CHILD DEVELOPMENT PERSPECTIVES

Missing Input: How Imbalanced Distributions of Textbook Problems Affect Mathematics Learning

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ABSTRACT—***Understanding how environments influence learning requires attending not only to what is present but also to what is absent. In the context of mathematics learning, this means attending not only to problems that children encounter frequently in textbooks but also to ones that appear rarely. We present research in this article showing that students perform surprisingly poorly on seemingly simple fraction and decimal arithmetic problems that are seldom seen in textbooks. Next, we describe imbalanced distributions in textbooks of mixed notation arithmetic and comparison problems, and we hypothesize similar relations between the frequency of those types of problems and student accuracy on those tasks. Finally, we review findings about relations between textbook input and student performance in whole number arithmetic and mathematical equality, and we propose a hypothesis regarding when imbalanced distributions of problems are most detrimental. We conclude that presenting more balanced distributions of problems and helping children understand mathematical principles that differentiate legitimate from flawed solution strategies offer promising ways of improving mathematics education.***

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The research reported in this article was supported by the Institute of Education Sciences, U.S. Department of Education (Grant No. R305A180514); the National Science Foundation (Grant No. 1844140); and the Schiff Foundations Chair in Psychology and Education at Columbia University. The opinions expressed are those of the authors and do not represent the views of the agencies that supported the research.

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DOI: 10.1111/cdep.12402

KEYWORDS—*cognitive processes; decimals; fractions; mathematics; textbooks*

In "The Adventures of Silver Blaze," Sherlock Holmes was asked to solve a mystery: The trainer of a champion racehorse had been murdered and the horse had disappeared (Doyle, 1894). After preliminary discussion, Detective Gregory asked if there was anything else to which Holmes would like to draw his attention. Holmes replied, "[t]o the curious incident of the dog in the night-time." Gregory responded, "The dog did nothing in the night-time." Holmes explained, "*That* was the curious incident" (Doyle, 1894, p. 11; emphasis added). The reasoning behind the comment was that if the culprit had been a stranger, the dog would have barked; the absence of barking meant that the dog knew the perpetrator.

Holmes's insight illustrates an important and broadly applicable point: In trying to explain observations, we should consider not just what we see but also what we do not see. This logic has led our research on how textbooks shape mathematics learning to focus on problems that are absent as well as on ones that are present. Our main hypothesis is that lack of experience with some types of problems hinders children's learning.

This research grows out of a general belief that the study of cognitive development would benefit from greater attention to the specifics of the environments in which learning occurs. Examining textbooks has several advantages for studying how the learning environment shapes children's thinking. The use of textbooks is nearly universal in the United States and internationally (Horsley & Sikorová, 2014; Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002). Textbooks bridge the gap between intended curricula and implementation of curricula in classrooms (Mullis, Martin, Foy, & Arora, 2012; Valverde et al., 2002). They also are publicly available, so researchers can replicate others' findings and test alternative hypotheses about them. Such considerations have led to several recent investigations of the relations between textbook characteristics and children's mathematics learning (e.g., Fagginger Auer, Hickendorff, van Putten, Béguin, & Heiser, 2016; Sievert, van den Ham, & Heinze, 2021). Textbook characteristics are certainly not the only determinant of mathematics learning—teaching approaches and many other variables also matter—but they are one important influence.

As Sherlock Holmes's tale cautions us, when we analyze textbooks, we need to attend to what is omitted as well as to what is included. In this article, we describe recent findings regarding imbalances in textbook problems, their relations to children's learning, and the conditions under which these imbalances affect learning most adversely.

RATIONAL NUMBER LEARNING AND TEXTBOOKS

In our recent research on relations between textbook content and mathematics learning, we have focused on how children learn about rational numbers (i.e., fractions, decimals, percentages). This domain is of considerable importance both in and out of school. In terms of school success, fraction knowledge in fifth grade uniquely predicts math achievement 5 years later in high school, even after controlling for socioeconomic status, IQ, reading comprehension, and whole number arithmetic knowledge (Siegler et al., 2012). High school and college math is used in relatively few occupations, but knowledge of rational numbers, taught in late elementary and middle school, is crucial for success in many fields, including nursing, pharmacy, and modern factory work (Douglas & Attewell, 2017; Handel, 2016).

Despite extensive instruction in rational numbers over several grades (Common Core State Standards Initiative, 2010), many older children and adults have a limited understanding of them (Stigler, Givvin, & Thompson, 2010; Tian, Braithwaite, & Siegler, 2020). For example, on the 1979 National Assessment of Educational Progress, only 24% of U.S. eighth graders who were presented with the equation 12/13 + 7/8 and asked to choose the closest answer from among 1, 2, 19, and 21 chose the correct answer, 2 (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980). Even after numerous mathematics education reform efforts since then, the percentage of eighth graders who answered the same problem correctly in 2014 was almost identical, 27% (Lortie-Forgues, Tian, & Siegler, 2015).

To understand more completely the sources of this weak learning, we examined the input from textbooks that children receive for rational number arithmetic. In particular, we coded problems from then-current editions (as of 2016, when our coding began) of three U.S. math textbook series: Pearson's *enVision Mathematics* (Charles et al., 2012), Houghton Mifflin Harcourt's *GO Math!* (Dixon, Adams, Larson, & Leiva, 2012), and McGraw-Hill's *Everyday Mathematics* (University of Chicago School Mathematics Project, 2015a, 2015b). These textbook series were chosen because they were among the most widely used in the United States (Opfer, Kaufman, Pane, & Thompson, 2018) and included volumes for grades four through six, the grades in which fractions and decimals are primarily taught in the United States (Common Core State Standards Initiative, 2010). We coded only problems that (a) were presented in purely numerical form (not word problems), (b) had two operands, at least one of which was a rational number, and (c) required an exact numerical answer (not an estimate). These were the large majority of rational number arithmetic problems in both the textbooks we coded and in three other series coded by Cady, Hodges, and Collins (2015).

MISSING INPUT AND ITS EFFECTS

Fraction Arithmetic

Table 1 shows the percentages of fraction arithmetic problems with each combination of the four operations (addition, subtraction, multiplication, and division) and equal/unequal denominators averaged across the three textbook series cited earlier. These variables were chosen for examination because together, they determine the correct procedure for all fraction arithmetic problems.

Certain combinations of operation and denominator equality proved to be rare in all three series. Multiplication and division problems with equal denominators were almost nonexistent. This scarcity was not attributable to either the type of operation or denominator equality alone. As seen in Table 1, addition and subtraction problems with equal denominators were common, as were multiplication and division problems with unequal denominators. Rather, it was the combination of equal denominator and the operation being multiplication or division that was uncommon.

These details of problem distributions might be assumed to be irrelevant to children's learning. After all, mathematics is about abstracting over irrelevant details, and the procedures for solving fraction multiplication and division problems are identical regardless of whether denominators are equal. However, examining children's performance indicated that their accuracy was considerably higher on the more common types of problems than on the less common ones. For example, sixth- and eighth-grade students correctly answered 58% of fraction multiplication

Table 1
Percentage of Fraction Arithmetic Problems in Fourth-, Fifth-, and Sixth-Grade Volumes of Three U.S. Textbook Series, Classified by Arithmetic Operation and Whether Denominators of Operands Are Equal.

Relation of denominators	Arithmetic operation			
	Addition	Subtraction	Multiplication	Division
Equal	17	19	**1**	**2**
Unequal	20	15	17	9

Note. Only problems with no whole number operands are included. Cells with very low values are bolded.

problems with unequal denominators, but only 36% of such problems with equal denominators (Siegler & Pyke, 2013).

Support for the hypothesis that scarcity of equal denominator multiplication and division problems reduces accuracy on such problems came from a computer simulation of fraction arithmetic learning (Braithwaite, Pyke, & Siegler, 2017). The model received as input all fraction arithmetic problems from a given textbook series in the order in which they appeared. As with the research we mentioned earlier (Siegler & Pyke, 2013), the simulation erred considerably more often on fraction multiplication problems with equal than unequal denominators. The most common type of error on the scarce type of problems was also the same one children made most frequently (e.g., 3/5 × 4/5 = 12/5). This error was very common both among the children in the aforementioned study (Siegler & Pyke, 2013) and in the simulation (about 40% of trials in each).

Within the simulation, and we believe within children, the mechanisms that produce such errors are association and generalization. More than 98% of problems with equal denominators in the three textbooks involved addition or subtraction. This input appears to lead to associations between equal denominator operands and the addition/subtraction strategy of performing on the numerators the operation specified in the problem and maintaining the common denominator in the answer (as in 3/5 × 4/5 = 12/5). The strong association between equal denominator operands and the addition/subtraction approach, the scarcity of equal denominator multiplication problems, and conceptual understanding not being used to override the association lead to overgeneralization of the addition/subtraction strategy on equal denominator multiplication problems. This analysis suggests that the mechanisms underlying U.S. children's learning rational number arithmetic are at a level much lower than deduction from mathematical principles.

Decimal Arithmetic

Analyses of distributions of decimal arithmetic problems in the same textbooks revealed similar imbalances (Tian, Braithwaite, et al., 2020). The most striking of these involved whether the operands were two decimals (DD problems, such as 0.2 + 1.47) or a whole number and a decimal (WD problems, such as 2 + 1.47), and whether the operation was addition/subtraction or multiplication/division.

As shown in Table 2, WD items involving addition and subtraction were very rare, accounting for only 1% of decimal arithmetic problems across the three textbook series. In contrast, WD items involving multiplication and division were 40% or more of total decimal arithmetic problems in each series. Although WD addition and subtraction procedures are arguably simpler than DD addition and subtraction procedures, researchers predicted that performance on WD addition and subtraction problems would be less accurate than on DD items because students rarely encounter them (Tian, Braithwaite, et al., 2020). The investigators tested this prediction using three datasets: a set collected more than 35 years earlier (Hiebert & Wearne, 1985), a set with more than 3,000 children collected on the web-based ASSISTments platform from 2013 to 2019 (Heffernan & Heffernan, 2014), and a pencil-and-paper study conducted in our lab in 2016.

Table 2

Percentage of Decimal Arithmetic Problems in Fourth-, Fifth-, and Sixth-Grade Volumes of Three U.S. Textbook Series, Classified by Arithmetic Operation and Whether Operands Include a Whole Number.

Type of operands	Arithmetic operation: Addition	Subtraction	Multiplication	Division
Decimal-decimal	14	14	15	12
Whole-decimal	**1**	**1**	21	22

Note. Bolding indicates rare types of problems.

In all these studies, the prediction about children's performance based on the textbook problem distributions was borne out: Children's accuracy was lower on WD addition and subtraction items than on DD items. As with fraction arithmetic, most errors involved overgeneralization of other arithmetic procedures. For example, in the older dataset (Hiebert & Wearne, 1985), only 42% of seventh-grade students correctly added 6 + .32 (as was common in decimal arithmetic problems at the time, no "0" before the decimal point was used in their stimuli). The most common error on 6 + .32 was .38, suggesting that students overgeneralized the whole number addition procedure of right adjusting the operands before adding them (Tian, Braithwaite, & Siegler, 2020). As with fraction arithmetic, these findings with decimal arithmetic indicate that omission of even seemingly simple types of problems opens the door to overgeneralizations of procedures that are correct for other types of problems. The decimal arithmetic data from the ASSISTments platform also indicate that relations between online problems and children's performance are similar to those between textbooks and children's performance.

Fraction and Decimal Arithmetic Assignments

Teachers do not assign all problems in textbooks, nor do they assign only problems from textbooks. In principle, they could compensate for the imbalanced distributions of textbook problems by assigning more of the scarce types. However, they do not appear to do so. A recent study examined all fraction and decimal arithmetic problems that 14 fourth-, fifth, and sixth-grade teachers assigned over the course of a school year (Tian, Leib, Griger, Oppenzato, & Siegler, 2020). The types of problems that were scarce in the textbooks proved to be equally scarce in the assignments, whether the assigned problems came from textbooks (70% of assigned items) or from other sources

(30% of assigned items). Analyses of video recordings of teaching from the Trends in International Mathematics and Science Study have shown similar strong relations between textbooks and teachers' coverage of topics in a wide range of areas of mathematics (Mullis et al., 2012).

Learning Across Rational Number Notations

Also rare in textbooks are problems that combine different rational number notations in a single item. Such problems are fairly common in the world outside the classroom. For example, anticipating the cost of 3/4 (or 75%) of a pound of tuna selling for $24.50/pound requires multiplying a fraction (or percentage) and a decimal, but textbooks almost never present such problems. In the aforementioned textbook series, only two of the almost 4,700 rational number arithmetic problems we coded involved mixed notations. Encountering so few problems is unlikely to produce an understanding of rational number arithmetic with multiple notations in a single problem. However, textbooks may help students understand relations among rational numbers with different notations by presenting tasks other than arithmetic, such as equivalence and comparison problems (e.g., .75 = ?/4; Is 4/5 > .6?).

To examine this type of input across rational number notations, we coded equivalence problems and comparison/ordering problems in which the operands had the same or different notations in the previously mentioned textbook series. The criteria for including problems were the same as those for the rational number arithmetic problems. We found that a substantial percentage of equivalence problems (38% across the three series) involved cross-notation relations. In contrast, the imbalance between within-notation and between-notation problems was much greater for comparison and ordering problems. Of them, only 9% of problems across the three series involved different notations.

These analyses of textbook problem distributions suggest two predictions. First, on equivalence problems, children will perform similarly on within-notation and between-notation problems. Second, on comparison and ordering problems, children will be more accurate on within-notation than on between-notation problems. These predictions need to be tested.

TEXTBOOK INFLUENCES IN OTHER AREAS

Whole Number Arithmetic Principles

Rational number arithmetic is not the only area of mathematics in which textbook characteristics influence children's learning. Another area for which evidence of such a relation is strong is basic arithmetic principles. Research has documented relations between German textbooks' coverage of basic arithmetic principles and first graders' learning of the principles (Sievert et al., 2021). The principles of interest were commutativity (a + b = b + a), inversion (if a + b = c, c − b = a), and what the authors labeled the *neighbor principle* (if a + b = c, a + (b + 1) = c + 1). The investigators used an idiosyncrasy of the educational system in the German state in which the study was conducted: The curriculum was the same throughout the state, but schools were free to choose among a range of textbooks for implementing it. Most schools chose one of four textbook series, which varied considerably in quantity and quality of coverage of the arithmetic principles and strategies for implementing them. One textbook had little discussion of the principles and strategies; coverage in the other three textbooks varied with the principle being considered. The amount of coverage in the textbook of the principles and strategies for implementing them predicted students' acquisition of that type of knowledge. Teachers' emphasis was similarly predictive, but neither accounted for the other. (See Sievert, van den Ham, Niedermeyer, & Heinze, 2019, for related findings.)

Mathematical Equality

The overwhelming majority of whole number arithmetic problems in elementary school textbooks present all operands and operations to the left of the equal sign (e.g., 4 + 5 = _). Almost no problems have operands and operations on both sides of the equal sign (e.g., 4 + 5 = 6 + ___) or only to the right of it (e.g., ___ = 9 + 5; Powell, 2012). As with rational number arithmetic, these omissions open the door to misconceptions and inaccurate performance on scarce types of problems. Even fourth graders usually err on mathematical equality problems with operands and operations on both sides of the equal sign (e.g., 3 + 4 + 5 = __ + 5; Falkner, Levi, & Carpenter, 1999). Most errors on these problems involve summing either all numbers to the left of the equal sign or all numbers anywhere in the problem, yielding answers on 3 + 4 + 5 = __ + 5 of 12 for the first approach and 17 for the second.

To test whether the scarcity of noncanonical problems adversely affects performance, researchers created two workbooks and randomly assigned them to second graders (McNeil, Fyfe, & Dunwiddie, 2015). One workbook had a typical distribution of problems; the other included numerous items of the types that are usually rare. Second graders who used the workbook that included many of the usually scarce types of problems were more accurate than peers who received the workbook with the typical distribution of problems on mathematical equality problems on a posttest given just after children completed the workbooks and on a delayed posttest 5–6 months later. Thus, encountering more rare types of problems improved children's performance on them. These findings are especially important because they demonstrate causal connections between the distribution of problems that children encounter and their learning of problems that are rarely presented in textbooks.

Limitations and Directions

Among the limitations of the research we have described are a scarcity of demonstrations of causal connections between problem distributions in textbooks and children's learning, and a lack of computer simulations of domains beyond fraction arithmetic. To overcome these limitations, researchers should test for causal connections between problem distributions and learning by randomly assigning children to receive primarily problems that rarely appear in textbooks or the same number of problems with a typical distribution. One domain in which researchers should perform this type of study is fraction multiplication, with children receiving either primarily equal or unequal denominator problems; another is decimal addition, with children practicing primarily on either WD or DD problems. Researchers should also formulate a computer simulation that receives textbook input for both decimal and fraction arithmetic, and learns both through the same mechanisms. Together, these extensions can considerably advance our understanding of the development of rational number arithmetic and provide a model for studying the role of problem input in other domains.

EDUCATIONAL IMPLICATIONS

Of all variables that contribute to weak mathematics learning, textbooks might be the easiest to change. Numerous contributors to inadequate learning are intractable; no one knows how to substantially reduce the deleterious effects on learning math of racism, socioeconomic disadvantages, inadequate teaching, uneven student motivation, weak prior knowledge, and limited cognitive capacities. In contrast, changing textbooks should be simpler. For example, textbooks could present more underrepresented types of rational number arithmetic problems and mathematical equality problems with numbers and operations on both sides of the equal sign. They also could provide more and better coverage of arithmetic principles, as well as strategies for implementing them. Learning also might be enhanced if textbooks featured problems to a greater extent than is typical, for example, interleaving multiplication and addition problems with equal and unequal denominators (Rohrer, Dedrick, & Hartwig, 2020). Changing from less to more effective textbooks is more cost-effective for increasing student achievement than investing equal amounts in professional development or class-size reductions (Chingos & Whitehurst, 2012; Koedel, Li, Polikoff, Hardaway, & Wrabel, 2017).

Even in the absence of such changes, many children learn elementary and middle school math reasonably well. This is especially true in countries such as China, where large majorities of middle school students master rational number arithmetic to high degrees of proficiency, despite the fact that distributions of problems in Chinese textbooks closely resemble those in U.S. textbooks (Bailey et al., 2015; Siegler, Im, & Braithwaite, 2020). Understanding relevant concepts, together with extensive practice, seems to allow children to overcome the influence of imbalances in textbooks and other instructional materials (Siegler, Im, Schiller, Tian, & Braithwaite, 2020). For example, a child who understood that multiplying two numbers between zero and one must result in an answer less than either of them would avoid the common error of claiming that $3/5 \times 4/5 = 12/5$. Similarly, a child who understood that both 12/13 and 7/8 were approximately 1 would avoid the common error of claiming that $12/13 + 7/8 \cong 19$.

Viewed from a different perspective, the findings we have presented suggest a general hypothesis about when imbalanced distributions of input problems are likely to have the most deleterious effects: This is likely to occur when (a) the scarce problems are very scarce, providing minimal opportunities for students to learn how to solve them, (b) a superficially similar strategy correctly solves other problems in the domain, and thus is available to be overgeneralized, and (c) conceptual understanding that could be used to override the overgeneralized strategies is absent or at least not applied. Fraction arithmetic, decimal arithmetic, and mathematical equality problems meet all three criteria, with predicted results. The hypothesis suggests teaching approaches to address the problem: provide more instances of otherwise-scarce problems, explicitly note differences between types of problems for which a strategy is appropriate and ones to which it is often overgeneralized, and emphasize mathematical principles that justify correct strategies and contradict incorrect ones.

As this analysis suggests, to improve math learning, we need not only more balanced distributions of textbook problems but also improved teaching. Teachers in China have a far deeper conceptual understanding of rational number arithmetic, as well as of what students typically do and do not know about the topic, than teachers in the United States (Ma, 1999; Zhou, Peverly, & Xin, 2006). The topic receives much more emphasis in Chinese teacher education programs than in U.S. programs, with the result that unlike in the United States, new teachers in China are as knowledgeable about teaching and learning about fractions as their more experienced peers (Zhou et al., 2006). It probably is not coincidental that Chinese children have a greater conceptual understanding of fraction arithmetic, as well as greater mastery of relevant procedures, than U.S. children (Bailey et al., 2015). Combining more frequent presentations of currently rare types of problems, clearer explanations of why tempting incorrect strategies are incorrect, and more effective communication of the principles that underlie correct procedures are promising ways to improve children's mathematics learning.

REFERENCES

Bailey, D. H., Zhou, X., Zhang, Y., Cui, J., Fuchs, L. S., Jordan, N. C., . . . Siegler, R. S. (2015). Development of fraction concepts and procedures in U.S. and Chinese children. *Journal of Experimental Child Psychology*, *129*, 68–83. https://doi.org/10.1016/j.jecp.2014.08.006

Braithwaite, D. W., Pyke, A. A., & Siegler, R. S. (2017). A computational model of fraction arithmetic. *Psychological Review*, *124*, 603–625. https://doi.org/10.1037/rev000072

Cady, J. A., Hodges, T. E., & Collins, R. L. (2015). A comparison of textbooks' presentation of fractions. *School Science & Mathematics*, *115*, 105–116. https://doi.org/10.1111/ssm.12108

Carpenter, T., Corbitt, M., Kepner, H., Lindquist, M., & Reys, R. (1980). Results of the second NAEP mathematics assessment: Secondary school. *The Mathematics Teacher*, *73*, 329–338. https://doi.org/10.5951/MT.73.5.0329

Charles, R., Caldwell, J., Cavanagh, M., Chancellor, D., Copley, J., Crown, W., ... Van der Walle, J. (2012). *enVisionmath* (Common Core ed.). Glenview, IL: Pearson Education.

Chingos, M. M., & Whitehurst, G. J. (2012). *Choosing blindly: Instructional materials, teacher effectiveness, and the common core*. Washington, DC: The Brookings Institution. Retrieved from Proquest Central http://ezproxy.cul.columbia.edu/login?url=https://www-proquest-com.ezproxy.cul.columbia.edu/docview/1792777788?accountid=10226

Common Core State Standards Initiative. (2010). *Common core state standards for mathematics*. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Dixon, J. K., Adams, T. L., Larson, M., & Leiva, M. (2012). *GO MATH!* (Common Core ed.). Orlando, FL: Houghton Mifflin Harcourt.

Douglas, D., & Attewell, P. (2017). School mathematics as gatekeeper. *The Sociological Quarterly*, *58*, 648–669. https://doi.org/10.1080/00380253.2017.1354733

Doyle, A. C. (1894). *The memoirs of Sherlock Holmes*. London, UK: George Newnes.

Fagginger Auer, M. F., Hickendorff, M., van Putten, C. M., Béguin, A. A., & Heiser, W. J. (2016). Multilevel latent class analysis for large-scale educational assessment data. Exploring the relation between the curriculum and students' mathematical strategies. *Applied Measurement in Education*, *29*, 144–159. https://doi.org/10.1080/08957347.2016.1138959

Falkner, K., Levi, L., & Carpenter, T. (1999). Children's understanding of equality: A foundation for algebra. *Teaching Children Mathematics*, *6*, 232–236. https://doi.org/10.5951/TCM.6.4.0232

Handel, M. J. (2016). What do people do at work? *Journal for Labor Market Research*, *49*, 177–197. https://doi.org/10.1007/s12651-016-0213-1

Heffernan, N. T., & Heffernan, C. L. (2014). The ASSISTments ecosystem: Building a platform that brings scientists and teachers together for minimally invasive research on human learning and teaching. *International Journal of Artificial Intelligence in Education*, *24*, 470–497. https://doi.org/10.1007/s40593-014-0024-x

Hiebert, J., & Wearne, D. (1985). A model of students' decimal computation procedures. *Cognition and Instruction*, *2*, 175–205. https://doi.org/10.1080/07370008.1985.9648916

Horsley, M., & Sikorová, Z. (2014). Classroom teaching and learning resources: International comparisons from TIMSS—A preliminary review. *Orbis Scholae*, *8*, 43–60. https://doi.org/10.14712/23363177.2015.65

Koedel, C., Li, D., Polikoff, M. S., Hardaway, T., & Wrabel, S. L. (2017). Mathematics curriculum effects on student achievement in California. *AERA Open*, *3*, 1–22. https://doi.org/10.1177/2332858417690511

Lortie-Forgues, H., Tian, J., & Siegler, R. S. (2015). Why is learning fraction and decimal arithmetic so difficult? *Developmental Review*, *38*, 201–221. https://doi.org/10.1016/j.dr.2015.07.008

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Erlbaum.

McNeil, N. M., Fyfe, E. R., & Dunwiddie, A. E. (2015). Arithmetic practice can be modified to promote understanding of mathematical equivalence. *Journal of Educational Psychology*, *107*, 423–436. https://doi.org/10.1037/a0037687

Mullis, I. V., Martin, M., Foy, P., & Arora, A. (2012). *TIMSS 2011 international results in mathematics*. Chestnut Hill, MA: TIMSS & PIRLS International Study Center, Boston College.

Opfer, V. D., Kaufman, J. H., Pane, J. D., & Thompson, L. E. (2018). *Aligned curricula and implementation of Common Core State Mathematics Standards: Findings from the American Teacher Panel*. Santa Monica, CA: RAND Corp. Retrieved from https://www.rand.org/pubs/research_reports/RR2487.html

Powell, S. (2012). Equations and the equal sign in elementary mathematics textbooks. *The Elementary School Journal*, *112*, 627–648. https://doi.org/10.1086/665009

Rohrer, D., Dedrick, D. F., & Hartwig, M. K. (2020). The scarcity of interleaved problems in mathematics textbooks. *Educational Psychology Review*, *32*, 873–883. https://doi.org/10.1007/s10648-020-09516-2

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., ... Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science*, *23*, 691–697. https://doi.org/10.1177/0956797612440101

Siegler, R. S., Im, S.-H., & Braithwaite, D. (2020). Understanding development requires assessing the relevant environment: Examples from mathematics learning. *New Directions for Child and Adolescent Development*, *2020*, 83–100. https://doi.org/10.1002/cad.20372

Siegler, R. S., Im, S.-H., Schiller, L. K., Tian, J., & Braithwaite, D. W. (2020). The sleep of reason produces monsters: How and when biased input shapes mathematical learning. *Annual Review of Developmental Psychology*, *2*, 413–435. https://doi.org/10.1146/annurev-devpsych-041620-031544

Siegler, R. S., & Pyke, A. A. (2013). Developmental and individual differences in understanding of fractions. *Developmental Psychology*, *49*, 1994–2004. https://doi.org/10.1037/a0031200

Sievert, H., van den Ham, A.-K., & Heinze, A. (2021). Are first graders' arithmetic skills related to the quality of mathematics textbooks? A study on students' use of arithmetic principles. *Learning and Instruction*, *71*, 1–14. https://doi.org/10.1016/j.learninstruc.2020.101401

Sievert, H., van den Ham, A.-K., Niedermeyer, I., & Heinze, A. (2019). Effects of mathematics textbooks on the development of primary school children's adaptive expertise in arithmetic. *Learning and Individual Differences*, *74*, 1–13. https://doi.org/10.1016/j.lindif.2019.02.006

Stigler, J. W., Givvin, K. B., & Thompson, B. (2010). What community college developmental mathematics students understand about mathematics. *The MathAMATYC Educator*, *10*, 4–16.Retrieved from http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.214.7819

Tian, J., Braithwaite, D. W., & Siegler, R. S. (2020). Distributions of textbook problems predict student learning: Data from decimal arithmetic. *Journal of Educational Psychology*. Advance online publication. https://doi.org/10.1037/edu0000618

Tian, J., Leib, E. R., Griger, C., Oppenzato, C. O., & Siegler, R. S. (2020). *Biased problem distributions in class assignments parallel those in textbooks: Evidence from fraction and decimal arithmetic*. Unpublished manuscript.

University of Chicago School Mathematics Project. (2015a). *Everyday mathematics student math journal* (4th ed.). Columbus, OH: McGraw-Hill Education.

University of Chicago School Mathematics Project. (2015b). *Everyday mathematics student reference book* (4th ed.). Columbus, OH: McGraw-Hill Education.

Valverde, G. A., Bianchi, L. J., Wolfe, R. G., Schmidt, W. H., & Houang, R. T. (2002). *According to the book: Using TIMSS to investigate the translation of policy into practice through the world of textbooks*. Dordrecht, The Netherlands: Kluwer.

Zhou, Z., Peverly, S. T., & Xin, T. (2006). Knowing and teaching fractions: A cross cultural study of American and Chinese mathematics teachers. *Contemporary Educational Psychology*, *31*, 438–457. https://doi.org/10.1016/j.cedpsych.2006.02.001