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Mr. Lee's Dilemma: An Instructional Minicase on Evaluating Student Approaches to Comparing Fractions

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Mr. Lee’s Dilemma:
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Abstract

There is a broad consensus that beginning teachers of mathematics need a strong foundation in mathematical knowledge for teaching (MKT), defined as the mathematical knowledge required to recognize, understand, and respond to the mathematical work of teaching one must engage in. One recurrent challenge in teacher education is how to provide support for preservice teachers (PSTs) to acquire such competencies. Recent trends toward practice-based teacher education support the idea of engaging novice teachers in activities that are purposefully constrained to a core teaching practice. “Mr. Lee’s Dilemma” is an abbreviated instructional case (i.e., a minicase) based on an assessment scenario in which PSTs are asked to attend to student reasoning about comparing the magnitude of fractions. PSTs are asked to judge the mathematical validity of students’ explanations as a way of further developing their own MKT.

Keywords: mathematics education, comparing fractions, student explanations, mathematical knowledge for teaching, teacher preparation

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We plan to increase the number of minicases in the coming years and to make further improvements based on feedback from those using the materials. If you would like to make suggestions, please contact Heather Howell at hhowell@ets.org.

On the following pages, we present the fruits of a line of work that has spanned multiple projects over multiple years and reflects the contributions of a number of individuals at different points in time. The rationale for the minicase's development is, in essence, quite simple. Much of recent scholarship on teachers' mathematical knowledge for teaching (MKT) has focused on the assessment of MKT via practice-based questions. Practice-based questions generally include a short introductory scenario whose features are critical in solving the task. These scenarios are not simply window-dressing for the task, but rather, along with the specified mathematical content, they codefine what is measured (Phelps & Howell, 2016). As such, these tasks can be understood to constitute abbreviated representations of teaching practice (Lai et al., 2013).

Because there has been intense interest in the field in assessing MKT, sample assessment tasks currently make up much of the field's description of specific MKT. Since such assessments became available, we have been approached by several teacher educators interested in integrating MKT assessment items into the curricular content of their mathematics and mathematics methods courses, not by using them as assessments, but rather by using them as exemplar instructional cases (see Lai & Howell, 2014, for example tasks). However, a number of obstacles to this kind of use have been noted, leading teacher educators to request publicly available full sets of materials that are aligned to instructional goals. Our goal in developing the minicases was to take on some of these challenges by developing a set of support materials designed to aid teacher educators in making use of the items as a curricular resource and, at the same time, to illustrate one type of support that could be developed more broadly out of such items.

The development team consists of researchers in mathematics and mathematics education, as well as current teacher educators. This work began as part of a 2011 project at Educational Testing Service (ETS) intended to investigate the design features of MKT items in hopes of identifying relationships between structural features of the items and how well they performed in measuring MKT. This project used released items from the Measures of Effective Teaching project and, for each item, created an analytic memo, the purpose of which was to document the reasoning a test taker might use in responding, clearly identifying in each case not just why the intended answer was best but also the logical basis on which each of the competing answer choices could be discarded and mapping that reasoning to types of specialized, common, and pedagogical knowledge, as described in the Ball et al. (2008) theory of MKT. Over the subsequent year, the team worked to refine these documents and tailor them to the possibility of serving multiple audiences, including item writers, researchers, teacher educators, and test takers themselves. We used this documentation in a validity study (Howell et al., 2013) and disseminated it at a number of conferences (Howell et al., 2017; Howell & Mikeska, 2016; Howell et al., 2016; Howell, Weren, & Ruiz Diaz, 2013; Lai & Howell, 2014; Phelps et al., 2013), where we received critical feedback but also an enthusiastic reception from teacher educators eager to see and use more of the items. In 2013, a separate National Science Foundation (NSF) funded project[1] created a set of secondary-level MKT items with accompanying documentation and collected similar validity evidence (Lai & Howell, 2016), and furthered our dissemination goals by creating a Google group in which the items and documentation are housed and available to interested parties.

With a critical mass of systematic assessment documentation at hand, we decided to further develop this material into a set of MKT minicases, documents designed to be used directly by teacher educators in supporting preservice teachers' (PSTs') development of MKT. We chose the name "minicase" to distinguish these materials from "instructional cases" (L. S. Shulman, 1986; Stake, 1987) because they differ from each other in structure and in degree of specificity (J. H. Shulman, 1992). The minicases are shorter than many cases used in professional preparation and are not structured to reveal additional information beyond the initial scenario. The minicases also target very specific knowledge about teaching and learning and are less open to interpretation than most instructional cases. In 2016 and 2017, ETS funded the development of four minicases (two at elementary level and two at secondary level) based on teacher educator input. In 2018, we solicited reviews of the materials from four researchers in the fields of mathematics and mathematics education, as well as from six practicing teacher educators. The feedback from these reviews was then used to revise the set of four minicases to improve mathematical accuracy and comprehensiveness, as well as usability.

Background

There is a broad consensus that beginning teachers of mathematics need a strong foundation in mathematical knowledge for teaching (MKT), defined as the mathematical knowledge required to recognize, understand, and respond to the mathematical work of teaching one must engage in (Ball et al., 2008). Standards call out, for example, competencies for beginning teachers such as "possessing robust knowledge of mathematical and statistical knowledge and concepts," "expanding and deepening [preservice teachers'] knowledge of students as learners of mathematics," and engaging in "effective and equitable mathematics teaching practice" (Association of Mathematics Teacher Educators, 2017, p. 6). One recurrent challenge in teacher education is how to provide support for PSTs to acquire such competencies. Recent trends toward practice-based teacher education support the idea of engaging novice teachers in activities that are purposefully constrained to a core teaching practice (Ball & Forzani, 2009). The MKT minicases we have developed represent one such example.

Research on using cases for subject-specific teacher learning goes as far back as the 1990s (Sykes & Bird, 1992). In mathematics and teacher education, cases can also provide a common language, explicit expectations of high-quality mathematics teaching, information about K–12 student development and common misunderstandings, and a means to interact with challenging content (Barnett, 1991).

Each minicase includes a situated task of teaching practice originally developed as part of teacher assessment efforts. Our guiding hypothesis is that these assessment scenarios, along with the accompanying documents that make up the minicases, form a set of resources for teacher educators. These resources are designed to support instructional goals, including developing PSTs' understanding of K–12 student and higher level mathematics, developing PSTs' orientations toward K–12 students and student work, helping PSTs understand what makes up the professional work of teaching mathematics, and providing them opportunities to engage in the cognitive work associated with addressing the given task.

Because each situated task was originally designed for assessment purposes and crafted to have a single best answer, the resulting minicases require users to take a stand with respect to the presented problem. These cases, unlike instructional cases that are more open-ended, invite response and disagreement in a way that can support rich but focused discussion. Our intention is to support teacher educators who are teaching math methods courses or math content courses for PSTs by providing a set of materials that can be used flexibly and adapted as appropriate.

Instructional Task: The Lee Item

Mr. Lee asked his students to compare $\frac{7}{8}$ and $\frac{6}{9}$. All of his students correctly answered that $\frac{7}{8}$ is greater than $\frac{6}{9}$, but they offered a variety of responses when asked to explain their reasoning. Of the following, which student responses provide mathematically valid explanations for why $\frac{7}{8}$ is greater than $\frac{6}{9}$? For each student response, indicate whether or not it provides a mathematically valid explanation.

Student response	Provides a mathematically valid explanation	Does not provide a mathematically valid explanation
(A) When you compare them, $\frac{7}{8}$ is greater than $\frac{6}{9}$ because 7 is greater than 6.		
(B) You can see that $\frac{7}{8}$ is greater than $\frac{6}{9}$ because ninths are smaller than eighths, which means that $\frac{6}{9}$ is less than $\frac{6}{8}$, which is less than $\frac{7}{8}$.		
(C) You just need to look at how many pieces are missing. $\frac{7}{8}$ is greater than $\frac{6}{9}$ because $\frac{7}{8}$ is only missing one piece from the whole, but $\frac{6}{9}$ is missing three pieces from the whole.		
(D) I think $\frac{7}{8}$ is greater than $\frac{6}{9}$ because $\frac{7}{8}$ has more pieces than $\frac{6}{9}$ and those pieces are larger.		
(E) $\frac{7}{8}$ is greater than $\frac{6}{9}$ because $\frac{6}{9}$ is equal to $\frac{2}{3}$, and because $\frac{1}{3}$ is greater than $\frac{1}{8}$, $\frac{2}{3}$ is farther away from 1 than $\frac{7}{8}$ is.		

Mathematical Content

The Lee minicase focuses on the mathematical content of fractions as well as on the practices of interpreting student work and evaluating K–12 students' mathematical reasoning. Fractions (along with proportions) are a substantial part of the late-elementary curriculum, accounting for two of the nine content domains in the Common Core State Standards (Grades 3–7). Fractions are to be seen as numbers in and of themselves (i.e., not only as parts of wholes). An incomplete understanding of fractions can negatively affect future mathematics learning (e.g., linear functions) and also learning in the sciences (e.g., physics and chemistry). Teachers need to be secure in their own knowledge of fractions in order to provide rigorous instruction to K–12 students.

Student Thinking and Learning

In the Lee minicase, elementary school students are asked to compare the magnitude of two fractions with different numerators and denominators. This task can be very difficult for elementary school students, primarily because they do not yet have a complete understanding of the idea of a fraction as part–whole, ratio, measure, quotient/division, decimal, and operator, and they may compare fractions using whole-number properties. For example, elementary school students may perceive the numerator and the denominator of a fraction as two separate whole numbers, each with a value independent of the other (i.e., "whole-number understanding"). This reasoning leads elementary school students to make magnitude judgments based on either the numerator or the denominator, but not both together as a number.

To make matters more complex, this misunderstanding can lead elementary school students to make correct or incorrect judgments about the magnitude of fractions, depending on the characteristics of the fractions involved. Elementary school students who draw correct conclusions about the magnitude of fractions but use erroneous thinking may face two potential negative consequences: (a) teachers may assume the elementary school student's reasoning is correct and may not explore the students' thinking, and (b) elementary school students may receive positive feedback on their answer and believe that this also applies to the approach, which reinforces their misunderstanding about the magnitude of fractions.

Work of Teaching

It is crucial that teachers learn how to interpret student work and evaluate mathematical reasoning in both spoken and written forms. Although these are skills that will continue to develop throughout a teacher's career, introducing PSTs to these expectations will improve their beginning instruction. For example, the first answer choice in the Lee minicase ("When you compare them, $\frac{7}{8}$ is greater than $\frac{6}{9}$ because 7 is greater than 6") is consistent with whole-number understanding of fractions. It is important for PSTs to recognize that (a) this misunderstanding is typical of elementary school students when they are first exposed to fractions, and (b) it can lead to both correct and incorrect judgments about the magnitude of fractions.

Elaborated Answer Key

This section provides teacher educators an explanation of the answer choices of the Lee item and a justification for the intended answer of the assessment item in terms of mathematically valid reasoning and generalizability.

What Is This Assessment Item Asking?

In this item, Mr. Lee has asked his students to compare two fractions. PSTs' task is to judge the mathematical validity of each of five separate student responses to determine if the elementary school student is demonstrating valid reasoning. In this item, *validity* means that the method is a correct way of obtaining the answer to this task. Although it is instructionally important for PSTs to attend to both the methods and the quality of the explanation, to answer the assessment item requires a focus on the methods.

What Information Is Important?

In this minicase, it is important to pay attention to each elementary school student's reasoning and evaluate it independently. It is also important to notice that in some cases, a teacher educator might make inferences to understand the underlying method the elementary school student has used. All elementary school students arrived at a correct answer, so the goal is not to figure out whether the method happened to generate a correct answer but, rather, whether the reasoning itself is valid. A method's validity can be thought of as a reflection of whether the method would work for tasks of a similar type, and part of answering this item is determining how broadly a method should have to work in order to be considered valid and whether it would be sufficient for it to work for a subset of cases. In general, when comparing fractions, elementary school students will need a method that accounts for the magnitude of the fraction. Elementary school students tend to reason independently about the numerator and the denominator, as shown in the given work samples, and this can be a correct line of reasoning as long as (a) the meaning of each is understood (e.g., numerators as number of pieces, denominator as size of pieces or a point on a number line), and (b) the elementary school student has a way to correctly coordinate information about the numerator with that of the denominator.

Mr. Lee asked his students to compare $\frac{7}{8}$ and $\frac{6}{9}$. All of his students correctly answered that $\frac{7}{8}$ is greater than $\frac{6}{9}$, but they offered a variety of responses when asked to explain their reasoning. Of the following, which student responses provide mathematically valid explanations for why $\frac{7}{8}$ is greater than $\frac{6}{9}$? For each student response, indicate whether or not it provides a mathematically valid explanation.

Student response	Provides a mathematically valid explanation	Does not provide a mathematically valid explanation
(A) When you compare them, $\frac{7}{8}$ is greater than $\frac{6}{9}$ because 7 is greater than 6.		✓
(B) You can see that $\frac{7}{8}$ is greater than $\frac{6}{9}$ because ninths are smaller than eighths, which means that $\frac{6}{9}$ is less than $\frac{6}{8}$, which is less than $\frac{7}{8}$.	✓	
(C) You just need to look at how many pieces are missing. $\frac{7}{8}$ is greater than $\frac{6}{9}$ because $\frac{7}{8}$ is only missing one piece from the whole, but $\frac{6}{9}$ is missing three pieces from the whole.		✓
(D) I think $\frac{7}{8}$ is greater than $\frac{6}{9}$ because $\frac{7}{8}$ has more pieces than $\frac{6}{9}$ and those pieces are larger.	✓	
(E) $\frac{7}{8}$ is greater than $\frac{6}{9}$ because $\frac{6}{9}$ is equal to $\frac{2}{3}$, and because $\frac{1}{3}$ is greater than $\frac{1}{8}$, $\frac{2}{3}$ is farther away from 1 than $\frac{7}{8}$ is.	✓	

Note.

What Is the Rationale for Selecting an Answer?

Row A: Not Mathematically Valid Reasoning

This student's answer is based on only a comparison of numerators. This is clearly not valid as a method except in the special case that the two fractions have equal denominators (and these do not), as it is the ratios that are being compared. A counterexample could easily be produced where a student might incorrectly identify the smaller fraction as the one with the smaller numerator (e.g., $\frac{3}{5}$ and $\frac{2}{3}$). In Row A, the student coincidentally arrived at the correct answer, but the reasoning is not mathematically valid.

Row B: Mathematically Valid Reasoning

This student initially states that ninths are smaller than eighths. The student uses this reasoning to conclude (correctly) that $\frac{6}{9} < \frac{6}{8}$ (because it has an equal number of smaller pieces). It is also correct to conclude that $\frac{6}{8} < \frac{7}{8}$ (because it has fewer pieces of the same size). Implicitly applying the transitive property for inequalities (if $a < b$ and $b < c$, then $a < c$), the student concludes that $\frac{6}{9} < \frac{7}{8}$. Although much of the reasoning is absent from this student's work, the work that is shown provides sufficient evidence of mathematically valid reasoning.

Row C: Not Mathematically Valid Reasoning

This student has considered the number of "missing pieces" of each fraction. This can be a reasonable method, but the student has not accounted for the sizes of the missing pieces, only the quantity of them. A counterexample can easily be produced—for example, $\frac{2}{3}$ is not greater than $\frac{99}{100}$ even though $\frac{97}{100}$ has more missing pieces (in this case, given the denominator, the 3 missing pieces in $\frac{97}{100}$ are very small compared to the 1 missing piece of $\frac{2}{3}$). For this reason, this student's work does not provide evidence of mathematically valid reasoning for this situation.

Row D: Mathematically Valid Reasoning

It is true as stated that $\frac{7}{8}$ has more pieces (a greater numerator) than $\frac{6}{9}$ and that those pieces are larger (a smaller denominator). It is also true that the fraction with *more* larger pieces is the larger fraction. This exact reasoning can be used only in cases in which there are "more larger" pieces (or "fewer smaller" ones). It is a valid coordination of the comparison of numerators and denominators, and an efficient method in cases where it works. It represents valid mathematical reasoning for this comparison.

Row E: Mathematically Valid Reasoning

The student begins by stating that $\frac{6}{9}$ is equivalent to $\frac{2}{3}$. By using this equivalence, it can be seen that $\frac{7}{8}$ and $\frac{2}{3}$ are each a unit fraction ($\frac{1}{8}$ and $\frac{1}{3}$, respectively) away from 1 or a whole. Because $\frac{1}{3}$ is greater than $\frac{1}{8}$, then $\frac{2}{3}$ must be farther from 1 than $\frac{7}{8}$ is. (Here, the student may have correctly applied the methodology suggested in Row C by taking only one piece from each fraction and comparing the relative sizes.) It is not clearly explained how the student knows that $\frac{1}{3}$ is greater than $\frac{1}{8}$, but it is appropriate to assume, based on the student's demonstrated reasoning, that the student is able to compare fractions with like numerators or that the student may be familiar with these as benchmark fractions. This method is a mathematically valid explanation of why $\frac{6}{9} < \frac{7}{8}$.

Instructional Objectives the Minicase Might Support

This section describes teacher educators' potential objectives of this minicase as a situated task to support variable instructional goals, including development of PSTs' understanding of student-level mathematical content of fractions and their practice of interpreting student work, conceptualizing the ideas of mathematical validity and generalizability, discussing student strategies, and evaluating the validity and adequacy of student explanations. Although this minicase lends itself to supporting the particular objectives below, teacher educators may find additional reasons to use this case.

Understanding Student-Level Content

Reviewing the mathematics of fractions, including the relationship between a numerator and a denominator, and judging the magnitude of fractions.

For PSTs whose understanding of the student-level content is weak, examining student approaches (correct and incorrect) may provide a safe place to address the mathematics without causing the PST to feel embarrassed.

Developing Productive Orientations Toward K–12 Students and Student Work: Emphasizing the Practice of Interpreting Student Explanations

To respond to the Lee item coherently requires PSTs to analyze the given student work. Each row provides a plausible sample of student work to discuss and provides an opportunity to practice that analysis as well as discuss why it is important to analyze student work in such a way.

This could provide PSTs a concrete context in which to discuss more general dispositions or instructional values, such as giving K–12 students opportunities to generate and discuss their own solutions, listening carefully, and considering how next instructional moves might vary depending on what the K–12 students have or have not understood.

Appreciating the Larger Mathematical Idea

Learning what constitutes a mathematically valid explanation and how this connects to the idea of generalizability.

The item asks PSTs whether the elementary school student approaches are valid, but all of them produce a correct answer. This could support a discussion with PSTs of what it means for an approach to be valid, what it means for it to be valid but only for certain cases, and what it means for it to be invalid but coincidentally produce a correct answer, pointing to the larger mathematical idea of generalization and how far a method must generalize for it to count as valid.

Understanding That the Work of Teaching Requires Attention to Student-Generated Strategies, Correct and Incorrect, and That Approaches Used By Adults May Not Be Appropriate for Children in Early Grades

Analyzing student-generated strategies is a different and more complex mathematical task than simply answering the student-level task.

This item could support a discussion with PSTs of the professional demands of the work, ways it is different from simply knowing how to solve student-level tasks, and what they should be ready to do as they enter classrooms. It also provides PSTs a context to discuss which given strategies are age appropriate and why common adult strategies, such as converting to decimals, might be less appropriate for elementary school students who are just learning about fractions.

Understanding How to Analyze Student Work in Terms of the Validity and Adequacy of the Explanations

Making sense of the student work and determining which student strategies work.

The item provides PSTs a context in which to practice these analytic skills, essentially serving as a practice exercise in the specific mathematics required by the situation. Additionally, it showcases common correct and incorrect solution patterns in comparing fractions that PSTs may not yet be familiar with.

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Appendix A. Sample Lesson Outline

This appendix provides teacher educators a sample lesson outline, including lesson goals, links to prior learning of and about fractions, and suggestions for lesson implementation to use with preservice teachers (PSTs). This sample lesson may provide an illustration of how a whole lesson can be planned around the Lee minicase, and is designed to be user ready, although it is only one example of how a lesson might be configured.

Comparing Fractions

Goals for This Lesson

For preservice elementary and middle-school teachers to

- Enhance elementary- and middle-school students' conceptual understanding of how fractions represent quantities formed by parts of a whole, especially in the context of comparing fractions.
- Recognize and analyze student thinking about comparing fractions, specifically in terms of how the elementary- and middle-school students interpret the fraction representation of part and whole.
- Consider what qualifies as valid mathematical reasoning about fraction comparison at the target age level.

Prior Experiences

This lesson assumes some familiarity with fraction equivalence, unit fractions, and how fractions can be mathematically defined as a representation of a quantity of parts of a whole.

Embedded Student Content

In this lesson, PSTs are asked to analyze three elementary school students' work on the task below:

> Compare $\frac{7}{8}$ and $\frac{6}{9}$.
>
> Is one larger than the other? If so, which one?

This task asks the elementary school students to compare the magnitudes of two fractions.

Elementary school students can take many valid approaches to this task, and the sample student work in the Lee scenario illustrates some of these:

- Student B said: *You can see that* $\frac{7}{8}$ *is greater than* $\frac{6}{9}$ *because ninths are smaller than eighths, which means that* $\frac{6}{9}$ *must be less than* $\frac{6}{8}$ *, which is less than* $\frac{7}{8}$. *Student B compared sizes of parts (eighths and ninths), compared fractions with same number of parts but different size parts (*$\frac{6}{9}$ *must be less than* $\frac{6}{8}$*), and used successive comparison (*$\frac{6}{9}$ *is less than* $\frac{6}{8}$*, the fraction* $\frac{6}{8}$ *is less than* $\frac{7}{8}$*, so* $\frac{6}{9}$ *is less than* $\frac{7}{8}$*).*
- Student D said: $\frac{7}{8}$ *is greater than* $\frac{6}{9}$ *because* $\frac{7}{8}$ *has more pieces than* $\frac{6}{9}$ *and those pieces are larger. Student D compared sizes of parts (ninths and eighths) and number of parts (7 and 6).*
- Student E said: $\frac{7}{8}$ *is greater than* $\frac{6}{9}$ *because* $\frac{6}{9}$ *is equal to* $\frac{2}{3}$*, and because* $\frac{1}{3}$ *is greater than* $\frac{1}{8}$*,* $\frac{2}{3}$ *is farther away from 1 than* $\frac{7}{8}$ *is. Student E used equivalent fractions (*$\frac{6}{9}$ *and* $\frac{2}{3}$*), unit fractions and sizes of parts (*$\frac{1}{3}$ *and* $\frac{1}{8}$*), and the idea of missing pieces or distance from the whole (there are just as many "missing pieces" in* $\frac{2}{3}$ *and* $\frac{7}{8}$*, namely 1 piece, and the* $\frac{1}{8}$ *piece is smaller than the* $\frac{1}{3}$ *piece).*

These ideas also surface in other elementary school students' work, though the thinking of Students A, C, and D is not completely mathematically valid because each of them considers parts or wholes independently of each other (i.e., saying that $\frac{7}{8}$ must be greater than $\frac{6}{9}$ only because 7 is greater than 6). This reasoning would not hold in general; for instance, $\frac{7}{11}$ is less than $\frac{6}{9}$, even though 7 is still greater than 6. Although not called out in the teacher-level Lee task, in the classroom elementary school students are often invited to use representations such as number lines or rectangles to compare two fractions with different denominators.

Opener: Comparing Fractions

We begin this lesson with a task of fraction comparison. The purpose of this opener is to discuss the notions of unit fractions, comparing sizes of parts, comparing fractions with same size parts, and missing pieces.

Compare the fractions below. Are they equal? Is one larger than another? Use symbols such as >, =, < to express your thinking.

1. $\frac{2}{6}$ and $\frac{5}{6}$
2. $\frac{1}{5}$ and $\frac{1}{7}$
3. $\frac{2}{5}$ and $\frac{2}{7}$
4. $\frac{3}{6}$ and $\frac{4}{8}$

Discussion for Opener

After working on this problem individually, ask PSTs to discuss in small groups:

- How do you know your comparison is true?
- Where do your representations use the definition of fraction?[2]
- Look at the representations your group used. Where are the wholes in comparison to each other? Where are the parts in comparison to each other? What are advantages and disadvantages of these representations?

Key ideas to surface from this discussion, and to record publicly, are these:

- When comparing fractions, we assume the wholes are the same size.
- When comparing the total quantity given by some number of parts, we have to keep in mind the size of the parts.
- Equivalent fractions are fractions that represent the same quantity.

Some strategies that may arise are these:

- Comparing unit fractions and sizes of parts using the definition of fraction (e.g., $\frac{1}{5} > \frac{1}{7}$ because dividing the same whole into 5 parts results in larger pieces than dividing that whole into 7 parts).
- Comparing different number of parts of the same size ($\frac{2}{6} < \frac{5}{6}$ because both describe copies of $\frac{1}{6}$, and $\frac{2}{6}$ is 2 copies, and $\frac{5}{6}$ is 5 copies, and 5 > 2.)
- Comparing same number of parts of different size ($\frac{2}{5} > \frac{2}{7}$ because one describes copies of $\frac{1}{5}$ and the other copies of $\frac{1}{7}$; because $\frac{1}{5} > \frac{1}{7}$, two copies of $\frac{1}{5}$ is greater than two copies of $\frac{1}{7}$.)
- Missing pieces ($\frac{2}{6} < \frac{5}{6}$ because both describe copies of $\frac{1}{6}$, both $\frac{2}{6}$ and $\frac{5}{6}$ lie between 0 and a whole of $\frac{6}{6}$, but $\frac{2}{6}$ is farther away than $\frac{5}{6}$ is from the whole.)

As these strategies arise, attend to how PSTs are referencing the key ideas, and make sure they are referring to them as much as possible.

Following this, the PSTs might debrief in small groups as a way to reflect and learn from their own work and each other's:

- Did your thinking change at all from before to now?
- Explain your thinking in comparing the fractions when you solved the problem.
- What strategies/models did you use?

Situating the Concepts in Teaching: Mr. Lee's Class

Now we examine a case where elementary school students are doing a fraction comparison task. Our goal is to understand how the elementary school students are thinking through the task, and how we know what they do or do not understand. Through this situation, PSTs will have a chance to practice interpreting student work and being sensitive to K–12 students' different strategies.

As PSTs read through the elementary school student responses, ask them to think about the following:

- What is the elementary school student's thinking? How might they have arrived at each step of their solution?
- What are you sure that each elementary school student understands? What are you sure that each elementary school student does not understand? What are you unsure that each elementary school student understands?
- Are the explanations mathematically sound? Why or why not?

The PSTs might fill out a table like this for each elementary school student, and then determine whether the elementary school student's work provides a mathematically valid explanation.

I am sure that Student A understands ...	I am sure that Student A does NOT understand ...	I am unsure whether Student A understands …

Discussion for the Case of Mr. Lee's Class, Part 1

Specific points to attend to for each elementary school student are

- Student A addresses only the numerator and not the denominator, so the strategy does not generalize to all fractions because the parts are not all the same size.
- Student B builds on Student A's strategy and acknowledges that the parts are different sizes, also using inequalities. Student B compares sizes of parts, compares fractions with same number of parts but different size parts, and uses successive comparison.
- Student C references missing pieces thinking, but does not acknowledge size of pieces. Noticing that this is the strategy, and that the strategy could be valid, is worthwhile.
- Student D compares sizes of pieces and number of pieces.

- Student E uses equivalent fractions ($\frac{6}{9}$ and $\frac{2}{3}$), unit fractions and sizes of parts ($\frac{1}{3}$ and $\frac{1}{8}$), and the idea of missing pieces or distance from the whole (there are just as many "missing pieces" in $\frac{2}{3}$ and $\frac{7}{8}$, namely 1 piece, and the $\frac{1}{8}$ piece is smaller than the $\frac{1}{3}$ piece).

Discussion for the Case of Mr. Lee's Class, Part 2

Based on your analysis above, do you think that the elementary school students demonstrate a mathematically valid approach to comparison? Take a moment to indicate what you are thinking.

Here, the PSTs in your class may broach discussion about partial validity as opposed to binary views of mathematical validity.

Student	Provides a mathematically valid explanation	Does NOT provide a mathematically valid explanation
Student A		
Student B		
Student C		
Student D		
Student E		

PSTs may also advocate that Student D's approach is the "best" because it is simplest. Although it is true that the approach is "simplest," it is still worthwhile to understand the thinking in all students' cases, because the approach that Student D used will not apply to all comparison problems. It is important when listening to K–12 students to be open to different approaches, to use approaches to help figure out what K–12 students may or may not understand, and to find ways they might move further along in understanding and using fractions.

Summary of Discussion of the Case of Mr. Lee's Class

In general, when attending to and making sense of student work, think about these points:

- What is the elementary school student's thinking? How might they have arrived at each step of their solution?

- What are you sure that each elementary school student understands? What are you sure that each elementary school student does not understand? What are you unsure that each elementary school student understands?
- Are the explanations mathematically sound? Why or why not?

When making sense of what the elementary school student understands or may not understand, attend to the core concepts of the task. In the case of fraction comparisons, such concepts may include the following:

- When comparing fractions, we assume the wholes are the same size.
- When comparing the total quantity given by some number of parts, we have to keep in mind the size of the parts.
- Equivalent fractions are fractions that represent the same quantity.

Appendix B. Additional Discussion Prompts

The following is a list of potential discussion prompts, extensions, or additional assignments teacher educators might use with PSTs around the Lee item. These prompts might be used instead of, in addition to, or to extend the sample lesson outline of the Lee minicase.

- Based on the approach shown, how might each elementary school student have responded differently to the original task if they were comparing $\frac{3}{5}$ and $\frac{2}{3}$? Do their methods extend in a systematic way? Do they still produce correct answers?
- What next pair of fractions might you propose if your goal were to …
  - … help Student A understand that the method does not always produce a correct answer?
  - … help Student C see that the method does not always produce a correct answer?
  - … help Student D see that the method might not always be possible to apply?
  - … help Student E see that the method might not always be possible to apply?
- Under what conditions would Student A's incorrect approach produce a correct answer? (For what set of fractions will it "work," even though it is not correct?) Student C's?
- What next instructional steps might you take to address the misunderstandings Students A and C may have? How would you expect those steps to play out?
- If you were placing these elementary school students in groups or pairs to discuss their work, how would you match them up, and what would your instructional objectives be in pairing them in that way?
- For each student work sample, write …
  - … what you think a high-quality explanation of that mathematical process would look like.
  - … what you think a minimally adequate explanation of that mathematical process would look like.
  - … an example of an explanation you would not consider adequate.

- What qualities does each explanation have that make it high quality/minimally adequate/ inadequate?
- Reflecting back on your work on the Lee item, what are some techniques you might have in mind in looking at student work samples in the future?
- How would you connect the reasoning each elementary school student has presented to other models you might want to use for fraction comparison, such as number lines or area models? Which methods connect more or less naturally, and what would you focus on if you wanted to support those connections? How would you help elementary school students see the advantages of drawing parallel number lines to illustrate the comparison (and the potential disadvantages for tricky or very close fractions)?
- Imagine a sixth student in the class has been taught by a parent to use cross multiplication to compare the fractions, and comments that this is the best way because it always works and the other ways you have to think about it more. How would you answer that student? What might your answer be in a parent/teacher conference if the parent wants to know why you aren't teaching cross multiplication?

Appendix C. Aligned Task–Franco

This appendix provides teacher educators an additional item (the Franco item focused on a fraction comparison, like the Lee item) with its elaborated answer key. This item asks preservice teachers (PSTs) to create some cognitive tasks examining student thinking of, and understanding about, the magnitude of fractions.

The Franco Item

The Franco item is similar to Lee in that it asks the respondent to focus on student-level content around the comparison of fractions. Unlike Lee, the work the respondent is asked to do is to generate a counterexample to a given incorrect student strategy that has coincidentally produced a correct answer to a problem. It is also different in that the task is open-ended, asking the respondent to consider their next move as a teacher by generating a new problem (as opposed to asking the respondent to make a discrete judgment about a given example of student work). The item might be helpful as a follow-up or homework assignment, or as a second lesson that would follow the Lee item. What follows is the Franco item and the elaborated answer key.

Elaborated Answer Key—Franco: Counterexample for Fraction Comparison

Ms. Franco was assessing students' work on comparing fractions.

She assigned the following problem.

> Put the following fractions in increasing order, and explain your reasoning. $\frac{4}{7}, \frac{5}{8}, \frac{2}{5}$

She noticed that Zachary got a correct answer with incorrect reasoning.

He explained that $\frac{2}{5} < \frac{4}{7} < \frac{5}{8}$ because $2 < 4 < 5$ and $5 < 7 < 8$.

To help Zachary understand that his reasoning is incorrect, Ms. Franco wants to give a similar problem using 3 different fractions. She wants to include fractions with 3 different numerators and 3 different denominators that, using Zachary's reasoning, would lead to ordering the fractions incorrectly, from greatest to least instead of least to greatest. List 3 such fractions in the boxes below in any order.

$$\frac{\square}{\square} < \frac{\square}{\square} < \frac{\square}{\square}$$

What Is This Assessment Item Asking?

This assessment item asks you to find a set of fractions that serves as a counterexample to a case in which an incorrect student approach has coincidentally produced a correct answer. It requires you to think simultaneously about the relationships between the values of fractions and the values of their numerators and denominators, and to be able to identify a set of fractions that satisfies multiple conditions. In particular, you are looking for three fractions which, when ordered using the elementary school student's method, will result in an incorrect ordering of the fractions.

What Information Is Important?

The elementary school student whose work is described, Zachary, has used the following strategy to order the fractions:

Given three fractions, that is, $\frac{n_1}{d_1}, \frac{n_2}{d_2}$, and $\frac{n_3}{d_3}$, when the fractions are arranged in increasing order of the numerators and denominators, that is, $n_1 < n_2 < n_3$ and $d_1 < d_2 < d_3$, then the fractions themselves will be in increasing order, that is, $\frac{n_1}{d_1} < \frac{n_2}{d_2} < \frac{n_3}{d_3}$.

The assessment task asks you to find three fractions such that when the fractions are arranged in increasing order of the numerators and denominators, that is, $n_1 < n_2 < n_3$ and $d_1 < d_2 < d_3$, then the fractions themselves will be in decreasing order $\frac{n_1}{d_1} > \frac{n_2}{d_2} > \frac{n_3}{d_3}$.

There are other types of counterexamples that might be useful. For example, choosing three fractions where some have common numerators might illustrate the point that Zachary's method is not clear enough in order to make a judgment of what he would do in this case. Choosing three fractions that demonstrate that ordering the numerators produces a different result than ordering the denominators would illustrate that it is not always possible to apply his method. But in the given task, you're asked to choose a set of fractions for which the method can be applied, but produces an incorrect answer, and in particular one that is incorrect by producing the reverse of the correct ordering.

There are a number of conditions the problem you are to generate must satisfy:

- The three numerators must be different.
- The three denominators must be different.

- When the fractions are arranged by increasing numerators, then denominators must also be increasing. (Likewise, if the fractions are arranged by increasing denominators, then the numerators must also be increasing.)
- When the fractions are arranged by increasing numerators and denominators, the fractions themselves will be in decreasing order—not just any order other than increasing.

It is also important to notice here that although you are not asked for a general method for finding such a set of fractions, it is much easier if you have a systematic strategy for finding them rather than guessing and checking until you find a set that works.

What Is the Rationale for Selecting an Answer?

A few of the methods that could be used to generate responses to this task are described below.

One method is to use common benchmark fractions in the reasoning process. (This method would be accessible to elementary school students as well.) You might start with $\frac{3}{4}$, which is equivalent to 0.75 but which has both a relatively small numerator and denominator. Your next fraction must have a numerator greater than 3, a denominator greater than 4, but an overall value that is less than $\frac{3}{4}$. You might then choose $\frac{5}{9}$ as your second fraction because it is a little greater than $\frac{1}{2}$. Finally, the third fraction could be $\frac{6}{13}$ as $6 > 5$, $13 > 9$, and $\frac{6}{13}$ is a little less than $\frac{1}{2}$.

A second method is to start with three fractions that are equal: $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$. You can then increase the denominators of the second and third fraction such that they lessen the overall value of the fractions: $\frac{1}{2} > \frac{2}{5} > \frac{3}{10}$.

A third method is to begin with decimals that one can easily identify as meeting the conditions: $0.1 > 0.02 > 0.003$ (in fractions form: $\frac{1}{10} > \frac{2}{100} > \frac{3}{1000}$) even though $1 < 2 < 3$ and $10 < 100 < 1000$.

A fourth method uses the fact that the assessment task does *not* state that the three fractions must each have a value less than 1. So consider any fraction greater than 1, $\frac{n}{d}$ where n > *d*, and the two fractions formed by adding 1 and 2 to both the numerator and denominator, $\frac{n+1}{d+1}$ and $\frac{n+2}{d+2}$. You can show that $\frac{n}{d} > \frac{n+1}{d+1} > \frac{n+2}{d+2}$, provided that none of the denominators equals 0, thus generating a set of fractions that meet the desired criteria. For example, start with $\frac{7}{3}$. Generate the next two fractions, $\frac{8}{4}$ and $\frac{9}{5}$. It is true that the numerators are increasing $7 < 8 < 9$, the denominators are increasing $3 < 4 < 5$, but the fractions are decreasing $\frac{7}{3} > \frac{8}{4} > \frac{9}{5}$.

Appendix D. Aligned Task–Richmond

This appendix provides teacher educators another item (the Richmond item focused on proportional reasoning) with its elaborated answer key. This item can be used to provide preservice teachers (PSTs) a further learning opportunity to practice interpreting student work.

The Richmond Item

The Richmond item is similar to Lee in that it asks the respondent to perform similar work in deciding if three student work samples provide evidence of mathematically valid reasoning. It is focused on different, but related, content. The item might be helpful as a follow-up or homework assignment, or as a second lesson that would follow the Lee item.

Elaborated Answer Key—Richmond: Proportional Reasoning

What Is This Assessment Item Asking?

In this assessment item, you'll need to evaluate each elementary school student's explanation separately and decide if there is enough evidence to conclude that the method shown is mathematically valid. All of the elementary school students have arrived at correct answers, so the task asks you to evaluate the methods, not the final answers. The main work of this assessment item can be thought of, then, as having two steps. First, figure out what general method is implied from the student explanation. Second, decide if that method is valid. Because what is asked for is "evidence," an explanation does not have to be completely clear or concise to qualify; there just has to be a plausible way of understanding what the elementary school student probably was thinking that is valid.

What Information Is Important?

The student-level task involves three quantities (amount of cocoa, amount of sugar, and number of brownies) that should remain in fixed proportionality relative to one another, no matter how the batch size is altered. So "mathematically valid" student thinking should involve manipulating these quantities in a way that maintains their relative proportionality. This can be done by scaling all three quantities at once (for example, an efficient solution is to multiply all three quantities by 1.5, although this is not a likely method for elementary school students to use) or by working with them in pairs as long as the third quantity is accounted for afterward.

In a unit on proportional reasoning, Ms. Richmond's class was discussing the following problem.

If 4 cups of cocoa and 2 cups of sugar yield 16 brownies, how many cups of cocoa and how many cups of sugar are needed to make 24 brownies?

Ms. Richmond's students used different strategies to solve the problem. For each strategy, indicate whether or not it provides evidence of mathematically valid student thinking.

Strategy	Provides evidence of mathematically valid student thinking	Does not provide evidence of mathematically valid student thinking
(A) 48 brownies need 12 cups of cocoa and 6 cups of sugar. To make 24 brownies, I need 6 cups of cocoa and 3 cups of sugar.		
(B) 4 and 2 both go into 16, 4 plus 2 is 6, half of 6 is 3, and 6 and 3 both go into 24, so you need 6 cups of cocoa and 3 cups of sugar to make 24 brownies.		
(C) 1 brownie needs $\frac{1}{4}$ cup of cocoa and $\frac{1}{8}$ cup of sugar. To make 24 brownies, I need to multiply by 24 for cocoa and sugar. Thus, I need 6 cups of cocoa and 3 cups of sugar.		
(D) 6 cups of cocoa and sugar makes 16 brownies, so 24 brownies need 9 cups of cocoa and sugar. Because the ratio of cocoa to sugar is 2:1, I need 6 cups of cocoa and 3 cups of sugar.		
(E) Because 1 cup of sugar is needed to make 8 brownies, I need 3 cups of sugar to make 24 brownies. The amount of cocoa is 2 times the amount of sugar in the recipe, so I need 6 cups of cocoa.		

What Is the Rationale for Selecting an Answer?

Row A: Mathematically Valid Reasoning

The student whose explanation is in Row A says that 48 brownies require 12 cups cocoa and 6 cups sugar, which is a true statement. The second statement, that you need 6 cups cocoa and 3 cups sugar to make 24 brownies, is also true. However, the explanation does not make clear how the student arrived at the first statement or how the student moved from the first statement to the second. Figuring out what the student was thinking requires a little imagination. It seems reasonable that the student tripled the given recipe to get a recipe for 48 brownies and then halved that one to get a recipe for 24 brownies. In doing so, the student provides a valid solution process because scaling (by a factor of 3 and then $\frac{1}{2}$) preserves proportionality. So there is evidence here of mathematically valid student thinking.

Row B: Not Mathematically Valid Reasoning

The explanation in Row B is a series of true statements with little connective tissue. It is true that 4 and 2 both go into 16, that 4 plus 2 is in fact 6, and that half of 6 is 3, but the student is not explicit about why this matters. Trying to imagine what the student might have been thinking, it seems reasonable to assume that the 4 and 2 are the cups of cocoa and sugar from the original recipe and that the 6 probably means 6 cups of cocoa/sugar mix, and in this case, 3 would represent half the mix. Six and 3 do indeed each go into 24, but it is not clear why being factors of 24 is adequate reason to expect that they would then represent the desired quantities of cocoa and sugar. In fact, if this line of reasoning does represent this student's thinking, the same amount of cocoa and sugar would have been required for a batch of 6, 12, 18, 24 . . . brownies (because in each case, 6 and 3 divide into the number of brownies). There does not, therefore, seem to be sufficient evidence here of mathematically valid thinking.

Row C: Mathematically Valid Reasoning

The student whose explanation is in Row C states that one brownie needs $\frac{1}{4}$ cup cocoa and $\frac{1}{8}$ cup sugar, and it is reasonable to imagine the student reached this conclusion by dividing each quantity by 16 as a means of finding the per unit (brownie) ingredient amounts. The student has been explicit that from there she multiplied the unit ingredients by the desired number of brownies to arrive at the solution. Scaling by a factor of $\frac{1}{16}$ and then by a factor of 24 preserves proportionality, so the method is valid, and this row has evidence of mathematically valid student thinking.

Row D: Mathematically Valid Reasoning

The statement that 6 cups of cocoa and sugar makes 16 brownies is unclear—it may mean 6 cups each of cocoa and sugar (which would be incorrect), or it may mean 6 cups combined from the 4 cups cocoa and 2 cups sugar (which would be correct). Assuming the second interpretation, it is also true that 24 brownies need 9 cups of ingredients, although the student did not specify how this was decided. The student states explicitly how the final cocoa and sugar amounts are calculated by breaking 9 cups into 2 parts with a ratio of 2:1, and this reading supports our assumption that the first statement was referring to 6 combined cups of cocoa and sugar. This student seems to think of this as (mix):(brownies) and then as (cocoa):(sugar) within the mix. Scaling from 6 to 9 cups of mix preserves proportionality, and the last step explicitly preserves the proportionality of cocoa to sugar, so this method is also valid, even though several steps of the process are not clearly explained. Although there is less explanation here than in some of the other rows, there is certainly evidence of mathematically valid student thinking.

Row E: Mathematically Valid Reasoning

This student states that 1 cup of sugar makes 8 brownies. This is correct, and a reasonable way the student might have reached this conclusion is to divide the given quantities by 2. Similarly, it is correct that 3 cups of sugar make 24 brownies and reasonable that the student would reach this conclusion by multiplying both quantities by 3. It is also correct that once having found the amount of sugar, the student can calculate the amount of cocoa by maintaining the constant ratio 2:1, as the student explains. Like the student in Row D, this student has chosen to consider things pairwise by looking at sugar in isolation first, then going back to calculate the amount of cocoa. This student has scaled by a factor of $\frac{1}{2}$ and then 3 correctly and has maintained the relative ratio of cocoa to sugar correctly, so the method is valid. There is evidence here of mathematically valid student thinking.

Appendix E. Resources

This appendix provides a few additional resources that are relevant to the mathematics and preteaching practices mentioned in this minicase. Articles listed below can be used as reading assignments for preservice teachers (PSTs).

Bray, W. S., & Abreu-Sanchez, L. (2010). Using number sense to compare fractions. *Teaching Children Mathematics, 17*(2), 90–97.

Common Core Standards Writing Team. (2013, September 19). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 6–8, The number system; High school, number.* Institute for Mathematics and Education, University of Arizona. http://commoncoretools.me/wp-content/uploads/2011/08/ccss_progression_nf_35_2013_09_19.pdf

McLeman, L. K., & Cavell, H. A. (2009). Teaching fractions. *Teaching Children Mathematics, 15*(8), 494–501.

Morrow-Leong, K. (2016). Evidence centered assessment. *Teaching Children Mathematics*, *23*(2), 82–89.

Olanoff, D., Lo, J.-J,, & Tobias, J. (2014). Mathematical content knowledge for teaching elementary mathematics: A focus on fractions. *The Mathematics Enthusiast*, *11*(5), 267–310.

Schwarz, V. J. (2006). *Fractions: Building a strong foundation based on conceptual understanding.* Yale National Initiative. http://teachers.yale.edu/curriculum/viewer/initiative_11.06.06_u

Appendix F. Frequently Asked Questions

Where did the assessment items come from?

These items were originally written for use on the Measures of Effective Teaching (MET) Project (https://k12education.gatesfoundation.org/blog/measures-of-effective-teaching-met-project/) as part of an assessment of content knowledge for teaching. The team that wrote the items included researchers at the University of Michigan and Educational Testing Service (ETS). At the conclusion of the study, the items were released for use, with some restrictions. Copies of the assessment forms can be requested from the ETS lead for the MET study, Geoffrey Phelps, gphelps@ets.org.

There's something I would like to change about the item. / I don't agree with the way the math is presented in the item. Would you consider changing it?

We decided in our work on the minicases to use the assessment items exactly as they were provided by the projects they came from (see FAQ #1). One goal of the further development work is to explore how existing intellectual capital in the form of assessments can be repurposed into material for teacher learning. The minicases have developed organically across a set of projects over a number of years, and there have been many contributors to them. The latest versions were reviewed by four experts in the field of mathematics and mathematics education, and their advice has been incorporated into revisions.

Part of what we want to illustrate is that the assessment item itself need not be above critique for it to be a useful starting point for PST learning. In fact, we think some critique might signal rich points for discussion as part of teacher development. That said, the point of the minicase is to be provocative, not prescriptive, and we encourage anyone who wishes to tweak, alter, subvert, delete, or completely rewrite the assessment item in service of their own instructional goals to do so. (And if it's an item from the Google drive, we hope you'll post your work back in the drive for others to use!)

I would like to use these items as a hiring screen for new teachers; where could I find more of them?

This is not an approved use of these items. Accessing these items (see FAQ #1) requires that you agree to terms of use that exclude high-stakes decision making.

Where could I find more minicases like these?

We have only a few exemplars ready for use at the current time, but we are more than happy to share them on request. To be added to our distribution list, contact Heather Howell, hhowell@ets.org. The minicases are a work in progress. If you have suggestions, please let us know!

Notes

[1] This material is based in part on work supported by the NSF under Grant No. 1445630/1445551. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF.

[2] Depending on the PSTs, it may be helpful to discuss a definition of fraction prior to the opener, e.g., the fraction $\frac{a}{b}$ represents a number when *b* is not zero. If some whole is divided into *b* equal pieces, then we write $\frac{1}{b}$ to name one of those parts. If a part is $\frac{1}{b}$ of a whole, then it takes *a* copies of that part to make the whole. $\frac{a}{b}$ is defined as *a* copies of $\frac{1}{b}$.