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Ms. Morgan's Interpretation: An Instructional Minicase on Evaluating Student Approaches to a Problem of Linear Functions

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Abstract

There is a broad consensus that beginning teachers of mathematics need a strong foundation in mathematical knowledge for teaching (MKT), defined as the mathematical knowledge required to recognize, understand, and respond to the mathematical work of teaching in which one must engage. One recurrent challenge in teacher education is how to provide support for preservice teachers (PSTs) to acquire such competencies. Recent trends toward practice-based teacher education support the idea of engaging novice teachers in activities that are purposefully constrained to a core teaching practice. *Ms. Morgan's Interpretation* is an abbreviated instructional case (i.e., a minicase) based on an assessment scenario in which PSTs are asked to attend to student reasoning about representing linear functions. PSTs are asked to judge the mathematical validity of students' explanations as a way of further developing their own MKT.

Keywords: mathematics education, linear functions, student explanations, mathematical knowledge for teaching (MKT), teacher preparation, preservice teachers (PST)

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We plan to increase the number of minicases in the coming years and to make further improvements based on feedback from those using the materials. If you would like to make suggestions, please contact Heather Howell at hhowell@ets.org.

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Maria DeLucia is now retired from Middlesex Community College.

On the following pages, we present the fruits of a line of work that has spanned multiple projects over multiple years and reflects the contributions of a number of individuals at different points in time. The rationale for the minicases' development is, in essence, quite simple. Much of recent scholarship on teachers' mathematical knowledge for teaching (MKT) has focused on the assessment of MKT via practice-based questions. Practice-based questions generally include a short introductory scenario whose features are critical in solving the task. These scenarios are not simply window dressing for the task, but rather, along with the specified mathematical content, they codefine what is measured (Phelps & Howell, 2016). As such, these tasks can be understood to constitute abbreviated representations of teaching practice (Lai et al., 2013).

Because there has been intense interest in the field in assessing MKT, sample assessment tasks currently make up much of the field's description of specific MKT. Since such assessments became available, we have been approached by several teacher educators interested in integrating MKT assessment items into the curricular content of their mathematics and mathematics methods courses, not by using them as assessments, but by using them as exemplar instructional cases rather than assessments (see Lai & Howell, 2014, for example tasks). However, a number of obstacles to this kind of use have been noted, leading teacher educators to request publicly available full sets of materials that are aligned to instructional goals. Our goal in developing the minicases was to take on some of these challenges by developing a set of support materials designed to aid teacher educators in making use of the items as a curricular resource and, at the same time, to illustrate one type of support that could be developed more broadly out of such items.

The development team consists of researchers in mathematics and mathematics education as well as current teacher educators. This work began initially as part of a 2011 project at ETS intended to investigate the design features of MKT items in hopes of identifying relationships between structural features of the items and how well they performed in measuring MKT. This project used released items from the Measures of Effective Teaching (MET) project (Bill and Melinda Gates Foundation, 2013) and, for each item, created an analytic memo. The purpose of the memo was (a) to document the reasoning a test taker might use in responding, clearly identifying in each case not just why the intended answer was best but also the logical basis on which each of the competing answer choices could be discarded, and (b) to map that reasoning to types of specialized, common, and pedagogical knowledge as described in Ball et al.'s (2008) theory of MKT. Over the subsequent year, the team worked to refine these documents and tailor them to the possibility of serving multiple audiences, including item writers, researchers, teacher educators, and test takers themselves. We used this documentation in a validity study (Howell, Phelps, et al., 2013) and disseminated it at a number of conferences (Howell et al., 2016; Howell et al., 2017; Howell & Mikeska, 2016; Howell, Weren, & Ruiz Diaz, 2013; Lai & Howell, 2014; Phelps et al., 2013) where we received not only critical feedback but also enthusiastic reception from teacher educators eager to see and use more of them. In 2013, a separate project funded by the National Science Foundation (NSF; https://www.nsf.gov/awardsearch/showAward?AWD_ID=1445630) created a set of secondary level MKT items with accompanying documentation, collected similar validity evidence (Lai & Howell, 2016), and furthered our dissemination goals by creating a Google group in which the items and documentation are housed and available to interested parties (see Appendix F).

With a critical mass of systematic assessment documentation at hand, we decided to further develop this material into a set of "MKT minicases," documents designed to be used directly by teacher educators in supporting preservice teachers' (PSTs) development of MKT. We chose the name *minicase* to distinguish these materials from *instructional cases* (L. S. Shulman, 1986; Stake, 1987) because they differ from each other in structure and in degree of specificity (J. H. Shulman, 1992). The minicases are shorter than many cases used in professional preparation and are not structured to reveal additional information beyond the initial scenario. The minicases also target very specific knowledge about teaching and learning and are less open to interpretation than most instructional cases.

In 2016 and 2017, ETS funded the development of four minicases (two at elementary level and two at secondary level) based on teacher educator input. In 2018, we solicited reviews of the materials from four researchers in the fields of mathematics and mathematics education and six practicing teacher educators. The feedback from these reviews was then used to revise the set of four minicases to improve mathematical accuracy and comprehensiveness as well as usability.

Background

There is a broad consensus that beginning teachers of mathematics need a strong foundation in mathematical knowledge for teaching (MKT), defined as the mathematical knowledge required to recognize, understand, and respond to the mathematical work of teaching in which one must engage (Ball et al., 2008). Standards call out, for example, competencies for beginning teachers such as possessing “robust knowledge of mathematical and statistical knowledge and concepts, . . . expanding and deepening [PSTs’] knowledge of students as learners of mathematics,” and engaging in “effective and equitable mathematics teaching practice” (Association of Mathematics Teacher Educators, 2017, p. 6). One recurrent challenge in teacher education is how to provide support for PSTs to acquire such competencies. Recent trends toward practice-based teacher education support the idea of engaging novice teachers in activities that are purposefully constrained to a core teaching practice (Ball & Forzani, 2009). The MKT minicases we have developed represent one such example.

Research on using cases for subject-specific teacher learning goes as far back as the 1990s (Sykes & Bird, 1992). In mathematics and teacher education, cases can also provide a common language, explicit expectations of high-quality mathematics teaching, information about K–12 student development and common misunderstandings, and a means to interact with challenging content (Barnett, 1991).

Each minicase includes a situated task of teaching practice originally developed as part of teacher assessment efforts. Our guiding hypothesis is that these assessment scenarios, along with the accompanying documents that make up the minicases, form a set of resources for teacher educators. These resources are designed to support instructional goals, including developing PSTs’ understanding of K–12 student and higher level mathematics, developing PSTs’ orientations toward K–12 students and student work, helping PSTs understand what makes up the professional work of teaching mathematics, and providing them opportunities to engage in the cognitive work associated with addressing the given task.

Because each situated task was originally designed for assessment purposes and crafted to have a single best answer, the resulting minicases require users to take a stand with respect to the presented problem. These cases, unlike instructional cases that are more open-ended, invite response and disagreement in a way that can support rich but focused discussion. Our intention is to support teacher educators who are teaching math methods courses or math content courses for PSTs by providing a set of materials that can be used flexibly and adapted as appropriate.

Instructional Task: The Morgan Item

During a lesson on writing equations of linear functions represented in tables, Ms. Morgan asked her students to write an equation of the linear function represented in the table below and to explain how they found their answers.

x	y
1	6
2	11
3	16
4	21

Students found the correct equation, but they gave different explanations of how they found their answers. For each of the following student explanations, indicate whether it demonstrates a mathematically valid approach to writing the equation of a linear function.

Student response	Demonstrates a mathematically valid approach	Does not demonstrate a mathematically valid approach
(A) Each time the value of x goes up by 1, the value of y goes up by 5, so the slope is 5. And if x goes down by 1, then y will have to go down by 5 so the y-intercept is 1. That means the equation is $y = 5x + 1$.		
(B) I just looked at the value of y and saw that it kept increasing by 5, so $m = 5$. Then I subtracted that number from the first value of y in the table, so $b = 1$. You always put m times x and add the b, so the equation is $y = 5x + 1$.		
(C) For this function, I saw that you can always multiply the value of x by 5 and then add 1 to get the value of y, so the equation is $y = 5x + 1$.		

Mathematical Content

The Morgan minicase focuses on the mathematical content of the different representations of linear functions as well as the practices of interpreting student work and evaluating mathematical explanations. Functions are heavily featured in Grade 8 of the Common Core State Standards, although middle school and high school students have been working on precursory skills such as analyzing patterns and relationships in elementary grades and using variables in Grades 6 and 7. Middle school and high school students must develop an understanding of function and specific types of functions. Working with linear functions is often used as the gateway to learning more mathematically challenging functions, such as exponential or quadratic functions, which appear in the Common Core State Standards for high school. Furthermore, understanding of linear functions supports the learning of many statistical models, such as correlation and linear regression. Of note, the Morgan minicase does not provide the students' grade level, and this is mathematical content that is relevant in both middle school and high school.

Student Thinking and Learning

The Morgan minicase presents students' conceptualizations of linear functions. For example, the *covariational view*, or noting the ways in which two quantities change in relation to one another (Carlson et al., 2002), is represented by Student A's response to the Morgan item whereas the *correspondence view*, which focuses on determining a *y*-value from a given *x*-value, is shown through the response that Student C presented in the Morgan item. Although both of these views are important for teachers to understand and identify, recent research indicates that PSTs' understanding of functions can benefit from activities designed to make the distinction between the covariational and correspondence views explicit (Yemen-Karpuzcu et al., 2015).

This minicase also reflects how students think about representing functions in multiple ways (e.g., a table and an equation). Fluently moving between representations is a difficult skill for many students to acquire (Nathan et al., 2002). In the Morgan item, for example, Student A provides an adequate explanation of correctly identifying the slope and *y*-intercept using values from the table whereas Student B's explanation focuses on one column of values to the relative neglect of the other.

Work of Teaching

It is also important that teachers learn how to interpret student work and evaluate mathematical reasoning in both spoken and written forms. Although these are skills that will continue to develop throughout a teacher's career, introducing PSTs to these professional expectations can improve their beginning instruction (Fennema et al., 1996). In examining the student explanations in the Morgan minicase, PSTs are prompted to evaluate the mathematical validity of student thinking and to consider what we accept as an adequate explanation.

Elaborated Answer Key

This section provides teacher educators with an explanation of the answer choices for the Morgan item and a justification for the intended answer in terms of a mathematically valid approach and generalizability.

What Is This Assessment Item Asking?

In this item, Ms. Morgan has asked her students to write an equation of a linear function. The PSTs' task is to evaluate each student's explanation separately and to decide whether there is enough evidence that the explanation demonstrates a mathematically valid approach to writing the equation of a linear function. All of the students have arrived at correct answers, so the point is to evaluate the methods, not the final answers. The main work of this assessment item can be thought of, then, as having two steps: First, determine what general method is implied from the student explanation. Second, decide whether the explanation demonstrates a mathematically valid approach.

What Information Is Important?

The given item involves a table of *x*- and *y*-values where the ratio of the change in *x*-values to the change in *y*-values is constant, which means there is a corresponding linear equation that can be written in standard form, $y = mx + b$, where *m* is the slope and (0, *b*) is the *y*-intercept. It remains to find the *y*-intercept since the value of *y* when *x* is zero is not given in the table. An explanation that illustrates a mathematically valid approach includes a description of a method that is generalizable, for instance, a method that the student could use to get a correct answer even if they were using a different table.

Student response	Demonstrates a mathematically valid approach	Does not demonstrate a mathematically valid approach
(A) Each time the value of *x* goes up by 1, the value of *y* goes up by 5, so the slope is 5. And if *x* goes down by 1, then *y* will have to go down by 5 so the *y*-intercept is 1. That means the equation is *y* = 5*x* + 1.	✓	
(B) I just looked at the value of *y* and saw that it kept increasing by 5, so *m* = 5. Then I subtracted that number from the first value of *y* in the table, so *b* = 1. You always put *m* times *x* and add the *b*, so the equation is *y* = 5*x* + 1.		✓
(C) For this function, I saw that you can always multiply the value of *x* by 5 and then add 1 to get the value of *y*, so the equation is *y* = 5*x* + 1.	✓	

What Is the Rationale for Selecting an Answer?

Student A: Mathematically Valid Approach

According to the student's given explanation in the Morgan item, Student A correctly looks at the change in *x* and the change in *y* to find the slope of the line that passes through the points on the table. Since the ratio is a constant value of 5, the student concludes that this is the value for slope, *m*, in the standard form, $y = mx + b$. Then the student seems to find the *y*-intercept by finding the value of *y* when *x* is equal to zero. That is, the student subtracts 1 from the first *x*-value in the table and 5 from the corresponding *y*-value using the concept of slope that has already been established. The student concludes that the *y*-intercept is (0, 1) and replaces *b* with 1. Because this student uses generalizable methods to find both the slope and *y*-intercept, this student demonstrates a mathematically valid approach to writing the equation of a linear function.

Student B: Not Mathematically Valid Approach

Student B looks only at the change in *y* and then subtracts that answer from the first *y*-value in the table with no reference to what the value of *x* would be in that ordered pair. There is not enough evidence to conclude that the student recognizes that the change in *y* is 1 or to determine whether the student always thinks that the change in *x* is 1 regardless of the given table or if the student thinks it is possible for the change in *x* to be something else. Additionally, the student's explanation lacks a clear understanding of the *y*-intercept because it does not state that *x* is equal to 0 when *y* is equal to 1. Because the student does not identify the change in *x* as the *y*-values change, nor the value of *x* when *y* is equal to 1, it is not clear that the method is generalizable to other problems. Thus, the explanation does not demonstrate a mathematically valid approach to writing the equation of a linear function.

Student C: Mathematically Valid Approach

Student C notices a consistent relationship between the values of *x* and *y* throughout the table and is able to write an equation that shows this relationship by relying on a *correspondence*, or input/output, view of functions. The explanation demonstrates a mathematically valid approach to writing the equation of a linear function for this situation. Presumably, given another table for a different linear function, this student would look for the pattern and would be able to come up with the appropriate equation. Although the evidence is less clear in this case as to how the student would generate a correspondence-based rule in response to a more difficult problem, the method itself is generalizable and, therefore, represents a mathematically valid approach.

Instructional Objectives the Minicase Might Support

This section describes teacher educators' potential objectives of this minicase as a situated task to support variable instructional goals, including development of PSTs' understanding of the mathematical content of linear functions and their practice of interpreting and evaluating student work in terms of mathematical validity and generalizability. Although this minicase lends itself to supporting the particular objectives, teacher educators may find additional reasons to use this case.

Understanding Student-Level Content

The relevant student-level content includes representing data provided in a numeric form (a bivariate data table) and in a symbolic form (a linear equation). It also includes understanding slope as a constant rate of change, understanding that the y-intercept of a linear equation is the value at which the corresponding x-value is 0, and explaining how to produce the equation. For PSTs whose understanding of the student-level content is weak, examining student errors may provide an unthreatening way to unpack the mathematics without causing the PST to feel embarrassed.

Developing Productive Orientations Toward K–12 Students and Student Work: Emphasizing the Practice of Interpreting Student Explanations

To respond to the Morgan item coherently requires PSTs to analyze the given student work. This item could provide PSTs with a concrete context in which to discuss more general dispositions or instructional values such as giving K–12 students opportunities to generate and discuss their own solutions, listen carefully, and consider how next instructional moves might vary depending on what the K–12 students have or have not understood. It could help PSTs appreciate how understanding student thinking can inform teachers' responses.

Appreciating the Larger Mathematical Idea

Functions can be thought of from a covariational or correspondence view, each of which has affordances and limitations. For PSTs, the given approaches demonstrate covariational (Student A) and correspondence (Student C) views of function, creating an opportunity to discuss and contrast those conceptualizations of linear functions and support them in seeing their affordances, limitations, and connections to later mathematics.

Understanding That the Work of Teaching Requires Close Analysis of Student-Generated Work and Attention Both to the Validity of the Underlying Mathematics and to the Adequacy of the Explanation

Analyzing student-generated strategies that are potentially unconventional or flawed requires teachers to engage in a different and more complex kind of mathematical analysis than is required to solve student-level problems. The item asks PSTs whether the written explanations demonstrate a mathematically valid approach. This item could support a discussion with PSTs of what it means for an approach to be valid, what it means for it to be adequately explained, and how those may be different. This item also illustrates that the work the teacher needs to do to make sense of and evaluate student approaches is really different from the work the teacher has to do to solve student-level problems, potentially supporting a discussion with PSTs of the professional demands of the work and what they should expect to be ready to do as they enter classrooms.

Understanding How to Analyze Student Work Samples in Terms of the Validity and Adequacy of the Explanations: Making Sense of the Student Work in This Item

The item provides PSTs with a context in which to practice these analytic skills, essentially serving as a practice exercise in the specific mathematics required by the situation and allowing PSTs to unpack the specific mathematics and evaluate the explanations that are given. In addition, this item might support a discussion with PSTs about what they need to know and notice to do that work and how they might approach similar situations differently as a result of that practice.

References

Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics.* www.amte.net/standards

Ball, D. L., & Forzani, F. M. (2009). The work of teaching and the challenge for teacher education. *Journal of Teacher Education*, *60*(5), 497–511. https://doi.org/10.1177/0022487109348479

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Barnett, C. (1991). Building a case-based curriculum to enhance the pedagogical content knowledge of mathematics teachers. *Journal of Teacher Education*, *42*(4), 263–272. https://doi.org/10.1177/002248719104200404

Bill and Melinda Gates Foundation. (2013). *Ensuring fair and reliable measures of effective teaching: Culminating findings from the MET Project's three-year study.* https://usprogram.gatesfoundation.org/-/media/dataimport/resources/pdf/2016/12/met-ensuring-fair-and-reliable-measures-practitioner-brief.pdf

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352–378. https://doi.org/10.2307/4149958

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education*, *27*(4), 403–434. https://doi.org/10.2307/749875

Howell, H., Garcia, N., Frank, S. S., & Phelps, G. (2017, March 30). Bringing content knowledge for teaching (CKT) into the elementary mathematics teacher education curriculum [Webinar]. In *American Association of Colleges for Teacher Education Webinar Series.* https://videos.aacte.org/2017-03-30-bringing-content-knowledge-for-teaching-ckt-into-the-elementary-mathematics-teacher-education-curriculum

Howell, H., Lai, Y., Miller, E., & Phelps, G. (2016, January 28–30). *Assessments of secondary level mathematical knowledge for teaching as opportunities for productive struggle* [Paper presentation]. Association of Mathematics Teacher Educators 20th Annual Conference, Irvine, California, United States.

Howell, H., & Mikeska, J. (2016, February 23–25). *Using content knowledge for teaching assessments as tools for learning and feedback to effect program improvement* [Paper presentation]. American Association of Colleges for Teacher Education 68th Annual Meeting, Las Vegas, Nevada, United States.

Howell, H., Phelps, G., Croft, A. J., Kirui, D., & Gitomer, D. (2013). *Cognitive interviews as a tool for investigating the validity of content knowledge for teaching assessments* (ETS Research Report No. RR-13-19). ETS. https://doi.org/10.1002/j.2333-8504.2013.tb02326.x

Howell, H., Weren, B., & Ruiz Diaz, S. (2013, April 17–20). *Mathematical knowledge for teaching measures as opportunities for professional learning* [Paper presentation]. National Council of Teachers of Mathematics 2013 Annual Meeting, Denver, Colorado, United States.

Lai, Y., & Howell, H. (2014). Deploying tasks assessing mathematical knowledge for teaching as tasks for professional preparation. In T. Fukawa-Connolly, G. Karakok, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 17th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 770–791). Mathematical Association of America. http://sigmaa.maa.org/rume/RUME17.pdf

Lai, Y., & Howell, H. (2016, October 3). Conventional courses are not enough for future high school teachers [Blog post]. *On Teaching and Learning Mathematics.* http://blogs.ams.org/matheducation/2016/10/03/conventional-courses-are-not-enough-for-future-high-school-teachers/

Lai, Y., Jacobson, E. D., & Thames, M. H. (2013, April 27–May 1). *The role of pedagogical context in measures of specialized and pedagogical content knowledge* [Paper presentation]. American Educational Research Association Annual Meeting, San Francisco, California, United States.

Nathan, M. J., Stephens, A. C., Masarik, K., Alibali, M. W., & Koedinger, K. R. (2002). Representational fluency in middle school: A classroom study. In D. S. Mewborn, P. Sztajn, D. Y. White, H. G. Wiegel, R. L. Bryant, & K. Nooney (Eds.), *Proceedings of the 24th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 463–472, ED471747). ERIC. http://files.eric.ed.gov/fulltext/ED471747.pdf

Phelps, G., & Howell, H. (2016). Assessing mathematical knowledge for teaching: The role of teaching context. *The Mathematics Enthusiast*, *13*(1–2), 52–70.

Phelps, G., Howell, H., Weren, B., & Ruiz Diaz, S. (2013, January 24–26). *Assessments of content knowledge for teaching (CKT) as opportunities for teacher learning* [Paper presentation]. Association of Mathematics Teacher Educators 17th Annual Conference, Orlando, Florida, United States.

Shulman, J. H. (Ed.). (1992). *Case methods in teacher education*. Teachers College Press.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4–14. https://doi.org/10.3102/0013189X015002004

Stake, R. E. (1987). An evolutionary view in programming and staff development. In M. F. Wideen & I. Andrews (Eds.), *Staff development for school improvement: A focus on the teacher* (pp. 55–69). The Falmer Press.

Sykes, G., & Bird, T. (1992). Teacher education and the case idea. *Review of Research in Education*, *18*(1), 457–521. https://doi.org/10.3102/0091732X018001457

Yemen-Karpuzcu, S., Ulusoy, F., & Işiksal-Bostan, M. (2015). Prospective middle school mathematics teachers’ covariational reasoning for interpreting dynamic events during peer interaction. *International Journal of Science and Mathematics Education*, *15*(1), 89–108. https://doi.org/10.1007/s10763-015-9668-8

Appendix A. Sample Lesson Outline

This appendix provides teacher educators a sample lesson outline to use with PSTs that includes lesson goals, links to prior learning of and about linear functions, and suggestions for lesson implementation. This sample lesson may provide an illustration of how a whole lesson can be planned around the Morgan minicase or broken up across multiple class sessions. The lesson plan is designed to be user ready, although it is only one example of how a lesson might be configured.

Representing Linear Data

Goals for This Lesson

For PSTs of secondary mathematics to do the following:

- Identify and understand the affordances and limitations of covariational and correspondence views of function. (See the Embedded Student Content section for brief descriptions of covariational and correspondence views.)
- Develop greater understanding of the practice of noticing student work, particularly attending to and making sense of student explanations and how satisfactorily the student explanations demonstrate mathematical validity and generalizability; attend to student understanding within covariational and correspondence views in the context of representing bivariate data with a linear equation.
- Identify and carefully analyze students' covariational approaches to representing bivariate data with a linear equation; analyze students' explanations of constant rate of change and use of the definition of y-intercept of the graph of a linear function in x and y as the y-value whose corresponding x-value is zero.

Embedded Student Content

In this lesson, PSTs are asked to analyze three students' work on this task:

Write an equation of the linear function represented in the table below and explain how you found your answer.

x	*y*
1	6
2	11
3	16
4	21

This task asks the students to represent bivariate data, with variables *x* and *y*, with a linear equation. One approach to solving the task involves understanding slope of a linear function as a constant rate of change—that the *y*-intercept of a linear function in *x* and *y* is the *y*-value whose corresponding *x*-value is zero—and explaining how to produce the equation. This approach relies on a covariational view of function: understanding how changing the value of one variable impacts the value of the other variable and coordinating changes in one variable with changes in the other.

Alternatively, solving the task can also be approached with the correspondence view of a function's equation: knowing that a value of *y* corresponds to a value of *x* in a solution to an equation in *x* and *y*, if the equation $y = f(x)$ when evaluated at those values makes a true statement.

Both covariational and correspondence views have their advantages and disadvantages; the covariational view can describe a function's behavior, but unlike the correspondence approach, it may not provide an immediate method for checking whether a value is correct or not.

Opener: Number Game

The lesson begins with two number games as a context to discuss the definition of constant rate of change and a definition of linear function in terms of constant rate of change. PSTs should begin by completing the number game.

Number Game

- Choose an integer.
- Subtract 1 from your number.
- Take what you have now and multiply it by 2.
- Take what you have now and add 4.
- Take what you have now and multiply it by 3.
- Take what you have now and multiply by 1/2.
- What value did you get? What values would you get if you started with the inputs of 1, 3, 5, 7?

Input number	Game output
N	G
1	
3	
5	
7	

In this game, what would the output be if the input were 0?

Give two explanations for how you know that this is a reasonable answer.

Discussion for Number Game: Covariational and Correspondence Views of Function

Discuss the number game with the PSTs. The purpose of this discussion is to elicit two kinds of explanations: one using covariational explanation using *constant rate of change,* and the other a correspondence explanation using the *definition of solution*. The reason for eliciting these two kinds of explanation are to motivate the definitions for constant rate of change, linear function, and solution, as well as to raise the ideas of covariational and correspondence views of functions.

An additional idea that may come up in this discussion is whether the explanations given by the PSTs are satisfying mathematical explanations. Because this is something PSTs are likely to attend to in student explanations, it is worth paying attention to this idea and separating it from the notion of mathematical validity. *Mathematically validity* refers to those approaches that are correct and can generalize to reasonable sets of similar problems; *satisfying* *mathematical explanations*, put briefly, are those that include adequate evidence of student reasoning, are expressed precisely, and are reasonably complete.

Covariational View, Featuring Constant Rate of Change

This explanation uses the fact that in the table, the G-values appear to have a constant rate of change with respect to the N-value. The key components of this explanation are as follows:

- Identifying the constant rate of change.
- Using the constant rate of change (in particular, having to divide the differences in consecutive table values by 2 to get the input value for 0).
- What this explanation might leave out—and it is critical to press the PSTs in your class about this—is the generalizability question: How does one know that the constant rate of change will hold for all possible N, including 0? It is okay to leave this as an open question because the next part of the lesson returns to this question, but the question needs to be opened at this point in the lesson.

The above explanation is based on a covariational view of the game functions.

Correspondence View, Featuring Definition of Solution

The solution explanation is that when you start with 0 and go through the steps, you obtain the previously predicted value. This explanation is based on a correspondence view of the game function.

Satisfying Mathematical Explanations

This is a quick discussion with the PSTs that can be revisited in later lessons as well as during the Morgan case; some courses build on this list as a public record over time. (For instance, using a thread on a discussion board if the course has an online platform.) There are many points that could be brought up here, but the key points for the Morgan case are as follows:

- A mathematical explanation accounts for all possibilities (here, the possibilities are of numerical input, e.g., is sufficiently complete).

- A mathematical explanation is based on mathematical facts that are known.
- A mathematical explanation answers the question asked.
- A mathematical explanation is mathematically accurate.

Covariational Versus Correspondence View

Both covariational and correspondence views have their advantages and disadvantages, depending on the situation. Here, the covariational view allows for more conceptual understanding of the pattern that emerges from the Number Game but does not provide an immediate method because there is more to explain; however, the correspondence view allows us to check immediately what the output of 0 is while not giving us a big picture. Putting them together and realizing they are two ways to approach functions helps us understand an idea more deeply. It most benefits middle-school and high-school students when they are fluent in both views, and so it is to a teacher's advantage to know and recognize both.

Defining Constant Rate of Change, Linear Function, and Solution

Provide the below definitions and discuss them with the PSTs, unpacking their understanding of and about these key mathematics concepts of this task.

Definition: A function has a constant rate of change if the ratio of change in the dependent variable: Change in the independent variable is constant.

Definition: A function is linear if it has a constant rate of change.

Linear functions are sometimes defined differently. Ask the PSTs: How does this definition relate to other definitions of linear function that you might be familiar with?

Example: $y = 3x$, $y = 3x + 3$, $y = 3x + 4x - 6$, $y = -5 + 7x + 8 - 1009x$. In each of these cases, *y* is a linear function of *x*.

Nonexample: *y* is the number you get by choosing *x*, multiplying by 5, adding 1, then multiplying the answer you get by itself. (This introduces a square factor: $y = (5x + 1)^2$.)

Quick Discussion Example

Prompt: The number games each define a function. For the inputs that we've seen, it looks like the functions might have a constant rate of change. But will they have a constant rate of change even for numbers that we haven't tried yet as inputs? How do you know?

Response: Each number game is a composition of linear functions, and the composition of linear functions is itself linear. One way to see this is by using the equations. This is because if you start with a linear function, it has a constant rate of change; adding or subtracting a number doesn't change the rate of change; and multiplying or dividing changes the rate of change by multiplying it, so it is still a constant rate of change.

Homework Assignment

Suggest the PSTs create their own 5-step number game that results in the described tables:

- Results in a table where inputs go up by 1
- Results in a table where inputs decrease by 1
- Results in a table where inputs increase by 2

Ask the PSTs to prove the following theorem: A function is a linear function if and only if it can be expressed in the form $f(x) = mx + b$, where $m, b \in \mathbb{R}$.

Definition: An ordered pair of values in N and G (or any other two variables) is a *solution* to an equation in those variables if evaluating the equation at those values makes a true statement. Example: The ordered pair (0,3) is a solution to the equation is G = Game(N), where Game is the function defined by performing the steps of the game.

Definition: The *y-intercept* or *vertical intercept* of the graph of a function is the output value *y* whose corresponding input value is zero. Example: The output G you found for the game when input is N = 0 is 3, so 3 is the vertical intercept of the graph of the function G = Game(N).

Situating the Concepts in Teaching: Ms. Morgan's Class

Now the PSTs examine a situation where students represent the values in a table with an equation. Ask the PSTs to solve the Morgan item. As PSTs read through the student responses, have them think about the following:

- What is each student's logic? How might they have arrived at each step of their solution?
- What would each student have to do to convince you that they understand? What makes you think that each student does not understand? What are you unsure that each student understands?
- Are the explanations mathematically sound? Why or why not?

Spend the next 10 minutes discussing the students' work with the PSTs. Suggest that they fill out a table for each student and then determine whether the student's work shows a mathematically valid approach.

I think that Student A understands ...	I think that Student A does NOT understand ...	I am unsure whether Student A understands …

I think that Student B understands ...	I think that Student B does NOT understand ...	I am unsure whether Student B understands …

I think that Student C understands ...	I think that Student C does NOT understand ...	I am unsure whether Student C understands …

Discussion for the Case of Ms. Morgan's Class

Based on PSTs' analysis above, ask the following question: Do you think that the students demonstrate a mathematically valid approach to writing the equation of a linear function? Take a moment to have the PSTs indicate what they are thinking.

Student	Demonstrates a mathematically valid approach	Does NOT demonstrate a mathematically valid approach
Student A		
Student B		
Student C		

The goal for the discussion is to situate the concepts of covariational and correspondence view as well as the concepts of constant rate of change, linear function, *y*-intercept, and solution. Ideas that should arise in the course of the discussion include the following:

- Student A (DOES demonstrate a mathematically valid approach)
    - Understands concept of constant rate of change, *y*-intercept, linear function
    - May not understand correspondence view of function
- Student B (does NOT demonstrate a mathematically valid approach)
    - Does know that linear functions have the form $y = mx + b$, where $m, b \in \mathbb{R}$
    - Does not understand constant rate of change
    - May not understand *y*-intercept
    - Does not necessarily understand correspondence view of function
- Student C (DOES demonstrate a mathematically valid approach)
    - Does know that linear functions have the form $y = mx + b$, where $m, b \in \mathbb{R}$
    - Does understand correspondence view of function
    - May not understand constant rate of change, *y*-intercept

Here, the PSTs may broach discussion about partial validity as opposed to binary views of mathematical validity. They may also comment on the adequacy of the explanation and raise the question of whether Student B, if prompted to explain in more depth, might understand more than the teacher can conclude based on the initial written answer. This is a nice point to spend time on, as it allows for a contrast between validity and adequacy of explanation as well as reinforcing the pedagogical point that, in real classrooms, teachers might capitalize on the opportunity to elicit more detail from the student in cases such as these. Valuable contrasts may also come up. For example, why do we have sufficient evidence to assume Student C's reasoning generalizes but Student B's does not? In addition, PSTs may ask, with respect to the mathematics, how precise or explicit the language needs to be to draw particular conclusions; for example, Student C implicitly knows the definition of a linear function, but we do not know whether Student C would describe it as a linear function or recognize the defining features, so the extent of the conclusions the teacher can draw about Student C's understanding of linear functions is limited.

Summary of Discussion of Morgan Case

In general, when attending to and making sense of student work, have the PSTs think about:

- What is each student's logic? How might they have arrived at each step of their solution?
- What would each student do to convince you that they understand? What makes you think that each student does not understand? What are you unsure that each student understands?

When making sense of what the student understands or may not understand, suggest that the PSTs attend to the core mathematics concepts of the task. In the case of linear functions, such concepts may include the following:

- constant rate of change
- definition of linear function
- form of linear function

- *y*-intercept
- correspondence view of function

Closing

Covariational and correspondence views have different advantages and disadvantages, and both are worth understanding. Sometimes you need to understand what's happening at each point of a function, in which case the correspondence view is very useful. Other times, the behavior of a function is the most important, in which case the covariational view is most important.

Appendix B. Additional Discussion Prompts

This appendix provides teacher educators more ideas to prompt and orchestrate discussion with PSTs with regards to the Morgan item in addition to the sample lesson plan for the Morgan minicase.

The following is a list of potential discussion prompts, extensions, or additional assignments teacher educators might use with PSTs around the Morgan item. Have the PSTs think about the following:

- How might each student have responded differently to the original task if the *x*-values had been 0, 2, 4, and so on?
- Based on the given student explanations, describe for each response what a student would need to have said to convince you that they understand linear functions, what makes you think the high school-student doesn't understand linear functions, and about what are you not sure you have enough evidence to draw a conclusion?
- What next question might you pose to the students if you wanted to . . .
  - find out whether Student A and Student B have the same understanding?
  - help Student C reason covariationally?
  - help Students A and B reason from a correspondence view?
- How would you create a similar problem in which Student C's approach would be noticeably easier or more efficient than Student A's? How would you create a similar problem in which it would be harder or less efficient?
- Imagine you are pairing up students to share their work with one another. How would you pair up these three students to share their thinking (A with B, B with C, or A with C)? What would your intention be in pairing them in that way? In other words, what would you expect each pair to understand as a result of sharing with one another what they did not understand initially?
- What next instructional steps might you take to address the misunderstanding Student B may have? How would you expect those steps to play out?

- Imagine it turns out that Student B did understand the mathematics but did a poor job of explaining. What more than what is written would they have to say to you to convince you that they did, in fact, understand?
- For each student work sample, ask the PSTs to write an example of the following, noting what qualities the explanation has that make it that type of explanation:
    - what you think a high-quality explanation of that mathematical process would look like
    - what you think the minimally adequate explanation of that mathematical process would look like
    - an example of an explanation you would not consider adequate
- Having PSTs reflect back on their work on the Morgan item, what are some techniques they might have in mind in looking at student work samples in the future?
- This item looked at two types of representations of data: a table and a symbolic representation. How might you bring in graphing? What might be more visible to students in a graphical representation, and what might be more difficult for them to understand in comparison?

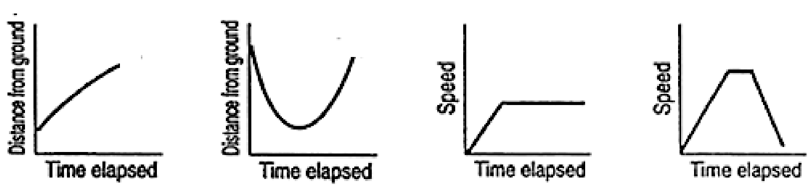
Appendix C. Aligned Task—Hillyard

This appendix provides teacher educators an additional item (the Hillyard item, focused on interpreting the meaning of graphs of functions) with its elaborated answer key. Like the Morgan item, the Hillyard item focuses on the concept of function but asks PSTs to complete different work than in the Morgan item. This item might be used to provide an opportunity for PSTs to practice interpretation of student work.

The Hillyard Task

The Hillyard item is similar to the Morgan item in that it focuses on a closely aligned topic (understanding the concept of function) but varies in the work PSTs are asked to complete. Rather than evaluating student work, this task asks PSTs to anticipate student thinking in response to a problem a teacher might pose and evaluate what the students would be likely to learn. The task might be helpful as a follow-up or homework assignment or could be used as a second lesson that would follow the Morgan item. What follows is the Hillyard item and the elaborated answer key.

Ms. Hillyard is preparing a lesson as part of a unit on functions and their graphs in her Algebra I class. While looking through the resources that accompany the textbook used in her class, she notices an activity in which students are asked to describe situations that could lead to the graphs shown here.



She has observed that the majority of the students in her class comfortably perform the procedural skills involved in graphing functions but that they do not yet have a well-developed concept of functions. She wonders whether it would be appropriate to use this activity with the students in her class. Which of the following *best* characterizes the use of the activity with the students in Ms. Hillyard's class?

A. The activity is *not* appropriate because none of the graphs include negative numbers in the domains or ranges of the functions.

B. The activity is *not* appropriate because the lack of numeric labels on the axes will prevent students from attending to important numeric values on the graphs, such as intercepts.

C. The activity *is* appropriate because real-world functions often do not include negative numbers in their domains and ranges.

D. The activity *is* appropriate because the lack of numeric labels on the axes will help students focus on the meaning of the graphs.

What Is This Assessment Task Asking?

This assessment task asks PSTs to analyze an activity that Ms. Hillyard is considering using in her class and to decide whether the proposed activity is appropriate for use with her students based on a characterization of how students might learn from the activity. Two relevant attributes of the activity are called out. The first is the restriction of the given graphs to the first quadrant, which excludes negative numbers from both the domains and the ranges of the graphs. The second is the lack of numerical labels on the axes. Therefore, the PSTs will need to consider the role of negative domain and range values as well as the relevance/importance of numerical labels to decide whether the activity is appropriate for her students.

What Information Is Important?

It is important to note that Ms. Hillyard's students are fluent in the procedural skills of graphing functions but do not yet have a well-developed concept of functions. An activity would be appropriate for her students if it encourages them to develop a conceptual understanding of functions without relying on their already strong procedural skills. Although the definition of function can vary and we do not know the grade level of the students, it is likely that the central focus would be on the concept of dependency of one variable on another.

It is also important to note that two attributes of the activity are mentioned in the characterizations: the graph excluding negative domain and range elements and the lack of numerical labels on the graph. The key consideration is how each of these supports (or does not support) the students' development of a conceptual understanding of functions without allowing a reliance on procedural skills. Other attributes of the functions might be of interest to a teacher; for example, that each could be taken to be representative of a real-life scenario, that some are curves and others piecewise linear, and so on. But these are not factors that the assessment task asks the PSTs to analyze.

The first attribute of the given graphs is that all four fall in the first quadrant, which excludes negative domain and range elements. Although familiarizing students with a variety of types of functions, including those with varied domain and range values, is desirable at some point, the key idea of dependency can be expressed clearly even for functions where the domain and range are nonnegative. Therefore, this attribute is not a reason to conclude that the activity is inappropriate.

The second attribute is that none of the four given graphs have numerical labels. The concept of a function as variable dependency is not made more or less visible by including (or not including) numerical labels on the graph. Not including them does make it less likely that students could use their procedural skills to calculate or approximate values in lieu of thinking about the relationship between the variables. Therefore, the lack of numerical labels makes the activity more appropriate for these students.

What Is the Rationale for Selecting an Answer?

Option A: Not the Best Answer

Option A states that the activity is not appropriate because negative values are not included in the domain or range for any of the given functions. Although there may be other reasons to include negative values in examples, a strictly positive domain and range does not detract from the key idea that one variable depends on the other. Therefore, this is not a reason to conclude that the activity would not be appropriate, and Option A is not the best answer.

Option B: Not the Best Answer

Option B states that the activity is not appropriate because without numeric labels, students will not attend to important values on the graphs such as intercepts. It is true that without numeric labels students will not know the values of such points as the intercepts, but this does not mean they will not attend to them, and it is also not necessary for students to attend to the intercepts in order to understand the concept of a function. Additionally, because there are no numeric labels, Ms. Hillyard's students will not be able to use their strong procedural skills and will be compelled to think about the graphs in a purely conceptual manner. Again, this is not a reason to conclude that the activity would be inappropriate. In fact, this reason is one that suggests the activity is appropriate. Therefore, Option B is not the best answer.

Option C: Not the Best Answer

Option C states that the activity is appropriate because real-world functions often do not include negative numbers in their domains and ranges. The activity is appropriate but not for this reason. Real-world functions sometimes do not include negative values in their domains and ranges, but they often do. And the central idea of functions is that of variable dependency, which can just as easily be considered in the first quadrant. Although it is true that the activity is appropriate, this characterization of the activity is not accurate, and Option C is not the best answer.

Option D: The Best Answer

Option D states that the activity is appropriate because the lack of numeric labels on the axes will help students focus on the meaning of the graphs. Not including numeric labels on the axes does make it more likely that students will pay attention to the shape of given graphs in terms of the concept of a function as variable dependency rather than looking at or calculating the numerical values on the graph, like intercepts, by using their procedural skills. The activity is appropriate, and the lack of numeric labels contributes to the appropriateness. Thus, Option D accurately characterizes both the appropriateness and the reason for doing this activity and is the best answer.

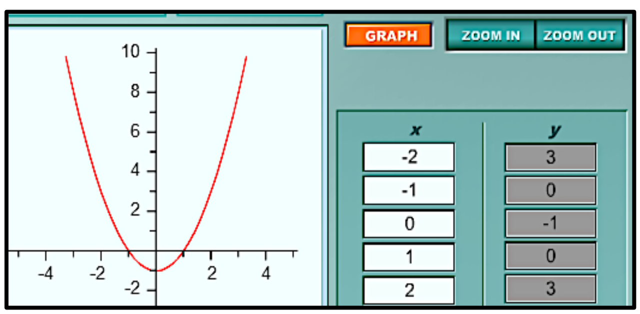
Appendix D. Aligned Task— Carlies

This appendix provides teacher educators another item (the Carlies item focuses on quadratic functions) with its elaborated answer key. This item can be used to provide PSTs a further learning opportunity to practice interpreting student work.

The Carlies Task

The Carlies item is similar to the Morgan item in that it asks the respondent to decide if three student work samples demonstrate or do not demonstrate mathematically valid reasoning. However, it is focused on different mathematical content: quadratic functions. The item might be helpful as a follow-up or homework assignment or as a second lesson that would follow the Morgan minicase. What follows is the Carlies task and the elaborated answer key.

During a lesson in his Algebra II class, Mr. Carlies asked his students to find the quadratic equation of the form $y = ax^2 + bx + c$ that corresponds to the graph and table shown below and explain how they determined their answers.



Three students each said the equation was $y = x^2 - 1$, but each gave a different explanation of how they determined their answers. For each of the following student explanations, indicate whether it demonstrates valid reasoning for why the equation must be $y = x^2 - 1$.

Representations of student work	Demonstrates valid reasoning	Does NOT demonstrate valid reasoning
Since the vertex of the graph is (0, −1), I knew that the equation had to be $y = x^2 + 0x - 1$, and then I simplified that equation to get $y = x^2 - 1$.		
I graphed $y = x^2$ on my graphing calculator, and I saw that if I translated it down 1 unit it would line up with the graph and table values, so I knew the equation was $y = x^2 - 1$.		
I can see that the zeros of the equation are −1 and 1, so I multiplied $(x + 1)(x - 1)$ and got $x^2 - 1$ so the equation was $y = x^2 - 1$.		

What Is This Assessment Task Asking?

This assessment task asks PSTs to determine if each student uses valid mathematical reasoning to find an equation for a quadratic function given its graphical and tabular forms. Three students have come to the same conclusion but have used different methods and techniques. The PSTs are asked to read through each explanation, determine what the high school student did, and then decide the mathematical validity of each student's work.

What Information Is Important?

To accurately solve the Carlies item, the PSTs will need to use information from both the graph and the table in at least some way. Since Mr. Carlies has asked for the quadratic equation in this assessment scenario, the PSTs can assume that the equation will be quadratic; it is not necessary to determine whether quadratic is the best fit for the given data.

What Is the Rationale for Selecting an Answer?

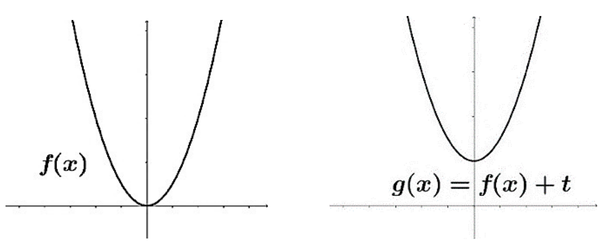
Student A: Not Mathematically Valid Reasoning

Although Student A's language is not fully precise, a reasonable interpretation is that the student is thinking that the coordinates of the vertex correspond directly to constants in the function, as they do in the vertex form, $y = a(x - h)^2 + k$. In this case, the student has let *b* = 0 and *c* = −1 and substituted directly into $y = ax^2 + bx + c$. There is a mismatch between the student's expectation that the vertex coordinates will match the constants and his choice of the form of equation. The coordinates would match if the student had used the vertex form, but in standard form, they do not. Even if the mismatch had not occurred, the student still would need to account for the leading coefficient *a.* Although the student's function is the correct one, $y = x^2 - 1$, the answer is correct only, and coincidentally, because of the special properties of this function. Student A does not use a valid mathematical technique to get the equation and, therefore, does not demonstrate valid reasoning.

Another possibility is that the student figured out *b* = 0 using the fact that *x*-coordinate for the vertex is equal to $\frac{-b}{2a}$. Since *x* = 0 then *b* = 0. The student may also have noticed that the graph is shifted down from the normal $f(x) = x^2$, so they made *c* = −1, which happens to work here because this equation is a special case. The student then forgot to incorporate the *a*-value; intentionally left out the *a*, thinking it was unnecessary; or just assumed that *a* = 1. Since in this situation *a* does in fact equal 1, the student happened to get the answer correct. This method would not work in all cases, however, so the student is not demonstrating mathematically valid reasoning.

Student B: Mathematically Valid Reasoning

Student B uses function transformations to generate a quadratic equation. The parent function $f(x) = x^2$ can be translated *t* units up or down by adding a constant term, as demonstrated in the equation $g(x) = f(x) + t$. If you know how to graph $f(x)$, you can add *t* to each of the *y*-coordinates to graph $g(x)$.



t can be a positive or negative value. When *t* is negative, the graph will be translated down $|t|$ units.

This student uses a calculator to graph $y = x^2$ and notices that translating it down by 1 unit will bring us to the desired graph and table. There are practical limitations on the generalizability of the technique, as seeing this shift by inspection could be challenging if it were not shifted by an integer value, but the student is using sound mathematics. Therefore, this student demonstrates mathematically valid reasoning.

Student C: Not Mathematically Valid Reasoning

Student C attempts to use the *x*-intercepts alone to determine the quadratic equation, substituting the roots corresponding to the *x*-intercepts $(r_1, 0)$ and $(r_2, 0)$ into the equation $y = C(x - r_1)(x - r_2)$. While the substitution and subsequent algebraic simplification are performed correctly, $r_1 = -1$ and $r_2 = 1$ into $y = C(x - r_1)(x - r_2)$ to get $y = C(x + 1)(x - 1) = C(x^2 - 1)$, the student fails to account for the constant term *C*. Because this function happens to have a leading coefficient of 1, the student does end up with the same equation, but a crucial step is omitted, and it is not clear that the student had a rationale for omitting it. Since the reasoning for the final formula of $y = x^2 - 1$ is not complete, this student does not demonstrate mathematically valid reasoning.

Appendix E. Resources and References

This appendix provides a few additional resources that are relevant to the mathematics and/or teaching practices mentioned in this minicase. In particular, the two articles from *Mathematics Teacher* can be potentially assigned as reading for PSTs.

Common Core Standards Writing Team. (2013, March 1). *Progressions for the Common Core State Standards in Mathematics (draft).* Institute for Mathematics and Education, University of Arizona. http://commoncoretools.me/wp-content/uploads/2013/07/ccss_progression_functions_2013_07_02.pdf

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352–378. https://doi.org/10.2307/4149958

Paoletti, T., Stevens, I. E., & Moore, K. C. (2017). Tricks may inhibit students' reasoning. *Mathematics Teacher*, *110*(6), 446–453. https://doi.org/10.5951/mathteacher.110.6.0446

Teuscher, D., & Reys, R. E. (2010). Slope, rate of change, and steepness: Do students understand these concepts? *Mathematics Teacher*, *103*(7), 519–524. https://pubs.nctm.org/view/journals/mt/103/7/article-p519.xml

Appendix F. Frequently Asked Questions

Where did the assessment items come from?

These items were produced by ETS staff in 2013 in an effort to determine how and how well item designs targeting elementary level MKT would extend to the secondary level, and the items were later utilized in a validity study (NSF grant number 1445630/1445551; https://www.nsf.gov/awardsearch/showAward?AWD_ID=1445630). The team that conducted the NSF work maintains an active Google Drive to provide access to items and elaborated answer rationale documents for the entire pool of items to interested scholars. If you are interested in joining this group, contact the ETS lead for the secondary MKT work, Heather Howell, hhowell@ets.org.

There's something I would like to change about the item. / I don't agree with the way the math is presented in the item. Would you consider changing it?

We decided in our work on the minicases to use the assessment items exactly as they were provided by the projects they came from (see the first FAQ). One goal of the further development work is to explore how existing intellectual capital in the form of assessments can be repurposed into material for teacher learning. The minicases have developed organically across a set of projects over a number of years, and there have been many contributors to them. The latest versions were reviewed by four experts in the field of mathematics and mathematics education, and their advice has been incorporated into revisions.

Part of what we want to illustrate is that the assessment item itself need not be above critique for it to be a useful starting point for PST learning. In fact, we think some critique might signal rich points for discussion as part of teacher development. That said, the point of the minicase is to be provocative, not prescriptive, and we encourage anyone who wishes to tweak, alter, subvert, delete, or completely rewrite the assessment item in service of their own instructional goals to do so. (And if it's an item from the Google Drive, we hope you'll post your work back in the drive for others to use!)

I would like to use these items as a hiring screen for new teachers, where could I find more of them?

This is not an approved use of these items. Accessing these items requires that you agree to terms of use, which exclude high-stakes decision making.

Where could I find more minicases like these?

We only have a few exemplars ready for use at the current time, but we are more than happy to share them on request. To be added to our distribution list, contact Heather Howell, hhowell@ets.org.