Efficacy of a Learning Trajectory Approach Compared to a Teach-to-Target Approach for Addition and Subtraction

Douglas H. Clements,* Julie Sarama,* Arthur J. Baroody,** and Candace Joswick***

*University of Denver
** University of Illinois at Urbana-Champaign
***University of Texas at Arlington

Corresponding author:

Douglas H. Clements
University of Denver
Katherine A. Ruffatto Hall Rm. 152/154
Denver CO 80208-1700
Douglas.Clements@du.edu
(303) 871-2046
FAX: (303) 871-6582

Julie Sarama
University of Denver
Katherine A. Ruffatto Hall Rm. 160
1999 East Evans Avenue
Denver CO 80208-1700
Julie.Sarama@du.edu
(716) 907-4552
FAX: (303) 871-6582

Arthur J. Baroody
University of Illinois At Urbana-Champaign
1310 S. Sixth St.
Champaign IL 61820
baroody@illinois.edu
(217) 333-4791

Candace Joswick
University of Texas at Arlington
5th Floor, Hammond Hall
701 Planetarium Place, Box 19227
Arlington TX 76019-0227
(817) 272-2591
FAX: (817) 272-2530

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Abstract

Although basing instruction on a learning trajectory (LT) is often recommended, there is little direct evidence to support the premise of a "LT approach"—that to be maximally meaningful, engaging, and effective, instruction is best presented one LT level beyond a child's present level of thinking. The present report serves to address the question: Is it necessary to teach each contiguous level of a LT or can instruction be similarly or more effective when skipping levels, provided the necessary exemplars are made? In a multimethod research study that included individual teaching experiments embedded inside of a quasi-experimental research design, one group of 13 kindergartners received instruction based on an empirically-validated LT for addition and subtraction (the "LT" treatment). The counterfactual, "skip" treatment (*n* = 12), received instruction focused mainly on levels at least two levels above their present level for the same amount of time as the LT treatment. More children in the LT treatment exhibited greater addition and subtraction learning during sessions and from pretest to posttest than children in the skip treatment. Implications for future study are discussed.

Keywords: Achievement, curriculum, early childhood, instructional design/development, learning trajectories, learning environments, mathematics education

1. Introduction

The use of learning trajectories (LTs) in early mathematics instruction has received increasing attention from policy makers, educators, curriculum developers, and researchers because they are generally deemed as a useful tool for guiding standards, curricula, instructional planning, teaching, and assessment (Baroody, Clements, & Sarama, 2019; Frye et al., 2013; Maloney, Confrey, & Nguyen, 2014; National Research Council, 2009). Despite these recommendations, little empirical evidence of the *efficacy* of a LT approach to teaching exists. That is, little research has directly tested the specific contributions of LTs to learning (Frye et al., 2013). To address this gap, we have planned multiple experiments comparing a LT-based intervention to a counterfactual that is similar but without a defining feature of the LT approach (see Clements, Sarama, Baroody, Joswick, & Wolfe, 2019). The present study compared the learning of kindergartners who receive instruction on addition and subtraction based on an empirically-validated LT, including any necessary support in corequisite counting concepts, to those who receive an equal amount of instruction focused only on predetermined target levels. That is, this counterfactual approach alters a critical attribute of the LT approach by essentially skipping instruction at intermediate *levels of thinking* in the LT. The present report focuses a preliminary study comparing kindergartners who received either LT-based or non-LT instruction on addition and subtraction.

2. Background and Theoretical Framework

LTs are not only under-researched, but often misunderstood. For example, some have confused LTs with a logical task analysis, hierarchies or sequences based solely on the structure of mathematics content (Resnick & Ford, 1981), or the on accretion of facts and skills (Carnine, Jitendra, & Silbert, 1997). Further, LTs are not just a “progression” of assessment tasks or cognitive patterns of thinking. Rather, a *complete* LT has three components: a goal, a developmental progression of levels of thinking, and instructional activities designed explicitly to promote the development of each level (Maloney et al., 2014; National Research Council, 2009).

LTs' goals are based on both the expertise of mathematicians and research on children's thinking about and learning of mathematics (Fuson, 2004; Sarama & Clements, 2009). LTs' developmental progressions are ordered levels of thinking, each more sophisticated than the last, through which children progress on their way to achieving the mathematical goal. Each level is characterized by specific concepts (e.g., mental objects) and processes (mental "actions-on-objects") that underlie mathematical thinking at level *n* and serve as a foundation to support successful learning of subsequent levels (Sarama & Clements, 2009; Steffe & Cobb, 1988). Specification of these actions-on-objects allows a degree of precision not achieved by previous theoretical or empirical works. Further, LTs address both thinking *and learning*–that is, achieving a higher level–are central (Steffe, Thompson, & Glasersfeld, 2000).

Some interpretations and appropriations of the LT construct emphasize only the developmental progressions of learning during the creation of a particular curricular or pedagogical context. In our theory, the efficacy and uniqueness of the LT construct largely stems from the inextricable interconnection between all three components. Thus, instructional activities must be included as the third component of LTs. Extant research is used to identify tasks as effective in promoting the learning of children at each level, by encouraging children to construct the concepts and skills that characterize the succeeding level. That is, we design tasks and teaching strategies that help children build the mental actions-on-objects underlying each level's pattern of thinking. The tasks include external objects and actions that mirror the hypothesized mental actions-on-objects as closely as possible.

2.1. Adding/Subtracting Learning Trajectory

A central topic for kindergarten mathematics is using informal methods to solve simple arithmetical problems such as “Al had 5 balls and gets 4 more. How many does he have in all?” Early arithmetic competencies are central to all early mathematical concepts from the earliest years of life (Gelman & Gallistel, 1978; Sarama & Clements, 2009; Wynn, 1992). Informal arithmetic competence is one of the best predictors of mathematical disabilities/difficulties and later achievement in not just mathematics but also in reading (Geary, 2011; Gersten, Jordan, & Flojo, 2005).

The following describes the three components of our LT for informal adding/subtracting, focusing on the levels most relevant to kindergarten age (all levels are available in Clements & Sarama, 2014; Sarama & Clements, 2009, and LearningTrajectories.org).

2.1.1. The goal

Young children informally view addition (subtraction) as adding more (taking away some) items from a collection and thus changing its total. This informal conception of addition (or subtraction) enables them to comprehend simple addition (or subtraction) word problems and use their existing counting knowledge to devise informal strategies for solving them. Given this informal strength, a developmentally appropriate goal for kindergartners is solving a variety of different addition and subtraction problems using informal, adaptive strategies.

An important aspect of the goal is the ability to solve different *problem types* with adaptive strategies (Carpenter & Fennema, 1992; NGA/CCSSO, 2010). Some problems are more difficult than others because they involve numbers in a larger range. The *type* of the word problem, which vary in mathematical structure, determines difficulty. Type depends on the situation and the unknown. There are four situations: Add-To (also called “Join”), Take-From (“Separate”), Part-Part-Whole, and Compare. For each, three quantities play different roles in the problem, any one of which could be the unknown quantity (for complete descriptions, see Appendix A; all appendices are online). For example, Add-To problems may have the Result Unknown (the typical "what is the total?" question), Change Unknown ("how many more do you need to have 9?"), or Start Unknown. Result Unknown are easiest for young children to model and solve; Start Unknown ("Maria had some marbles, she got 5 more and now has 12. How many did she start with?") are the most difficult for Add-To and Take-From problems.

2.1.2. The developmental progression

The second component of a LT is the developmental progression of levels for adding/subtracting LT, shown in Figure 1. Three features distinguishing the levels are (a) the *problem type* previously discussed, (b) the size of the numbers in the problem (both of these are summarized in the third column), and (c) the increasingly sophisticated strategies children use to solve those problems (illustrated in the first column, with the mental actions-on-objects underlying them in the second column).

Progressing through the eight levels, then, children develop increasingly sophisticated strategies to solve increasingly difficult arithmetical problem types. For example, most initially use a concrete *counting-all* procedure, to directly model and solve simple Result Unknown addition or subtraction problems, first with sums to 5 (**1–Small Number +/-** level in Fig. 1), then with sums to 10 or beyond using strategic shortcuts (**2–Find Result +/-** level)[1]. To illustrate concrete *counting-all*, given a situation of 6 + 3, children count out objects to form a set of 6 items, then count out 3 more items, and finally count all those objects, starting at one, and reporting "nine." Children can use such *counting-all* methods to solve Result Unknown story

[1] Note that the names of levels of the learning trajectory always end with "+/-"; however, because many similar terms appear in the problem type names, which are likewise capitalized in the literature, we also distinguish the learning trajectory names with **boldface**. The names were not elaborated for changed to keep consistency between this paper and the sources of the names (Clements & Sarama, 2014; Sarama & Clements, 2009, and LearningTrajectories.org).

situations as long as they understand the language in the story.

Next, children figure out that they can shortcut the laborious procedure of *counting-all* by representing one or both of the addends immediately, such as with finger patterns. For example, to solve 6 + 2, children might extend 2 fingers, verbally count up to 6, then continue this count, pointing in turn to each of the previously extended fingers: "7, 8" (note the frequent use of finger patterns detailed in Fig. 1). Children progress to the **4–Counting Strategies +/-** level by curtailing such of *counting-all* methods to *counting-on*; for example, solving 6 + 2 concretely by making piles containing 6 and 2 objects, but then pointing at the former, saying either "One-two-three-four-five-six" quickly without re-counting the objects or just "Siiiiix… " and then pointing at the other 2, counting, "7, 8. *8!*" The "running through" the counts quickly or the elongated pronunciation substitutes for the act of counting the initial set one-by-one. It is *as if* they counted the set of 6 items. A more abstract version is *counting-on* through the first addend (also known as "abstract counting all"), keeping track of the number added via objects ("one, two, three, four, five, six, *seven* [puts up one finger], *eight* [puts up a second finger, and stops, recognizing the finger pattern as two]. Eight!"

At the **5–Part-Whole +/-**level, children build an embedded-addends concept, in which they can explicitly represent both addends simultaneously within the sum count, supporting them in adopting the *counting-on-from-larger* strategy (given 2 + 7, they count on from 7 instead of 2). This also enables them to devise indirect-modeling procedures, such as double-counting to keep track of the second addend—how much is added on to the first addend (e.g., 2 + 7: Seven, eight *[is one more]*, nine *[is two more]*. At this level, children can solve Find Result and Change Unknown problems with flexible strategies—which may also include some familiar combinations (e.g., to solve 5 + 8, they may say, "5 + 5 is 10, and 3 more is [putting up fingers until they see a 3 pattern]…11, 12, 13").

Children's part-part-whole knowledge is extended at the **6–Numbers-in-Numbers +/-** level in which they can keep the two parts and the whole in mind simultaneously. A defining behavior is their ability to solve Start Unknown problems (e.g., "You have some balls. I give you 4 more balls, and now you have 11. How many did you start with?") because they can understand that the number started with is a part of the whole in this case.

At the **7–Deriver +/-** level, children's ability to hold three numbers in mind simultaneously allow them to use derived combinations (e.g., "6 + 6 = 12, so 6 + 7 = 13") and sophisticated strategies such as including *break-apart-to-make-ten* (Clements & Sarama, 2014; Murata & Fuson, 2006); for example, "9 + 6–I take one of the 6 to make a 10, then I know 10 plus 5 is 15." At this level, children may also start solving multidigit addition and subtraction—by increasing or decreasing by tens and/or ones. Such multidigit addition and subtraction is formalized in the **8–Problem Solver +/-**level, along with the use of flexible strategies and known combinations for solving all problems types.

The developmental progression for adding/subtracting was based on many empirical studies (e.g., Carpenter & Fennema, 1992; Fuson, 1992; Steffe & Cobb, 1988) and have been supported, albeit with small differences in descriptions of levels and strategies and developmental order, by others, including international research (see Baroody et al., 2019; Clements & Sarama, 2014; Sarama & Clements, 2009, for reviews).

2.1.3. The instructional tasks

Instructional tasks in LTs are a way to guide children to achieve the levels of thinking described in the developmental progression; those in the last column of Figure 1 are specific examples of the type of instructional activity that help promote thinking at that level. For this topic, the children are of course asked to solve arithmetic problems fitting each level as shown in the third column of Figure 1, with increasingly sophisticated strategies, described in the first column of Figure 1. Learning of strategies is supported both by posing the arithmetic problems and by instructional activities that research indicates are especially effective for each level. As an example, our sample instructional activity for teaching *counting-on* skills in the **4–Counting Strategies +/-** level is based on theory and empirical work (a) Baroody and others, showing the relationship between the number-after rule for adding one, then two (Baroody, 1995; Clements & Sarama, 2014) and (b) El'konin (1975), providing effective instructional procedures to teach *counting-on* to children who have not yet developed this skill. Next, presenting problems such as 3 + 22, where the most *counting-on* work is saved by reversing the problem, often prompts children to start counting with the 22, *counting-on-from-larger* (Siegler & Jenkins, 1989), which may prompt the use of the strategy initially, but must be followed up with smaller numbers emphasizing the conceptual foundation.

2.2. Assumptions of a LT Approach

To evaluate whether instruction based on LTs is significantly more efficacious than plausible alternatives, we must avoid confounding assumptions of the former with various other instructional factors. The LT approach involves using *formative assessment* (National Mathematics Advisory Panel, 2008; Shepard & Pellegrino, 2018) to provide instructional activities aligned with empirically-validated developmental progressions (Clarke et al., 2001; Fantuzzo, Gadsden, & McDermott, 2011; Gravemeijer, 1999; Jordan, Glutting, Dyson, Hassinger-Das, & Irwin, 2012). Perhaps the main assumption of such an instructional approach is that instruction should move children from their present level of thinking (level *n*) to the following level (level *n*+1) to the target level (level *n* +2), with a combination of teaching strategies that evoke children's natural patterns of thinking at each level (Sarama & Clements, 2009).

Level and Strategies	Mental "Actions-on-Objects"	Problem Types and Numbers/Quantities	Sample Instructional Activities
1–Small Number +/- Finds sums for Add-To problems up to 3 + 2 and finds difference for "Take-From" problems up to 3 - 2, by direct modeling via *counting-all* with objects such as counters or fingers.	Experience provides implicit scheme of situations of changing a group by adding another group to it and determining the numerosity of the composite set. Uses the counting competencies to produce each set, then count the total.	*Add-To, Result Unknown or Take-From, Result Unknown* Numbers ≤ 5	Pose simple addition problems with toys that represent the objects in the problems, totals up to 5 and ask children to explain their answers. Differentiate by decreasing the numbers in the problem and providing support with counting, or by increasing the numbers in the problem.
2–Find Result +/- Finds sums for Add-To problems with sums to 10 or more and differences for "Take-From" problems with minuends of 10 or more by concrete *count-all-groups* and with *counting-all* shortcuts such as using finger patterns to shortcut counting one or more of the sets. Also uses these strategies to find the whole in Part-Part-Whole problems with similar size numbers.	Competencies are extended to larger sets. Child forms scheme for combing groups, counts out 5, then counts as adds 2 more to the pile (or makes separate pile and combines piles), then counts all 7. May attenuate the counting process with finger patterns, such as putting up 5 on one hand and 2 on the other immediately (subitizing), then counting 7.	*Add-To, Result Unknown or Take-From, Result Unknown or Part-Part-Whole, Whole Unknown* Numbers < 21	Give the group a number cube with the numerals 1-3. Demonstrate how to roll the number cube, take that number of toy bears, and put each toy bear on a block. Have children take turns rolling the number cube and adding toy bears to their own block rows. After each roll, children should figure out and announce how many more toy bears they need to fill their rows.
3–Find Change +/- Finds the missing addend (e.g., 5 + _ = 7) by adding on or separating from objects by *add-to (or separate-to) and count-all-groups.* Compares simple situations by *match–count rest*.	The scheme for Add-To operations is sufficiently re-represented to allow creation of a mental "placeholder" for the collection of objects that must be added to another group to make the required total. With perceptual support, this allows the separation of the added collection. May use fingers, and attenuate counting by using finger patterns.	*Add-To, Change Unknown or Take-From, Change Unknown or Part-Part-Whole, Part Unknown, or Compare, Difference Unknown* Numbers < 21	On each of several whiteboards place a tens frame with any number from 1-9 shown. Under the tens frame, write an addition problem for the children to solve based on the number of spaces filled in. Extend this by asking children to solve problems that do not perfectly fill the tens frame. For example, 4 + ___ = 7.
4–Counting Strategies +/- Finds sums for Add-To, Take-From,	The counting scheme is elaborated so that a number is intuitively conceived	*All previous* **using counting strategies.**	Set up the problem situation, then guide children to connect the numeral signifying

Level and Strategies	Mental "Actions-on-Objects"	Problem Types and Numbers/Quantities	Sample Instructional Activities
and Part-Part-Whole (assimilated by children to their Add-To scheme) problems with *counting-up-to, and/or by counting-on, often using finger patterns in these strategies*.	simultaneously as a cardinal amount and a part of the total. The starting number therefore represents the number of counting acts it would take to reach that number without perceptual support and the counting continued via a cardinal-to-count transition so as to constitute the second number, with temporal subitizing or perceptual tracking used to keep track of the numerosity of this second number. Commutativity, initially a theorem-in-action, is used to reorder addends to save effort (*counting-on-from-larger*); initially this may be recognized only when adding one.		the first addend to objects in the first set. This helps children learn to recognize that the final object in a set is assigned the counting word of the cardinality of the set. Afterwards, help children understand that the first object in the second set (second addend) will always be assigned the next counting number after the first addend.
5–Part-Whole +/- Has initial part-whole understanding. Solves all previous problem types using flexible strategies and may use some known combinations (such 5 + 5 is 10). Can sometimes can do "Start Unknown" problems, but only by trial and error.	Schemes for Add-To, Take-From, and Part-Part-Whole situations are sufficiently re-represented and related to form an explicit, although nascent, part-whole scheme. This embedded-addend concept support the develop and use of various arithmetic strategies, allowing more flexible counting strategies, including some derived combinations and inverse operations.	*All previous* *Add-To, Start Unknown*, or *Take-From, Start Unknown* Often numbers < 10	Hide 4 counters under a dark cloth and show children 7 counters. Tell them that 4 counters are hidden and challenge them to tell you how many there are in all. *Or*, tell them that there are 11 in all and ask how many are hidden. Have them discuss their solution strategies.
6–Numbers-in-Numbers +/- Can solve "Start Unknown" problems with *counting strategies*.	Recognizes when numbers are part of a whole and can keep parts and whole in mind simultaneously–the embedded-addends concept.	*Add-To, Start Unknown*, or *Take-From, Start Unknown* Usually numbers < 21	One child rolls two number cubes. Child sums the two numbers (e.g., 2 + 4 = 6). Child then turns over any combination of numeral cards that equal the sum of the cubes (e.g., 6; 2 + 4 = 6; 1 + 5 = 6). The next child rolls the two number cubes and then turns over any combination of numeral cards that equal their cube sum.

Level and Strategies	Mental "Actions-on-Objects"	Problem Types and Numbers/Quantities	Sample Instructional Activities
			Each child adds the remaining numeral cards (the ones left face up) and the child with the lowest number wins.
7–Deriver +/- Solves all types of problems using *flexible strategies* and *derived combinations* (e.g., "7 + 7 is 14, so 7 + 8 is 15"), including *break-apart-to-make-ten*. Solves simple cases of multidigit addition (sometimes subtraction) by *incrementing tens and/or ones*.	Can simultaneously think of 3 numbers within a sum, and can move part of a number to another, aware of the increase in one and the decrease in another.	*All types* of single-digit problems, **using derived strategies.**	Using a regular deck of cards, the dealer gives everyone 2 cards, including themselves. For point values, an ace is worth either 1 or 11, 2 to 9 are worth their values, and face cards are worth 10. (Face cards can be removed for this activity). Each player finds the sum of their own cards. The goal of the game is to get as close to 21 as possible without going over. On each round, if player's sum is less than 21, they can request another card, or "hold." If any new card makes the sum more than 21, the player is out. Continue the activity until everyone "holds." The player whose sum is closest to or exactly 21 wins.
8–Problem Solver +/- Solves all types of problems, with *flexible strategies* and *known combinations*. Multidigit may be solved by *incrementing* or *combining tens and ones* (latter not used for "Add-To, Change Unknown" problems).		*All types* of problem structures for single-digit problems and multidigit numbers **using strategies and known combinations.**	Present addition (and then subtraction) problems under an empty number line and have children "talk aloud" to solve the problem. After talking aloud, the children can represent their thinking on the empty number line. Be sure to use some numbers where the ones place adds up to more than 10. Focus on asking children how they compose the numbers together and what strategy they used to solve the problem.

Figure 1. Relevant Levels from the Learning Trajectory for Adding/Subtracting (adapted from Sarama & Clements, 2009).

The competing hypothesis is that it is more efficient and mathematically rigorous to teach the target level immediately by providing accurate definitions and demonstrating accurate mathematical procedures, obviating the need for potentially slower movement through each level (see Carnine et al., 1997; Clark, Kirschner, & Sweller, 2012; Clements & Sarama, 2014; Wu, 2011). That is, such instruction is deemed more efficient because it skips one or more of a LT's levels (level n and level n+1) and explicitly focuses on a target competence (level $n + 2$ or $n + 3$) that is assumed to enable the student to perform tasks associated with that and all previous levels.

In contrast, LT-based approaches justify the assumption that each contiguous level be developed consecutively because each level is characterized by actions-on-objects that hypothetically must be built at level n as a foundation for effective learning of level $n + 1$ (and thus, if skipped, leave gaps that impede learning). Further, these progressions play a special role in children's meaningful learning because they are particularly consistent with children's intuitive knowledge and patterns of thinking and learning at various levels of development, implying a combination of pedagogical strategies emphasizing sense-making and problem solving at each level (Sarama & Clements, 2009).

2.3. Efficacy Evaluation of Learning Trajectories

Although there is research on some LTs' developmental progressions, and successful early mathematics intervention projects based on LTs, there is no research besides our own of which we are aware that directly tests the theoretical assumptions of LTs. That is, often developmental progressions are used, but are not linked to instruction explicitly. Further, multiple studies using LTs have shown strong results, but the designs are limited in what they claim about LTs per se because they confound the use of LTs with other factors (Frye et al., 2013). Thus, these studies support the efficacy of the use of LTs but cannot identify their unique contribution, particularly beyond that of other instructional approaches (Clarke et al., 2001;

Fantuzzo et al., 2011; Gravemeijer, 1999; Jordan et al., 2012).

In our earlier study of the efficacy of our 2D position LT (Clements et al., 2019), we evaluated whether one group of preschoolers (n = 82), who were at least two levels below the target instructional LT level, received instruction based on an empirically-validated LT, would outperform the preschoolers (n = 63) in the counterfactual skip treatment. The counterfactual received an equal amount of instruction focused only on the target level. Although instruction was brief, consisting of an average of a little more than eight 9-minute sessions over five weeks, we found that LT instruction was more efficacious than skip instruction (effect size, .55). There were no significant differences on outcomes for the variables of gender, age, ethnicity, or time on task, indicating a robust and general result. Further, examination of individual items confirmed that the LT group made more completely correct solutions to the assessment items and used strategies at higher levels of sophistication than children in the skip group. These effects were especially pronounced on tasks similar to the target level, that is, on near transfer tasks. This is notable, as the target level was achieved more frequently by LT children who experienced fewer tasks and less instructional time at that level than did the skip children.

2.4. Purpose of the Current Study

The present study rigorously tests the same basic assumption of a LT approach that meaningful instruction should identify a child's present level and teach each successive level up to the target level. Specifically, we addressed the following research question: Does instruction in which LT levels are taught consecutively (e.g., for children at level n, instructional tasks from level $n + 1$, then $n + 2$) result in greater learning than instruction that immediately and solely targets level $n + x$, where x is > 1 (the skip approach)? In particular, for the LT treatment, we evaluated whether level-n children given $n + 1$ tasks learned the mental actions-on-objects hypothesized to enable thinking at that level and, if so, if these new constructions then supported *n* + 2 thinking. For the counterfactual group, the skip treatment, we evaluated whether level-*n* children could learn higher (*n* + *x*) levels of thinking, with instruction at least skipping at least one intermediate (*n* + 1) level~~s~~. We further investigated whether such potential efficiency could be realized when two levels (*n* + 3) were skipped, or whether only one (*n* + 2) could be skipped. We used data from both pre- and posttesting and session-by-session behaviors to examine progress on individual problem-solving tasks and on achievement at multiple levels of thinking.

3. Method

We conducted a multimethod research study that included individual teaching experiments (Steffe et al., 2000) embedded inside of a quasi-experimental research design. Half of the children were taught using the LT approach, and half with the skip approach. We pre- and posttested children and also followed each child's learning session-by-session.

3.1. Participants

We recruited and received consent from all children enrolled in both kindergarten classes in a University-affiliated and nationally recognized program for children "who demonstrate exceptional, differentiated abilities and learning needs." Children identified as gifted were selected as the sample because such children are the most likely to be able to skip levels and benefit from immediate instruction of a target level. In other words, the unrepresentative sample worked against our hypothesis, providing a rigorous test of it. Table 1 presents children's demographics.

Table 1
Participant Demographics by Treatment Group

Treatment	*N*	Age (months)	Male/Female	Asian	Black	Latinx	White (non-Latinx)
LT Class 1	13	5.4	8/5	0	1	0	12
Skip Class 2	13	5.5	7/6	2	1	2	8
Total	26	5.5	15/11	2	2	2	20

Note: One male in Class 2 attritted, leaving school before the completion of the pretest.

3.2. Procedures

Class 1 was randomly assigned to the LT treatment. After completing the pretest, each child completed the treatment in four weeks or less. Approximately 180 minutes of instruction was completed one-on-one with the same one or two instructors (splitting instruction equally), up to 4 days a week, for about 20 minutes a session. Posttests were administered the week following their last treatment session.

LT instruction was based on the learning trajectory detailed in Figure 1. Instruction began one level higher than that indicated by the pretest. Each instructional session focused on a level until it was attained and then proceed to the next higher level, with the intent of reaching the target level **6–Numbers-in-Numbers +/-**, but following the child's developmental pace. Daily curriculum-embedded assessment was used to determine when to move instruction to a more sophisticated level of the LT.

To determine the number of levels that instruction can skip productively, if any, children in the skip treatment were assigned a predetermined instruction plan broken into three sections. Section A, the first 60 minutes of instruction, presented problems of the Start/Part Unknown type using numbers 21-100 (at the **6–Numbers-in-Numbers +/-** LT level). This was a skip of 2 to 4 levels from the children's pretest LT level. Sections B and C, also 60 minutes each, were at the same or a lower level, with Section C including some n + 1 (no levels skipped) when the child displayed considerable frustration. Problem types were Start/Part Unknown in section A, Change/Difference Unknown in section B, and Result/Whole Unknown in section C. For complete instructional plan assignments, see Appendix B.

3.3. Data Collection

3.3.1. Pretest/posttest content assessment

The pretest and posttest were comprised of arithmetic-relevant counting and adding/subtracting items from the REMA (Clements, Sarama, Wolfe, & Day-Hess, 2008/2019) and TEMA-3 (Ginsburg & Baroody, 2003). Counting consisted of 24 items including items assessing set production and counting forward from any number. There were 45 adding/subtracting items covering all problem types (Appendix A) and aligned with all levels of the LT (Fig. 1).. Each assessment was administered individually to each child by a trained assessor (who also served as instructors but not of children they tested). At posttest, children were assessed by graduate research assistants who did not know their treatment condition.

3.3.2. Session data

Instructors kept daily field notes from each session and reviewed a video recording immediately after to add details, emphasizing the child's accuracy and strategic competence on each task. LT group children who showed behaviors consistent with a level on three consecutive tasks were considered to be functioning at that level and ready for instruction at the next level. Assessment of the children in the skip group was more limited because the target levels of instructional tasks were fixed by the design, but correctness and any use of strategies was recorded. Instructor summaries of LT level were checked once a week by the fourth author.

3.4. Analyses

3.4.1. Pretest/posttest content assessment

As stated, we examined progress on both problem-solving items within levels and on achieving higher levels of thinking. For the former, percentage correct on items in each adding/subtracting LT level were calculated and growth within each LT level (e.g., from 30% correctness in a given level to 67% correctness, with a minimum of 20% change) was labeled an *incremental* improvement. Also, average correct across all items was compared for each child between pretest and posttest to determine overall change.

To identify each children's level of thinking, we determined the highest level on which the child was consistently accurate (all but one item correct) on problem types for that level and all previous levels in the developmental progression. *Developing* at a level indicates some behaviors at the level but not consistent in accuracy and in strategy use for all tasks at that level. Each child's pretest LT level at (i.e., "level *n*") was determined as the highest level for which the child correctly answered all or all-but-one of the three or more items assigned to the level. Change in LT level attainment (e.g., from **1–Small Number +/-** to **2–Find Result** +/- or from Developing **2–Find Result** +/- to Developing **3–Find Change** +/-) is described as a *level transition*.

4. Findings

We present the results of the pre- and posttesting, analyzed by level transitions and incremental growth within levels, followed by analyses of session-by-session changes.

4.1. Pretest to Posttest LT Level Change

Table 2 presents pretest-to-posttest change in LT levels for adding/subtracting by treatment group (Appendix C includes details). An increase indicates a transition from, for example, **1–Small Number +/-** to **2–Find Result +/-**; a partial increase might be from **2–Find Result +/-** to Developing **3–Find Change +/-.**

Table 2
Children's Pretest to Posttest LT Level Change by Treatment

	Treatment

	LT	Skip
Two-level increase	2	
One-level increase	4	
Partial level increase		1
Neutral	7	5
Partial level decrease		5
One-level decrease		1

More children in the LT than in the skip treatment increased in their level of thinking from pretest to posttest although about half did not.

Overall, every child in both treatments experienced incremental growth in at least one adding/subtracting level (Appendix C), more so for LT children (3 children in 1 level, 3 children in 2 levels, 4 children in 3 levels, 3 children in 4 LT levels) than skip (1 child in 1 level, 8 in 2 levels, and 1 at each of 3, 4, and 5 levels). Contrary to the rationale for the skip treatment, most skip children did not master or even perform well on their targeted level, although about half did make incremental improvements at one or more of the higher levels of the LT. The total pretest and posttest means of the LT group (*M* = 0.18, *SD* = 0.29 to posttest *M* = 0.28, *SD* = 0.36) showed greater increase than those of the skip group (*M* = 0.19, *SD* = 0.31 to posttest *M* = 0.25, *SD* = 0.34).

4.2. Session by Session Levels, Bracketed by Pre- and Posttests

Although the pre-post differences are clear, there is reason to believe that they underestimate children's learning, especially for those in the LT group. Figure 2 shows a pattern of increasing levels of thinking of the LT children from the pretest assessment through all of the instructional sessions that was not observed in the skip group. However, there is a precipitous drop for the posttest, more so for the LT than skip group. (Figures in Appendix D separate children who progressed three or more levels at some point during the instructional sessions from children who progressed only one or two. The former not only showed signs of more progress, but a more precipitous decline for the posttest.) This could indicate that the posttest was less sensitive than instructional tasks in identifying incremental growth (it also may mean children's learning in these brief interventions was fragile and not consolidated). For example, an examination of the videos of the sessions shows increasingly high performance in the LT with no scaffolding. Similar examination of the videos of the skip sessions shows children did not show competence during instruction at their targeted levels. Indeed, skip children gave few correct answers and their strategies were overwhelmingly guesses; therefore, there were little or no signs of growth. Further, the posttest occurring at the end of the school semester by an unknown adult and these factors may have substantially curtailed performance of children in both groups.

4.1. Commonalities and Differences Within and Between Groups

4.1.1. Consistencies with hypotheses

Six LT children developed from pretest to posttest in their instructed adding/subtracting LT level (see Appendix C). For example, Child 01 transitioned from solving some problems with numbers up to 5 using counting-all (Developing **1–Small Number +/-**) at pretest to similar performance on problems up to 10 with some use of counting shortcuts (Developing **2–Find Result +/-**) at posttest. Instruction targeted levels from **1–Small Number +/-** up to **5–Part-Whole +/-** (Levels 1 to 5) except for **4–Counting Strategies +/-**, which the instructor skipped by mistake. The instructor anticipated posttest performance at the **5–Part-Whole +/-** (see Fig. 2), but the skipped instructional level may have affected child 01's learning. Nevertheless, consistent with our hypotheses, the child progressed to the level immediately above the child's pretest level.

Like Child 01, Child 06 transitioned from Developing **1–Small Number +/-** at pretest to Developing **2–Find Result +/-** at posttest. Instruction targeted those two levels specifically, and the **3–Find Change +/-** level, including adding or subtracting by counting forward or back and identifying unknowns using composition (e.g., "this number is two more than three"). Although

Child 06 did not increase in the **3–Find Change +/-** level, they did in the **5–Part-Whole +/-** and **6–Numbers-in-Numbers +/-** levels (increases in correctness of 40% and 100% respectively), using an abstract *counting-on* strategy that entails a keeping-track process to find missing sums on the posttest. Accurate use of this strategy represents a substantial increase in Child 06's level of thinking.

Recall, 11 of the 12 skip children did not show growth from their assigned pretest level to posttest LT levels. Child 14, for example, did not change correctness at their pretest LT level (Developing **1–Small Number +/-**) to posttest. She did, though, show minimal incremental improvement at other levels—she made more answers correct in the **Derive +/-**level (though only 1 item correct, and she did not show any strategy use, stating that she just guessed) and **8–Problem Solver +/-**levels (though again only 1 item correct, again stated she guessed–we credited it nonetheless to avoid biasing results against the skip group). The skip children who show incremental growth at the most levels (15, 23, and 25) all experienced $n + 1$ instruction in Section C.

Child 20 of the skip group would often say the last number word said as an answer to a problem, sometimes answering before the problem was entirely said by the instructor. Her small incremental growth from pretest to posttest in the **5–Part-Whole +/-**and **8–Problem Solver +/-** levels appear not attributable to changes in strategy use or understanding of problem types and solution strategies. Instead, the child's significant instructional time on counting-specific support might be related. That is, this child was often distracted and somewhat frustrated, so to keep interest, the instructors would move to easier problems. She used trial-and-error counting to get two items correct at higher levels.

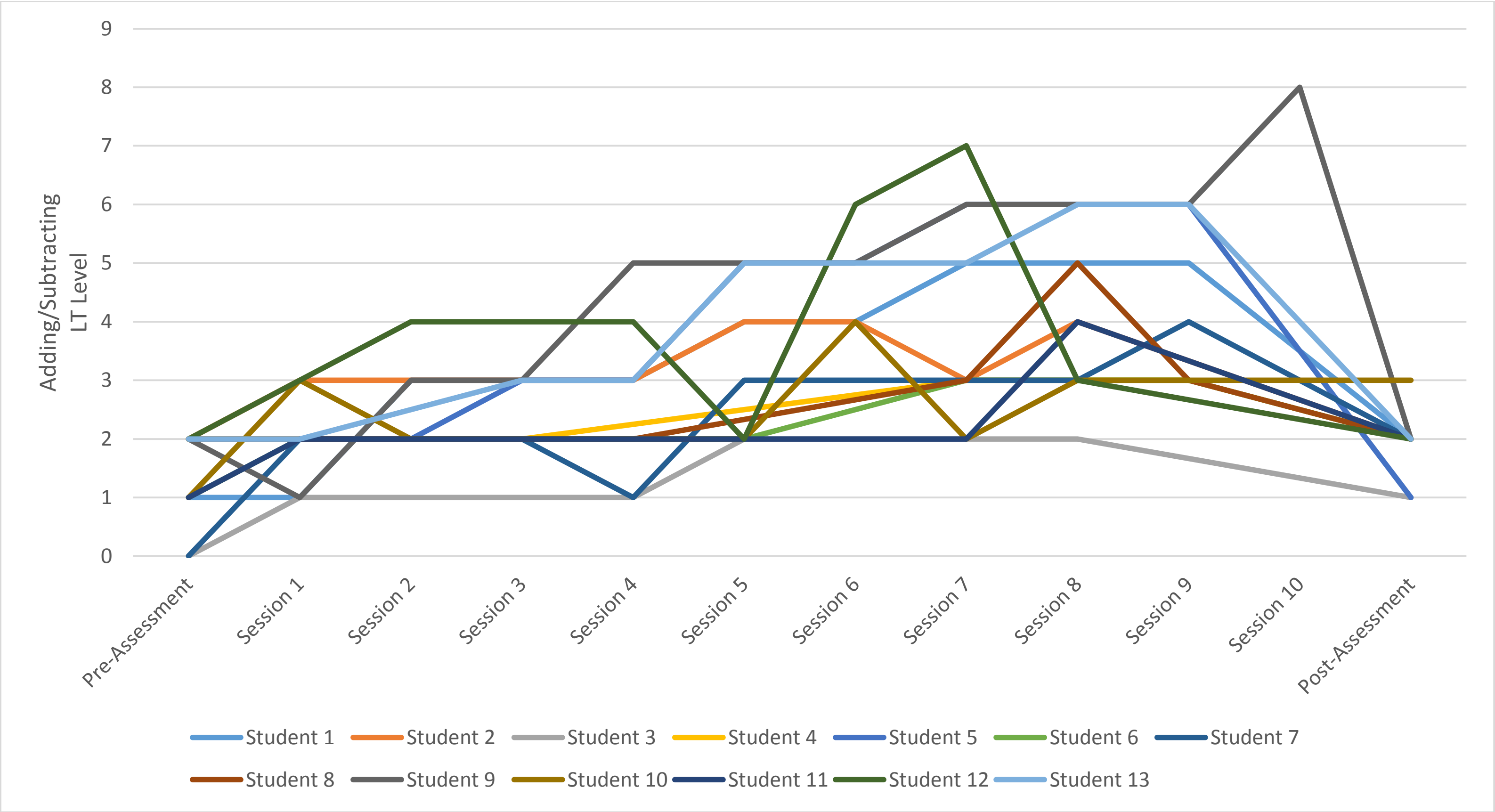


Figure 2. Adding/Subtracting LT levels for Pretest, Each Instructional Session, and the Posttest, for the LT Group (note that some children participated in 9 or 10 sessions). Levels, defined in Figure 1, are 1–Small Number +/-, 2–Find Result +/-, 3–Find Change +/-, 4–Counting Strategies +/-, 5–Part-Whole +/-, 6–Numbers-in-Numbers +/-, 7–Deriver +/-, and 8-Problem Solver +/-.

4.1.2. Inconsistencies with hypotheses

Recall, six LT children did not progress in their pretest to posttest LT levels. For example, Child 12 did not improve at his initial Developing **2–Find Result +/-** level, although he did evince incremental growth in four other adding/subtracting LT levels (**3–Find Change +/-**, **4–Counting Strategies +/-**, **6–Numbers-in-Numbers +/-, 7–Deriver +/**), which align with his instruction. In some cases, a more sophisticated strategy was used but not skillfully. For example, Child 02 decreased from 60% correctness at pretest to 40% correctness at posttest in the **2–Find Result +/-** level. At pretest, she answered all **2–Find Result +/-** questions using concrete *counting-all* strategies; at posttest, she attempted to use the most sophisticated *counting-on* strategy, but made errors in execution. Child 08 similarly dropped from 60% to 20% correctness at that level, getting only one item correct using *counting-all* and missing others using *counting-on* but with errors.

Child 23 was the only child in the skip treatment who displayed pretest to posttest increase in LT level and that was partial. His instructional plan path was n + 2 for section A, n + 2 for section B, and n + 1 for section C, and he was most successful in this final section, mainly on Add-To, Result Unknown problems and those with doubles (e.g., "Elf had 10 snowflakes on his tongue then she put 10 more, how many in all?"). He also demonstrated gains in **3–Find Change +/-** (+33%), **5–Part-Whole +/-** (+60%), **6–Numbers-in-Numbers +/-**(Level +100%), and **7–Deriver +/-** (+29%). His growth within the **3–Find Change +/-** level might be explained by the third of instruction given at that level—during which they would use trial-and-error to often correctly answer problems or give reasonable estimates. However, the other increases are inconsistent with our LT hypothesis, an issue to which we return.

4.1.3. Effects of skipping different numbers of levels

Skip children who had instruction at levels ($n + x$; $x > 1$) rarely made progress at the targeted level, although there were exceptions. Of the six children who experienced $x > 2$ instruction in at least one of the sessions, three showed a partial decrease and three showed no change in LT level. Of the six children who experienced at the most $x = 2$ instruction, three showed a partial decrease, two showed no change, and one showed a partial increase in LT level. Thus, there was little difference between these subgroups, with only a slight trend that skipping fewer levels was more beneficial.

5. Discussion

This study rigorously evaluated the specific contributions of learning trajectories (LTs) to young children's learning of arithmetic. Initial analyses of the children's pretest to posttest level change were somewhat disappointing. That is, considering previous research and theory, we hypothesized that LT children would make a level transition, but only about half did so. This contrasted with positive signs of level transitions during the instructional sessions (Fig. 2 and online Appendix D). Instructors and assessors noted differences between the assessment and instruction environments (the unfamiliar person offering no emotional or strategic support and the limited amount of time given to a child to try to solve a problem); further, we hypothesize that indications of growth were also limited by our assessment items. For example, many children were able to complete more sophisticated and difficult problems at higher levels of the adding/subtracting LT (in which they showed incremental growth)—but for small numbers only. To minimize time of testing, most assessment items confounded those two characteristics, because higher conceptual levels usually involved larger numbers. Finally, there was an indication in LT children that when internalizing nascent strategies, some may make more mistakes, resulting in seemingly stunted learning on tests of accuracy.

Testing the theoretical assumption of LTs that to be maximally effective instruction should follow the developmental progression for each child, we hypothesized that most skip treatment children would not grow substantially in their level of thinking. Only one children did show a partial positive transition. The *n* + 1 instruction that this child received in the final section may have contributed to this increase. No other skip child increased in learning trajectory level from pretest to posttest, and half decreased, suggesting that instruction that is not given in level-to-level order may be detrimental to learning or engagement.

However, two findings contradict an unmitigated negative evaluation of the skip treatment. First, all children in both treatments evinced some incremental growth within one or more levels of the LT, including levels high in the LT. Thus, the skip children did learn. Second, the child who made a partial level transition also made incremental growth in five levels of the learning trajectory, an impressive gain. Such increases are inconsistent with our LT hypothesis (although note that 1/3 of his instruction was at the *n* + 1 level) and indicate that at least some children do not need to following LT levels to learn to solve increasingly sophisticated problem types. This study's population was selected because children identified as gifted are the most likely to be able benefit from instruction that skipped levels, and there are indications that at least some could do so. Thus, we do not conclude that following the LT approach exactly is *necessary* for learning. We do conclude that, even for this group of exceptional children, the fewer levels skipped, the more learning was observed.

In summary, consistent with previous work (Clements et al., 2019), our findings indicate that a LT-based approach to teaching early arithmetic will facilitate greater learning than from instruction that skips levels, at least for most children. There were few signs of progress on levels of thinking in the skip group during the sessions. However, there are several limitations to our study. Our sample was one of convenience—two kindergarten classes at a private, gifted school, limiting generalization, although it was chosen for a principled reason (optimizing the chances for success of the counterfactual, skip approach). There are possible differences in classroom instruction due to the small numbers of participants and the lack of random assignment. The time we were allowed for both assessments instruction was confined by the teachers' and school schedule. Finally, comparing treatment groups on assessment items from all levels might be viewed as biasing results in favor of the LT group; however, the skip instruction spent more time on higher-level problems and the assumption of this approach (Carnine et al., 1997; Clark et al., 2012; Clements & Sarama, 2014; Wu, 2011) is that tasks from earlier levels will then be easily solved.

The next study in this sequence, designed based on the results of this study, is a large-scale test, cluster randomized trial from which we will be able to more conclusively address our questions regarding the efficacy of the learning trajectory for adding/subtracting. We also are testing such moderators as gender, age, language, and race. At scale, we will be able to investigate more thoroughly any relationship between prior knowledge—both in adding/subtracting and in counting—and adding/subtracting learning from each treatment. We will also examine the levels of the LT, as some children's inconsistent performance on levels (such as performing well on higher levels but not on a lower one) may indicate needed revisions to the LT's developmental progression. Finally, we will continue to collect session-by-session data to compare to pre- and posttest results that are collected using procedures designed to be more sensitive to children's learning.

References

Baroody, A. J. (1995). The role of the number-after rule in the invention of computational short cuts. *Cognition and Instruction, 13*, 189–219.

Baroody, A. J., Clements, D. H., & Sarama, J. (2019). Teaching and learning mathematics in early childhood programs. In C. Brown, M. B. McMullen & N. File (Eds.), *The Wiley Handbook of Early Childhood Care and Education* (1st ed., pp. 329-353). Hoboken, NJ: Wiley Blackwell Publishing.

Carnine, D. W., Jitendra, A. K., & Silbert, J. (1997). A descriptive analysis of mathematics curricular materials from a pedagogical perspective: A case study of fractions. *Remedial and Special Education, 18*(2), 66–81.

Carpenter, T. P., & Fennema, E. H. (1992). Cognitively guided instruction: Building on the knowledge of students and teachers. *International Journal of Educational Research, 17*(5), 457–457. doi: 10.1016/S0883-0355(05)80005-9

Clark, R. E., Kirschner, P. A., & Sweller, J. (2012). Putting students on the path to learning: The case for fully guided instruction. *American Educator, 36*(1), 6–11.

Clarke, D. M., Cheeseman, J., Clarke, B., Gervasoni, A., Gronn, D., Horne, M., . . . Sullivan, P. (2001). Understanding, assessing and developing young children's mathematical thinking: Research as a powerful tool for professional growth. In J. Bobis, B. Perry & M. Mitchelmore (Eds.), *Numeracy and beyond (Proceedings of the 24th Annual Conference of the Mathematics Education Research Group of Australasia, Vol. 1)* (pp. 9–26). Reston, Australia: MERGA.

Clements, D. H., & Sarama, J. (2014). *Learning and teaching early math: The learning trajectories approach* (2nd ed.). New York, NY: Routledge.

Clements, D. H., Sarama, J., Baroody, A. J., Joswick, C., & Wolfe, C. B. (2019). Evaluating the efficacy of a learning trajectory for early shape composition. *American Educational Research Journal, 56*(6), 2509-2530. doi: 10.3102/0002831219842788

Clements, D. H., Sarama, J., Wolfe, C. B., & Day-Hess, C. A. (2008/2019). *REMA—Research-based Early Mathematics Assessment* Denver, CO: Kennedy Institute, University of Denver.

El'konin, D. B., & Davydov, V. V. (1975). Children's capacity for learning mathematics. In L. P. Steffe (Ed.), *Soviet studies in the pyschology of learning and teaching mathematics* (Vol. 7, pp. 1 – 11). Chicago: University of Chicago Press.

Fantuzzo, J. W., Gadsden, V. L., & McDermott, P. A. (2011). An integrated curriculum to improve mathematics, language, and literacy for Head Start Children. *American Educational Research Journal, 48*, 763–793.

Frye, D., Baroody, A. J., Burchinal, M. R., Carver, S., Jordan, N. C., & McDowell, J. (2013). *Teaching math to young children: A practice guide*. Washington, DC: National Center for Education Evaluation and Regional Assistance (NCEE), Institute of Education Sciences, U.S. Department of Education.

Fuson, K. C. (1992). Research on learning and teaching addition and subtraction of whole numbers. In G. Leinhardt, R. Putman & R. A. Hattrup (Eds.), *Handbook of research on mathematics teaching and learning* (pp. 53–187). Mahwah, NJ: Erlbaum.

Fuson, K. C. (2004). Pre-K to grade 2 goals and standards: Achieving 21st century mastery for all. In D. H. Clements, J. Sarama & A.-M. DiBiase (Eds.), *Engaging young children in mathematics: Standards for early childhood mathematics education* (pp. 105–148). Mahwah, NJ: Erlbaum.

Geary, D. C. (2011). Cognitive predictors of achievement growth in mathematics: A 5-year longitudinal study. *Developmental Psychology, 47*(6), 1539–1552. doi: 10.1037/a0025510

Gelman, R., & Gallistel, C. R. (1978). *The child's understanding of number*. Cambridge, MA: Harvard University Press.

Gersten, R., Jordan, N. C., & Flojo, J. R. (2005). Early identification and interventions for students with mathematical difficulties. *Journal of Learning Disabilities, 38*, 293–304.

Ginsburg, H. P., & Baroody, A. J. (2003). *Test of Early Mathematics Ability (3rd ed.).* (3rd ed.). Austin, TX: Pro Ed.

Gravemeijer, K. P. E. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning, 1*, 155–177.

Jordan, N. C., Glutting, J., Dyson, N., Hassinger-Das, B., & Irwin, C. (2012). Building kindergartners' number sense: A randomized controlled study. *Journal of Educational Psychology, 104*(3), 647–660. doi: 10.1037/a0029018

Maloney, A. P., Confrey, J., & Nguyen, K. H. (Eds.). (2014). *Learning over time: Learning trajectories in mathematics education*. New York, NY: Information Age Publishing.

Murata, A., & Fuson, K. C. (2006). Teaching as assisting individual constructive paths within an interdependent class learning zone: Japanese first graders learning to add using 10. *Journal for Research in Mathematics Education, 37*(5), 421–456. doi: 10.2307/30034861

National Mathematics Advisory Panel. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*. Washington DC: U.S. Department of Education, Office of Planning, Evaluation and Policy Development.

National Research Council. (2009). *Mathematics learning in early childhood: Paths toward excellence and equity*. Washington, DC: National Academy Press.

NGA/CCSSO. (2010). *Common core state standards*. Washington, DC: National Governors Association Center for Best Practices, Council of Chief State School Officers.

Resnick, L. B., & Ford, W. W. (1981). *The psychology of mathematics for instruction*. Hillsdale, NJ: Erlbaum.

Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. New York, NY: Routledge.

Shepard, L., & Pellegrino, J. W. (2018). Classroom assessment principles to support learning and avoid the harms of testing. *Educational Measurement: Issues and Practice*, *37*(1), 52–57.

Siegler, R. S., & Jenkins, E. (1989). *How children discover new strategies*. Mahwah, NJ: Erlbaum.

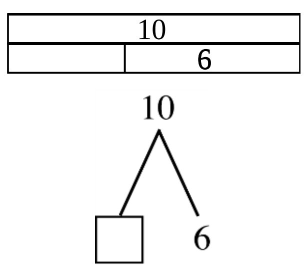
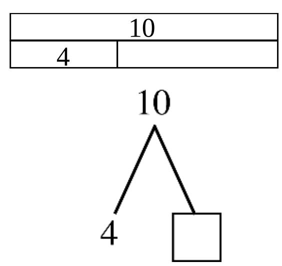
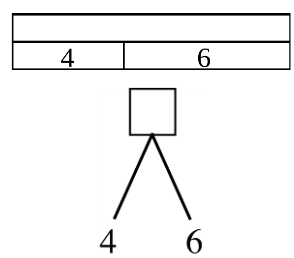
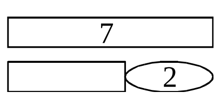
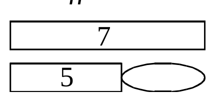
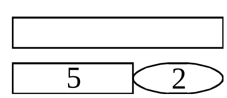
Steffe, L. P., & Cobb, P. (1988). *Construction of arithmetical meanings and strategies*. New York, NY: Springer-Verlag.

Steffe, L. P., Thompson, P. W., & Glasersfeld, E. v. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–306). Mahwah, NJ: Erlbaum.

Wu, H.-H. (2011). *Understanding numbers in elementary school mathematics*. Providence, RI: American Mathematical Society.

Wynn, K. (1992). Addition and subtraction by human infants. *Nature*, *358*, 749–750.

Appendix A: Addition and Subtraction Problem Types (Carpenter & Fennema, 1992; adapted from Clements & Sarama, 2014; NGA/CCSSO, 2010).

	Unknown		
Situation	**Start/Part/Smaller**	**Change/Part/Difference**	**Result/Whole/Larger**
Add-To (Join) An action of adding increases the number in a set.	*Start Unknown* □ + 6 = 11 Al had some balls. Then he got 6 more. Now he has 11 balls. How many did he start with?	*Change Unknown* 5 + □ = 11 Al had 5 balls. He bought some more. Now he has 11. How many did he buy?	*Result Unknown* 5 + 6 = □ Al had 5 balls and gets 6 more. How many does he have in all?
Take-From (Separate) An action of taking away decreases the number in a set.	*Start Unknown* □ - 5 = 4 Al had some balls. He gave 5 to Barb. Now he has 4. How many did he have to start with?	*Change Unknown* 9 - □ = 4 Al had 9 balls. He gave some to Barb. Now he has 4. How many did he give to Barb?	*Result Unknown* 9 - 5 = □ Al had 9 balls and gave 5 to Barb. How many does he have left?
Part-Part-Whole (Put Together/Take Apart) Two parts make a whole, but there is no action—the situation is static.	*Part Unknown*  Al has 10 balls. Some are blue, 6 are red. How many are blue?	*Part Unknown*  Al has 10 balls; 4 are blue and the rest are red. How many are red?	*Whole Unknown*  Al has 4 red balls and 6 blue balls. How many balls does he have in all?
Compare The numbers of objects in two sets are compared.	*Smaller Unknown*  Al has 7 balls. Barb has 2 fewer balls than Al. How many balls does Barb have?	*Difference Unknown*  Al has 7 blocks. Barb has 5. How many more does Al have than Barb?	*Unknown*  Al has 5 marbles. Barb has 2 more than Al. How many balls does Barb have?

References

Carpenter, T. P., & Fennema, E. H. (1992). Cognitively guided instruction: Building on the knowledge of students and teachers. *International Journal of Educational Research, 17*(5), 457–457. doi:10.1016/S0883-0355(05)80005-9

Clements, D. H., & Sarama, J. (2014). *Learning and teaching early math: The learning trajectories approach* (2nd ed.). New York, NY: Routledge. doi:10.4324/9780203520574

NGA/CCSSO. (2010). *Common core state standards*. Washington, DC: National Governors Association Center for Best Practices, Council of Chief State School Officers.

Appendix B: Skip Condition Children's Pretest and Posttest Adding/Subtracting LT Level and Assigned Instruction Plan by Problem Type and Problem Size

Child	Adding/Subtracting Pretest LT Level (x = 0)	Instruction A: Problem Type Problem Size ($n + x$)	Instruction B: Problem Type Problem Size ($n + x$)	Instruction C: Problem Type Problem Size ($n + x$)	Adding/Subtracting Posttest LT Level
14	Developing *1–Small Number +/-*	Start/Part Unknown 20-100 $n + 3$	Start/Part Unknown 0-20 $n + 3$	Change/Difference Unknown 0-10 $n + 2$	Developing *1–Small Number +/-*
15	Developing *1–Small Number +/-*	Start/Part Unknown 20-100 $n + 3$	Change/Difference Unknown 20-100 $n + 2$	Result/Whole Unknown 20-100 $n + 1$	Developing *1–Small Number +/-*
16	1–Small Number +/-	Start/Part Unknown 20-100 $n + 3$	Change/Difference Unknown 20-100 $n + 2$	Result/Whole Unknown 20-100 $n + 1$	Developing *1–Small Number +/-*
17	Developing *1–Small Number +/-*	Start/Part Unknown 20-100 $n + 3$	Start/Part Unknown 0-10 $n + 3$	Change/Difference Unknown 20-100 $n + 2$	None
19	2–Find Result +/-	Start/Part Unknown 20-100 $n + 2$	Start/Part Unknown 0-10 $n + 2$	Change/Difference Unknown 20-100 $n + 1$	Developing 2–Find Result +/-
20	None	Start/Part Unknown 20-100 $n + 4$	Change/Difference Unknown 20-100 $n + 3$	Change/Difference Unknown 0-10 $n + 2$	None
21	Developing 2–Find Result +/-	Start/Part Unknown 20-100 $n + 2$	Start/Part Unknown 20-100 $n + 2$	Result2Whole Unknown 20-100 $n + 1$	Developing 2–Find Result +/-
22	1–Small Number +/-	Start/Part Unknown 20-100 $n + 3$	Change/Difference Unknown 0-10 $n + 2$	Change/Difference Unknown 0-10 $n + 2$	Developing *1–Small Number +/-*
23	Developing 2–Find Result +/-	Start/Part Unknown 20-100 $n + 2$	Start/Part Unknown 20-100 $n + 2$	Change/Difference Unknown 20-100 $n + 1$	2–Find Result +/-

24	Developing 2–Find Result +/-	Start/Part Unknown 20-100 $n + 2$	Start/Part Unknown 0-10 $n + 2$	Change/Difference Unknown 0-10 $n + 1$	Developing 2–Find Result +/-
25	2–Find Result +/-	Start/Part Unknown 20-100 $n + 2$	Change/Difference Unknown 20-100 $n + 1$	Change/Difference Unknown 0-10 $n + 1$	Developing 2–Find Result +/-
26	Developing 2–Find Result +/-	Start/Part Unknown 20-100 $n + 2$	Start/Part Unknown 0-10 $n + 2$	Change/Difference Unknown 20-100 $n + 1$	1–Small Number +/-

Appendix C: Each Child's Pretest and Posttest LT Levels, Anticipated Level Transition, and Incremental Improvement within Levels

Child	Pretest LT Level	Anticipated Posttest LT Level based on Session Performance	Posttest LT Level	Level Change*	Incremental Improvement
LT Treatment					
1	Developing *1–Small Number +/-*	5–Part-Whole +/-	Developing 2–Find Result +/-	+	1–Small Number +/- 3–Find Change +/- 4–Counting Strategies +/- 5–Part-Whole +/-
2	Developing 2–Find Result +/-	3–Find Change +/-	Developing 2–Find Result +/-		7–Deriver +/-
3	None	2–Find Result +/-	Developing *1–Small Number +/-*	+	1–Small Number +/-
4	Developing 2–Find Result +/-	3–Find Change +/-	Developing 2–Find Result +/-		3–Find Change +/-
5	1–Small Number +/-	6–Numbers-in-Numbers +/-	*1–Small Number* +/-		2–Find Result +/- 4–Counting Strategies +/- 5–Part Whole +/-
6	Developing *1–Small Number +/-*	3–Find Change +/-	Developing 2–Find Result +/-	+	1–Small Number +/- 2–Find Result +/- 5–Part-Whole +/- 6–Numbers-in-Numbers +/-
7	None	4–Counting Strategies +/-	Developing 2–Find Result +/-	+ +	1–Small Number +/- 3–Find Change +/-
8	Developing 2–Find Result +/-	3–Find Change +/-	Developing 2–Find Result +/-		1–Small Number +/- 2–Find Result +/- 3–Find Change +/-
9	Developing 2–Find Result +/-	8–Problem Solver +/-	Developing 2–Find Result +/-		3–Find Change +/- 5–Part-Whole +/- 7–Deriver +/-
10	Developing *1–Small Number +/-*	5–Part-Whole +/-	Developing 3–Find Change +/-	++	1–Small Number +/- 2–Find Result +/- 4–Counting Strategies +/-
11	1–Small Number +/-	3–Find Change +/-	2–Find Result +/-	+	2–Find Result +/- 4–Counting Strategies +/-
12	Developing 2–Find Result +/-	7–Deriver +/-	Developing 2–Find Result +/-		3–Find Change +/- 4–Counting Strategies +/- 6–Numbers-in-Numbers +/- 7–Deriver +/-
13	Developing 2–Find Result +/-	6–Numbers-in-Numbers +/-	Developing 2–Find Result +/-		3–Find Change +/- 7–Deriver +/-

Skip Treatment					
14	Developing *1–Small Number +/-*	Developing *1–Small Number +/-*	Developing *1–Small Number +/-*		7–Deriver +/- 8–Problem Solver +/-
15	Developing *1–Small Number +/-*	Developing *1–Small Number +/-*	Developing *1–Small Number +/-*		2–Find Result +/- 4–Counting Strategies +/- 5–Part-Whole +/- 6–Numbers-in-Numbers +/-
16	1–Small Number +/-	1–Small Number +/-	Developing *1–Small Number +/-*	(–)	2–Find Result +/- 3–Find Change +/-
17	Developing *1–Small Number +/-*	Developing *1–Small Number +/-*	None	(–)	2–Find Result +/- 7–Deriver +/-
19	2–Find Result +/-	2–Find Result +/-	Developing 2–Find Result +/-	(–)	2–Find Result +/- 8–Problem Solver +/-
20	None	None	None		5–Part-Whole +/- 8–Problem Solver +/-
21	Developing 2–Find Result +/-	Developing 2–Find Result +/-	Developing 2–Find Result +/-		2–Find Result +/- 7–Deriver +/-
22	1–Small Number +/-	1–Small Number +/-	Developing *1–Small Number +/-*	(–)	1–Small Number +/- 2–Find Result +/-
23	Developing 2–Find Result +/-	Developing 2–Find Result +/-	2–Find Result +/-	(+)	2–Find Result +/- 3–Find Change +/- 5–Part-Whole +/- 6–Numbers-in-Numbers +/- 7–Deriver +/-
24	Developing 2–Find Result +/-	Developing 2–Find Result +/-	Developing 2–Find Result +/-		4–Counting Strategies +/- 5–Part-Whole +/-
25	2–Find Result +/-	2–Find Result +/-	Developing 2–Find Result +/-	(–)	2–Find Result +/- 3–Find Change +/- 4–Counting Strategies +/-
26	Developing 2–Find Result +/-	Developing 2–Find Result +/-	Developing *1–Small Number +/-*	–	5–Part-Whole +/-

* + indicates level increase; – indicates level decrease; (–) indicates partial level decrease.

Appendix D: Adding/Subtracting LT levels for Pretest, Each Instructional Session, and the Posttest, for the LT Group, Separated by Amount of Level Change (note that some children participated in 9 or 10 sessions)

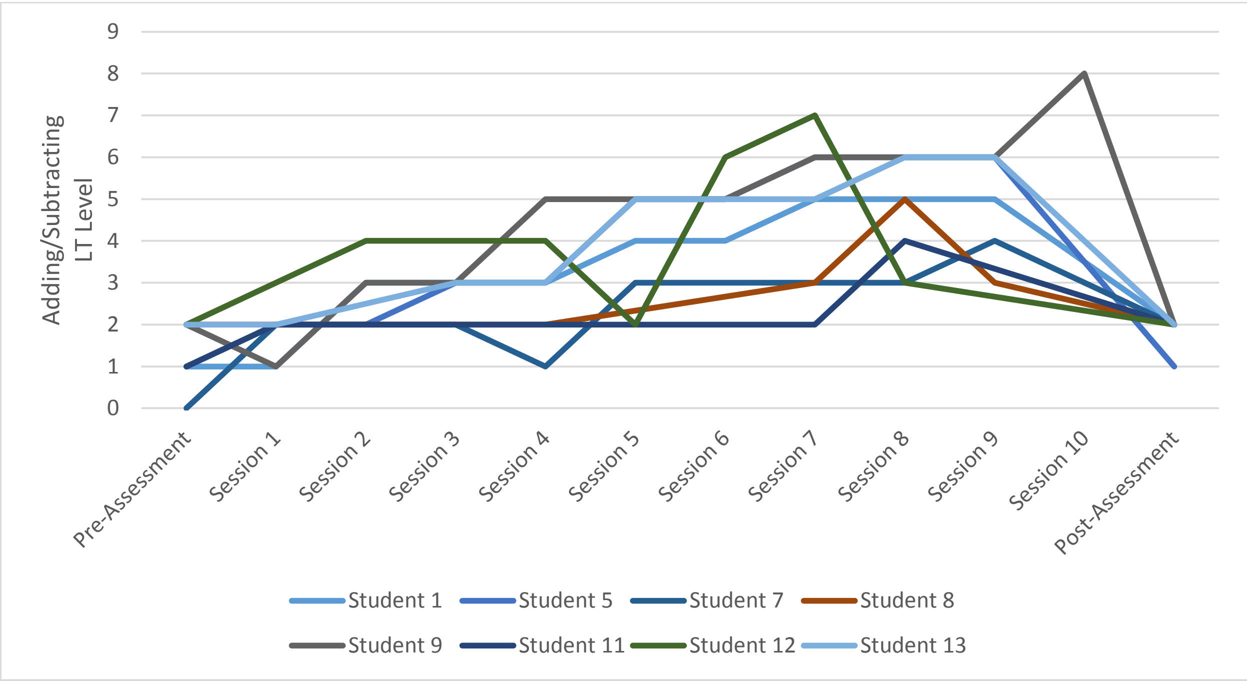


Figure D-1. For children who advanced at least 3 levels at some point in instruction, Adding/Subtracting LT levels for Pretest, Each Instructional Session, and the Posttest, for the LT Group (note that some children participated in 9 or 10 sessions). Levels, defined in Figure 1, are 1–Small Number +/-, 2–Find Result +/-, 3–Find Change +/-, 4–Counting Strategies +/-, 5–Part-Whole +/-, 6–Numbers-in-Numbers +/-, 7–Deriver +/-, and 8-Problem Solver +/-.

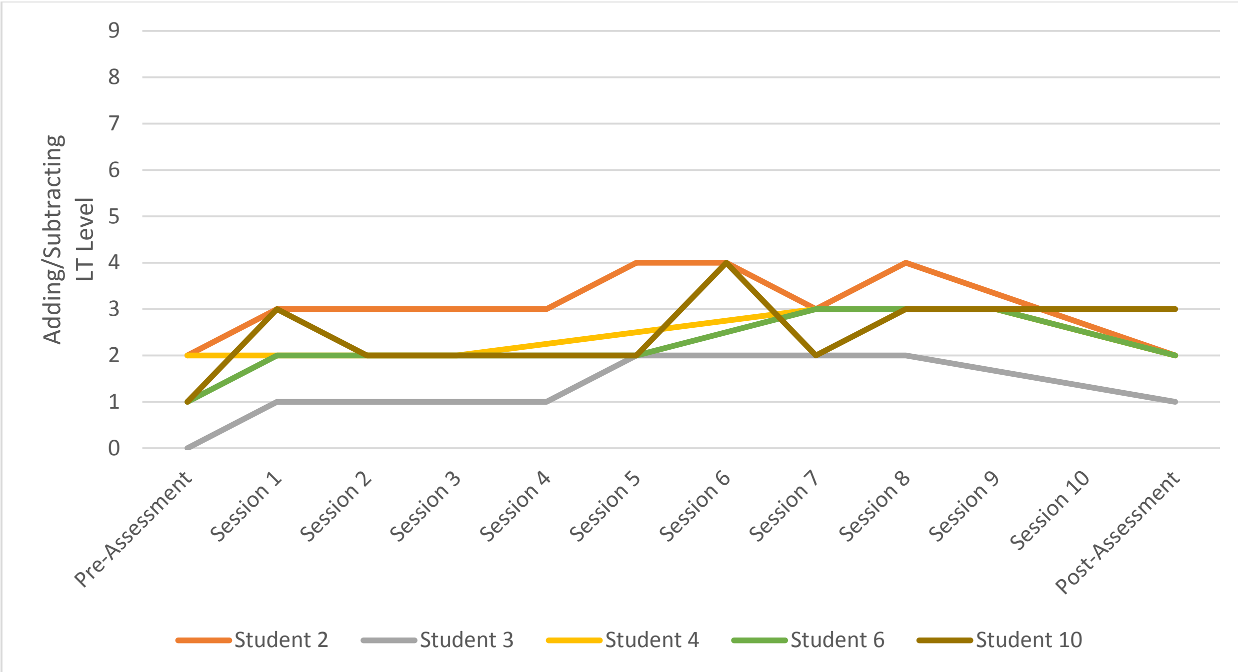


Figure D-2. For children who advanced only one or two levels at some point in instruction, Adding/Subtracting LT levels for Pretest, Each Instructional Session, and the Posttest, for the LT Group (note that some children participated in 9 or 10 sessions). Levels, defined in Figure 1, are 1–Small Number +/-, 2–Find Result +/-, 3–Find Change +/-, 4–Counting Strategies +/-, 5–Part-Whole +/-, 6–Numbers-in-Numbers +/-, 7–Deriver +/-, and 8-Problem Solver +/-.