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Compression is Evident in Children's Unbounded and Bounded Numerical Estimation: A Reply to Cohen and Ray

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Abstract

Kim and Opfer (2017) found that number-line estimates increased approximately logarithmically with number when an upper bound (e.g., 100 or 1000) was explicitly marked (bounded condition) and when no upper bound was marked (unbounded condition). Employing procedural suggestions from Cohen and Ray (this issue), we examined whether this logarithmicity might come from restrictions on the response space provided. Consistent with our previous findings, logarithmicity was evident whether tasks were bounded or unbounded, with the degree of logarithmicity tied to the numerical value of the estimates rather than the response space per se. We also found a clear log-to-linear shift in numerical estimates. Results from Bayesian modeling supported the idea that unbounded tasks are qualitatively similar to bounded ones, but unbounded ones lead to *greater* logarithmicity. Our findings support the original findings of Kim and Opfer (2017) and extend their generality to more age groups and more varieties of number-line estimation.

Keywords: Cognitive development, numerical estimation, mathematical cognition

Introduction

Cohen and Ray (this issue; C&R henceforth) argue that the logarithmic compression in number-line estimation observed by Kim and Opfer (2017) came from flawed methods—i.e., testing numbers too large to overestimate in the space provided by conventional computer displays. This methodological argument relies on a crucial, untested assertion—viz., that compression (negative acceleration) is simply related to the space provided for estimates, rather than knowledge of numeric magnitude. In our reply, we test their idea empirically, following their methodological suggestions. To presage our results, we found that our original conclusions stand, that compression exists even when children are given space to radically overestimate numbers (e.g., if estimates increase with $n^{1.5}$), and that C&R's calculation of the response space required (and thus choice of numerical stimuli) relies on an ill-fitting, biased statistical model.

According to C&R, unbounded number-line estimation is characterized by *expansion* (positively acceleration; Cohen et al., 2018) in estimates. To record an exponential increase, researchers must leave enough space to the right of a unit (0-1 line) by obtaining computer displays that do not yet exist or by limiting stimuli to small numbers. This is important in developmental studies, C&R claim, because children's estimates are *extremely* expansive in unbounded tasks. For example, Cohen and Sarnecka (2014) reported that children overestimated numbers two to three times larger than actual magnitudes, which they modeled using a power function with a large exponent ($n^{1.5}$). Given this model (and a computer display with a horizontal resolution of 1280 pixels), the highest number that could be presented for children to estimate would be 58 (Eq. A6 in C&R, this issue). The Cohen Ray Number-Line task limits numbers even further (to 17).

To our knowledge, however, most studies have found children's unbounded estimates to be *compressive* or *linear* (see Table B1 for summarized studies), reflecting children's developing representations of numerical magnitude. For C&R, however, compression can be explained by researchers not leaving adequate space for accelerating overestimation.

To address whether unbounded number-line estimation is expansive or compressive, we conducted a new developmental study, using the methods of C&R to ensure that adequate space was provided to capture an accelerating pattern of overestimates (if they exist). In our study, children completed four estimation tasks on identically sized number lines. The three unbounded tasks—where children estimated small (2-56), medium (2-128), or large numbers (2-427)—successively provided children with less and less room for radically over-estimating numeric magnitude. The one bounded task, where large numbers (2-427) were estimated on 0-538 number lines, allowed us to revisit the conclusions from Kim & Opfer (2017). (For full details on methods, see Appendix A).

Following C&R's reasoning, an expansive-to-compressive pattern would be expected as the range of numbers increased in value (small, medium, and large) and room for overestimates decreased. When estimating small numbers (2-56) in the response space that could capture exponential increase (i.e., enough space to accurately estimate numbers up to 538), expansion would be expected. When estimating large numbers (128-427), compression would be evident, putatively for lack of space.

In contrast, if compressive patterns in unbounded estimates mirror logarithmic representation of numbers, compression would be evident in small, medium, and large number estimation. Further, the amount of logarithmic compression would be expected to decrease with age and increase with number, replicating the log-to-linear shift observed in previous studies of bounded number-line estimation (Berteletti et al., 2010; Dehaene et al., 2008; Heine et al., 2010;

Friso-van den Bos et al., 2015; Kim & Opfer, 2017, 2018; Laski & Yu, 2014; Opfer et al., 2011, 2016; Opfer & Martens, 2012; Opfer & Siegler, 2007; Opfer & Thompson, 2008; Sasanguie et al., 2012; Sella et al., 2015; Siegler & Booth, 2004; Siegler & Opfer, 2003; Thompson & Opfer, 2008; White & Szűcs, 2012; Yuan et al., 2019).

Results

Does compression in unbounded estimates stem from a lack of space for overestimation? To examine this, we used the mixed log-linear model (MLLM) to compute the degree of logarithmicity in number-line estimates (Cicchini et al., 2014; Kim & Opfer, 2017, 2018; Opfer et al., 2016; Yuan et al., 2019). The MLLM includes a parameter (λ) that is equal to 1 when estimates are perfectly logarithmic and to 0 when estimates are perfectly linear.

Compression versus expansion. As shown in Figure 1, median estimates in all conditions were compressive and well fit by the MLLM (R^2s=.93 to .97), with non-zero logarithmicity (λs=.09 to .44). In all conditions, fits of the compressive MLLM was better than fits of the expansive, single-scallop power model (1SPM) favored by Cohen et al. (2018), whether using R^2 or AICc as a criterion (ΔAICc from 1SPM-MLLM=38.15, 34.89, and 28.04 for unbounded-small, medium, and large respectively). This finding indicates that compressive estimation in Kim and Opfer (2017) did not result from a lack of space for children to overestimate.

Intriguingly, the 1SPM presented the poorest fit in the unbounded-small condition, despite this condition providing children with the most room for overestimation. The model-fitting of the 1SPM was poor (R^2=.40 for 1SPM vs. .93 for MLLM) and highly biased (i.e., 1SPM residuals correlated with to-be-estimated numbers, r= -.69, p<.01). Thus, under the most favorable methodological condition, the degree of bias in the expansion model (the systematicity of its errors) was greater than the degree to which the expansion model fit the data. This means the model C&R used to calculate the response space required for their number-line task is flawed; number-line tasks can use much larger numbers than recommended by C&R because children's estimates are not expansive.

Stability of individual differences in compression. We next examined whether logarithmic compression in individual participants was consistent across conditions. If number-line tasks index the degree of compression in numerical representation, one would expect children who were most logarithmic in a task with lots of room for overestimates to also be most logarithmic in a task with little room for overestimates. To address this, identical analyses were conducted on estimates of individual children (Table 1). Logarithmicity (λ) was significantly greater than 0 in all conditions, $t(28)=4.08$ to 9.47, $ps<.001$, Cohen's ds=.76 to 1.09. A majority of children were better fit by the MLLM than the 1SPM in unbounded conditions (52-86% of children by AICc). Replicating previous research (Kim & Opfer, 2017; Qin et al., 2017), logarithmicity components were also strongly correlated with one another, showing stable individual differences in number-line estimation regardless of boundedness or number size.

Logarithmicity in all conditions decreased with age, replicating log-to-linear shifts. We employed a mixed-effects model to test the effect of age on logarithmicity, with participants as a random effect and age and condition as fixed effects. Significance of fixed effects was tested using Satterthwaite's approximation method (Satterthwaite, 1941). Even after controlling for the effects of condition, age was a significant predictor of logarithmicity, $b=-.61$, $p<.001$, indicating decrease in logarithmicity with age.

Effect of task boundedness on compression. Would unbounded estimation be qualitatively different from bounded estimation? To test this, we conducted Bayesian modeling with the hierarchical Bayesian MLLM (Kim & Opfer, 2017) using RStan (Stan Development Team, 2018).[1]

Figure 2 illustrates the posterior distributions of logarithmicity for all conditions. First, all conditions had a non-zero mean of posteriors, *M*=.18 to .57, with a 95% highest posterior density interval (HPDI) without zero included, [.09, .32], [.12, .26], [.45, .68], [.34, .51] for the unbounded-small, medium, large, and bounded-large condition respectively. This indicates that estimates across conditions had logarithmic compression significantly distinct from perfect linearity. We next computed the posterior distributions of difference in logarithmicity among conditions using the HPDI method (Kruschke, 2014). Consistent with non-Bayesian results, the unbounded tasks with small and medium numbers did not differ from each other (95% HPDI [-.14, .11]), whereas they were substantially less logarithmic than the unbounded- and bounded-large tasks (95% HPDIs [.08 to .25, .36 to .55]). Importantly, unbounded-large estimates were more logarithmic than bounded-large estimates, 95% HPDI [.01, .28]. This is consistent with the re-analyzed results of Kim and Opfer (2017) using Bayesian modeling (see Appendix C for details). Together, the results provide supportive evidence that unbounded estimation is

[1] The Bayesian MLLM is formalized as follows: $y_{ijk} = a_k\left(\lambda_k \frac{U}{\ln(U)}\ln(x_{ijk}) + (1-\lambda_k)x_{ijk}\right) + e_{ijk}$, where i denotes trials that child j was given in condition k. The λ_k was transformed into the inverse of a logistic function for better parameter constraint: $\lambda_k = \frac{\exp(\beta_k)}{\exp(\beta_k)+1}$. The model parameters were assumed to be normally distributed (for details on priors, see Kim & Opfer, 2017). On 8 chains, 30,000 iterations were run in total, including 5,000 burn-in and 1,000 sample iterations with thining of the chains by 25 iterations.

qualitatively similar to bounded estimation with the logarithmic characteristics, but quantitatively *more logarithmic* than bounded estimation if estimated magnitudes are the same.

Discussion

We sought to examine if logarithmic compression in children's unbounded estimation came from insufficient space for overestimates, as argued by C&R. The results of the study were straightforward: children's estimates were logarithmically compressive regardless of space left for overestimation. As in Kim and Opfer (2017), we also found log-to-linear improvement with age in both bounded and unbounded tasks, even when unbounded task settings followed C&R's recommendations (unbounded-small condition). Further, we found that bounded and unbounded estimates were both characterized by a significant degree of logarithmicity and fit poorly by Cohen et al.'s (2018) 1SPM.

The degree of logarithmicity that we observed was within normally expected values. Because children learn the meaning of large numbers after they learn the meaning of small numbers, compression in number-line estimates typically declines with age and increases with scale (e.g., 0-100 vs. 0-1,000). In the current study, our 0-538 task elicited a λ-value of .23, which was smaller than the .73 value elicited by the 0-1,000 task in Kim & Opfer (2017). Apparently, use of one 'atypical' endpoint (538) did not make much difference, though when lower- and upper-bounds are atypical (e.g., 1639-2897 number lines as in Hurst et al., 2014), there may be more than a simple age x scale trade-off in compression.

What might have driven different conclusions in studies by Cohen and colleagues and our own? Broadly speaking, the differences among studies fall into two major categories – methodological and analytic. We have included Table B1 listing some of the major differences among the unbounded number-line studies. We will focus on methodological differences here but we refer the reader to our discussion of analytic differences in Kim and Opfer (2017, p.

1096). Among the various methodological differences for unbounded tasks, certain features do not seem to matter much, including the monitor size, the maximum number-line length, and the physical distance between 0 and 1. These factors play a large role in C&R's explanation for why we reach different conclusions, but here we manipulated these variables and found estimates were always compressive rather than expansive.

Other differences for unbounded tasks might matter. In our version, the 0-1 number-line remained in the same location with a fixed length; in theirs, the number-line changed location and length on every trial. In our version, children's first responses were recorded and could not be changed; in theirs, children could mark and modify their response as many times as they wished. In our version, children were presented each number only once; in theirs, children could be presented the same number repeatedly. In our version, all children were presented with the same numbers; in theirs, numbers were presented randomly. In our version, children were verbally encouraged to persist in the task; in their version, a cartoon image, a unique facial image, or an animation played between trials on a variable reinforcement schedule.

These same methodological differences carry over to differences in how the *bounded* version of the tasks were administered. What this means is that Cohen and Sarnecka (2014) never compared the classic number-line task to an unbounded version of the same, as we did in Kim and Opfer (2017). Thus, their bounded number-line changed size from trial to trial, appeared in a new location on every trial, and presented a unique set of (possibly identical) numbers to every child – with a dynamic distractor appearing between each and every response.

Quite possibly this buzzing, blooming, bounded number-line task was distracting, and children had difficulty sustaining focus. From research with adults, we know that presenting a visuospatial distractor increases logarithmicity (Anobile et al., 2012). This finding might explain why Cohen and Sarnecka (2014)'s 4- to 6-year-olds appeared much more logarithmic on bounded 0-20 tasks than expected. Although they did not use a MLLM, we fit the MLLM to data we recovered from their published figure and found strong compression (λ = .80, R^2 = .94) – much higher than observed among 4-year-olds (λ = .09) and 5-year-olds (λ = .07) on 0-20 tasks (Opfer et al., 2019). If anything, their children--recruited from a university-based center--looked more like the 3-year-olds (λ = .73) that Opfer et al. (2019) recruited from community-based centers. Further, Cohen and Sarnecka (2014) failed to counter-balance the order of bounded and unbounded tasks, always presenting unbounded tasks first while children were still fresh. The fact that 24 of 62 children could not complete their study adds to the impression that children were getting fatigued. Thus, a simple methodological explanation for our discrepant findings is that their procedure creates greater cognitive load, which leads to biased results when failing to counter-balance the order of conditions.

If differences between the bounded and unbounded tasks arose chiefly because Cohen and Sarnecka (2014) confounded fatigue and boundedness, one would not expect to find differences using a between-subjects design, especially among adults. Data from Cohen et al. (2018) speak to this issue, where data were collected from two different groups of adults. Using their preferred designs, evidence for differences between bounded and unbounded estimation "is extremely weak and therefore little confidence should be assigned to it" (p. 18). We agree.

In conclusion, compression in children's number-line estimates appear in both bounded and unbounded versions of the number-line task, whether much or some or little space is allocated for overestimates, and whether children are given small, medium, or large numbers. As long as these task variants are controlled and systematically manipulated, we would expect to find a developmental shift from greater compression to less compression as children learn the magnitude of numbers.

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Table 1. Mean (SD) of age and logarithmicity (λ) in four conditions, their correlations, and the percentages of children who were better fit by a MLLM.

	Mean (*SD*)	Correlation				% children by MLLM
		2	3	4	5	
1. Age	8.71(2.05)	-.73***	-.38*	-.72***	-.75***	
2. Unbounded small	.27 (.34)	-	.54**	.67***	.74***	86%
3. Unbounded medium	.23 (.30)		-	.52**	.39*	66%
4. Unbounded large	.53 (.30)			-	.86***	52%
5. Bounded large	.42 (.38)				-	66%

Note. * $p < .05$; ** $p < .01$; *** $p < .001$.

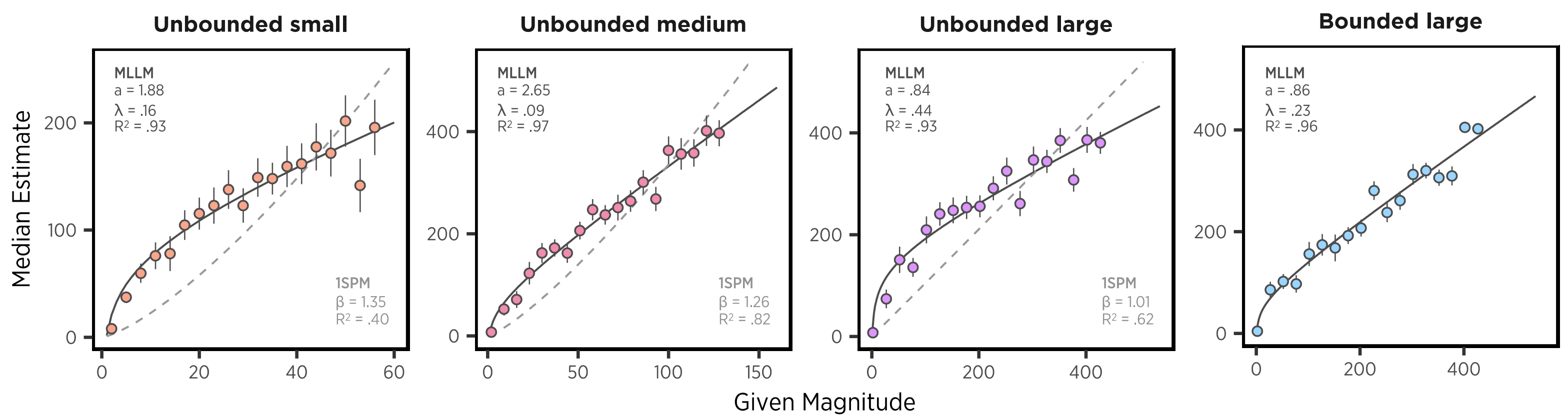

Figure 1. Median responses in each condition. The best-fitting parameters of the MLLM for all conditions are shown at the top left, whereas those of the 1SPM for unbounded conditions are located at the bottom right. Error bars represent standard errors of the mean.

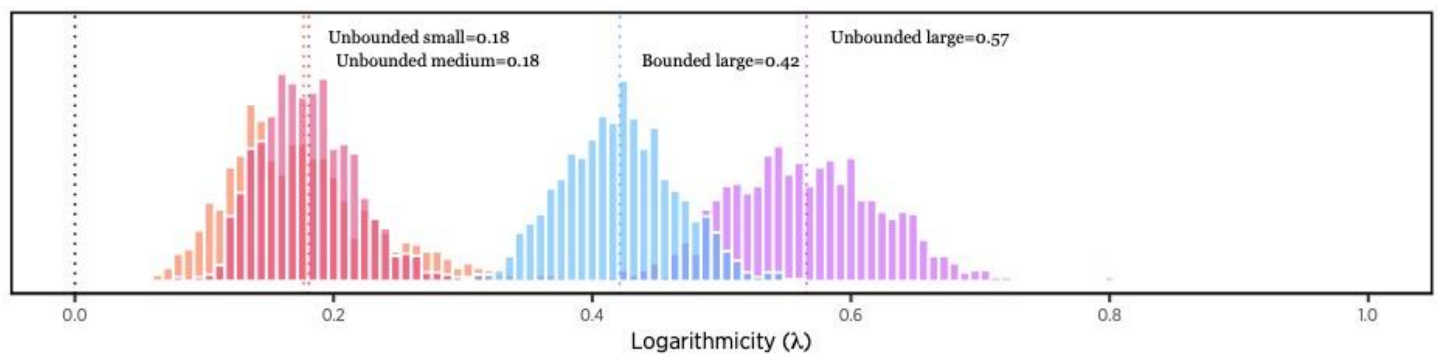

Figure 2. Posterior distributions of logarithmicity (λ) in four tasks: unbounded-small (orange), unbounded-medium (red), unbounded-large (purple), and bounded-large conditions (blue).

Appendix A

Method

Participants. Thirty-one children were recruited and tested in a quiet and dedicated room at a local science museum in Columbus, OH (19 female children; age M=8.71 years, SD=2.05 years; age range=4.25 to 12.21 years) in July 2019. Two children (a 5- and a 9-year-old) who were unable to complete tasks were removed from analyses.

Materials and Procedure. All procedures were approved by Behavioral and Social Sciences Institutional Review Board (IRB) of [blinded] (2005B0192-[blinded]). In the study, participants were asked to complete three unbounded and one bounded number-line tasks that were given in a counterbalanced order. Because the Cohen Ray Number-Line task does not allow researchers to choose the numbers to be tested, we programmed the number-line tasks to choose different number ranges (see below). To avoid inadvertently biasing children's estimates, the tasks were designed with the physical and spatial constraints recommended in C&R (this issue). Specifically, in the current experiment we used computer monitors with a horizontal screen resolution (SR_{Hpx}; See Appendix A in Cohen & Ray, this issue) of 1,280 or 1,440 pixels (px). A margin (M_{Lpx}) was fixed to 100px for 1280px resolution and to 180px for 1,440px resolution, such that all the monitors had the same response range from 0 to 1080px. A response-line width (R_{Wpx}) and a boundary-line width (B_{Wpx}) were set to 1px, and a single-unit length (U_{Lpx}) to 2px. Within these constraints, the upper-bound number in a bounded task was 538 (Eq. A4 in Cohen & Ray, this issue).

Depending on the size of to-be-estimated numbers, there were three unbounded conditions: the unbounded-large, medium, and small conditions. The three unbounded conditions were identical, presenting participants with a single-unit number line (0-1) on which to base the estimate of the target number (e.g., 27).

For the unbounded-large condition, numbers were chosen under the assumption that there would be no expansion in unbounded estimates (β=1), and thus no extra space would be required for overestimates. Using Eq. A6 in C&R (this issue), the largest number that could be tested in an unbounded number-line with our physical settings was 448. Eighteen numbers were evenly sampled from the 0-448 range: 2, 27, 52, 77, 102, 127, 152, 177, 202, 227, 252, 277, 302, 327, 352, 377, 402, 427. To make the unbounded-large condition parallel to the bounded task, we used identical stimuli in the bounded task—hereafter, the bounded-large condition. Therefore, the stimuli were estimated on 0-538 number lines in the bounded-large condition, whereas they were estimated on 0-1 number lines in the unbounded-large condition.

For the unbounded-medium condition, numbers were chosen under the assumption that there would be moderate expansion (β=1.25), requiring a moderate amount of extra space for overestimates. With this moderate exponent, the largest number that could be tested in an unbounded number-line with our physical settings was 132. Nineteen numbers were evenly sampled from the 0-132 range: 2, 9, 16, 23, 30, 37, 44, 51, 58, 65, 72, 79, 86, 93, 100, 107, 114, 121, 128.

For the unbounded-small condition, numbers were chosen under the assumption that there would be large expansion (β=1.5), as reported in Cohen and Sarnecka (2014) and as assumed in the C&R task. With this large exponent, the largest number that could be tested in an unbounded task with our physical settings was 58. Nineteen numbers were evenly sampled from the 0-58 range: 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35, 38, 41, 44, 47, 50, 53, 56.

In all conditions, to-be-estimated numbers were presented in random order. Participants were asked to drag the response line to the position that represented a to-be-estimated number that was displayed below 0 at the lower end of a number line. As in Kim and Opfer (2017), a to-be-estimated number was presented for 2,000 ms and disappeared afterward. Although the presentation time was long enough for most participants to encode stimuli, a target number was verbally provided by an experimenter if participants missed the number. On every trial, the location of the response line was reset to be at 0. There was no time limit in responses. Neither feedback nor practice was provided.

Appendix B

Table B1. Features Among Unbounded Number-Line Studies

Study	Participant	Display width	Max response line length	Unit size	Largest number tested	Model comparisons	Head-to-head comparison with MLLM	Best-fitting model
Cohen & Blanc-Goldhammer (2011)	Adults	1,920 px	1,720 px	2 to 32 px	25	Linear vs. 1SPM *or* 2SPM *or* multi-SPM	No	One of SPMs
Cohen & Sarnecka (2014)	Children aged 3-8 years	1,920 px	1,720 px	10 to 30 px	19	Log vs. linear vs. 1SPM *or* 2SPM *or* multi-SPM	No	One of SPMs
Link, Huber et al. (2014)	Children in 1st to 4th grade and Adults	29.7 cm	20 cm	NA	19	Linear vs. 1SPM vs. 2SPM vs. multi-SPM vs. 1CPM vs. 2CPM	No	1SPM[2]
Link, Nuerk et al. (2014)	Children in 4th grade	29.7 cm	20 cm	NA	19	Linear	No	Linear
Ebersbach et al. (2015)	Children in K to 2nd grade	NA	100 cm	1 cm	92	Log vs linear vs. 1CPM	No	Log or linear for K, log for 1st graders,

[2] In Link, Huber et al. (2014), this model was called the unbounded power model with a single free parameter. However, this model could be the zero-cyclic power model (0CPM) with a fixed scaling factor as in Slusser et al. (2013). Because there is no model equation in the study, it is ambiguous which model between 1SPM and 0CPM was the best-fitting model.

								linear for 2nd graders
Kim & Opfer (2017)	Children in K to 2nd grade	1,280 px	1,100 px	36 px for K, 11 px for 1st, and 1 px for 2nd graders	30 for K, 100 for 1st, 1,000 for 2nd graders	MLLM vs. MSPM	Yes	MLLM
Qin et al. (2017)	Children in K to 2nd grade and adults	1,440 px	1,000 px	33 px	29	MLLM vs. MSPM	Yes	MLLM
Cohen et al. (2018)	Adults	1,920 px	1,720 px	2 to 32 px	21	1SPM vs. 2SPM vs. multi-SPM	No	1SPM
Reinert et al. (2019)	Adults	1,204 px	18 cm	.3 cm	49	Linear	No	Linear
Current study	Children aged 4-12 years	1,280 or 1,440 px	1,080 px	2 px	56, 128, and 427	MLLM vs. 1SPM	Yes	MLLM

Note. px denotes pixels; CPM = cyclic power model; MLLM = mixed log-linear model; MSPM = mixed scallop power model.

Appendix C

Were results using Kim and Opfer's (2017) methods similar to those using methods recommended by C&R? To address this, we re-fit the Bayesian MLLM to children's number-line data from Kim and Opfer (2017). In the re-analysis, logarithmicity components were computed by boundedness (unbounded or bounded) × number range (0-30, 0-100, or 0-1,000) (Fig. C1). As in the current study, we found that logarithmicity components were substantially greater than 0 in both unbounded and bounded conditions across ranges—i.e., no 0 in 95%HPDIs. Moreover, unbounded estimation was more logarithmic than bounded estimation in all ranges, ΔM=.32, .17, .10 for 0-30, 0-100, and 0-1, 000 range respectively, and the differences in logarithmicity between bounded and unbounded tasks were statistically significant, 95% HPDI [.13, .48] for 0-30, [.05, .28] for 0-100, and [.01, .21] for 0-1,000 range. The results are consistent with findings of the unbounded- vs. bounded-large conditions in the current study, implying that unbounded estimation is logarithmically compressive, and that if there is anything different between unbounded and bounded tasks, that might be greater logarithmicity in unbounded tasks.

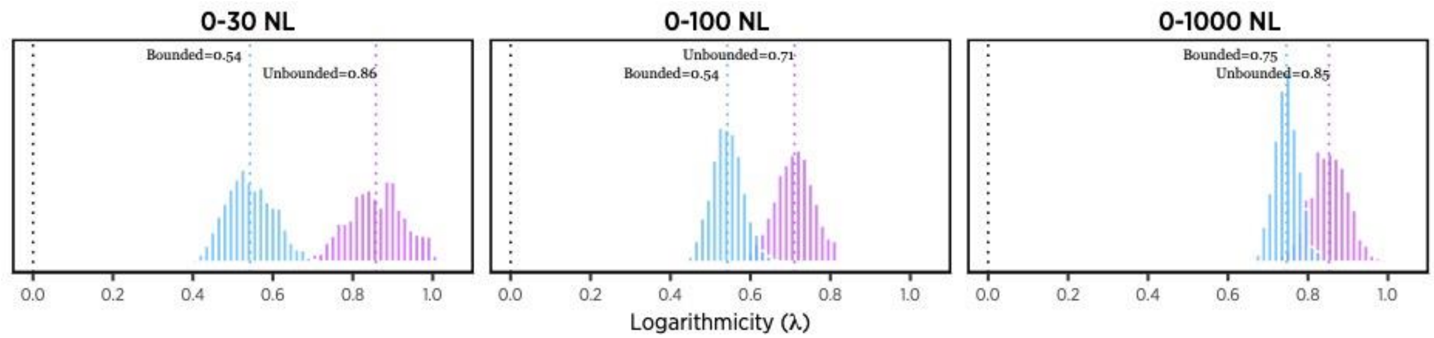

Figure C1. Posterior distributions of logarithmicity in Kim and Opfer (2017), in which children completed both unbounded (purple) and bounded (blue) tasks. The 0-30 range was given to kindergartners, the 0-100 range to 1st graders, and the 0-1,000 range to 2nd graders.