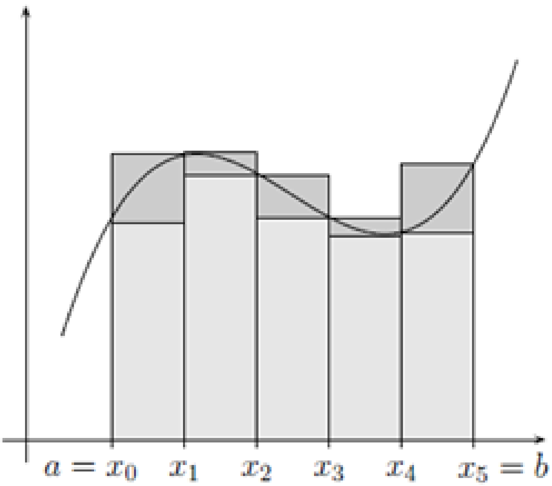
Figure 6
The Diagram Used by Newton to Prove Lemma II.



Note: Retrieved from Newton, 1726, p. 28

In this Lemma the concept of integral is clearly stated. For, the teacher, relying upon clear and intuitive Newton's explanations might give a first visual idea of what the integral of a function is as follows: you have to calculate the area of the curvilinear region included between the graphic of the drawn function and the *x*-axis (Fig. 7) in the interval (*a,b*). Such interval is divided into a certain number of partitions. A series of rectangles is drawn whose sum approximates by defect the searched area and a series of rectangles whose sum approximate by excess. Under appropriate conditions, when the interval $x(i)$-$x(i+1)$ tends to 0 and the number of the infinitesimal rectangles tend to infinity, the sum of the series of the rectangles which approximate by defect and the series of that approximating by excess tend to the same value. Such a value is assumed as the required area.

Figure 7
A Figure which can be Used to Give an Intuitive Idea of the Concept of Integral



All these steps are already present in Newton, thought in a less formalised manner than ours because the concept of function was not yet defined and that of limit was, consequently, conceived in a manner slightly different from ours. But all the fundamental ingredients of the integral calculus exist in Newton. Starting from the work of this great scientist it is, hence, possible to propose an intuitive, but, however, rather precise approach to the notion of integral. Here, you have a positive idea which can be developed into a technique or a series of techniques to calculate curvilinear areas and volumes. By the way, Newton himself developed many of such techniques. The stage in which the infinitesimal concepts are used as demonstrative means to which no technique of calculus was associated has been overcome, thanks to Newtons' idea to consider the area of a curvilinear figure like in the Lemma II.

Newton introduced two other lemmas on the integral calculus, the Lemma III, and Lemma IV. The former is particularly worth being mentioned: Newton shows that the same conclusions drawn in Lemma II hold if not all the bases of the parallelograms are equal, granted that all decrease at infinity. Some corollaries follow, the most interesting of which is the fourth one, which, from our standpoint is remarkable because Newton claims that the figure obtained increasing more and more the number of the majorant and the minorant parallelograms and whose perimeter is *AcE* (Fig. 6) is not anymore a rectilinear figure, but the curvilinear limit of a series of rectilinear figures. In this context he used explicitly the Latin term *limites* (plural), that is limits, which means that, also from a terminological point of view, his conceptions are similar to ours, or better ours are similar to his!

At this point, once understood the fundamental characteristics of the concept of integral thanks to Newton's work, the pupils will also realize that a formal definition of function, limit and integral will be necessary to fully guess to what extend the concept and the techniques of integration developed by Newton can be extended. Now the entire preparatory work to introduce the notion of limit and of integral in a genetic, problem-centred and intuitive way is finished and the learners will be ready to accept more formal definitions, which will be easy to be understood after this conceptual introductory work.

Conclusions and Implications

The effort of this research has been to show in the concrete case of integral calculus the way in which history of mathematics could be used in an educative context. We have seen the advantages of a historical approach insofar as it allows the pupils to understand not only the chronological origin of the concepts, but also the genetic-conceptual one. Through this approach, they will comprehend the progressive constructive steps through which the edifice of mathematics has been built. The advantage of this educative standpoint relies also upon the possibility to develop an interdisciplinary approach with other subjects such as physics, history and history of philosophy. Furthermore, the teacher has a great freedom to choose the historical itinerary he prefers. For example, in reference to integral calculus other authors different from Archimedes and Newton might have been chosen. The didactical itinerary, if well conceived, would have been equally valid. It is important to show how the educative path here proposed might be continued so as to include the differential calculus: for, following Leibniz's train of thoughts, it will be possible to show that the operations of drawing the tangent to a curve and of identifying the points of maxima and minima are opposite to that of calculating an area. This is very interesting because it will allow the pupils to intuitively understand what today we call the fundamental theorem of integral calculus, that is (to simplify) the fact that derivation and integration are two opposite operations. Such result will be achieved through the analysis of concrete problems (calculations of areas, of tangents and of maxima and minima), which could represent a stimulus for the pupils to increase their interest towards mathematics.

For all the reasons expounded in this research, a historical approach to mathematics education is highly recommendable.

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