# CANADIAN MATHEMATICS EDUCATION STUDY GROUP

# GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

## PROCEEDINGS / ACTES
## 2019 ANNUAL MEETING / RENCONTRE ANNUELLE 2019

St. Francis Xavier University
Antigonish, Nova Scotia
May 31 – June 4, 2019

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# PROCEEDINGS OF THE 2019 ANNUAL MEETING OF THE CANADIAN MATHEMATICS EDUCATION STUDY GROUP / ACTES DE LA RENCONTRE ANNUELLE 2019 DU GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

43rd Annual Meeting
St. Francis Xavier University
Antigonish, Nova Scotia
May 31 – June 4, 2019

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# INTRODUCTION

Peter Liljedahl – President, CMESG/GCEDM
*Simon Fraser University*

*(Texte en français suit.)*

The 43rd meeting of CMESG was held at St. Francis Xavier University in Antigonish, Nova Scotia (May 31–June 4, 2019). The quaint setting served as an idyllic location for us to engage in a rigorous scientific program that both enlightened and pushed us to think differently about issues that are all around us in mathematics education. From the opening reception to the stunning excursion and conference dinner, Lisa Lunney Borden, Evan Throop-Robinson, Ellen Carter, Tara Taylor, and their team saw to our every need.

This meeting marked the only the third time CMESG/GCEDM had been held in Nova Scotia (1996, 2003), and the first time it had been held at St. Francis Xavier University. The meeting was attended by 116 participants, 25 of which were first timers. As a group we were stimulated by a scientific program featuring plenary sessions by Jean-Marie De Koninck and Rochelle Gutierrez. While Jean-Marie moved us to be passionate about discovering mathematics with our students, Rochelle pushed us to think about mathematics as dispossession and to seek out forms of mathematics that respects mental sovereignty. We also had a plenary panel, chaired by Frédéric Gourdeau, composed of three pairs of pairs of presenters from across Canada brought together to answer the question of how to initiate and nurture collaboration between mathematicians and mathematics educators. The scientific program also offered us a choice among six working groups, four topic sessions, five new PhD presentations, 18 gallery walk presentations, and 12 AdHoc sessions. This meant that at every moment of the program there were sessions happening that were just as good as the one you were in.

In this regard, the proceedings of the 43rd meeting of CMESG/GCEDM is an opportunity to catch up on all the things that we missed when we were forced to choose one amazing session over another as well as remind us of the things we experienced in the sessions we did attend. On behalf of the CMESG/GCEDM membership I would like to thank the contributors to this proceeding as well as the proceeding editor, Jennifer Holm, for their dedication in creating a written record of our meeting. As you read these proceedings, I hope that you will be able to relive some of the stimulating conversations that takes place when so many dedicated people come together to discuss the direction of mathematics education across Canada and around the world.

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La 43e rencontre annuelle du CMESG/GCEDM s'est tenue à l'Université St. Francis Xavier à Antigonish, en Nouvelle-Écosse (31 mai–4 juin 2019). Ce cadre pittoresque nous a servi de lieu idyllique pour nous engager dans un programme scientifique rigoureux qui nous a à la fois éclairés et poussés à penser différemment aux questions qui nous entourent et qui concernent l'enseignement des mathématiques. De la réception d'ouverture à la superbe excursion et au banquet, Lisa Lunney Borden, Evan Throop-Robinson, Ellen Carter, Tara Taylor et leur équipe ont su répondre à tous nos besoins.

Pour la première fois tenue à l'Université St. Francis Xavier, cette rencontre annuelle du CMESG/GCEDM est la troisième à avoir eu lieu en Nouvelle-Écosse (1996, 2003). Au total, 116 personnes ont participé à la rencontre et 25 d'entre elles se sont jointes à nous pour la première fois. En tant que groupe, nous avons été stimulés par un programme scientifique comprenant des séances plénières de Jean-Marie De Koninck et de Rochelle Gutierrez. Alors que Jean-Marie nous a incités à être passionnés par la découverte des mathématiques avec nos étudiantes et étudiants, Rochelle nous a poussés à penser les mathématiques comme une dépossession et à rechercher des formes de mathématiques qui respectent la souveraineté mentale. Nous avons également réuni un panel, présidé par Frédéric Gourdeau et composé de trois paires de présentatrices et présentateurs provenant de différentes régions du Canada, pour discuter de différentes façons d'initier et de nourrir la collaboration entre les mathématiciennes et mathématiciens et les didacticiennes et didacticiens en mathématiques. Le programme scientifique nous a également offert le choix entre six groupes de travail, quatre séances thématiques, cinq présentations de nouveaux titulaires de doctorats, 18 présentations dans le cadre de la Galerie mathématique et 12 séances Ad Hoc. Cela signifie qu'à tout moment de la journée, il y avait des séances qui se déroulaient, aussi intéressantes les unes que les autres.

À cet égard, les actes de la 43e rencontre annuelle du CMESG/GCEDM représentent une occasion pour découvrir tout ce que nous avons manqué lorsque nous avons été contraints de choisir une excellente séance plutôt qu'une autre, ainsi que de nous rappeler ce que nous avons vécu lors des séances auxquelles nous avons participé. Au nom des membres du CMESG/GCEDM, j'aimerais remercier les personnes qui ont contribué à ces actes, ainsi que la rédactrice en chef des actes, Jennifer Holm, pour leur dévouement pour créer un compte rendu écrit de notre rencontre. Quand vous lirez ce compte rendu, j'espère que vous revivrez certaines des conversations stimulantes qui ont lieu lorsque tant de personnes dédiées se réunissent pour discuter de l'orientation de l'enseignement des mathématiques au Canada et dans le monde entier.

# Horaire

| Vendredi 31 mai | Samedi 1 juin | Dimanche 2 juin | Lundi 3 juin | Mardi 4 juin |
|---|---|---|---|---|
| **Réception des amis de FLM** vendredi, 31 mai 15h30 – 16h20 | 8h45 – 10h15 **Groupes de travail** | 8h45 – 10h15 **Groupes de travail** | 8h45 – 10h15 **Groupes de travail** | 8h45 – 9h30 **N$^{lles}$ thèses** |
| 14h30 – 18h45 **Inscription** | 10h15 – 10h45 Pause café | 10h15 – 10h45 Pause café | 10h15 – 10h45 Pause café | 9h45 – 11h00 **Panel** |
| 17h00 – 18h30 **Souper** | 10h45 – 12h15 **Groupes de travail** | 10h45 – 12:15 **Groupes de travail** | 10h45 – 12h15 **Groupes de travail** | 11h00 – 11h15 Pause café |
| 18h30 – 19h30 **Séance d'ouverture** | 12h30 – 13h45 Dîner | 12h30 – 13h30 Dîner | 12h30 – 13h45 Dîner | 11h15 – 12h30 **Séance de clôture** |
| 19h30 – 20h30 **Plénière I** | 13h45 – 14h15 **Petits groupes** | 13h00 – 14h00 **Café, dessert et galerie mathématique** | 13h45 – 14h15 **Petits groupes** | |
| 20h30 – 22h00 **Réception** | 14h25 – 15h25 **Discussion de la plénière I** | 14h00 – 15h00 **Plénière II** | 14h25 – 15h25 **Discussion de la plénière II** | |
| | 15h30 – 16h15 **Séances thématique** | 15h30 **Excursions** | 15h30 – 16h00 **Temps pour préparer las présentation de « clôture »** | |
| | 16h15 – 16h45 Pause café | 19h00 – ? **Souper** | 16h00 – 16h45 **Séances thématique** | |
| | 16h45 – 17h25 **N$^{lles}$ thèses** | | 16h55 – 17h25 **Séances ad hoc** | |
| | 17h30 – 18h45 **Assemblée générale annuelle** | | 17h30 – 22h00 **Souper Social/Danse** | |
| | 17h30 – ? **Souper libre** | | | |

## Schedule

| Friday May 31 | Saturday June 1 | Sunday June 2 | Monday June 3 | Tuesday June 4 |
|---|---|---|---|---|
| **Friends of FLM**<br>Friday, May 31<br>3:30 – 4:20 | 8:45 – 10:15<br>**Working Groups** | 8:45 – 10:15<br>**Working Groups** | 8:45 – 10:15<br>**Working Groups** | 8:45 – 9:30<br>**New PhDs** |
| 12:30 – 6:45<br>**Registration** | 10:15 – 10:45<br>Break | 10:15 – 10:45<br>Break | 10:15 – 10:45<br>Break | 9:45 – 11:00<br>**Panel** |
| 5:00 – 6:30<br>**Dinner** | 10:45 – 12:15<br>**Working Groups** | 10:45 – 12:15<br>**Working Groups** | 10:45 – 12:15<br>**Working Groups** | 11:00 – 11:15<br>Break |
| 6:30 – 7:30<br>**Opening Session** | 12:30 – 1:45<br>Lunch | 12:30 – 1:30<br>Lunch | 12:30 – 1:45<br>Lunch | 11:15 – 12:30<br>**Closing session** |
| 7:30 – 8:30<br>**Plenary I** | 1:45 – 2:15<br>**Small groups** | 1:00 – 2:00<br>**Dessert, coffee, and math gallery** | 1:45 – 2:15<br>**Small groups** | |
| 8:30 – 10:00<br>**Reception** | 2:25 – 3:25<br>**Plenary I discussion** | 2:00 – 3:00<br>**Plenary II** | 2:25 – 3:25<br>**Plenary II discussion** | |
| | 3:30 – 4:15<br>**Topic sessions** | 3:30<br>**Excursions** | 3:30 – 4:00<br>**Preparation for the closing session** | |
| | 4:15 – 4:45<br>Break | 7:00 – ?<br>**Dinner** | 4:00 – 4:45<br>**Topic sessions** | |
| | 4:45 – 5:25<br>**New PhDs** | | 4:55 – 5:25<br>**ad hoc sessions** | |
| | 5:30 – 6:45<br>**Annual general meeting** | | 5:30 – 10:00<br>**Dinner Social/Dancing** | |
| | 5:30 – ?<br>**Dinner on your own** | | | |

# Plenary Lectures

# Conférences plénières

# DÉCOUVRIR LES MATHÉMATIQUES ENSEMBLE AVEC LES ÉTUDIANTS

Jean-Marie De Koninck
*Université Laval*

Nous sommes tous convaincus que les mathématiques représentent une science que tout jeune a intérêt à connaitre, voire à étudier en profondeur. En effet, celui ou celle qui compte se tailler un jour une bonne place sur le marché du travail de demain a besoin aujourd'hui d'une bonne formation en mathématiques, peu importe le métier auquel il ou elle aspire. Ainsi, selon moi, il incombe à chaque enseignant en mathématiques d'établir une stratégie pour amener les jeunes qu'il ou elle encadre à découvrir par eux-mêmes le plaisir de faire des maths. En fait, c'est précisément ce message que j'ai voulu transmettre lors de l'exposé que j'ai livré au colloque du Groupe canadien d'étude en didactique des mathématiques en juin 2019. Concrètement, le principal objectif de ma présentation était de partager avec les participants les expériences acquises au cours de ma carrière académique en matière d'éveil des jeunes et moins jeunes à l'aventure mathématique.

## L'ÉTINCELLE

Pour moi, tout a commencé à l'âge de 9 ans. Ma passion pour les mathématiques est née en cinquième année, lorsque la religieuse, qui était mon enseignante de mathématiques, affichait un grand plaisir à écrire et à résoudre des équations de premier degré au tableau noir. D'une part, cela a certainement éliminé de mon esprit la possibilité d'avoir un jour des préjugés à l'égard des femmes en mathématiques. D'autre part, avec le temps, je suis devenu de plus en plus convaincu que, en plus d'aider ses étudiants à acquérir des connaissances de base en mathématiques, le principal rôle d'un enseignant de maths devrait être de partager sa passion pour les mathématiques. Certes, cela a été ma philosophie durant mes 44 années de carrière. De plus, afin de partager ma passion avec un plus grand public, j'ai aussi choisi d'être actif à l'extérieur de ma communauté universitaire. C'est principalement cette expérience que je veux raconter dans ce texte.

## C'EST MATHÉMATIQUE !

Cette stratégie d'aller répandre la bonne nouvelle à l'extérieur du monde universitaire a été provoquée par une intervention externe. En effet, en l'an 2000, j'ai été approché pour animer une émission de télé sur les mathématiques. J'ai d'abord refusé cette offre parce que je ne me voyais pas devenir animateur à la télévision. Cependant, encouragé par mes collègues universitaires qui y voyaient là une opportunité de faire aimer les mathématiques à un public élargi, j'ai finalement accepté la proposition. Ainsi, pendant deux trimestres d'hiver, j'ai animé une émission de télévision hebdomadaire intitulée *C'est mathématique* ! Elle était diffusée au Canal Z aux heures de grande écoute et on y traitait des applications des mathématiques dans notre vie de tous les jours. L'aventure s'est avérée un succès, et c'est ainsi que j'ai reçu une foule de commentaires encourageants, la plupart venant d'enseignants des niveaux secondaire et collégial. C'est à ce moment que j'ai réalisé qu'il était possible de faire aimer les mathématiques en sortant du cadre académique. Ce n'est donc pas un hasard si en 2002, on m'invitait à donner la conférence grand public lors du colloque d'été de la Société mathématique du Canada. Fort de la publicité autour de l'événement, tout près de 400 personnes ont assisté à mon exposé. Une fois de plus, j'ai beaucoup appris sur la manière d'intervenir devant des non-mathématiciens pour faire valoir l'importance des mathématiques dans le fonctionnement de notre société, tout en mettant en évidence le grand plaisir que l'on peut avoir à faire des maths.

## LA NAISSANCE DE SCIENCES ET MATHÉMATIQUES EN ACTION

Après quelques interventions disparates dans quelques écoles et auprès du grand public, j'ai été encouragé par les autorités de l'Université Laval à consacrer davantage de temps à la diffusion des mathématiques à l'extérieur du campus universitaire. C'est dans ce contexte qu'en 2005, j'ai créé *Sciences et mathématiques en action* (SMAC), un programme dont la mission est encore aujourd'hui de susciter chez les jeunes un intérêt pour les mathématiques et de démystifier les mathématiques auprès de la population en général.

Dès la première année de SMAC, fort de l'appui d'étudiants enthousiastes, nous avons créé *Show Math*, un spectacle multimédia destiné aux jeunes du secondaire ainsi qu'au grand public, où l'humour côtoie les mathématiques. Animé par des mordus de mathématiques et appuyé par des vidéos et des sketches humoristiques réalisés par des comédiens professionnels, le spectacle *Show Math* aborde de manière simple et amusante une foule de sujets mathématiques qui touchent de près tout le monde. On y aborde ainsi des questions bien intrigantes, comme par exemple comment, il y a plus de 2000 ans, le mathématicien grec Eratosthène s'y est pris pour mesurer la circonférence de la Terre sans avoir accès aux instruments modernes ou encore, du côté plus ludique, comment se fait-il que lors d'une soirée à laquelle participent 23 de vos amis, il y a 50 % de chances qu'au moins deux personnes de ce groupe soient nées le même jour de l'année. La conférence-spectacle *Show Math* s'est mise à sillonner les routes du Québec, de l'Ontario et du Nouveau-Brunswick, et s'est même rendue jusqu'à Vancouver. Au total, plusieurs centaines de représentations ont été offertes depuis l'automne 2005. Le Ministère de l'Économie et de l'Innovation nous a même soutenus financièrement pour créer une trousse éducative destinée aux enseignants souhaitant préparer leurs élèves à la venue de *Show Math* dans leurs écoles respectives et aussi pour prolonger les retombées positives du spectacle bien au-delà de la journée de sa présentation. Ensuite, afin de répondre à la demande des enseignants qui souhaitaient offrir à leurs élèves une deuxième expérience mathématique « amusante » nous avons créé un deuxième spectacle, celui-là appelé *Show Math 2*. Suite aux commentaires recueillis auprès des enseignants qui avaient assisté au premier *Show Math*, nous avons décidé d'inclure dans le nouveau spectacle des thèmes davantage présents dans notre vie de tous les jours. Ainsi, parmi les sujets abordés, on trouve l'utilisation des fractales pour créer des effets spéciaux au cinéma. Aussi au programme de *Show Math 2*, on trouve la cryptographie, soit la science du codage et du décodage de l'information secrète, laquelle a entre autres grandement contribué à accélérer la fin de la 2e guerre mondiale. Également dans ce nouveau spectacle, on décrit les mathématiques toutes simples, mais essentielles, qui permettent le fonctionnement du GPS.

## PETIT SHOW MATH

De l'avis de plusieurs enseignantes et enseignants des écoles secondaires, l'idée de créer une activité mathématique à caractère ludique aura permis de changer la perception des mathématiques dans la tête des jeunes qu'ils encadrent. Néanmoins, ayant le goût de créer un plus grand impact auprès des jeunes, en discutant avec mon équipe d'enseignants et d'étudiants engagés dans le processus d'élaboration de nos activités, nous avons décidé de revoir notre approche créative. Concrètement, pour répondre à la demande des directeurs et directrices des écoles primaires qui souhaitaient la venue de notre spectacle dans leurs établissements, nous avons mis sur pied un comité chargé de concevoir un tout nouveau spectacle, celui-là adapté au niveau de connaissances des élèves du primaire. L'idée était de bien cerner l'approche et le contenu susceptibles d'intéresser les jeunes et de les engager davantage. Ce comité était composé d'étudiants et étudiantes en enseignement primaire et préscolaire, d'une graphiste et bien sûr de quelques membres de mon équipe. Ainsi est né, en 2009, *Petit Show Math*, un spectacle multimédia destiné aux élèves des 4e, 5e, et 6e années. Présenté par un animateur dynamique et mettant en vedette son invité virtuel *Smat*, l'auditoire est amené à découvrir l'univers fantastique des mathématiques. Pour ce faire, l'auditoire est plongé dans une aventure spatio-temporelle qui lui permet d'explorer et d'admirer les maths de l'Âge de pierre jusqu'à celles de l'exploration spatiale. Comment avons-nous écrit les nombres à travers l'histoire ? Comment décrire le son avec les maths ? Qu'est-ce que les mathématiques nous ont permis d'apprendre à propos de notre système solaire ? Voilà quelques-unes des questions abordées dans *Petit Show Math* et qui font le plus grand bonheur des enfants.

## S'AMUSER SUR INTERNET EN FAISANT DES MATHS

Parallèlement à ces spectacles, l'équipe de SMAC a décidé d'aller rejoindre les jeunes et les moins jeunes dans leurs foyers… devant leurs ordinateurs ! Comment ? En créant un jeu en ligne appelé tout simplement *Math en jeu*, un jeu multimédia interactif accessible gratuitement en ligne et destiné aux étudiants des niveaux primaire, secondaire et collégial. Pour débuter une partie, chaque joueur doit d'abord se choisir un avatar et ensuite répondre à des questions mathématiques (adaptées à son niveau académique), et cela afin de pouvoir se déplacer sur une sorte d'échiquier virtuel et ainsi gagner des points. Le gagnant ou la gagnante est le joueur qui a accumulé le plus de points. Le jeu contient 5721 questions, de sorte que chaque joueur risque fort peu de se retrouver avec la même question, même en jouant plusieurs parties. Le jeu est également disponible en anglais sous le nom de *MathAmaze* et contient 6663 questions. L'organisme national de recherche *Mitacs* a reconnu la valeur et la portée des activités de SMAC et a donc choisi d'investir temps et ressources financières dans le développement de *MathAmaze*, d'où l'immense popularité que connait le jeu aujourd'hui. Qui plus est, le jeu contient une plateforme pour les enseignants qui voudraient utiliser *Math en Jeu* dans le cadre de leurs cours.

## DE RETOUR APRÈS LA PAUSE

Compte tenu de la notoriété acquise grâce aux nombreuses activités de SMAC, en particulier celles très médiatisées, le monde radiophonique m'a approché pour animer une chronique de nature scientifique à la radio. C'est ce que j'ai fait tous les samedis matins de 2009 à 2012 sur les ondes de la radio de Radio-Canada dans le cadre de l'émission radiophonique de Catherine Lachaussée. Une chronique de 13 minutes au cours de laquelle je présentais aux auditeurs trois ou quatre sujets scientifiques, souvent à saveur mathématique, allant de la manière dont s'y prennent les astronomes pour découvrir les exoplanètes jusqu'à l'explication mathématique de la raison pour laquelle il arrive souvent qu'une même personne remporte deux fois le gros lot à la loterie. Durant les années 2013 et 2014, j'ai aussi eu le privilège d'animer une chronique sur la science du sport durant l'émission hebdomadaire de Robert Frosi. C'est ainsi que j'ai pu expliquer aux auditeurs pourquoi, si vous arrivez en retard à un match de soccer et qu'une équipe mène 1 à 0, cette équipe perdra la partie seulement une fois sur sept.

## POUR CEUX ET CELLES QUI AIMENT LIRE

En 2009, la maison d'édition *Septembre Éditeur* m'a approché pour écrire un livre de vulgarisation sur les mathématiques pour les jeunes du secondaire et le grand public. Quatre bouquins sont ainsi nés : *En chair et en maths*, *En chair et en maths 2*, *The Secret Life of Mathematics*, et *Cette science qui ne cesse de nous étonner*. Les trois premiers ouvrages ont été écrits en collaboration avec le journaliste scientifique Jean-François Cliche. Chacun de ces ouvrages avait pour objectif d'amener les jeunes, par le biais d'histoires amusantes, à découvrir que les mathématiques sont utiles et en fin de compte très accessibles. Le quatrième ouvrage est constitué de 32 rubriques scientifiques puisées à même les centaines de chroniques radiophoniques déjà diffusées à la radio de Radio-Canada alors que j'étais l'invité de l'émission matinale de Catherine Lachaussée.

## PLUTON VA EN APPEL !

Pour être en mesure de bien apprécier l'exploration spatiale, il est fort utile de connaitre quelques notions mathématiques. Ou disons-le d'une autre manière. Quoi de mieux que l'exploration spatiale pour bien faire saisir aux jeunes le rôle important des mathématiques dans la compréhension de l'univers qui nous entoure ? Personnellement, j'ai toujours été fasciné par notre système solaire et le mouvement des planètes qui le composent. Or, un jour, alors que j'étais à feuilleter les livres scientifiques de la Librairie Gibert à Paris, je tombe sur un petit livre intitulé *The Case for Pluto*. On y racontait toute l'histoire de la petite planète Pluton depuis sa découverte en 1930 jusqu'à la controverse créée en août 2006 par une décision de l'Union d'astronomie internationale (UAI). En effet, ayant constaté qu'il y avait dans notre système solaire des corps célestes de même taille que Pluton—voire plus gros et plus massif, tel que l'astéroïde Eris—les membres de l'UAI décidèrent, suite à un vote serré, d'enlever à Pluton son titre de « planète » et de lui attribuer à la place le statut de « planète naine ». Maigre consolation pour Pluton qui grossit ainsi les rangs d'astres moins célèbres comme Cérès (un astéroïde orbitant entre Mars et Jupiter) et 70 000 autres objets. Cette décision n'a pas du tout fait l'unanimité dans la communauté scientifique, ni auprès du grand public. Cela m'a alors donné l'idée de créer un nouveau spectacle, celui-là relevant plutôt du domaine de l'astronomie et de l'exploration spatiale : *Pluton va en appel* ! Trois acteurs sont sur scène : Mercure (qui souhaite que Pluton redevienne une planète), Neptune (qui souhaite qu'en aucune façon Pluton ne soit autorisée à retrouver son nom de « planète ») et le Soleil (qui agit en quelque sorte comme un arbitre qui devra éventuellement décider du sort de Pluton). À force d'arguments mathématiques, Mercure et Neptune débattent devant le Soleil pour déterminer si Pluton pourra ou non réintégrer le groupe. Vers la fin du spectacle, le Soleil, se sentant incapable de trancher, demande à l'auditoire de voter pour décider du sort de Pluton. Selon le résultat du vote, le spectacle a alors deux issues possibles, toutes deux hilarantes. Certes, plusieurs jeunes auront découvert par cette présentation que les mathématiques se retrouvent dans des domaines insoupçonnés et qu'elles constituent une matière essentielle à la compréhension du monde dans lequel on vit. Les enseignants ayant apprécié la venue de *Pluton va en appel* dans leurs écoles, une suite a été créée : *Pluton a disparu*, une occasion supplémentaire de faire apprécier le rôle central des mathématiques dans les sciences appliquées.

## LA BONNE MANIÈRE DE CRÉER UNE ACTIVITÉ QUI REJOINDRA LES JEUNES

La mise en place de *Show Math*, notre tout premier spectacle, s'est faite très rapidement. Nous ne savions pas exactement comment cette activité serait reçue par les jeunes du secondaire. Certains enseignants nous ont signalé que certaines notions abordées dans le spectacle gagneraient à être revues. Suivant la suggestion de mon adjointe Andrée-Anne Paquet, nous avons dès lors mis en place des comités de création pour chacun des nouveaux spectacles destinés aux écoles. Chaque comité était composé d'enseignants, de conseillers pédagogiques, de comédiens et bien sûr de mathématiciens professionnels. De plus, pour chacun de ces spectacles, une version bêta était ensuite offerte dans certaines écoles « volontaires », et cela afin de tester et ensuite peaufiner le produit. Ces sorties préliminaires se sont avérées extrêmement utiles, parce qu'elles ont permis de recueillir les commentaires des enseignantes et enseignants et de mesurer le niveau d'attention et de participation des élèves, et ainsi d'améliorer le spectacle en conséquence.

## HIER, J'AI VIEILLI DE 4 SECONDES…

En 2015, j'ai été invité à donner une conférence portant sur l'innovation à la maison mère du *Cirque du Soleil* à Montréal devant 200 employés et en visioconférence pour des centaines d'autres. Ayant alors suscité l'intérêt et la confiance de la vice-présidente à la création du *Cirque du Soleil*, j'ai été en mesure de solliciter leur appui pour la création d'un nouveau spectacle fortement teinté d'astronomie et traitant tout particulièrement de la théorie de la relativité. Du coup, deux employés du Cirque du Soleil ont accepté de faire partie du comité de création du nouveau spectacle. Ainsi, toute l'expertise de ce prestigieux organisme était mise à contribution, en particulier pour savoir comment on doit s'y prendre pour susciter l'intérêt et maintenir l'attention d'un public varié. Le synopsis de *Hier, j'ai vieilli de 4 secondes…* va comme suit. Une adolescente part à la recherche de son père disparu il y a 9 ans. Astronome réputé, il rentrait de l'observatoire du Nouveau-Mexique avant que son avion ne s'écrase dans le mystérieux triangle du Nevada. À travers des notes et des livres qui lui appartenaient, et des rencontres pour le moins inusitées (le P'tit Prince de St-Exaspéré, le « cerveau » de Copernic…), elle tente trouver un chemin qui la mènera à lui. Un prétexte pour aborder des notions de la théorie de la relativité avec humour autour d'une intrigue qui nous tient en haleine du début jusqu'à la fin.

## PEUT-ON ÉMERVEILLER LES ÉLÈVES DE 1ÈRE ET 2E ANNÉES AVEC LES MATHS ?

Nos spectacles mathématiques se sont avérés une manière originale d'entrer dans les écoles pour aller répandre la bonne nouvelle que les mathématiques peuvent être amusantes et surtout qu'elles sont essentielles au bon fonctionnement de notre société. Or, jusqu'en 2018, une clientèle scolaire n'avait pas encore été rejointe : le premier cycle du primaire. En effet, comment s'y prendre pour offrir un spectacle à caractère mathématique à des enfants qui ont peine à rester en place ? Notre comité de création, cette fois composée d'enseignantes du premier cycle du primaire, d'étudiants et d'étudiantes en enseignement primaire et préscolaire, d'une comédienne professionnelle de l'art clownesque, d'un spécialiste du décor, d'une spécialiste des costumes, un compositeur musical et d'une metteure en scène professionnelle, a relevé le défi et conçu le spectacle clownesque *Compte sur moi*. Pourquoi l'approche clownesque ? Le clown est un personnage naïf, spontané, curieux, enthousiaste et c'est un grand adepte de l'apprentissage par l'essai et l'erreur. Un tel archétype permet de valoriser le savoir-faire de l'enfant qui devient alors l'expert et corrige le clown, le tout favorisant l'estime de soi chez l'enfant. De plus, le clown est un modèle qui vient renforcer les attitudes souhaitées chez l'élève : demeurer confiant face à l'inconnu, rester calme devant les problèmes, valoriser l'apprentissage par les erreurs et être créatif dans la recherche de solutions. Nous avons ainsi constaté qu'aborder les mathématiques de manière ludique alors qu'ils sont encore très jeunes contribue grandement à leur épanouissement scientifique.

## L'OUTIL LE PLUS PUISSANT POUR REJOINDRE LES JEUNES ET LEURS ENSEIGNANTS

Outre les cours, les visites dans les écoles et les livres, il existe une manière extrêmement efficace pour rejoindre les jeunes et leurs enseignants. Je fais bien sûr allusion à l'Internet. C'est pourquoi, en collaboration avec l'*Association Québécoise des Jeux Mathématiques* (AQJM), nous avons implanté *La semaine des maths* (en anglais, *Amazingmaths*), soit le site web www.semainedesmaths.ulaval.ca (en anglais, www.amazingmaths.ulaval.ca), lequel offre du matériel pédagogique, dont en particulier des vidéos de magie mathématique, ainsi que des énigmes et des défis que l'on peut relever individuellement ou en groupe.

## LA PARTIE HUMAINE DE L'ÉQUATION

Je n'aurais jamais pu réaliser toutes ces activités en travaillant en solo. En réalité, il y avait et il y a encore toute une équipe dans le programme SMAC. Depuis les tous débuts de SMAC en 2005, plus d'une cinquantaine de jeunes et moins jeunes ont contribué—chacun à sa façon—au succès du programme. Quelle que soit l'activité, le point de vue de chaque membre de chacun des comités de conception est pris en compte. Il y a là toute une dynamique qui en fin de compte porte ses fruits. Sans tous ces esprits innovateurs et dévoués à la tâche, mes expériences en diffusion des mathématiques auraient été bien moins retentissantes. Somme toute, afin de réussir des opérations de sensibilisation majeures, comme celles que l'on a énumérées ici, il est important d'être bien entouré et de respecter les opinions de tout un chacun.

## QUE FAUT-IL RETENIR DE TOUT ÇA ?

En conclusion, je souhaite que mon auditoire retienne de ma présentation que nous avons tous un immense potentiel de créativité que nous pouvons mettre au service des jeunes. Il ne faut surtout pas avoir peur de sortir des sentiers battus afin de concevoir des activités susceptibles d'éveiller chez les jeunes un intérêt pour les maths. Nous l'avons fait avec des spectacles, un jeu en ligne, des interventions médiatiques, des conférences sur les applications des maths et des ouvrages amusants, mais on aurait sans doute pu davantage diversifier notre approche, sans compter qu'il existe certainement plusieurs autres façons de procéder. Je veux citer comme exemple une idée qui a fait un bon bout de chemin, soit l'AQJM créé en 1998 par mon collègue Frédéric Gourdeau (et déjà mentionnée ci-dessus), dont la mission est de faire la promotion des mathématiques dans les écoles et auprès du grand public, par le biais de concours mathématiques, auxquels participent maintenant chaque année plus de 20 000 jeunes et moins jeunes. Certes, il y aura toujours place pour d'autres initiatives. Et, à mon avis, on en aura besoin. En effet, notre société fait face à d'immenses défis. Qu'il s'agisse des changements climatiques, des problèmes de santé liés à une population vieillissante, de la préservation de notre vie privée et de la recherche de nouvelles sources d'énergie propre, ce sont les jeunes d'aujourd'hui qui auront à relever ces défis. Comme chacun de ces défis nécessite une formation à tout le moins minimale en mathématiques, il nous incombe de fournir à cette jeunesse les outils dont elle aura besoin pour s'y attaquer.

# MATHEMATICS AS DISPOSSESSION: RECLAIMING MENTAL SOVEREIGNTY BY LIVING MATHEMATX

Rochelle Gutierrez
*University of Illinois*

Paper was not available at the time of publication.

# Working Groups

# Groupes de travail

# PROBLEM-BASED LEARNING IN POSTSECONDARY MATHEMATICS

# L'APPRENTISSAGE PAR PROBLÈMES EN MATHÉMATIQUES AU NIVEAU POSTSECONDAIRE

Laura Broley, *Concordia University*
Jeff Hooper, *Acadia University*

## PARTICIPANTS

| | | |
|---|---|---|
| Ayman Aljarrah | Parker Glynn-Adey | Paul Muir |
| John Bosica | Frédéric Gourdeau | Tyler Pattenden |
| Olive Chapman | John Hodder | David Poole |
| Amenda Chow | Richard Hoshino | Patrick Reynolds |
| Michelle Davidson | Lisa Lunney-Borden | Robert van den Hoogen |
| Ryan Gibara | Joe MacDonald | Zach Wolske |
| Marianne Girard | Asia Matthews | |

*(Texte en français suit.)*

## INTRODUCTION

Can problem-based learning (PBL) work in post-secondary mathematics? The year 2019 marked the 50th anniversary (!!) of the implementation of a PBL curriculum in McMaster University's Medical Education program (Servant-Miklos, 2019). Despite its successes, similar programs have not emerged widely and especially not in post-secondary mathematics.

The term 'problem-based learning' has been used to describe a pedagogy that

> *begins with a problem to be solved, and the problem is posed is such a way that students need to gain new knowledge before they can solve the problem. Rather than seeking a single correct answer, students interpret the problem, gather needed information, identify possible solutions, evaluate options, and present conclusions.* (Roh, 2003, p. 1)

This working group set out to explore the use of the PBL approach in the teaching of post-secondary mathematics. More specifically, we aimed to reflect on three broad topics. First, the conceptualization of PBL: what PBL means, and what is gained or lost by adopting PBL as opposed to other kinds of pedagogies. Second, the implementation of PBL: in particular, the development of 'good' PBL problems and units for different types of undergraduate mathematics, and the integration of PBL throughout a post-secondary mathematics curriculum. And third, research on PBL: i.e., questions, theoretical frameworks, and methodologies that might serve to help us better understand the use of PBL in post-secondary mathematics.

To take into account participants' varied backgrounds, interests, and views in relation to PBL, the working group developed a more specific list of burning questions. This list, which remained on the walls surrounding us over the three days, served as a representation of where we started and a springboard to future conversation. On the first day, our participants wondered:

1. How does PBL (in mathematics) connect to the case-based model (in medicine)?
2. What is a or *the* framework for PBL? Does using this framework hinder or assist?
3. How is PBL different from 'inquiry-based learning' in mathematics?
4. Are there exemplars or concrete models that can help us?
5. How can we determine if tasks are appropriate, approachable, and worthwhile?
6. How can we deal with mathematics problems in the age of Google?
7. How do we bridge student expectations?
8. How does PBL connect with assessment?
9. Are we all talking about the same thing?
10. Is group work essential to PBL?
11. Can PBL work in every setting (with limited resources, in large classrooms, etc.)?
12. Can a case-based approach work in general in all mathematics courses?
13. How can we ensure we are covering *all* topics?
14. How is interdisciplinarity related to PBL?
15. How can we ensure that we are teaching effectively to mixed audiences?
16. What is the big idea behind doing this?

## PART 1: WHAT IS PBL?

The morning session of Day 1 was dedicated to a reflection on the nature of PBL: what it entails, what it provides, and how we can recognize it. In small groups, participants discussed their prior experiences with PBL (as an instructor or as a student), which led us to realize that we may have all had slightly different ideas of what constitutes 'PBL.' To make these ideas explicit, and try to develop a collective image of PBL, each small group was tasked with focusing on one component of pedagogy—classroom culture, types of problems, teacher's role, student's role, or valued outcomes—and offering a description of the component in the case of PBL. Groups recorded their descriptions on chart paper, displayed them on the walls, and took time to observe and comment on the descriptions offered by other groups. The resulting content is summarized in Figure 1.

At the end of Day 1, the working group considered another possible image of PBL, which has been developed over many years within the medical education domain. Southern Illinois University has built a School of Medicine curriculum similar to the McMaster case-based model, and they have a fly-on-the-wall video showing an example of how they use PBL: www.youtube.com/watch?v=CZeP1a7xQdY. The video depicts a small group of medical students working together to analyze a ‘case’: i.e., a new patient’s complaint. A facilitator is present, but it is the students who do the work to solve the problem: they brainstorm a list of potentially relevant medical conditions, slowly refine the list by asking questions and analyzing responses about the patient (e.g., their medical history or certain test results), and, eventually, determine a diagnosis and action plan.

**CLASSROOM CULTURE**

Students are engaged in the active process of creating and reproducing knowledge, instead of simply being passive consumers of knowledge.


It’s essential that the classroom culture be high-trust and high-accountability [How to build high-trust and high-accountability?], where the expectations are set high and there is plenty of support given to the students. Teachers need to trust students that they can succeed, and they’ll be fully engaged in the problem.

**TYPES OF PROBLEMS**

- “Real” [What does “real” mean?];
- Comprehendible (language, culturally, student level);
- Investigative, interpretable;
- Communicatable understanding;
- Multiple paths and solutions (opportunity for meaningful failure; possibility of no solution) [Examples?];
- Undirected, independent thinking;
- Time frames: in class, or short/long term with research.

**STUDENT’S ROLE**

- To be an active learner;
- To be a good teammate;
- To be willing to specialize and teach others;
- To push the boundaries of problems;
- To [take chances,] make mistakes, [and get messy!];
- To learn from mistakes;
- To communicate;
- To take ownership/authority [Love this!];
- To be persistent.

**TEACHER’S ROLE**

Design:

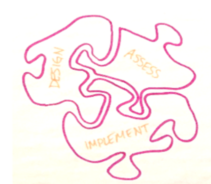
- Connecting learning outcomes;
- Appropriate (formal topics), approachable (student readiness), and worthwhile (systematic and creative ways of thinking?).

Implement:

- Making groups (or not) [When to use groups and why?];
- Guiding learning (asking questions, answering questions);
- Assessing student understanding throughout – *formative* – observation and conversation to guide learning;
- Moderating [Ensure groups are effective and that “all” students are learning and engaging].

Assess:

- Continuous assessment and readjusting implementation;
- Formative assessment of students’ understanding and direction.




**VALUED OUTCOMES**

- Deep understanding;
- Intrinsic motivation;
- Mathematical thinking [What is “mathematical thinking”?];
- Understanding that maths is not all done.

Figure 1. Descriptions of five components of PBL created by members of the working group.

The video was inspirational for the working group. Among shared observations, one nugget was the identification of gaps in student knowledge as ‘Learning Issues’ or 'L.I.’s’: i.e., topics which need to be researched between class meetings. For the medical students, the phrase "I don't know" (e.g., "I don't know what a normal thyroid exam would find") gets translated into an action item; and at the end of a session, each student takes ownership of some action items, uses time in between sessions to independently learn relevant knowledge, and then reports back their findings. In other words, not knowing and needing to learn (both independently and collectively) become normal parts of the problem-solving process.

When we imagined using the medical education approach in post-secondary mathematics, there were some positive takeaways. One suggestion was that a case-based model of learning might offer students a setting for encountering 'meaningful' mathematics and may be more attractive to students who currently choose to leave mathematics because they cannot see the use of it. Another suggestion was that it may be possible to use a full-blown PBL approach in a mathematics course by focusing on two or three large, interesting cases that tie all the content together.

Many constructive questions were also raised. One participant wondered if it would be realistic to distribute mathematical L.I.'s among a mixed-level group of mathematics students. Others questioned the appropriate nature of an L.I. in mathematics: for example, would we want students to independently learn and report on simple facts, challenging mathematical concepts, or proofs? We were unsure if it would be possible (or desirable) to create a sort of 'fact book' for mathematics problems in PBL (like the 'fact book' for a medical case, which students in the video used to look up answers to questions about the patient's medical history or test results). Finally, we considered the question of whether one could really teach all mathematics this way—not only all types of mathematics but also all topics within a particular type. This issue was revisited several times throughout our work.

During the other sessions of the working group, participants were left to further develop their own image of PBL through engaging in activities that did not directly address a definition of the term. One consensus seemed to be that 'PBL' means "beginning with a problem that provides a rich context for learning." Beyond that, we came to question the necessity of confining the meaning of PBL and tended to see it as a spectrum of approaches satisfying the descriptions in Figure 1, any of which could be called upon depending on the outcomes one hopes to achieve and the constraints one faces.

## PART 2: PBL IN ACTION

In an attempt to free participants from their own institutional contexts and engage them in a form of PBL, a fictive setting was introduced: Problems-Based Learning University (or PBLU). PBLU is a new university with 8 000–10 000 undergraduates and approximately 500 graduate students. Of course, PBLU is based around a PBL curriculum. The institution offers programs in standard areas: e.g., life science, pre-health, physical science and engineering, humanities, social science, commerce, and kinesiology. For some reason or other, PBLU has never considered mathematics, and they are now planning a new Mathematics Department. So they are hiring, and working group members are applicants. To inform applicants of the interview process, PBLU sent out letters (for an example, see Figure 2). The first round of interviews would involve the creation of problems for use in PBL. If successful, applicants would then be required to transform some of those problems into PBL units.

**Faculty of Scientific Constructivism**

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pblu
what's the problem?

1 June, 2019

**Re: Position 3-14159**

Dear Applicant,

The Faculty of Scientific Constructivism here at PBLU is pleased to invite you to participate in the first round of interviews for tenure-track positions in our new Department of Mathematics. At PBLU we interview in groups, and hire based on how the group works collectively. As part of the interviewing process, we require you to show that you can come up with appropriate problems for PBL.

In order to prepare for your interview, your group has time now to complete the following pre-interview task. **For each of the following 6 scenarios, come up with 1-2 "good" problems and think about how they might work in a PBL context.**

The Scenarios:

1. Senior theoretical mathematics or statistics course (e.g., Real Analysis, Topology, Probability,. . . )
2. Junior-level course for mathematics majors or physical science students (e.g., Calculus, Linear Algebra, Intro Stats,. . . )
3. Applied mathematics or Statistics course (e.g., Differential Equations, Operations Research,. . . )
4. Mathematics Course designed for future teachers
5. Transition course such as Pre-calculus
6. Service course for non-math majors (e.g., Stats for Life Science, Calculus for Business, Discrete Math for Computer Science,. . . )

[Your group has permission to substitute its own scenario(s) if deemed appropriate.]

To be completely fair, and since you may not have extensive experience with group interviews, we offer a suggestion for this part of the process: your group might wish to distribute the work to different "experts," who can take lead on specific scenarios. We will not offer further suggestions.

Your Interview will officially take place during the morning of **Sunday, June 2nd**. Good luck!

Sincerely,

Dr. J. Piaget, Dean
Faculty of Scientific Constructivism
PBL University

Figure 2. The letter describing the first round of interviews for a faculty position at PBLU.

ROUND 1 INTERVIEWS: DEVELOPING GOOD PROBLEMS FOR PBL

The letter in Figure 2 was handed out in the afternoon session of Day 1. Thus began the heart of the working group, which consisted of discussing problems and how they fit in post-secondary curricula. Small groups began by working to develop 'good' PBL problems for a variety of scenarios: senior theoretical mathematics, junior-level mathematics, applied mathematics, mathematics for future teachers, and mathematics for transition or service courses.

When invited to share how the process was going, several participants commented on the groupwork component of the activity. On the one hand, there was optimism that generating ideas with others and hearing about what they do in their classrooms could be helpful. One member noted that it was particularly difficult to imagine how to develop a PBL problem without having experienced PBL before. On the other hand, the diversity of perspectives, backgrounds, and thoughts about PBL was seen by some as making the process of developing problems much more demanding. Recall that the working group was engaging in its own PBL experience (the problem: develop 'good' PBL problems for a range of scenarios in post-secondary mathematics). Students engaging in PBL in post-secondary mathematics might also experience groupwork as both constructive and challenging.

Participants seemed to agree that one important part of developing a 'good' PBL problem is clearly identifying the intended learning outcomes. This means answering three questions First, where are students now (point A)? Second, where do we want them to go (point B)? And third, how do we get them from point A to point B? Participants highlighted the challenge of answering these questions in the working group due to one crucial missing link: the particular institutional context, including, for example, the curriculum of certain programs and the syllabi of certain courses. An interesting debate ensued concerning the nature of point B. For some working group members, the development of PBL problems was not constrained by the need to teach specific mathematical content, since content was seen as a medium to teach 'ways of thinking' or 'habits of mind.' Others, in contrast, stressed the importance of having content goals that reflect the syllabi of their home institution: 'problem-solving competencies', for example, could not be the main goal of a PBL problem.

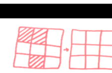
One key outcome of the first round of interviews was the sharing of a broad collection of problems (Figure 3 provides a subset). At the beginning of Day 2, the groups from the previous day wrote out their problems on chart paper, displayed them on the walls, and then observed the 'gallery' created.

**MATHEMATICS FOR TRANSITION COURSES**

| | |
|---|---|
| Some "special math" is shown below. $\frac{16}{64} = \frac{1}{4}, \frac{26}{65} = \frac{2}{5}$. For what other fractions does this "special cancellation" lead to a correct result? | Two players select digits (i.e., numbers from the set {1, 2, 3, 4, 5, 6, 7, 8, 9}), taking turns until one player has selected three digits adding up to 15. For example, 8 + 2 + 5 = 15. Why is this game identical to Tic Tac Toe? |

**JUNIOR-LEVEL MATHEMATICS**

In a 3 x 3 grid, several lights are turned on. Your goal is to turn all the lights off. For each square you touch, that square, plus all its adjacent squares, flip their orientation (on to off, off to on). Can you always turn off all the lights, no matter what its initial configuration?



At the present time there is a large construction of a set of power lines from Labrador to the US. The current design is across the Strait of Belle Isle, along the west coast of Newfoundland, across Cabot Strait, and through Nova Scotia and New Brunswick to the US. It is known that the cost of building power lines over land is less (how much?) than the cost of laying lines along the ocean floor. Ignoring the current infrastructure in Quebec, is the current pathway the most cost-effective way of transporting electricity from Labrador to the US? How does your answer change if we consider the infrastructure in Quebec which will need some upgrading to meet the new supply?

The goal is to place $n$ points on a sphere so that they are all as far away from each other as possible (an application might be in placement of cell phone towers). What does it mean to say that the points are all as far away from each other as possible? Consider a few cases for small $n$. Discuss the questions that arise in attempting to find solutions. Discuss an algorithm for determining an optimal point placement for larger $n$. Implement it and try it out for several larger $n$ values and discuss the process and results.

**MATHEMATICS FOR FUTURE TEACHERS**

You're in a rectangular gym. Starting at a point on one wall, you must touch all four walls and return to your starting point. How can you do this walking the shortest total distance?

**MATHEMATICS FOR SERVICE COURSES**

The health practitioners in a certain small community suspect that the average blood sugar measurements of those coming into the local community health centre have increased beyond the normal acceptable range in recent years. They are concerned that there might be an increased risk of diabetes within this community. The community health leaders have reached out to you in an effort to develop a healthy living program (dietary and physical activity) that will hopefully stop this increasing trend. The health practitioners have the ability to share with you their blood sugar measurements of everyone who visited the health centre. Is there currently a potential health problem? At the end of the year, how will you determine the effectiveness of your program? How might you improve the validity of your results?

**APPLIED MATHEMATICS**

You get an email from someone in an office in a high-rise: "We are still having problems with our employees getting bottlenecked at the elevators and being late for work. We have calculated that we are losing about one salary. Can you propose a solution?"

**SENIOR THEORETICAL MATHEMATICS**

What is $2^{\sqrt{3}}$?

Figure 3. Some of the 'good' PBL problems shared by members of the working group.

In doing this, the general image painted in Figure 1 became more concrete, and we were able to deepen our reflections on what makes a 'good' PBL problem. The list below summarizes (and explains) the key characteristics that were mentioned by participants.

A PBL problem should be

- appropriate for the post-secondary mathematics scenario (e.g., in advanced courses, maybe the goal is not to solve a problem, but to determine why a solution solves a problem);
- interesting to students;
- adaptable to different student needs (low floor, high ceiling);
- 'de-scaffolded' (not necessarily purposefully ill-defined, but with less guidance than we usually give; ambiguity can be good; you might even give extraneous information; let the students figure out what they need, and which of many possible solution paths they will take; instructors can also "throw in wrenches" as the solving processes unfold);
- problematic (if you know how to solve the 'problem', then it is not a problem); and
- productive (learning should be maximized; learning of mathematics, not just applying mathematics; it is the journey, not the destination; problems might develop over time, lead to generalizations or more problems).

We all seemed to agree that "a good problem for PBL" is different from "a problem," or even "a good problem." Nevertheless, an interesting question that arose was whether any problem could be "PBLarized." In other words, is it possible to develop a good problem for PBL by simply choosing a (good) problem and modifying it to meet the criteria presented above?

Several working group members highlighted the difficulty of engaging in the process of developing good problems for PBL without also thinking about issues related to implementation. Indeed, a PBL environment is defined not only by the type of problem used, but also by how that problem is implemented with students. This was the focus of our work for the rest of Day 2.

### ROUND 2 INTERVIEWS: DEVELOPING PBL UNITS

After the morning break of Day 2, exciting news was given to the working group: all members had progressed to the second round of interviews at PBLU! Participants received another letter and were invited to form new small groups based on a particular area of post-secondary mathematics. Their task was to choose a PBL problem for that area and develop it into a PBL unit, taking into consideration various issues related to implementation. At the end of Day 2, each group presented the various design elements they had considered. Short descriptions of the four units that were presented are given in Figure 4.

Each presentation of a PBL unit was followed by a period of questioning, during which a number of interesting points were made. It was noted that the units serve as excellent examples of 'de-scaffolding': that is, they allow students to make pertinent problem-solving decisions on their own and require them to figure out numerous facts and assumptions before they can even begin solving the problem. This kind of situation is normal for mathematicians and other professional problem-solvers, but it can be very different from most 'problem-solving' experiences encountered by mathematics students. There was also a discussion about the pros and cons of inviting mathematics students to go on field trips and collect experimental data themselves (as opposed to, for example, providing students with the information they need in a 'fact book' about the situation). If a PBL unit is given in a *mathematics* course, then 'science-like' activities might take too much time away from learning *mathematics*. This said, some participants suggested that offering mathematics students opportunities to get their hands dirty and conduct their own scientific experiments could help to humanize the mathematics they are studying. Finally, the unit presentations brought about the question of the necessity and nature of assessment in the process. We wondered if a PBL unit should always include some sort of evaluation. The possibility of a unit contributing to formative (rather than summative) assessment was also highlighted.

**JUNIOR-LEVEL MATHEMATICS AND PHYSICS (e.g., Calculus with Physics)**

A performer walks across a gorge on a slack line so that when they are in the middle of the line, they have descended far enough to no longer be visible to spectators standing some distance from the cliff edge.

Students will need to suggest appropriate angles of descent/ascent, spectator distance from the cliff edge, height of the performer, and lengths of some gorges to formulate the problem. Physics students may be able to derive an appropriate curve of the slack line, while others can discover it through historical information. Solving the problem may involve computing arclengths and tangent slopes algebraically, numerically, or by measuring physical models with enough accuracy.

**MATHEMATICS FOR FUTURE TEACHERS**

Your local planning department (*) is looking for help in planning for its snow removal process. The area has a number of streets and highways, and the new planning manager is responsible for designing and implementing a snow removal strategy. Your team's task is to help the manager find the most effective strategy. But there are constraints: some routes need priority cleaning, and may need to be cleaned multiple times during a storm; the department has a fixed number of drivers and plows; and so on. Teams will produce a report, and will present their findings to the planning manager and staff.

(*) this problem should be tailored to the community in which the students are studying. Students would be required to find any missing information (street maps, numbers of plows, numbers of drivers, priority routes, etc.).

**APPLIED MATHEMATICS (e.g., Differential Equations for Science Students)**

A fish farming business in the Waycobah First Nation wishes to increase the number of fish it farms from approximately 300,000 to 1,000,000 fish per year. Can the Whycocomagh Bay handle this increase in production?

In teams, students will produce a written and oral report, including a literature review, model, assumptions, findings, graphs, etc. To solve the problem, they will need to learn a combination of mathematics (advanced mathematical modelling, classic mixing problems in differential equations), statistics (parameter estimation, experimental design), aquatic ecology (fish, bacteria, water flow, aquatic biology and chemistry), and interdisciplinary teamwork skills. They may learn these by going on a field trip to Whycocomagh Bay and using library materials. They will be assessed based on their set-up of the problem (what do you know, what do you need to find out), and on their creativity, intuition, and persistence in solving the problem.

**SENIOR THEORETICAL MATHEMATICS**

Students are given a collection of polyhedra with regular faces and various properties (convex or nonconvex; all similar faces or two or more distinct faces; vertex transitive or not). For example: some of the Platonic, Archimedean, Catalan, and Johnson solids; prisms and antiprisms; stellated and toroidal polyhedra. They are told which solids are Platonic and asked to determine their characterizing properties in order to give a definition which specifically identifies them.

Figure 4. Descriptions of the four PBL units presented by members of the working group.

The discussion of the specific units in Figure 4 led quite naturally to a more general reflection on the process of developing units for use in PBL. A key point concerned the initial need to take a step back and consider the larger picture when planning a PBL unit. For instance, one must think carefully about the big ideas one wishes to emphasize in the unit and the benefits a PBL approach may provide. One should also take the time to think about how a PBL unit might elicit *processes* of mathematical thinking (not only *products* of mathematical thought).

Another important theme that arose in our reflections was the significant role of the instructor in designing and implementing an effective PBL unit. The instructor must not only choose a good PBL problem; they must also consider how the problem will be presented to and solved by students. The statement of the problem is a key consideration since it places the problem in context and sets expectations. For example, the instructor needs to decide if there should be a story or situation behind the problem. Given the de-scaffolded nature of a PBL problem, the instructor must also spend time playing with the problem and thinking about the directions students may take. In light of this, the instructor will need to make several important decisions. For instance, they will need to set a reasonable timeline (students need to be able to accomplish the various steps required, but the unit must not creep into the time needed for other parts of the course); determine when and what kinds of guidance should be given (instructor intervention will be necessary and helpful, but instructors must resist the urge to over-guide); and plan for the use of breakout sessions or tutorials (teaching assistants may offer additional support, especially in larger universities, but this may require additional effort for training).

It was clear that the process of developing and implementing PBL units could be daunting. Instructors would need to have courage to take this on. They would also need to realize that mistakes are inevitable and sources for making improvements. Working group members made some suggestions for easing the process. An instructor could start small (e.g., trying one PBL unit in a small class) or work with a group of colleagues. One participant pointed out how helpful it was to design a PBL unit in a small group setting in the working group. The collection, maintenance, and further development of resources was also emphasized. A list of some resources known to members of the working group is provided at the end of this report.

## PART 3: PBL IN CONTEXT

On Day 3, participants received the wonderful news that they had been hired at PBLU. But the euphoria was short-lived: according to a memo sent by the dean, members had new, wider issues to consider, including the appropriateness of variations of PBL in different institutional contexts, how PBL could fit into a university curriculum, and research questions about the use of PBL in post-secondary mathematics. Since we did not spend a significant amount of time talking about research, we mention some potential research questions later in the conclusion.

### WORK TASK 1: IMPLEMENTING PBL AT TRADITIONAL U

The first task for new professors was to imagine how to bring PBL into an existing, traditionally flavoured educational culture at a neighbouring university: Traditional U. It is possible that this task enabled some participants to think about how they might implement PBL under the conditions and constraints of their home institution.

We brainstormed several suggestions for introducing PBL to the faculty at Traditional U. Although there is a need to be bold about the approach (and what it offers), working group members stressed that it should not be forced on faculty. Instead, we could inquire into the needs of Traditional U and convince faculty of the potential usefulness of PBL through well-developed examples. In addition to providing resources about PBL, we could invite faculty to observe PBL in action and learn from those with some expertise in the pedagogy. It could also help to create a network of interested people to share ideas and offer support.

The working group also reflected on how to implement PBL within the courses and curriculum at Traditional U. We felt that the work to implement PBL in any course or any program (at any university) could be modelled by at least four fundamental steps:

1. Determine a clear set of 'learning outcomes' or 'graduation attributes' for the course or the program;
2. Identify and map the 'big ideas', 'gems', and 'transition points' within the course or the program;
3. Select (important, fascinating, challenging, etc.) problems that match these (i.e., that can reach the aims and drive the content); and
4. Use the problems to motivate the learning.

At a 'traditional' university, working group members pointed out that assessment may need to be reconsidered. In particular, if the development of certain 'soft skills' is considered as a learning outcome, then it will be necessary to figure out how to meaningfully evaluate these. It was also noted that since students tend to spend limited time in classes with professors, it might be easier to implement PBL using a 'flipped' model: i.e., by giving students work (e.g., research, readings, and exercises) to do outside and in between class meetings.

### WORK TASK 2: DEVELOPING A CURRICULUM AT PBLU

The second task for new professors was to reflect on the design of a mathematics curriculum at PBLU. In other words, members were invited to imagine the ideal "PBL mathematics curriculum," ignoring (as much as possible) the constraints of their home institutions.

A pertinent question driving our reflections was "Should a problems-based approach be ubiquitous in a 'PBL mathematics curriculum'?" Although we might be able to envision a full-blown PBL curriculum, we wondered if it would be preferable. Even in the medical education example we considered, the PBL cases were supplemented with other kinds of learning activities, including more standard lectures and tutorials. Seeing the usefulness of other pedagogies, we discussed the possibility of a partial use of PBL in a curriculum. We thought, for example, of having PBL-specific courses that occur once a year and a larger capstone PBL project at the end of a program. Such courses and projects could pose problems that require students to (a) call upon tools learned in previous years, and (b) push this learning further (as is required of a PBL approach). Given that mathematics students can have very different backgrounds, interests, and learning styles, we also wondered if students should sometimes have a choice of learning through PBL or not: for instance, a curriculum could include some multi-section courses where only some sections implement PBL.

No matter how prevalent PBL is in a mathematics curriculum, we identified at least two crucial issues that would need to be considered by curriculum developers. First, there may be students who have never learned in a PBL environment before and need assistance in adapting to new expectations (e.g., taking ownership of the learning process, collaborating with peers, having persistence in problem-solving, experimenting and making mistakes). While it may be ideal to set such expectations as soon as possible in a university program, thought needs to be given to the rollout of PBL with identified starting and/or transition points. It may be helpful, for example, for the first 'PBL course' to focus on general problem-solving skills and offer problems that rely only on basic mathematics. Second, curriculum developers will need to face the fact that choosing to implement PBL courses or projects may require them to drop some mathematical content goals. PBL is not only more time-consuming than 'traditional' approaches, but it may also give more weight to learning outcomes of a more general nature (e.g., the development of 'soft skills'). Gaps in the mathematical knowledge traditionally intended to be developed by students will necessarily be created, and plans may need to be made for dealing with those. Of course, if more time is spent on helping students to become independent learners of mathematics (a potential outcome of PBL), then perhaps students will be able to fill in the gaps on their own.

As we discussed curriculum development, the potential need and interest in collaborating with other departments was mentioned. More specifically, we considered service courses typically offered by mathematics departments, which would seem natural places to use PBL to engage students in applying mathematics to 'real world' situations. We considered the potential of working with other departments to develop the approach in such courses. Some participants noted, nonetheless, that service courses are often seen by other departments as mathematics content courses; straying too far from this (e.g., by reducing content and focusing on problem-solving skills) may encourage other departments to teach the courses themselves.

## CONCLUSION

As our working group came to an end, there was some clarification of the reasons we spent three days thinking about PBL. This involved a recognition that universities need to adapt to the world we live in now. A generation or two ago, knowledge was scarce and difficult to obtain. This is no longer the case, and with the rise of web-based resources like YouTube and Khan Academy, we must question what we offer in university mathematics courses. PBL shows promise of offering something different: the opportunity for students to gain skills in authentic mathematical problem-solving. Note that this does not mean that students should not also be exposed to the logic and structure considered essential to mathematical knowledge.

Although we found many answers over our three days of working together, we also raised several questions, some of which might serve to inspire future research: e.g.,

- Do other fields (other than medicine) effectively use approaches like PBL and offer frameworks that could be adapted for use in mathematics?
- What are the 'big ideas' and/or problems in post-secondary mathematics that could drive the curriculum?
- How can PBL be effective at large universities or within large university classes?
- What is the actual impact of being involved (as a student) in a PBL-style course? For instance, do students really become better learners through PBL?
- What are students' attitudes towards PBL?
- How do exemplary practitioners successfully implement PBL with students?

The last question participants answered within the working group was "Where will you go next?" Many suggested that they would focus on the collection, development, and/or testing of PBL resources (in particular, good problems). Some identified specific topics or courses of interest (e.g., arithmetic, geometry, advanced algebra, a problem-solving course, courses for students in other disciplines), while others expressed a general desire to try to modify any course or tutorial to be more problems-based. A couple of working group members re-emphasized the strategies of (a) moving some student learning to occur outside of class; and (b) starting small (for instance, trying out one or two PBL experiences in a single course or tutorial). A few participants specified that they would be considering heavily-structured multi-section courses or huge classes as potential places for PBL activities, which would require them to reflect on the kinds of resources and/or training to provide to TAs. All in all, it seemed that participants were inspired to take more steps in exploring the use of PBL in post-secondary mathematics.

In closing, we would like to thank the members of our working group for their enthusiasm, generosity, and hard work. It was our colleagues who participated who made this group as successful as it was.

*(References and resources follow the French version.)*

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## INTRODUCTION

L'apprentissage par problèmes (APP), peut-il fonctionner en mathématiques au niveau postsecondaire ? L'année 2019 a marqué le 50e anniversaire (!!) du programme APP mis en œuvre à l'École de médecine de l'Université McMaster (Servant-Miklos, 2019). Malgré son succès, des programmes semblables n'ont pas vu le jour à grande échelle, notamment dans le domaine des mathématiques au niveau postsecondaire.

L'expression « apprentissage par problèmes » a été utilisée pour décrire une pédagogie qui :

> *commence avec un problème à résoudre, et le problème est posé de telle sorte que les étudiants doivent acquérir de nouvelles connaissances avant de pouvoir résoudre le problème. Plutôt que de chercher une seule bonne réponse, les étudiants interprètent le problème, recueillent des informations, identifient des solutions possibles, évaluent les options et présentent des conclusions.* (Roh, 2003, p. 1, notre traduction)

Ce groupe de travail avait pour objectif général d'explorer l'utilisation de l'APP dans l'enseignement des mathématiques au niveau postsecondaire. Plus précisément, nous avons prévu d'aborder trois grands thèmes. Premièrement, la conceptualisation de l'APP : c'est-à-dire, les éléments clés de l'APP, ainsi que les avantages et les inconvénients d'une telle approche par rapport à d'autres. Deuxièmement, la mise en œuvre de l'APP : en particulier, le développement d'un « bon problème » et d'un « bon module » pour l'APP dans différents types de cours, et l'intégration de l'APP dans un programme d'études. Et troisièmement, la recherche sur l'APP : y compris les questions, les cadres théoriques, et les méthodologies qui pourraient nous aider à mieux comprendre l'utilisation de l'APP en mathématiques au niveau postsecondaire.

Afin de tenir compte de la diversité des expériences, des intérêts, et des points de vue des participants concernant l'APP, le groupe de travail a élaboré une liste plus précise de questions.

Cette liste, qui est restée sur les murs autour de nous pendant les trois jours, a servi de représentation de notre point de départ et de tremplin pour de futures conversations. Le premier jour, nos participants se demandaient :

1. Comment l'APP (en mathématiques) est-il lié au modèle d'apprentissage basé sur des cas (en médecine) ?
2. Quel est un ou *le* cadre pour l'APP ? Ce cadre est-il une entrave ou une aide ?
3. En quoi l'APP est-il différent de « l'apprentissage par enquête » en mathématiques ?
4. Y a-t-il des exemples ou des modèles concrets qui peuvent nous aider ?
5. Comment déterminer si les tâches sont appropriées, accessibles et profitables ?
6. Comment traiter des problèmes mathématiques à l'ère de Google ?
7. Comment combler les attentes des étudiants ?
8. Quel est le lien entre l'APP et l'évaluation des étudiants ?
9. Parlons-nous tous de la même chose ?
10. Le travail de groupe est-il essentiel à l'APP ?
11. Cette approche peut-elle fonctionner dans tous les contextes (par exemple, avec des ressources limitées, dans de grandes salles de classe) ?
12. Une approche basée sur des cas peut-elle fonctionner en général dans les cours de mathématiques ?
13. Comment assurer le traitement de *tous* les sujets ?
14. Quel est le lien entre l'APP et l'interdisciplinarité ?
15. Comment assurer un enseignement efficace à des clientèles mixtes ?
16. Quelle est la grande idée derrière tout cela ?

## LA PARTIE 1 : QU'EST-CE QUE L'APP ?

La session du matin du premier jour a été consacrée à une réflexion sur la nature de l'APP : ce que l'approche implique, ce qu'elle apporte, et comment nous pouvons la reconnaître. En petits groupes, les participants ont discuté de leurs expériences antérieures avec l'APP (en tant qu'enseignant ou étudiant), ce qui nous a amenés à réaliser que nous avions peut-être chacun une idée différente de ce qu'est « l'APP ». Pour rendre ces idées explicites, et pour essayer de développer une image collective de l'APP, chaque petit groupe a été chargé de se concentrer sur un élément de la pédagogie—la culture de la classe, les types de problèmes, le rôle de l'enseignant, le rôle de l'étudiant, ou les résultats valorisés—et d'en proposer une description pour l'APP. Chaque groupe a noté sa description sur une grande feuille de papier, l'a affichée sur un mur, et a pris le temps d'observer et de commenter les descriptions proposées par les autres groupes. Le contenu résultant est résumé dans la Figure 1.

À la fin de la première journée, le groupe de travail a examiné une autre image possible de l'APP, qui a été développée pendant de nombreuses années dans le domaine de formation médicale. L'Université de l'Illinois du sud a mis en place un programme d'études à son École de médecine qui est semblable à celui de McMaster, et elle a publié une vidéo qui montre un exemple de son utilisation de l'APP : www.youtube.com/watch?v=CZeP1a7xQdY. Dans la vidéo, un petit groupe d'étudiants en médecine travaillent ensemble pour analyser un « cas » : c'est-à-dire, une plainte d'un nouveau patient. Un animateur est présent, mais ce sont les étudiants qui font le travail pour résoudre le problème : ils établissent une liste initiale de conditions médicales potentiellement pertinentes, ils affinent la liste en posant des questions et en analysant les réponses concernent le patient (par exemple, ses antécédents médicaux ou les résultats de certains de ses examens médicaux), et, finalement, ils établissent un diagnostic et mettent en place un plan d'action.

**CULTURE DE LA CLASSE**

Les étudiants s'engagent dans un processus actif de création et de reproduction des connaissances, au lieu d'être simplement des consommateurs passifs des connaissances.


Il est essentiel que la culture de la classe soit caractérisée par une grande confiance et une grande responsabilité [Comment créer une grande confiance et une grande responsabilité?], où les attentes sont élevées et où les étudiants bénéficient d'un soutien important. Les enseignants doivent faire confiance aux étudiants, qu'ils seront pleinement impliqués dans le processus et qu'ils réussiront.

**TYPES DE PROBLÈMES**

- « Réel » [Que signifie « réel »?];
- Compréhensible (langue, culture, niveau de l'étudiant);
- D'enquête, ouvert à l'interprétation;
- Compréhension communicable;
- Voies et solutions multiples (possibilité d'un échec significatif; possibilité de ne pas trouver de solution) [Exemples?];
- Pensée indépendante et non dirigée;
- En classe, ou à court/long terme avec recherche.

**RÔLE DE L'ÉTUDIANT**

- Être un apprenant actif;
- Être un bon coéquipier;
- Être prêt à se spécialiser et à enseigner aux autres;
- Repousser les limites des problèmes ;
- [Prendre des risques,] faire des erreurs, [et se salir!];
- Apprendre de ses erreurs;
- Communiquer;
- S'approprier de / prendre autorité sur l'apprentissage [J'adore ça!];
- Être persistant.

**RÔLE DE L'ENSEIGNANT**

Conception :

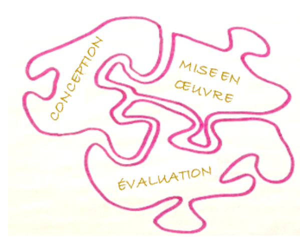
- Reliée aux objectifs d'apprentissage;
- Appropriée (sujets officiels), accessible (préparation des étudiants), et profitable (modes de pensée systématiques et créatifs?).

Mise en œuvre :

- Former des groupes (ou non) [Quand utiliser des groupes, et pourquoi?];
- Guider l'apprentissage (poser des questions, répondre aux questions);
- Évaluer la compréhension des étudiants pendant le processus – *formative* – observation et conversation pour guider l'apprentissage;
- Modérer [S'assurer que les groupes sont efficaces et que « tous » les étudiants apprennent et s'engagent.].

Évaluation :

- Continue, pour réajuster la mise en œuvre;
- Formative, de la compréhension et l'orientation des étudiants.




**RÉSULTATS VALORISÉS**

- Compréhension profonde;
- Motivation intrinsèque;
- Pensée mathématique [Qu'est-ce que la « pensée mathématique »?];
- Compréhension du fait que les mathématiques ne sont pas toutes achevées.

Figure 1. Les descriptions de cinq éléments de l'APP proposées par le groupe de travail.

La vidéo était une source d'inspiration pour le groupe de travail. Un aspect particulièrement marquant était la transformation des lacunes dans les connaissances des étudiants en « questions d'apprentissage » (Q.A.) : c'est-à-dire, des sujets qui doivent être recherchés entre les réunions de classe. Pour ces étudiants en médecine, la phrase « je ne sais pas » (par exemple, « Je ne sais pas quel serait un résultat normal pour un examen de la thyroïde. ») est traduite en un point d'action; et à la fin d'une session, chaque étudiant choisit certains de ces points d'action, utilise le temps entre les sessions pour acquérir de manière indépendante des connaissances pertinentes, et, à la prochaine session, présente ses conclusions au groupe. Alors, le fait de ne pas savoir et le besoin d'apprendre (indépendamment et collectivement) deviennent des éléments normaux du processus de résolution des problèmes.

Lorsque nous imaginions l'utilisation de l'approche basée sur des cas dans le contexte des mathématiques au niveau postsecondaire, quelques suggestions optimistes ont été formulées. L'une des suggestions était que l'approche pourrait offrir aux étudiants un cadre pour rencontrer des mathématiques « significatives, » et qu'elle pourrait être plus attrayante pour les étudiants qui choisissent actuellement de quitter les mathématiques parce qu'ils ne peuvent pas en voir l'utilité. Une autre suggestion était qu'il pourrait être possible de réaliser un cours complet de mathématiques en utilisant uniquement une approche d'APP si on se concentre sur deux ou trois grands cas intéressants qui relient tout le contenu.

De nombreuses questions constructives ont aussi été soulevées. Un participant s'est demandé s'il serait réaliste de répartir les Q.A. mathématiques entre des étudiants de niveau mixte. D'autres se sont interrogés sur la nature appropriée d'une Q.A. en mathématiques : par exemple, voudrions-nous que les étudiants étudient et présentent des faits simples, des concepts difficiles, ou des preuves ? Nous n'étions pas sûrs qu'il soit possible (ou souhaitable) de fournir un « livre de faits » pour les problèmes mathématiques utilisés dans l'APP (comme le « livre de faits » pour le cas médical dans la vidéo, qui contient des informations sur les antécédents médicaux du patient et les résultats de certains de ses examens médicaux). Enfin, nous avons considéré la question de la possibilité d'enseigner toutes les mathématiques de cette manière; non seulement tous les types de mathématiques, mais aussi tous les sujets dans un type de mathématiques particulier. Cette question a été réexaminée à plusieurs reprises tout au long de nos travaux.

Pendant les autres sessions du groupe de travail, chaque participant continuerait à développer sa propre image de l'APP en s'engageant dans des activités qui ne portaient pas directement sur une définition du terme. Il semblait y avoir un consensus sur le fait que « l'APP » signifie, à tout le moins, « commencer par un problème qui offre un contexte riche d'apprentissage. » Au-delà de cette définition de base, nous nous sommes interrogés sur la nécessité de limiter la signification du terme « l'APP. » Nous avons eu tendance à le considérer comme un éventail d'approches conformes aux descriptions dans la Figure 1, chacune pouvant être sollicitée à un moment différent en fonction des résultats que l'on espère obtenir et les contraintes auxquelles on est confronté.

## LA PARTIE 2 : L'APP EN ACTION

Afin de libérer les participants de leurs propres contextes institutionnels et de les inviter à vivre une expérience de l'APP, un cadre fictif a été introduit : l'Université de l'apprentissage par problèmes (ou l'U de l'APP). L'U de l'APP est une nouvelle université qui compte 8 000 à 10 000 étudiants de premier cycle et environ 500 étudiants de deuxième cycle. Bien sûr, l'U de l'APP est fondée sur des programmes d'études qui intègrent pleinement l'APP. L'institution offre des sujets d'études assez standards : par exemple, en sciences de la vie, la préparation en sciences de la santé, sciences physiques et ingénierie, sciences humaines, sciences sociales, commerce, et kinésiologie. Pour une raison ou une autre, l'U de l'APP n'a jamais envisagé l'enseignement des mathématiques et le temps est venu de créer un Département de mathématiques. Alors, l'université est en train d'embaucher des professeurs de mathématiques, et les membres du groupe de travail sont des candidats. Pour informer les candidats de la procédure d'entretien, l'U de l'APP a envoyé des lettres (pour un exemple, voir la Figure 2). La première série d'entretiens porterait sur la création des problèmes qui pourraient être utilisés dans l'APP. En cas de succès, les candidats seraient alors invités à transformer certains de ces problèmes en modules.

## LA PREMIÈRE SÉRIE D'ENTRETIENS : LE DÉVELOPPEMENT DE BONS PROBLÈMES POUR L'APP

**Faculté de Constructivisme Scientifique**

L'université de l'APP
Antigonish, Nova Scotia
Canada B2G 2L4

Téléphone: (902) 555-1234
Facsimile: (902) 555-1074
science.con.dean@pblu.ca

u de l'app

1er juin 2019

**Re: Position 3-14159**

Chère candidate, Cher candidat,

La Faculté de Constructivisme Scientifique de l'Université de l'APP a le plaisir de vous inviter à participer à la première série d'entrevues pour les postes menant vers la permanence dans notre nouveau Département de mathématiques. Chez l'Udel'APP, nous conduisons des entrevues de groupe et notre sélection se base sur la manière dont le groupe travaille ensemble. Pendant cette partie du processus d'entrevue, nous demandons que vous démontriez votre capacité à développer des problèmes appropriés pour l'APP.

Afin de vous préparer à l'entrevue, votre groupe devrait compléter la tâche suivante. **Pour chacun des six scénarios ci-dessous, proposez un ou deux « bons » problèmes et penser de comment ils pourraient être utilisés dans un contexte de l'APP.**

Les scénarios:

1. Un cours avancé théorique de mathématiques ou de statistiques (p.ex., Analyse réelle, Topologie, Probabilités,. . . )
2. Un cours moins avancé destiné aux étudiants en mathématiques ou en physique (p.ex., Calcul, Algèbre linéaire, Introduction aux statistiques . . . )
3. Un cours appliqué de mathématiques ou de statistiques (p.ex., Équations différentielles, Recherche opérationnelle,. . . )
4. Un cours de mathématiques destiné aux futurs enseignants
5. Un cours de transition, comme ceux qui préparent à étudier le calcul
6. Un cours de service pour ceux ou celles qui ne sont pas étudiant(e)s de mathématiques (p.ex. Statistiques pour les sciences de la vie, Calcul pour le programme d'administration, Mathématiques discrètes pour le programme d'informatique,. . . )

[Votre groupe peut substituer un autre scénario si vous le jugez nécessaire.]

Nous savons qu'il est possible que vous n'ayez pas eu d'expérience en entrevue de groupe. Alors, nous vous offrons une suggestion: votre groupe pourrait distribuer le travail pour que différents « experts » prennent en charge différents scénarios. Nous n'offrons pas d'autres suggestions.

Votre entrevue aura lieu officiellement dans la matinée du **dimanche 2 juin**. Bonne chance !

Cordialement,

Dr. J. Piaget
Doyen, Faculté de Constructivisme Scientifique
L'université de l'APP

Figure 2. La lettre décrivant la première série d'entretiens pour un poste de professeur de mathématiques à l'U de l'APP.

Au début de l'après-midi du premier jour, les participants ont reçu la lettre dans la Figure 2. C'est ainsi qu'a débuté la partie principale de notre travail, qui consistait à discuter des problèmes appropriés à l'APP et de la manière dont ils pourraient être intégrés dans les cours et les programmes d'études de mathématiques au niveau postsecondaire. En petits groupes, les participants ont commencé par travailler à l'élaboration de « bons problèmes » pour l'APP dans divers scénarios : les mathématiques avancées théoriques, les mathématiques moins avancées, les mathématiques appliquées, les mathématiques destinées aux futurs enseignants, et les mathématiques dans les cours de transition ou des cours pour les étudiants dans des disciplines autres que les mathématiques.

En partageant leurs réflexions sur le déroulement du processus, plusieurs participants ont fait des commentaires sur le volet « travail de groupe » de l'activité. D'une part, on était optimiste quant à l'utilité de générer des idées avec d'autres et d'entendre parler de ce qu'ils font dans leurs classes. Un participant a noté qu'il était particulièrement difficile de développer un problème pour l'APP sans avoir fait l'expérience de l'approche auparavant. D'autre part, la diversité des expériences antérieures, des points de vue, et des pensées en ce qui concerne l'APP a été considérée par certains comme rendant le processus du développement des problèmes beaucoup plus exigeant. Rappelons que le groupe de travail était en train de s'engager dans sa propre expérience de l'APP (le problème : développer de « bons problèmes » pour l'APP dans différents types de cours de mathématiques au niveau postsecondaire). Il est probable que les étudiants qui s'engagent dans l'APP en mathématiques au niveau postsecondaire trouveront également que le travail de groupe est à la fois constructif et exigeant.

Les participants ont semblé s'accorder sur le fait qu'un élément clé du développement d'un « bon problème » pour l'APP est l'identification des résultats attendus d'apprentissage. Alors, il faut répondre à trois questions. Premièrement : Où sont les étudiants actuellement (point A) ? Deuxièmement : Où voulons-nous qu'ils aillent (point B) ? Et troisièmement : Comment les faire passer du point A au point B ? Il était difficile de répondre à ces questions au sein du groupe de travail en raison de l'absence d'un contexte institutionnel particulier ; y compris, par exemple, les programmes d'études spécifiques et le contenu spécifique que l'on est censé enseigner dans des cours. Un débat intéressant s'est ensuivi concernant la nature du point B. Pour certains membres du groupe de travail, le développement des problèmes pour l'APP n'était pas limité par la nécessité d'enseigner du contenu mathématique spécifique, puisque le contenu était considéré comme un moyen d'enseigner des « modes de pensée » ou des « habitudes mentales. » D'autres participants, en revanche, ont souligné l'importance de fonder l'élaboration de leurs problèmes pour L'APP sur des objectifs de contenu spécifiques qui reflètent les programmes de leurs institutions d'origine : « les aptitudes à résoudre des problèmes, » par exemple, ne pouvaient pas être l'objectif principal des problèmes.

Un résultat important de la première série d'entretiens a été le partage d'un vaste ensemble de problèmes (la Figure 3 en fournit un sous-ensemble). Au début du deuxième jour, les petits groupes ont écrit les problèmes qu'ils avaient élaborés sur de grandes feuilles de papier, les ont affichés sur les murs, et ont parcouru « la galerie » ainsi créée.

Ce faisant, l'image générale présentée dans la Figure 1 est devenue plus concrète, et nous avons pu approfondir nos réflexions sur ce qui constitue un « bon problème » pour l'APP. La liste ci-dessous résume (et explique) les caractéristiques principales qui ont été mentionnées par les participants.

**MATHÉMATIQUES DANS LES COURS DE TRANSITION**

Voilà des mathématiques « spéciales » :

$$\frac{16}{64} = \frac{1}{4}, \quad \frac{26}{65} = \frac{2}{5}.$$

Pour quelles autres fractions cette annulation « spéciale » conduit-elle à un résultat correct?

Deux joueurs sélectionnent à tour de rôle un chiffre (c'est-à-dire un nombre de l'ensemble {1, 2, 3, 4, 5, 6, 7, 8, 9}) jusqu'à ce qu'un joueur ait sélectionné trois chiffres dont la somme est de 15. Par exemple, 8 + 2 + 5 = 15. Pourquoi ce jeu est-il identique au Tic Tac Toe?

**MATHÉMATIQUES MOINS AVANCÉES**

Dans une grille 3 x 3, plusieurs lumières sont allumées. Votre objectif est d'éteindre toutes les lumières. Pour chaque carré que vous touchez, ce carré, ainsi que tous les carrés adjacents, inversent leur orientation (allumé à éteint, éteint à allumé). Pouvez-vous toujours éteindre toutes les lumières, quelle que soit leur configuration initiale?

À l'heure actuelle, il y a une grande construction d'un ensemble de lignes électriques du Labrador vers les Etats-Unis. La conception actuelle prévoit de traverser le détroit de Belle Isle, la côte ouest de Terre-Neuve, le détroit de Cabot, la Nouvelle-Écosse, et le Nouveau-Brunswick pour arriver aux États-Unis. On sait que le coût de la construction de lignes électriques terrestres est inférieur (de combien?) à celui de la pose de lignes au fond des océans. Si l'on fait abstraction de l'infrastructure actuelle du Québec, la voie actuelle est-elle la plus rentable pour transporter l'électricité du Labrador vers les États-Unis? En quoi votre réponse change-t-elle si l'on considère l'infrastructure du Québec qui devra être modernisée?

L'objectif est de placer $n$ points sur une sphère afin qu'ils soient tous aussi éloignés les uns des autres que possible (une application pourrait être le placement de tours de téléphonie cellulaire). Qu'est-ce que cela signifie de dire que les points sont tous aussi éloignés les uns des autres que possible? Considérez quelques cas pour un petit $n$. Discutez des questions qui se posent lorsque vous essayez de trouver des solutions. Discutez d'un algorithme permettant de déterminer un placement optimal des points pour un grand $n$. Mettez-le en œuvre et essayez-le pour plusieurs valeurs de grand $n$. Discutez du processus et des résultats.

**MATHÉMATIQUES DESTINÉES AUX FUTURS ENSEIGNANTS**

Vous êtes dans un gymnase rectangulaire.
En partant d'un point sur un mur, vous devez toucher les quatre murs et revenir à votre point de départ.
Comment pouvez-vous le faire en transversant la distance la plus courte?

**MATHÉMATIQUES DESTINÉES AUX ÉTUDIANTS D'AUTRES DISCIPLINES**

Dans une petite communauté, les professionnels de la santé soupçonnent que le taux moyen de glycémie des gens qui viennent au centre de santé a augmenté au-delà d'une limite acceptable ces dernières années. Ils craignent qu'il y ait un risque accru de diabète au sein de cette communauté. Les responsables de la santé de la communauté ont pris contact avec vous dans le but d'élaborer un programme de vie saine (comprenant un régime alimentaire et de l'activité physique) qui pourrait mettre fin à cette tendance croissante. Les praticiens de santé peuvent vous donner les mesures de glycémie de toutes les personnes qui se sont rendues au centre de santé. Y a-t-il actuellement un problème de santé potentiel? À la fin de l'année, comment allez-vous juger l'efficacité de votre programme? Comment pourriez-vous améliorer la validité de vos résultats?

**MATHÉMATIQUES APPLIQUÉES**

Vous recevez un courriel d'une personne qui travaille dans un bureau situé dans un gratte-ciel : "Nous avons un problème : à cause des ascenseurs, nos employés arrivent en retard au travail. Nous avons calculé que nous perdons environ un salaire! Pouvez-vous proposer une solution?"

**MATHÉMATIQUES AVANCÉES THÉORIQUES**

Qu'est-ce que $2^{\sqrt{3}}$?

Figure 3. Quelques-uns des problèmes pour l'APP partagés dans le groupe de travail.

Un problème pour l'APP devrait être

- approprié au type de cours de mathématiques (par exemple, dans les cours avancés, il se peut que le but n'est pas de résoudre un problème, mais de déterminer pourquoi une solution résout un problème) ;

- intéressant pour les étudiants ;
- adaptable aux différents besoins des étudiants (« plancher bas, plafond haut ») ;
- « sans échafaudage » (pas nécessairement mal défini, mais avec moins d'indications que ce que nous donnons habituellement ; l'ambiguïté peut être une bonne chose ; vous pouvez même donner des informations non pertinentes ; laissez les étudiants déterminer ce dont ils ont besoin, et laquelle des nombreuses démarches possibles ils adopteront ; les enseignants peuvent aussi « mettre des bâtons dans les roues » au fur et à mesure que les processus de résolution se déroulent) ;
- problématique (si vous savez comment résoudre « le problème, » alors ce n'est pas un problème) ; et
- productif (l'apprentissage doit être maximisé ; l'apprentissage des mathématiques, et pas seulement de l'application des mathématiques ; c'est le voyage, pas la destination ; les problèmes peuvent se développer avec le temps, conduire à des généralisations ou à plus de problèmes).

Nous avons tous semblé être d'accord sur le fait qu'un « bon problème pour l'APP » est différent d'un « problème, » voire d'un « bon problème. » Néanmoins, une question intéressante qui s'est posée était de savoir si n'importe quel problème pourrait être « APParisé. » En d'autres termes : Est-il possible de développer un bon problème pour l'APP en choisissant simplement un (bon) problème et en le modifiant pour répondre aux critères présentés ci-dessus ?

Plusieurs participants ont souligné la difficulté de s'engager dans le processus de développer de bons problèmes pour l'APP sans considérer également des questions liées à la mise en œuvre. En effet, l'APP est défini non seulement par le type de problème utilisé, mais aussi par la façon dont ce problème est mis en œuvre avec des étudiants. C'est ce sur quoi nous nous sommes concentrés le reste de la deuxième journée.

### LA DEUXIÈME SÉRIE D'ENTRETIENS : LE DÉVELOPPEMENT DES MODULES POUR L'APP

Après la pause matinale du deuxième jour, les participants ont tous été félicités : ils avaient réussi la première série d'entretiens et ont été invités à participer à la deuxième série ! Les participants ont reçu une autre lettre et ont formé de nouveaux petits groupes basés sur des domaines particuliers des mathématiques. Leur tâche consistait à choisir un problème pour l'APP dans un domaine et à en concevoir un module pour l'APP, tout en prenant en considération diverses questions liées à la mise en œuvre. À la fin du deuxième jour, chaque groupe a présenté les différents éléments de conception qu'il avait envisagés. La Figure 4 montre des descriptions des quatre modules qui ont été présentés.

Chaque présentation d'un module a été suivie d'une période de questions, au cours de laquelle plusieurs points intéressants ont été soulevés. On a noté que les modules servent d'excellents exemples de « problèmes sans échafaudage » : c'est-à-dire qu'ils permettent aux étudiants de prendre des décisions pertinentes par eux-mêmes, et leur demandent de découvrir de nombreux faits et de déterminer plusieurs hypothèses avant même de pouvoir commencer à résoudre le problème. Une telle situation est normale pour les mathématiciens et autres professionnels qui résolvent des problèmes « du monde réel », mais elle peut être très différente de la plupart des expériences de « résolution de problèmes » rencontrées par les étudiants. Il y avait également une discussion sur les avantages et les inconvénients d'inviter les étudiants à faire des excursions et à collecter des données expérimentales (par opposition, par exemple, on pourrait fournir aux étudiants les informations dont ils ont besoin dans un « livre de faits » sur la situation problématique). Si un module est donné dans un cours de *mathématiques*, il se peut que des activités « scientifiques » laisseront trop peu de temps pour l'apprentissage des *mathématiques*. Cela dit, certains participants ont fait remarquer qu'offrir aux étudiants la possibilité de se salir les mains et de mener leurs propres expériences scientifiques pourrait contribuer à humaniser les mathématiques qu'ils étudient. Enfin, les présentations des modules ont soulevé la question de la nécessité et de la nature de l'évaluation. Nous nous sommes demandé si un module pour l'APP devait toujours inclure une sorte d'évaluation. La possibilité qu'un module contribue à l'évaluation formative (plutôt que l'évaluation sommative) a également été soulignée.

| **MATHÉMATIQUES ET PHYSIQUE MOINS AVANCÉS (par exemple, calcul avec physique)** |
|---|
| Un artiste traverse une gorge sur une ligne lâche, de sorte que lorsqu'il se trouve au milieu de la ligne, il est descendu suffisamment loin pour ne plus être visible pour les spectateurs qui se tiennent à une certaine distance du bord de la falaise.<br><br>Pour formuler le problème, les étudiants devront suggérer des angles de descente et d'ascension, la distance des spectateurs par rapport au bord de la falaise, la hauteur de l'artiste, et la longueur de certaines gorges. Les étudiants de physique pourraient être en mesure de dériver une courbe appropriée de la ligne, tandis que d'autres pourraient la découvrir en recherchant des informations historiques. Pour résoudre le problème, il pourrait être nécessaire de calculer des longueurs des arcs et des pentes tangentes de différentes manières (par exemple, algébrique, numérique, ou en mesurant des modèles physiques avec suffisamment de précision). |
| **MATHÉMATIQUES DESTINÉES AUX FUTURS ENSEIGNANTS** |
| Le service local d'urbanisme (*) cherche de l'aide pour planifier son processus de déneigement. La zone compte un certain nombre de rues et d'autoroutes, et le nouveau directeur de la planification est chargé de concevoir et de mettre en œuvre une stratégie de déneigement. La tâche de votre équipe est d'aider le directeur à trouver la stratégie la plus efficace. Mais il y a des contraintes : certaines rues doivent être nettoyées en priorité, d'autres doivent être nettoyées plusieurs fois pendant une tempête; le département a un nombre fixe de chauffeurs et de chasse-neige; etc. Les équipes produiront un rapport et présenteront leurs conclusions au directeur de la planification et à son personnel.<br>(*) ce problème doit être adapté à la communauté dans laquelle les étudiants étudient. Les étudiants devront trouver les informations manquantes (plans de ville, nombre de chasse-neige, nombre de chauffeurs, rues prioritaires, etc.) |
| **MATHÉMATIQUES APPLIQUÉES (par exemple, équations différentielles pour les étudiants en sciences)** |
| Une entreprise de pisciculture de la Première nation Waycobah veut augmenter le nombre de poissons qu'elle cultive : d'environ 300 000 à environ 1 000 000 de poissons par an. La baie de Whycocomagh peut-elle supporter cette augmentation de la production?<br><br>En équipes, les étudiants produiront un rapport écrit et un rapport oral, comprenant une revue de la littérature, un modèle, des hypothèses, des conclusions, des graphiques, etc. Pour résoudre le problème, ils devront apprendre des mathématiques (modélisation mathématique avancée, problèmes classiques en équations différentielles), de la statistique (estimation des paramètres, conception expérimentale), de l'écologie aquatique (poissons, bactéries, écoulement de l'eau, biologie et chimie aquatiques) et des compétences de travail d'équipe interdisciplinaire. Ils pourront utiliser divers documents de la bibliothèque et/ou faire une excursion à la baie de Whycocomagh. L'évaluation portera sur la manière dont ils élaborent le problème (que savez-vous, que devez-vous découvrir), ainsi que sur leur créativité, leur intuition, et leur persistance dans la résolution du problème. |
| **MATHÉMATIQUES AVANCÉES THÉORIQUES** |
| Les étudiants reçoivent une collection de polyèdres à faces régulières et aux propriétés diverses (convexes ou non convexes; toutes les faces similaires ou deux ou plusieurs faces distinctes; sommet transitif ou non). Par exemple : certains des solides de Platon, d'Archimède, de Catalogne, et de Johnson; des prismes et des antiprismes; des polyèdres stellaires ou toroïdaux. On leur montre quels sont les solides de Platon et on leur demande de déterminer leurs propriétés caractérisantes afin de donner une définition qui les identifie spécifiquement. |

Figure 4. Descriptions des quatre modules pour l'APP présentés dans le groupe de travail.

La discussion des modules spécifiques dans la Figure 4 a tout naturellement conduit à une réflexion plus générale sur le processus du développement des modules pour l'APP. Une remarque clé concernait la nécessité initiale de prendre un peu de recul et de considérer la situation dans son ensemble. Par exemple, il faut bien réfléchir aux grandes idées que l'on souhaite mettre en avant dans le module et aux avantages que l'APP peut apporter. Il faut également réfléchir à la manière dont un module pour l'APP pourrait susciter des *processus* de la pensée mathématique (et pas seulement des *produits* de la pensée mathématique).

Un autre thème important qui est ressorti de nos réflexions est le rôle significatif de l'enseignant dans la conception et la mise en œuvre d'un module efficace pour l'APP. L'enseignant ne doit pas seulement choisir un bon problème pour l'APP ; il doit aussi considérer la manière dont le problème sera présenté aux étudiants et résolu par eux. L'énoncé du problème est un élément important, car il situe le problème dans un contexte et communique les attentes. Par exemple, l'enseignant doit décider si le problème sera introduit par une histoire ou une situation réelle. Étant donné qu'un problème pour l'APP est généralement « sans échafaudage, » l'enseignant doit aussi passer du temps à jouer avec le problème et à imaginer les directions que les étudiants pourraient prendre. En fin de compte, l'enseignant doit prendre plusieurs décisions importantes. Notamment, il doit fixer un calendrier raisonnable (les étudiants doivent disposer de suffisamment de temps pour accomplir les différentes étapes requises, mais le module ne doit pas s'immiscer dans le temps nécessaire pour d'autres parties du cours) ; déterminer quand il convient de fournir du soutien et quels types de soutien à fournir (l'intervention des enseignants sera nécessaire et utile, mais les enseignants devront résister à la tendance à trop guider) ; et prévoir l'utilisation de séances de groupe ou de travaux dirigés (les auxiliaires d'enseignement peuvent apporter un soutien supplémentaire, en particulier dans les grandes universités, mais cela peut nécessiter des efforts supplémentaires pour la formation des auxiliaires).

Il semblait que le processus du développement et de la mise en œuvre des modules pour l'APP pouvait être intimidant. Pour assumer cette tâche, il faut que les enseignants aient du courage. Ils devraient également comprendre que les erreurs sont inévitables et qu'elles sont des sources d'amélioration. Les membres du groupe de travail ont fait des suggestions pour faciliter le processus. Un enseignant pourrait commencer modestement (par exemple, en essayant un module pour l'APP dans une petite classe), ou il pourrait travailler avec un groupe de collègues. Un participant a suggéré qu'il était extrêmement utile de concevoir un module pour l'APP dans le cadre d'un petit groupe au sein du groupe de travail. La collecte, l'entretien, et le développement des ressources ont aussi été soulignés. Une liste de quelques ressources connues des membres du groupe de travail est fournie à la fin de ce rapport.

## LA PARTIE 3 : L'APP EN CONTEXTE

Le troisième jour, les participants ont reçu les bonnes nouvelles qu'ils ont tous été embauchés par l'U de l'APP. Mais l'euphorie a été éphémère : selon un mémo envoyé par le doyen, les participants avaient de nombreuses questions plus larges à examiner, y compris la possibilité d'introduire des variations de l'APP dans différents contextes institutionnels, la manière dont l'APP pourrait s'intégrer dans un programme d'études, et les questions de recherche potentielles sur l'utilisation de l'APP dans l'enseignement des mathématiques au niveau postsecondaire. Vu que nous n'avons pas consacré beaucoup de temps à la discussion de la recherche, nous indiquons quelques questions de recherche potentielles plus tard, dans la conclusion.

## LA PREMIÈRE TÂCHE DU TRAVAIL : L'INTÉGRATION DE L'APP À L'U TRADITIONNELLE

La première tâche pour les nouveaux professeurs était d'imaginer comment transposer l'APP dans une culture d'enseignement existante et « traditionnelle » à une université voisine : l'U Traditionnelle. Il est possible que la tâche ait permis certains participants de réfléchir à la mise en œuvre de l'APP sous les conditions et les contraintes de leurs institutions d'origine.

Nous avons formulé plusieurs suggestions pour présenter l'APP aux professeurs de l'U Traditionnelle. Bien qu'il soit nécessaire d'être audacieux quant à l'approche (et tout ce qu'elle offre), les membres du groupe de travail ont insisté sur le fait qu'elle ne devait pas être imposée au corps professoral. Nous pourrions plutôt interroger l'U Traditionnelle sur ses besoins et convaincre le corps professoral de l'utilité de l'APP en leur montrant des exemples bien développés. En plus de fournir des ressources sur l'APP, nous pourrions inviter les professeurs à observer l'APP en action et à apprendre de ceux qui ont une certaine expertise dans la pédagogie. Il pourrait également être utile de créer un réseau de personnes intéressées qui partageraient des idées et offriraient du soutien.

Le groupe de travail a aussi fait des suggestions sur la façon de mettre en œuvre l'APP dans les cours et les programmes d'études à l'U Traditionnelle. Nous avons modélisé le travail pour mettre en œuvre l'APP dans n'importe quel cours ou programme (dans n'importe quelle université) en quatre étapes fondamentales :

1. Déterminer un ensemble clair d'apprentissages ou d'attributs attendus pour la réussite du cours ou du programme ;
2. Identifier et schématiser « les grandes idées », « les joyaux » et « les points de transition » au sein du cours ou du programme ;
3. Sélectionner des problèmes (importants, fascinants, stimulants, etc.) qui correspondent aux attentes et à la schématisation (c'est-à-dire, qui peuvent atteindre les objectifs et orienter le contenu du cours ou du programme) ; et
4. Utiliser les problèmes pour motiver l'apprentissage.

À une université « traditionnelle, » il pourrait être nécessaire de repenser l'évaluation. En particulier, si le développement de certaines « compétences non techniques » est considéré comme un résultat d'apprentissage important, il sera alors nécessaire de trouver comment évaluer de manière significative ces types de compétences. Étant donné le temps limité que les étudiants passent généralement en classe avec des professeurs, nous avons aussi conclu qu'il pourrait être plus facile de mettre en œuvre l'APP en utilisant le modèle d'une « classe inversée » : c'est-à-dire en donnant aux étudiants du travail (par exemple, des tâches de recherche, des lectures, des exercices) à faire en dehors et entre les réunions de classe.

## LA DEUXIÈME TÂCHE DU TRAVAIL : LE DÉVELOPPEMENT D'UN PROGRAMME D'ÉTUDES À L'U DE L'APP

La deuxième tâche pour les nouveaux professeurs était de réfléchir à la conception d'un programme de mathématiques à l'U de l'APP. En d'autres termes, les participants ont été invités à imaginer le « programme de mathématiques APP » idéal, en faisant abstraction (autant que possible) les contraintes imposées par leurs institutions d'origine.

Une question pertinente guidant nos réflexions était : Est-ce que l'APP doit être omniprésente dans le « programme de mathématiques APP » idéal ? Bien que nous puissions envisager un programme d'études entièrement basé sur l'APP, nous nous sommes demandé si cela serait préférable. Même à l'école de médecine dont nous avons examiné l'approche basée sur des cas, il y a d'autres activités d'apprentissage, y compris des cours et des travaux dirigés plus standardisés. Conscients de l'utilité d'autres types de pédagogie, nous avons discuté de la possibilité d'une utilisation partielle de l'APP dans un programme d'études : par exemple, en y incluant un cours par an et un grand projet à la fin basés complètement sur l'APP. Ces cours et projets pourraient poser des problèmes qui obligent les étudiants à (a) faire appel aux outils appris pendant les années précédentes, et (b) pousser cet apprentissage plus loin (comme l'exige l'APP). Puisque les étudiants peuvent avoir des formations, des intérêts, et des styles d'apprentissage très différents, nous nous sommes aussi demandé s'ils devraient parfois avoir le choix d'apprendre par problèmes ou non : par exemple, un programme d'études pourrait inclure des cours avec plusieurs sections où seules certaines intègrent l'APP.

Quelle que soit la prévalence de l'APP dans un programme de mathématiques, nous avons identifié au moins deux questions qui devraient être considérées par les concepteurs de programmes. Premièrement, certains étudiants n'ont peut-être jamais appris dans un environnement fondé sur l'APP et auront besoin pour s'adapter aux nouvelles attentes (comme s'approprier le processus d'apprentissage, collaborer avec ses pairs, persister dans la résolution de problèmes, expérimenter et faire des erreurs). Il pourrait être idéal d'établir de telles attentes le plus tôt possible dans un programme d'études ; mais il est important de réfléchir à comment mettre en place l'APP, y compris l'identification des points de départ et de transition. Par exemple, le premier cours qui met en œuvre l'APP pourrait se concentrer sur des compétences générales de résolution de problèmes et proposer des problèmes qui ne reposent que sur des mathématiques de base. Deuxièmement, les concepteurs de programmes devront faire face au fait que choisir de mettre en œuvre de nouveaux cours ou projets fondés sur l'APP peut les obliger à abandonner certains objectifs liés à l'enseignement du contenu mathématique spécifique. L'APP est non seulement plus longue que les approches « traditionnelles, » mais elle peut aussi accorder plus de poids aux résultats d'apprentissage de nature plus générale (par exemple, le développement de « compétences non techniques »). Des lacunes seront nécessairement créées dans les connaissances mathématiques traditionnellement visées, et il peut être nécessaire d'élaborer des plans pour combler ces lacunes. Par ailleurs, si l'on consacre plus de temps à aider les étudiants à devenir des apprenants indépendants des mathématiques (un résultat potentiel de l'APP), alors peut-être que les étudiants seront capables de combler les lacunes par eux-mêmes.

Au cours de notre discussion de l'élaboration des programmes d'études, nous avons évoqué l'intérêt et la nécessité potentiels de collaborer avec d'autres départements. Plus précisément, nous avons considéré les cours typiquement donnés par les départements de mathématiques aux étudiants dans d'autres disciplines, qui semblaient être des endroits naturels pour utiliser l'APP afin de faire participer les étudiants à la résolution des problèmes du « monde réel. » Nous avons envisagé l'utilité de travailler avec d'autres départements pour développer l'approche et le contenu dans ces cours. Certains participants ont néanmoins noté que ces cours sont généralement vus par les autres départements comme des cours visant principalement l'enseignement du contenu mathématique ; s'en écarter trop (par exemple, en réduisant le contenu et en se concentrant sur les aptitudes générales à résoudre des problèmes) pourrait encourager les autres départements à enseigner ces cours eux-mêmes.

## CONCLUSION

À la fin de notre groupe de travail, les raisons pour lesquelles nous avons passé trois jours à parler de l'APP ont été clarifiées. Il s'agissait notamment de reconnaître que les universités doivent s'adapter au monde dans lequel nous vivons aujourd'hui. Il y a une ou deux générations, les connaissances étaient difficiles à obtenir. Ce n'est plus le cas, et avec l'essor des ressources en ligne telles que « YouTube » et « Khan Academy, » nous devons nous interroger sur ce que nous offrons dans les cours de mathématiques universitaires. L'APP est prometteur car il offre quelque chose de différent : la possibilité pour les étudiants de s'engager dans des processus authentiques de résolution de problèmes. Notons que cela n'implique pas que les étudiants ne doivent pas également être exposés à la logique et à la structure considérées comme essentielles à la connaissance des mathématiques.

Bien que nous ayons trouvé beaucoup de réponses pendant les trois jours, nous avons aussi soulevé plusieurs questions, certaines desquelles pourraient inspirer de futures recherches :

- D'autres domaines (autres que la médecine) utilisent-ils efficacement des approches comme l'APP et offrent-ils des modèles qui pourraient être adaptés pour les mathématiques ?
- Quelles sont les « grandes idées » et/ou les problèmes en mathématiques au niveau postsecondaire qui pourraient orienter un programme d'études ?
- Comment l'APP peut-il être efficace dans les grandes universités ou dans les grandes classes universitaires ?
- Quel est l'impact réel de la participation (en tant qu'étudiant) à l'APP ? Par exemple, les étudiants deviennent-ils vraiment de meilleurs apprenants grâce à cette approche ?
- Quelles sont les attitudes des étudiants par rapport à l'APP ?
- Comment des enseignants exemplaires réussissent-ils à mettre en œuvre l'APP avec leurs étudiants ?

La dernière question à laquelle les participants ont répondu au sein du groupe de travail était : Où irez-vous ensuite ? Beaucoup ont suggéré qu'ils se concentreraient sur la collecte, le développement, et/ou l'essai des ressources pour l'APP (en particulier, de bons problèmes). Certains ont identifié des sujets ou des cours d'intérêt spécifiques (par exemple, l'arithmétique, la géométrie, l'algèbre avancée, un cours de résolution de problèmes, et des cours destinés aux étudiants d'autres disciplines), tandis que d'autres ont exprimé un désir général d'essayer de modifier n'importe quels cours ou travaux dirigés pour qu'ils soient axés sur des problèmes. Quelques participants ont de nouveau mis l'accent sur les stratégies consistant à (a) déplacer l'apprentissage de certaines choses en dehors de la classe ; et (b) commencer modestement (par exemple, essayer une ou deux expériences de l'APP dans un cours ou une séance de travaux dirigés). Il y avait aussi des participants qui ont précisé qu'ils envisageraient la mise en œuvre de l'APP dans des cours très structurés avec plusieurs sections ou dans des classes énormes, ce qui les obligerait à réfléchir sur les types de ressources et/ou de formation à fournir aux auxiliaires d'enseignement. Dans l'ensemble, il semble que les participants aient été inspirés à continuer leurs explorations sur l'utilisation de l'APP en mathématiques au niveau postsecondaire.

En conclusion, nous tenons à remercier les membres de notre groupe de travail pour leur enthousiasme, leur générosité et leur travail acharné pendant les trois jours. Ce sont nos collègues, qui ont participé au groupe, qui ont fait de ce groupe un succès.

## RESOURCES / RESSOURCES

- UMAP—ILAP (books / livres)
- https://firstyearmath.ca/ (First Year Math and Stats in Canada Database / base de données pour les mathématiques et le statistique de premier cycle au Canada)
- https://nrich.maths.org (for everything from early years mathematics to undergraduate mathematics / pour tout, des mathématiques pour les jeunes aux mathématiques pour les étudiants de premier cycle universitaire)
- Inquiry-Oriented Linear Algebra / Algèbre linéaire orientée par l'enquête : (http://iola.math.vt.edu/index.php)
- Active Learning Calculus / Apprentissage actif en calcul : (http://math.colorado.edu/activecalc/)
- http://mathpickle.com
- https://wild.maths.org
- https://callysto.ca
- https://richardhoshino.com/inspiring-students

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# TEACHING PRIMARY SCHOOL MATHEMATICS… WHAT MATHEMATICS? WHAT AVENUES FOR TEACHER TRAINING?

# ENSEIGNER LES PREMIERS CONCEPTS MATHÉMATIQUES ÀL'ÉCOLE PRIMAIRE… QUELLES MATHÉMATIQUES? QUELLES AVENUES POUR LA FORMATION À L'ENSEIGNEMENT?

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## INTRODUCTION/MISE EN CONTEXTE

While we considered planning our working group around mathematics in the early years, we decided to combine our experiences working with pre-service teachers as the focal point for the discussion over the three days. We both could recall future teachers who were worried about the prospect of teaching mathematics to those who admitted a hatred for the subject area.

Alors que nous envisagions de planifier notre groupe de travail autour des mathématiques des premières années scolaires, nous avons décidé de combiner nos expériences auprès des enseignants en formation initiale comme point focal de la discussion au cours des trois jours. Nous avions des souvenirs de futurs enseignants inquiets à l'idée d'enseigner les mathématiques à des élèves qui disent détester cette discipline.

We also discussed how sometimes there is a misconception that early mathematics is 'easy' and the complexities of those early concepts are often overlooked by future teachers. As we considered the future of mathematics in the early years, we decided that the goal we would discuss in our working group was around supporting those teachers in changing their conceptions.

Nous avons également discuté du fait qu'il existe une fausse croyance voulant que les mathématiques des premières années scolaires soient « faciles » et que la complexité de ces premiers concepts est souvent négligée par les futurs enseignants. En considérant l'avenir des mathématiques des premières années scolaires, nous avons décidé que l'objectif qui serait discuté dans le groupe de travail serait d'aider ces enseignants à changer leurs conceptions.

In planning the three days, we decided to choose themes that mirrored some of the activities that we consider in our teacher education classrooms in supporting future teachers. We decided moving from 'Getting pre-service teachers to buy in', 'Building their confidence with a complexification of understandings', and, finally, 'Moving towards the more abstract concepts and ideas' would be the flow through the three days.

En planifiant les trois jours, nous avons décidé de choisir des thèmes qui reflètent certaines situations auxquelles nous recourons dans la formation initiale à l'enseignement afin de soutenir les futurs enseignants. Pour le déroulement des trois jours, nous avons décidé d'une progression qui a d'abord abordé. « Faire en sorte que les enseignants en formation adhèrent », puis « Renforcer leur confiance avec une complexification de leur compréhension » et, enfin, « Aller vers des concepts et des idées plus abstraits ».

We decided to use many activities that we have used in our own teacher education classrooms as the starting points to spark discussion and open a dialogue about early years pre-service teachers. Our discussions in this report highlight the activities we chose, and the very important discussions that resulted from participating in the activities together.

Nous avons décidé d'utiliser des situations utilisées dans nos propres cours pour la formation initiale comme points de départ pour susciter la discussion et ouvrir un dialogue sur la formation des enseignants des premières années scolaires. Dans ce rapport, nous mettons en évidence les situations retenues et les discussions importantes qui ont découlé de leur réalisation au sein du groupe.

## DESCRIPTIONS OF THE THREE DAYS/DESCRIPTIONS DES TROIS JOURNÉES

### PREMIÈRE JOURNÉE (SAMEDI)/DAY 1 (SATURDAY)

First activity

We began the working group in the same way that Jennifer begins her pre-service mathematics methods classes: with a Snowball activity. Participants were given the opportunity to create their own 'snowball' prior to investigating the one created by the pre-service teachers. For the activity, the instruction was to use up to five words or phrases that immediately come to mind when you hear the word 'mathematics.' The words and phrases were then used to create a word cloud in *Mentimeter* to show which words and phrases were the most common among the working group participants. Figure 1 shows the word cloud created on this first day by the participants of the working group before any discussion or activities had begun.

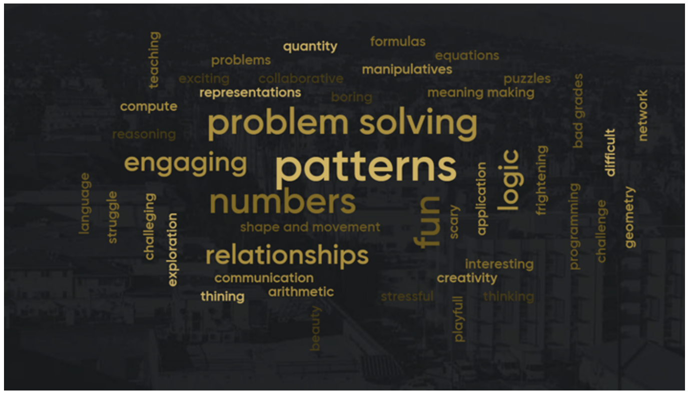


Figure 1. Word cloud created by working group participants.

Participants discussed how most of the terms were positive (e.g., “fun,” “playful”), not all of them were (e.g., “scary,” “frightening”), despite mathematics being the chosen field of work or study of all the individuals in the room. The image in Figure 2 was then shown to the group. This word cloud was created by the students in two of Jennifer’s primary pre-service classrooms. What struck the participants were how overwhelmingly negative the responses were in the pre-service group. The following questions were then asked to the group: Is it important to consider these feelings in working with future teachers? Why or why not? What is important in ensuring strong future early years mathematics teachers?


Figure 2. Word cloud created from the activity with Jennifer’s pre-service teachers.

Many important discussions arose from the image starting with the need to reduce the fear of mathematics and calming the future teachers in order to reduce the risks of transmitting fears about mathematics to future students. The group discussed how there was a need to help future teachers develop *grit*: they need to have the courage to face difficulties and make them aware of the resources at their disposal when they encounter difficulties. In the end, we decided that making everything easy and obvious was not the best way because we wanted our future teachers to be able to persevere when they encounter struggles. Giving challenges and obstacles are extremely important in learning and working with mathematics, but we must consider how those challenges are not destructive to the future teachers.

We also discussed that we could not focus solely on the feelings of the future teachers but needed to see the links with the mathematics understanding that would help to mitigate the concerns held by the future teachers. There is a need for us to work in pre-service classrooms on having future teachers reflect and understand the links between processes and mathematics concepts and that meaning and understanding needs to be important in these classes. One essential piece of the discussion was moving away from the need to be fast and equating being good at mathematics as getting the correct answer quickly. We wondered how we could work to change the idea that there is value in the slowness and the time required to come to understand. We problematized how the idea of speed even came to be so important in classrooms when Mathematicians are slow…and that is acceptable!

We also problematized the idea that mathematics must be 'fun' to be enjoyable. We talked about how the place of play is important and must be valued instead of only valuing the right answer. It is okay to meet challenges in mathematics but boiling it down to just being 'fun' seemed problematic to us. Participants decided words like "engaging," "challenging," and "enjoyable" were preferable instead of just fun (which is sometimes equated with easy in pre-service classrooms).

Deuxième activité

La deuxième partie de la séance du samedi a été dédiée à la réalisation d'un atelier de jeux de rôles reliés à l'enseignement des mathématiques qui a été proposé par Vincent. Ainsi, en nous inspirant de travaux existants (Lajoie, Maheux, Marchand, Adihou, et Bisson, 2012 ; Marchand, Adihou, Lajoie, Maheux, et Bisson, 2012), nous avons réalisé une série de quatre tours de jeux de rôles décentralisés. Ces jeux de rôles ont été orchestrés autour d'une trentaine de difficultés reliées aux différents domaines des mathématiques.

Voici trois exemples de difficultés jouées durant la séance : a) un élève rencontre une difficulté à réaliser une division lorsqu'il y a un zéro au dividende (opérations sur les entiers) ; b) un élève rencontre une difficulté à représenter des couples de fractions équivalentes avec un modèle de collection (nombres rationnels) ; c) sur une roulette, un élève se questionne si la probabilité associée à un type de secteur si les secteurs de ce type sont collés ou dispersés (probabilités).

Ainsi, chacune des personnes participantes s'est vue attribuer au hasard une difficulté et s'est préparée individuellement à mettre en scène cette difficulté mathématique pendant une séance de jeux de rôles de huit minutes.

Puis, à tour de rôle, chacun des membres des équipes a joué le rôle d'élève en difficulté auprès d'un autre membre de l'équipe qui agissait à titre de personne enseignante, tandis que les deux autres membres de l'équipe observaient silencieusement (personnes observatrices) l'interaction entre l'élève et la personne enseignante. Dans ce contexte, le rôle de l'élève consistait à simuler les difficultés ou les stratégies de l'élève et à jouer le jeu pour toute la durée du jeu de rôle. Le rôle de la personne enseignante visait à simuler une intervention et à offrir un soutien à l'apprentissage. Le rôle des personnes observatrices demandait d'analyser les échanges, les interactions, la gestion de la situation et les aspects didactiques dans le but d'alimenter le retour sur les jeux de rôles.

Après chaque jeu de rôles, les personnes participantes changeaient de rôle, et ce, jusqu'à ce que chacun des membres du groupe ait joué, au sein de son équipe, une fois son rôle d'élève, une fois son rôle de personne enseignante et deux fois un rôle de personne observatrice.

Cet atelier de jeux de rôles est utilisé dans un des cours donnés par Vincent auprès de personnes futures enseignantes au sein du baccalauréat à l'éducation préscolaire à l'enseignement primaire au campus de Drummondville de l'Université du Québec à Trois-Rivières. Les objectifs visés dans ce contexte sont notamment de réfléchir à la compréhension et la maîtrise des contenus mathématiques, d'analyser des raisonnements d'élèves en action, de développer une réflexion face à la pertinence de certaines actions (ou approches et matériels), de développer l'habileté à anticiper les réactions d'élèves, de développer l'habileté à intervenir à partir de ce qui vient de l'élève, ainsi que de donner l'occasion de pratiquer la communication mathématique devant un groupe.

Figure 3. Des jeux de rôles en actions.

Dans les jeux de rôles, les personnes enseignantes se trouvaient en contexte d'improvisation, car elles ne connaissaient pas les difficultés rencontrées par les élèves. En effet, le contenu mathématique ciblé et la nature de l'erreur ou de la difficulté en jeu n'étaient pas connus à l'avance par la personne enseignante. Ainsi, avant de démarrer les tours de jeux de rôles, la perspective de se dévoiler aux yeux des autres membres de l'équipe et d'être jugé à travers une intervention improvisée a manifestement généré une tension dans l'air, de la nervosité et une certaine anticipation au sein du groupe. Cependant, l'atmosphère a changé assez rapidement dans les premières minutes du premier tour de jeux de rôles et un rythme généralement confortable s'est installé par la suite. La Figure 3 présente des moments pendant un de jeux de rôles au sein de chacune des équipes.

À la suite des jeux de rôles, nous avons discuté de l'importance à accorder aux difficultés proposées aux personnes jouant le rôle d'élève. Le choix de les réfléchir en amont et de les imposer aux personnes participantes a été remis en question, même en sachant que la création de difficultés ou d'erreurs réalistes et pertinentes constitue un facteur déterminant pour la réussite des jeux de rôles. Nous nous sommes donc interrogés sur la possibilité de laisser les personnes jouant les rôles d'élèves choisir elles-mêmes des erreurs ou des difficultés, ce qui pourrait permettre qu'elles soient plus facilement comprises par les élèves et que ces derniers puissent plus aisément les raisonner, les expliquer et les jouer. Certaines personnes participantes du groupe ont souligné que devoir réfléchir et cibler des difficultés à mettre en scène dans le cadre des jeux de rôles représentait une tâche riche et complexe, mais que certaines personnes futures enseignantes ayant un fort bagage mathématique pourraient trouver difficile de cibler des difficultés à mettre en scène.

Par ailleurs, nous avons collectivement constaté que des interventions faites par des personnes enseignantes sont apparues embrouillées (« clear as mud ») et que des réflexions d'élève (qui devaient être des erreurs) se sont avérées surprenamment vraies (« impressively true »). La perspective à partir de laquelle les échanges sont observés, à savoir celle de l'élève, celle de la personne enseignante ou celle d'une personne observatrice, est apparue comme importante, alors qu'elle influence le regard porté et les constats dégagés. Ainsi, une explication proposée par la personne enseignante peut lui paraitre claire, mais sans amener l'élève à mieux comprendre. De plus, il a été souligné par plusieurs que les personnes observatrices—avec un certain recul par rapport aux échanges—voyaient souvent des pistes d'intervention que la personne enseignante elle-même ne semblait pas arriver à voir dans l'action.

Même si un moment de huit minutes dédié à un seul élève est long par rapport au rythme d'une classe ordinaire, plusieurs personnes enseignantes ont constaté leur incapacité à « enseigner » dans le cadre des jeux de rôles, se retrouvant prises dans des boucles où les « pourquoi » entrainaient des logiques circulaires qui n'aboutissent jamais au fond des choses. Toutefois, si certains cercles peuvent être vicieux, d'autres peuvent au contraire être vertueux et mener à des apprentissages chez l'élève !

De plus, nous avons réfléchi à la distinction entre « dire comment faire » à l'élève vivant une difficulté et lui poser des questions pour l'amener à progresser vers une solution. Quel est le but derrière le fait de donner des explications à l'élève ? Ces explications visent-elles à occulter la difficulté (ou l'inconfort) vécue par l'élève en lui montrant comment se rendre à la solution, en lui exposant la procédure attendue ? L'objectif est-il la réponse ou la compréhension ? Et si la compréhension de l'élève constitue le but de l'intervention, alors comment faire pour que les interventions de la personne enseignante ne soient pas trop directement orientées vers la solution ? Nous avons constaté qu'il pouvait parfois y avoir un écart entre les croyances de la personne enseignante et ses actions. Par exemple, celle-ci pourrait définir les mathématiques comme étant plus qu'un ensemble de procédures, mais tout de même arriver à mettre en lumière une procédure menant à la solution lorsque la pression d'intervenir adéquatement se fait sentir auprès de l'élève. Ainsi, comme personne enseignante, il nous est apparu bien facile de tomber dans le rôle de « dire » plutôt que de « questionner », peut-être pour nous rassurer nous-mêmes devant l'élève rencontrant une difficulté en le guidant implicitement vers la réussite, c'est-à-dire vers la solution.

DEUXIÈME JOURNÉE (DIMANCHE)/DAY 2 (SUNDAY)

First activity

To continue the general conversation from the first day around future early years teachers, we provided three case studies comprised of information from Jennifer's research in order to bring the conversation to a specific focus about different 'types' of pre-service teachers experiences in mathematics. The three cases were Harley, Peyton, and Taylor and were meant to be an amalgamation of different trends in stories that have been heard across the research. Taylor represented a future teacher who had poor experiences in mathematics, was terrified of teaching mathematics, and lacked confidence in their own understandings while holding a view that mathematics is about speed and right answers. Peyton represented future teachers who had successful stories in mathematics because they were able to quickly memorize formulas and give answers and held the view that mathematics was about right and wrong answers and procedures. Harley represented future teachers who loved mathematics but saw it as being vastly different from boring 'school mathematics' and held a view that school mathematics was something to survive so that the real beauty of mathematics could be appreciated. The cases were first read and discussed in small groups and then brought to the entire group with the following questions in mind: What does the case tell you about what the individual believes about mathematics? What challenges might this belief present in thinking as a teacher? How might you disrupt this belief in an Education program?

The initial conversation focused on a concern of moving into the realm of psychologists as mathematics educators being problematic because of some of the differences in theories. We discussed the need to pay attention to the emotions, not just the mathematics, in order to help the problem with future teachers but noted that maybe it cannot be solved by mathematics educators or just psychologists. We proposed a need for the two fields to work together so that we could find a solution playing to the strengths of the theories of both fields. The overall concern was that there is a need for some theoretical background in psychology to be able to deal with the emotional issues so that as mathematics educators we are not misapplying labels or ideas from outside the field while also being presented with some ideas that would not readily occur to us within the field of mathematics education.

As a result of the activity, a few important salient issues were brought into our thoughts about future work with pre-service teachers. The first was around ensuring that future teachers need to be able to distinguish that their own story as a pupil will be different from their teaching story, and this story will differ from the experiences of their future students. What we decided was important is to ensure that a change of perspective is noted by the future teachers. We also felt it was important that the future teachers need to be aware that it is necessary to know more than what needs to be taught to be able to teach. We had some wonderings about using these case studies in our future pre-service classrooms: Would it be interesting to use the context of initial teacher training to learn more about the different types of students? Could the fictitious students be presented to future teachers to make them aware of stereotypical cases that may arise in their future classrooms? A suggestion was also made to present our (as mathematics educators) mathematics histories, or case studies, to our future teachers to talk about how we went through the system. As was noted on the first day, just because we teach in mathematics does not mean we have had a positive past with mathematics. It was noted by a couple of participants in the group that they would have assumed that given the chosen careers of the participants that we all would have had positive past experiences.

The case studies themselves brought up some interesting questions and conversations about what it would mean for these individuals as future teachers. The query was made about if Peyton had ever faced open problems, and it was asked if experiencing more open problems would push beyond thinking about mathematics as so 'black and white' as was noted in the case study. Concern was raised that Harley could very well become the kind of teacher who will offer a class they would hate themselves, so we wondered what types of experiences would help Harley to expand their belief about mathematics. It was questioned that possibly cases like Harley are not able to realize big mathematical ideas because they have missed basic skills or if there was too much of a focus on 'basics' as being all of mathematics. To further push the conversation, a participant asked, "If we had to place one of our children in one of these classes, which one would we choose?" Maybe it would be Taylor's class because they usually become a teacher who can really empathize with children's experiences having had negative experiences. The concern over Peyton's class was the rigorous attention to right answers and getting them in the exact right way and that maybe there would be some missing understandings about what mathematical concepts mean beyond being able to use a formula.

Gale shared an experience of having future teachers write themselves a letter about what they learned and what they wanted to try in their own classrooms and then mail it to them in the future. This would allow them to remind themselves of the experiences in the program once the need for 'survival' in those first years had worn off. As we noted, real experiences are so important for provoking transformations of teachers' conceptions and sometimes the actual experiences provide that reinforcement.

Deuxième activité

Pour la seconde partie de la séance, nous avons proposé au groupe de se pencher sur des enregistrements vidéos (en français) d'entrevues d'environ 20 minutes réalisées par Vincent auprès de quatre enfants âgés de 4 ans et demi à 6 ans autour du développement du sens du nombre. Les entrevues ont été réalisées à partir du guide d'entrevue élaboré par Bednarz et Janvier (1988). Chaque enfant a été amené à réaliser différentes tâches relatives au développement du sens du nombre, par exemple réciter la comptine numérique, dénombrer et transformer une collection d'objets, créer une collection d'objets et comparer deux collections d'objets. Une tâche vise également à vérifier si l'enfant a acquis la conservation du nombre. Une description de chacune des tâches réalisées par les enfants accompagnait également les vidéos présentés aux personnes participantes.

Au sein du groupe de travail, chacune des équipes a pu regarder au moins un entretien et prendre connaissance des différentes tâches demandées à l'enfant. Des discussions en équipe (et ensuite, en grand groupe) ont permis de (re)plonger dans le développement du sens du nombre. Ce contenu mathématique dont l'enseignement est prescrit au début de l'école primaire a alors constitué un contexte pour revenir sur les trois profils de personnes futures enseignantes (Harley, Peyton, et

Taylor) et réfléchir à la manière avec laquelle celles-ci pourraient gérer les tâches et les mathématiques en jeu dans l'entretien.

Au fil des discussions en plénière, il a été évoqué que Peyton et Taylor traiteraient probablement la situation de la même manière, c'est-à-dire en utilisant un manuel ou un cahier d'apprentissage pour expliquer une façon de réaliser une certaine tâche (par exemple une stratégie de comparaison de deux collections). Ces deux personnes pourraient recourir à un genre de liste de vérification (ou de contrôle) pour amener l'élève à réaliser la tâche. Cependant, certaines personnes se sont questionnées à savoir si Taylor ne pourrait pas en arriver à manquer de connaissances mathématiques nécessaires pour soutenir ses interventions face à certaines difficultés rencontrées par les élèves dans la réalisation de la tâche. Dans le même sens, il a été perçu que Peyton, qui adopte une posture dichotomique (voire manichéenne) face aux mathématiques, pourrait rencontrer des obstacles dans le contexte de cette tâche, par exemple en ne voyant pas que la récitation de la comptine numérique jusqu'à $n$ ne signifie pas que l'enfant est nécessairement capable de dénombrer une collection de $n$ objets. Cela dit, il a été proposé que Peyton pourrait bénéficier de réaliser de tels entretiens avec des enfants, afin de s'assoir devant eux et les regarder réagir différemment, de les voir réaliser les tâches avec des stratégies diversifiées, l'amenant à constater que sa manière n'est pas nécessairement la « seule et unique bonne manière » de faire les choses. Du côté d'Harley, le groupe a exprimé l'idée que cette personne pourrait plutôt adopter une posture d'ouverture et explorerait la tâche ou des déclinaisons de celle-ci, par exemple des énigmes basées sur la tâche. Toutefois, il semble possible qu'une ouverture trop grande puisse entrainer certains dérapages comme perdre la focalisation sur ce que la tâche demande et s'éparpiller dans la résolution.

Les discussions ont également permis d'aborder la question des tâches ouvertes impliquant plusieurs stratégies de résolution et plusieurs réponses. Dans ce contexte, Gale a présenté le problème de la main, en référence aux travaux de Burns (1989), qu'elle utilise dans un cours donné à des personnes futures enseignantes. La tâche consiste à déterminer le nombre de carrés recouverts par la main de Gale dans un quadrillage (Figure 4).

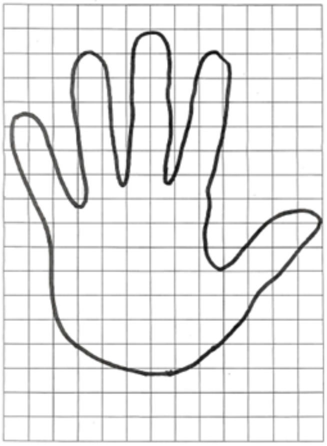

Figure 4. Le quadrillage superposé à une main de la tâche proposée par Gale.

Elle a témoigné que les échanges qui entourent la réalisation de cette tâche ne mènent pas seulement à déterminer de quelle manière doit être réalisée la tâche (convergence), mais plutôt sur les différentes voies permettant de réaliser cette tâche (divergence). L'idée devient alors d'amener les personnes futures enseignantes à s'ouvrir aux différentes possibilités offertes par une tâche et d'apprendre à anticiper les solutions qui pourraient découler de différentes stratégies de résolution (et vice-versa).

De plus, il est ressorti des discussions qu'il pourrait être intéressant d'exposer les personnes futures enseignantes à des enfants qui tentent de comprendre (ou de s'approprier) un concept mathématique et donc, qui exposent des réflexions et solutions davantage teintées de gris que de noir et blanc. Les mathématiques sont vivantes et vécues comme un processus en mouvement et non seulement comme un produit finalisé. Dans ce contexte, il devient important d'apprendre à poser des questions pour alimenter l'activité mathématique en cours et non pas seulement lors de moments d'arrêt singuliers, qui risquent de couper le fil de résolution.

Au final, plusieurs ont exposé (voire réalisé) que l'enseignement des mathématiques se fait auprès d'enfants et qu'ainsi, il ne suffit pas de maitriser les mathématiques, mais qu'il est impératif d'apprendre à connaitre les enfants (pour et par) eux-mêmes. Capturant l'essence de cette idée, Ann a dit : « Les enfants sont des individus singuliers, beaux et en développement qui nous apportent beaucoup ».

## TROISIÈME JOURNÉE (LUNDI)/DAY 3 (MONDAY)

### Première activité

Pour débuter la séance de la troisième journée, l'atelier immersif *Cinq-Pint-Boune* (ou *Five-Pent-Boune*) a été proposé aux personnes participantes du groupe. Il a été élaboré et mis sur pied par Louis Côté (chargé de cours à l'Université de Sherbrooke). Cet atelier, dont les particularités conceptuelles ont été présentées précédemment (Côté et Martin, 2017), vise la (re)construction d'un système de numération pour ébranler les conceptions relatives à l'apprentissage de la numération par l'enfant. Nous l'avons utilisé dans le groupe de travail pour susciter une réflexion vers des concepts et des idées mathématiques plus abstraits en proposant aux personnes participantes de se (re)placer dans la posture d'une personne apprenante qui rencontre des obstacles dans l'apprentissage de contenus mathématiques qui pourraient être jugés simples et allant de soi. L'adoption de cette posture nous a permis de réfléchir aux défis cognitifs et émotionnels rencontrés par les personnes apprenantes (élèves ou personnes étudiantes) dans un parcours d'apprentissage et en conséquence, aux choix didactiques que nous faisons ou pouvons faire et à leurs effets potentiels. Deux questions ont été posées aux personnes participantes pour conclure l'atelier :

- Quels sont les potentialités et les défis associés à l'utilisation de cet atelier dans la formation des futurs enseignants du primaire ?
- Quel peut être la puissance de transformation de ce type de tâches dans le contexte de la formation à l'enseignement, notamment en termes de sentiments et de croyances des futurs enseignants ?

Pour lancer l'atelier, des problèmes typiques qui sont utilisés pour l'enseignement de la numération dans les classes ont été exposés ont été proposés ont personnes participantes. Cependant, comme ces problèmes contenaient des mots et des symboles spécifiques à l'atelier, ils ont provoqué un conflit cognitif chez les personnes participantes et ainsi, initié une réflexion sur l'importance des symboles et des mots dans l'apprentissage d'un système de numération.

Puis, une comptine numérique réinventée[1] a été présentée afin de mettre en lumière les caractéristiques de la comptine numérique associée à notre système de numération en base 10 (Figure 5).

- one, two, three, four, five
- six, seven, five-three, five-four, two-five
- two-five-one, two-five-two, two-five-three, two-five-four, three-five
- three-five-one, three-five-two, three-five-three, three-five-four, four-five
- four-five-one, four-five-two, four-five-three, four-five-four, pent
- pent-one, pent-two, pent-three, pent-four, pent-five, pent-six, pent-seven, pent-five-three, pent-five-four, pent-two-five…
- boune…

- un, deux, trois, quatre, cinq
- six, sept, cinq-trois, cinq-quatre, deux-cinq
- deux-cinq-une, deux-cinq-deux, deux-cinq-trois, deux-cinq-quatre, trois-cinq
- trois-cinq-un, trois-cinq-deux, trois-cinq-trois, trois-cinq-quatre, quatre-cinq
- quatre-cinq-un, quatre-cinq-deux, quatre-cinq-trois, quatre-cinq-quatre, pent
- pent-one, pent-deux, pent-trois, pent-quatre, pent-five, pent-six, pent-seven, pent-cinq-trois, pent-cinq-quatre, pent-deux-cinq…
- boune

Figure 5. Les grandes lignes de la comptine numérique utilisée pour l'atelier.

Les personnes participantes se sont ensuite mises au travail pour arriver à créer une collection de *boune* objets, c'est-à-dire jusqu'au troisième niveau de groupement du système de numération, et ce, à l'aide d'un matériel au groupement apparent et accessible, c'est-à-dire des macaronis et des sacs transparents de trois tailles différentes.

Lorsque les équipes se sont mises au travail, une personne participante a rapidement rencontré une difficulté avec la tâche en jeu (compter *boune* macaronis), alors que les autres membres de son équipe arrivaient progressivement à fonctionner. Elle n'arrivait pas à s'approprier la nouvelle comptine numérique, avec ses règles et ses mots-étiquettes, ainsi qu'à se plonger dans la logique d'un système de numération en base 5. Cette situation a créé pour cette personne un environnement négatif, parce que les autres membres disaient « J'ai compris » les uns après les autres, mais qu'elle restait prise avec les difficultés de langage et de symboles. Des personnes participantes prenaient plaisir à réaliser la tâche et exprimaient leur enthousiasme, mais elles ne semblaient pas conscientes des effets potentiellement délétères de leurs attitudes et actions sur les personnes participantes vivant une expérience plus négative dans le contexte. De plus, la vitesse avec laquelle les membres

[1] Une des personnes participantes du groupe de travail a signalé le besoin d'ajouter le zéro dans la liste des mots-étiquettes formant la comptine numérique.

de l'équipe se sont mis au travail et se sont « précipités » vers la réponse a accentué les sentiments d'incompréhension et d'impuissance vécus par la personne participante. Un fossé entre elle et le reste de l'équipe s'est creusé.

Avec l'idée reçue selon laquelle, en tant que personne enseignante, elle devait en savoir plus que les élèves (voire tout savoir) à propos des mathématiques, sa lenteur à comprendre la tâche et ses exigences lui est apparue mauvaise dans le contexte et l'a amené à se sentir « stupide » de ne pas arriver à réaliser la tâche aussi rapidement que les autres. Les membres de l'équipe, voulant aider leur collègue, ont cherché à lui réexpliquer les exigences de la tâche, mais l'effet de répétition n'a pas été aidant. Elle a alors ressenti le besoin de s'éloigner du groupe, de s'asseoir seule avec la situation pour y réfléchir et tenter d'arriver à la comprendre.

L'expérience vécue par cette personne participante a généré une discussion dans l'ensemble du groupe au sujet des difficultés d'apprentissage vécues par un élève dans une classe. Par exemple, nous nous sommes questionnés sur le fait que la difficulté de l'élève, plutôt que de simplement venir d'elle-même et de ses caractéristiques, pouvait peut-être découler (en partie) du travail au sein d'un groupe et des différents rythmes d'apprentissage de ses membres. Ainsi, ayant besoin de plus de temps pour réfléchir et comprendre, la personne apprenante se retrouve en difficulté. Elle n'arrive pas à « avancer » assez rapidement et ne peut pas « maintenir le rythme ».

En tant que personnes enseignantes, comment peut-on accompagner les élèves plus « lent » pour éviter que le fossé ne se creuse trop par rapport au rythme et aux apprentissages du reste du groupe ? D'où ce rythme d'apprentissage plus lent peut-il venir ? S'agit-il d'un problème relevant du système didactique, du type de savoirs en jeu, de l'élève et de ses caractéristiques, de la personne enseignante et de ses choix didactiques ?

Les membres du groupe ont souligné que la personne enseignante, sachant qu'elle doit considérer les mathématiques et les contenus dans l'enseignement des mathématiques, doit également prendre en considération les sentiments positifs (comme l'excitation) et négatifs (comme l'anxiété) et les croyances des élèves pour arriver à piloter adéquatement une situation d'enseignement-apprentissage. Toutefois, des contraintes logistiques comme le temps peuvent limiter la marge de manoeuvre de la personne enseignante, qui doit aménager du temps et de l'espace et davantage d'opportunités et de façons différentes de réfléchir pour un groupe d'élèves qui sont tous uniques.

En tant que personne enseignante, comment « pousser » ces élèves rencontrant des difficultés et accumulant souvent un retard par rapport au groupe afin de les aider à progresser, mais sans alimenter leur sentiment d'anxiété à l'égard des mathématiques et finir par provoquer du dégoût (voire de la haine) envers cette discipline ? Comment faire pour ne pas accentuer le sentiment d'être en retard par rapport au groupe que certains élèves vont ressentir ?

Dans le contexte de l'atelier, la tâche de constitution d'une collection de *boune* macaronis a semblé provoquer un conflit cognitif d'une telle magnitude chez une des personnes participantes du groupe qu'une forme de panne ou de rupture (cognitive et émotionnelle) a été provoquée face à la tâche. Devant ce constat, des questions ont été soulevées : y aura-t-il toujours des personnes en panne ? Quels sont les points de rupture possibles dans une tâche mathématique ? Quelles stratégies peuvent être mises en œuvre pour permettre à la personne apprenante vivant cette rupture ou cette panne de se remettre en mouvement ?

Sans nécessairement identifier de réponses formelles à ces questions, il est ressorti des discussions que la personne formatrice a le devoir d'agir dès qu'il est possible de croire que la rupture ou la panne vécue dans le contexte d'une tâche va provoquer des dégâts irréversibles, voire détruire une personne apprenante.

En réfléchissant à l'utilité ou à la pertinence de provoquer une telle rupture dans l'activité mathématique d'une personne apprenante (possiblement au profit de l'avancement du plus grand nombre possible de personnes apprenantes au sein d'un groupe), Egan a évoqué un article intitulé « The absolutely true confession of a prospective elementary school math teacher » et ayant été rédigé en collaboration avec une future enseignante au sujet du rapport négatif qu'elle avait développé à l'égard des mathématiques et des effets d'un cours de didactique des mathématiques (Merkowsky et Chernoff, 2014). L'expérience vécue par la future enseignante expose comment le rapport négatif aux mathématiques développé au fil de son parcours scolaire a pu évoluer dans le contexte d'un cours de didactique des mathématiques ayant provoqué un conflit cognitif, voire une panne ou une rupture dans l'expérience vécue.

Une réflexion s'est aussi faite au sujet de l'idée de forcer le travail en équipe pour l'apprentissage des mathématiques et des conséquences potentielles qui peuvent en découler. La personne ayant rencontré une difficulté dans le cadre de l'atelier a considéré que son expérience a été transformative, car elle a pu se voir à l'œuvre avec un regard complètement différent sur la difficulté qu'elle a rencontré dans la réalisation de la tâche en contexte d'un travail en équipe. En effet, elle a réalisé que cette organisation en équipe, qui se voulait bénéfique, voire nécessaire pour la réalisation de l'atelier, n'est pas toujours la meilleure façon de faire pour favoriser l'apprentissage des mathématiques.

Ne serait-il pas possible d'offrir la possibilité aux élèves de travailler individuellement ou en groupe, et ce, pour la durée totale de l'atelier ou simplement pour une partie de celui-ci ? Qu'arriverait-il si les équipes pouvaient même changer au fil de l'atelier, alors qu'une personne pourrait choisir de s'isoler temporairement, puis de revenir travailler au sein de l'équipe, voire même de se joindre à une nouvelle équipe ? Ces choix, qui pourraient être offerts aux élèves dans le cadre de l'atelier, pourraient être réfléchis a posteriori par les élèves, afin que ceux-ci puissent prendre conscience des émotions vécues au fil de l'atelier et de l'évolution de leurs besoins dans ce contexte.

<u>Second activity</u>

We decided to end the third day by taking measure of what the participants were thinking or feeling at the end of the working group. We asked them to use chart paper to pull their thoughts together about what the activities had caused them to think about or consider for their future practices. Figure 6 shows images of what the participants produced.

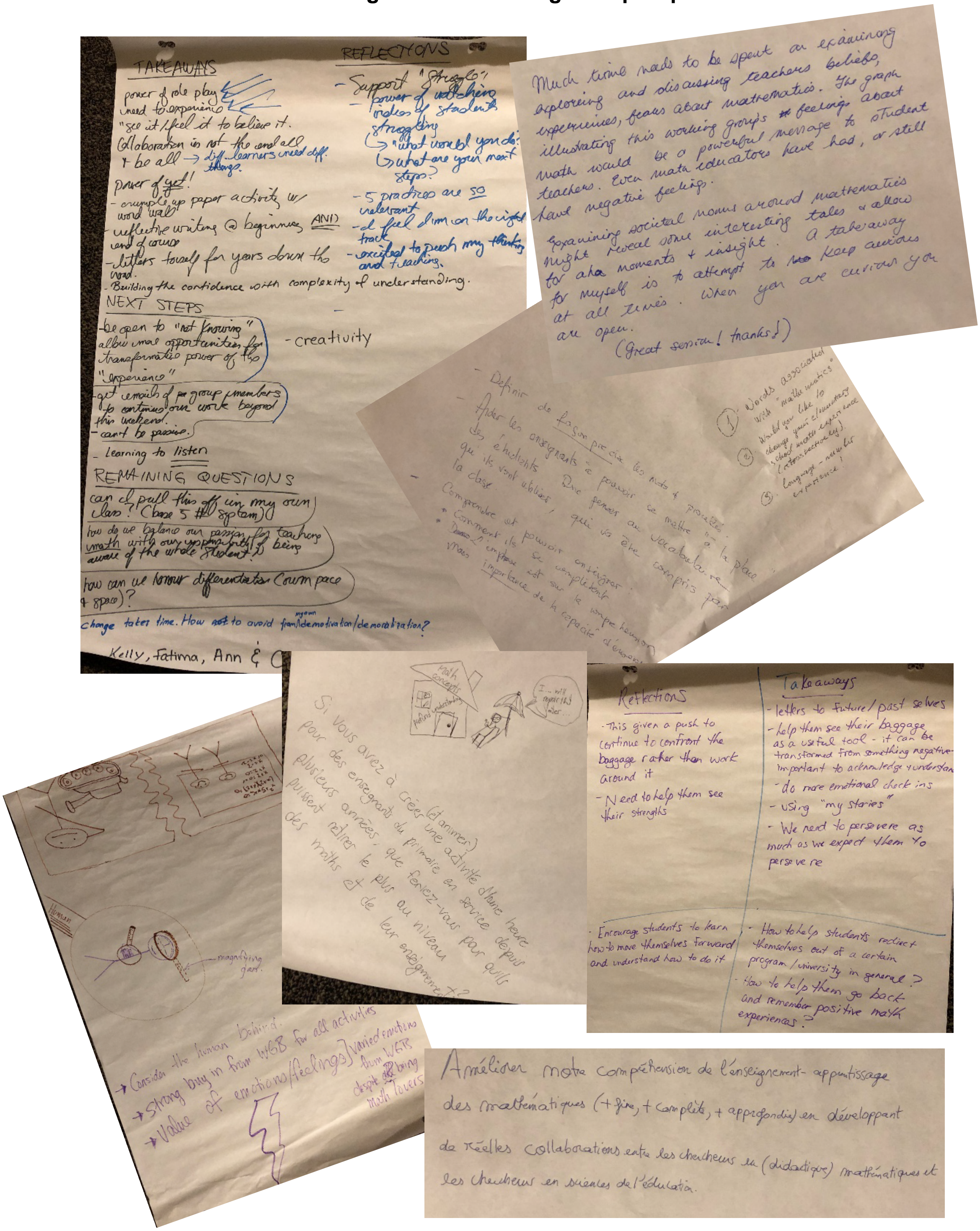


Figure 6. Final reflections of the participants.

## CONCLUSION

On our last session to plan for the presentation, we asked our participants to summarize the three days in a single thought:

1. Doing in the presence of the others with your (co-chairs) guidance allowed me to query and elevate “taken for granted” practices and re-affirmed the need to see our TCs as “whole” learners with experiences, emotions, etc. (more than those who don’t know math content.)
2. I will strive to help my students not pass their (negative) baggage on to future generations and to help them transform their baggage into an asset in their teaching. Preservice elementary teachers is where the cycle will break.
3. On ne peut omettre de prendre l’humain en compte dans le processus d’enseignement et d’apprentissage de l’enseignement des mathématiques. Il y a des humains derrière les mathématiques.
4. A door from within me to out of me, from me as a teacher to me as a student, from me as a Ph.D. student to me as a teacher, from me as a teacher to me as a researcher.
5. Value and importance of emotions in a math education classroom—yours, students, learners, teachers. How to access these: role play, self story, case studies and workshop like “Five, pent, boune”.
6. Enriching; sharing of experiences: same issues everywhere; Being pushed beyond the comfort zone is so instructive
7. Importance of stories—“my math story”, for students and educators; value of exploring baggage and to use as a positive tool
8. It is important to focus on pre-service teachers’ challenges, but vital to amplify their strengths.
9. Au début, je croyais : « Si quelqu’un est conscient qu’on doit connaitre plus que A pour enseigner A, alors cette personne devrait être très motivée à apprendre A et un peu plus, malgré ses connaissances initiale de A ». Ensuite, j’ai compris que (merci à Vincent pour ça) ce n’est pas parce que tu sais que tu devrais faire le ménage de ta maison que tu vas nécessairement le faire.
10. We need to learn to be destabilized, to destabilize our students and to understand how the mathematics (elementary) we teach can destabilize kids.
11. Try to “think” as one of the students when you teach. Be open to different approaches from students themselves.
12. As a preservice mathematics teacher educator, I need to remember to persevere at least as much as I expect the future teachers to persevere when working with their students.
13. Experiencing authentic “struggle” allowed me to consider a necessary shift in my teaching. I also was encouraged by my time with the group that I am on the right path.
14. “Dear Peyton STOP I hope you are enjoying your new career path STOP We are loving life in the classroom STOP Sincerely STOP Harley and Taylor STOP”

As we consider the three days of activities of the working group, the final thoughts of the participants from the chart paper images, and the final quotes, it is clear that a lot of interesting thoughts and conversations

En considérant les trois jours d’activités du groupe de travail, les réflexions finales des participants à partir des schémas synthèses sur les grandes feuilles et leurs citations finales, il est clair que beaucoup de réflexions et resulted from the discussions. What is clear as well is that there are still many more questions to answer, and even though we came away with great ideas, we continue to be perplexed by how to support future teachers in a way that will help them to be the kind of mathematics teachers that will support future students in avoiding the issues that we are currently facing.

We all acknowledge that supporting future early years teachers in being teachers who embrace mathematics as this rich, exciting subject that is more than worksheets, speed drills, and tests, but teachers who also have the depth of knowledge and vision and pedagogical skills for the classroom that will be able to support future students in truly understanding mathematics.

We acknowledge that there is still a lot of work to be done in this area and that the issue is bigger than just Teacher Education programs. We noted that there needs to be communication between all levels of mathematics teaching in order to favour a more cohesive vision and a lasting, long-term plan to help support future teachers and to change the stories of everyone. We also noted how vitally important relationships are in teaching, but especially in the Teacher Education mathematics classes.

d'échanges intéressants ont résulté des discussions. Il ressort également qu'il reste encore beaucoup de questions sans réponse. Dans ce sens, même si nous sommes repartis avec de bonnes idées, nous restons perplexes quant à la manière de soutenir les personnes futures enseignantes d'une manière qui les aidera à devenir des personnes enseignantes de mathématiques qui sauront aider les futurs élèves à éviter ou surmonter les problèmes auxquels nous sommes actuellement confrontés.

Nous sommes tous conscients du fait qu'il faut amener les personnes futures enseignantes qui auront à enseigner les premiers concepts mathématiques du primaire à voir les mathématiques comme une discipline riche et passionnante qui va au-delà des feuilles d'exercice, des exercices de vitesse et des tests.

Nous croyons également qu'il est nécessaire de soutenir les personnes enseignantes afin qu'elles maitrisent les contenus mathématiques à enseigner, ainsi qu'elles développent des compétences pédagogiques et une vision pour la classe qui leur permettront d'amener les futurs élèves à développer une compréhension approfondie des mathématiques.

Nous reconnaissons qu'il y a encore beaucoup de travail à faire dans ce domaine et que les enjeux vont au-delà des programmes de formation à l'enseignement.

Nous avons noté qu'il doit y avoir une communication entre tous les niveaux d'enseignement des mathématiques afin de favoriser une vision plus cohérente et un plan durable et à long terme pour aider à soutenir les personnes futures enseignantes et à transformer positivement chaque histoire personnelle.

Nous avons également relevé l'importance des relations dans l'enseignement, tout particulièrement dans les cours de didactique des mathématiques au sein de la formation initiale à l'enseignement.

Future teachers need the safety to make mistakes and heal past wounds, but also the confidence to go out and support future generations in finding their love for mathematics. In the end, we want future teachers to inspire their students that mathematics is not just about the dusty chalkboards in the movies (even if this could be part of it) and that everyone is capable of doing 'mathematics' in their daily lives. It is with this vision that we want to continue to strive to foster and support future teachers and students in our own practices.

Les personnes futures enseignantes ont besoin d'un sentiment de sécurité pour accepter leurs erreurs et guérir leurs blessures passées, mais ils ont aussi besoin d'une confiance qui leur permettra d'aller dans les écoles et d'aider les générations futures à trouver leur amour pour les mathématiques.
Enfin, nous souhaitons que les personnes futures enseignantes puissent montrer à leurs élèves que les mathématiques ne se font pas seulement sur des tableaux poussiéreux comme dans les films (même si cela peut en faire partie) et que tout le monde est capable de faire des mathématiques dans leur vie quotidienne.
C'est avec cette vision que nous voulons continuer à nous efforcer d'accompagner et de soutenir les personnes futures enseignantes et les éléves dans nos propres pratiques de personnes formatrices.

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# HUMANIZING DATA

# HUMANISER LES DONNÉES

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## INTRODUCTION

We think of humanizing data as fluid, a process that is in constant construction. At its core, that process is a communicative process reflecting human choices. As part of the communication process, there could be a human or a group choosing to represent data to present a certain message to other humans. In particular, humanizing data is a choice in how to use data and a choice in how to represent the data. The communication process is reciprocal and fluid. Subsequently there is often a human or group of humans experiencing and interpreting data presented by other humans. When reading others' data, we may choose to read it in the way that it is presented, or we may **re-**present the data for ourselves to facilitate our understanding of the phenomena represented in the data. Importantly, whenever there is data there are humans behind that data making choices, choosing (consciously or not) which phenomena to represent, which data to collect for this re-presentation and how to re-present.

Our working group's focus was on the communicative nature of humanizing data and particularly how the choices involved in this communication illuminate tensions. Importantly, the choices made with data highlight people living in tension. This interaction in-tension directs our attention to intention (which we will refer to as *in-tension*) in data representation. We express our intentions through navigating tensions among mathematics, self, and others.

Dans ce qui suit, nous racontons une histoire de notre groupe de travail à travers les thèmes qui ont émergé de nos discussions. Nous les utilisons pour présenter et encadrer nos activités et nos réflexions. Tout au long de nos trois jours de travail, nous avons cherché à développer une nouvelle compréhension de ce que l'on pourrait vouloir dire par « humanisation des données ».

## DATA INTENSION (IN-TENSION)

In what follows, we tell a story of our working group through the in-tensions that were illuminated in our discussions. We introduce each tension with an artefact or narrative from our working group. Each of the activities in the working group and the choices made by participants in producing data representations was aimed at developing new understandings of what could be meant by humanizing data. Of course, other intentions emerged alongside our primary aim. In the next section, we will look back across these tensions first through the artefacts that were produced by our group and then through the lenses of the four statements that the participants settled on for our presentation to the conference plenary in the closing session.

### ABSTRACTION AND RICH EXPERIENCE

To open the working group, we sought to infuse our introductions with an experience of data. Thus, we asked each person to introduce themselves with a piece of data that describes something about themselves. For example, one of us leaders gave the numbers 20-20 as a re-presentation of ourselves. Other group members gave numbers for example, 1, 4, 96, and 1243. Someone gave the name of a town, reminding us that data may be non-numerical. We all guessed what the provided data re-presented for each person. Some re-presentations were easier to guess than others. Some of the possibilities discussed included vision, age of the oldest grandparent, number of times at CMESG, and place of birth. We saw that when a person is reduced to a single piece of data, the data communicates, or re-presents, only a small aspect of that person no matter how precise this data is thought to be.

Sometimes we choose to re-present ourselves with data. We might self-identify with a gender, an ethnicity, a position, or our age, amongst many choices. Sometimes others re-present us with data. We may be part of a larger re-presentation—one person amongst many who depict a balance or imbalance in gender re-presentation, for example. Or the data may measure us as individuals, as with a grade re-presenting our performance in a school subject. Whatever data is used by us or others to re-present us, it communicates a small part of a much larger whole. We noticed that the data depicting ourselves can feel oppresive or opportunistic when someone else chooses which data to draw from us. When we self-identify data about ourselves it can be useful for our purposes and even empowering.

Les données sont par nature une abstraction, donc les humaniser consiste à les rendre plus matérielles (moins abstraites). L'humanisation des données représente la navigation explicite des tensions entre expérience et représentation. Ces tensions ont émergé dans le travail que nous avons fait.

If we want our data re-presentations to honour humanity more, we would look for ways

- to provide richer connection to the complexities of experience and being,
- to involve people in the choices of what data identifies them,

- to make representations that remind audiences of the limited (abstract) nature of the data, and
- to read data with awareness of its limited (abstract) nature.

DATA IN MOTION

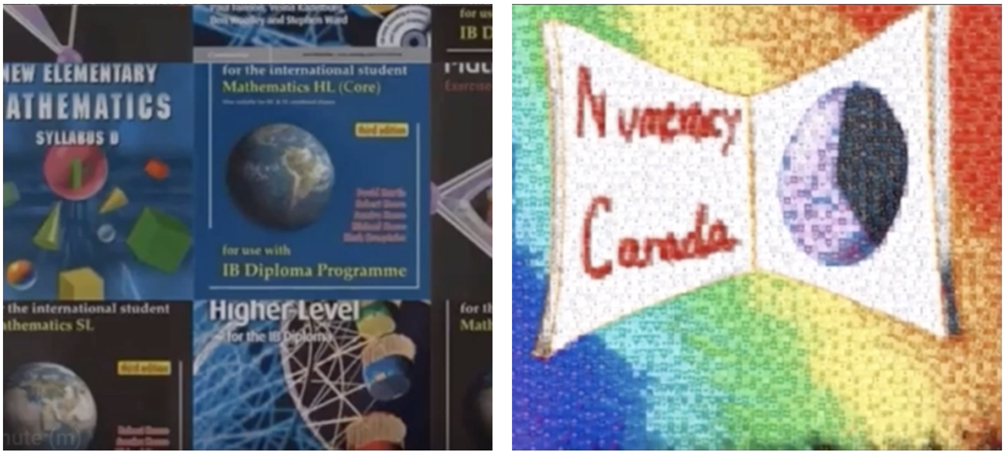


Figure 1. Re-presenting large numbers—zooming in and zooming out.

Humanizing data is a reciprocal activity. The data is constantly presenting and re-presenting us as humans, and reciprocally, as humans, we are constantly presenting and re-presenting the data. The act of humanizing, then, is the act of making this reciprocity explicit. We can make reciprocity explicit by, on the one hand, identifying the human within the data (zooming in) or, on the other hand, identifying the data within the humans (zooming out). We attempt to make sense of any large number that re-represents data through the human within that large number.

Les représentations dynamiques étaient importantes. Grâce à l'intégration de différents outils technologiques, les re-présentations qui ont permis le « zoom-in » et « zoom-out » des ensembles de données ont attiré notre attention, en se concentrant à la fois sur le particulier et le général. La nature fluide de la technologie dynamique utilisée reflète la nature fluide du processus d'humanisation des données et met en évidence certaines des tensions entre la partie et le tout. Un exemple peut être trouvé dans le travail et re-présentation de la photo-artiste Chris Jordan, dont le travail nous nous sommes inspirés dans nos discussions :
http://www.chrisjordan.com/gallery/rtn/#building-blocks.

For re-presentations of zooming in and zooming out we first looked to Chris Jordan before creating our own images. Chris Jordan's artwork depicts data in motion. For example, upon encountering Chris Jordan's image called Building Blocks (Jordan, 2013), we first saw 11 rows of 14 columns of blocks. The blocks were easily identifiable to us. We could easily compute and comprehend the number of blocks we saw in front of us. However this comprehension turned to tension as we pressed a button and the field of view zoomed out. In the zoomed out image we saw the re-representation of "1.2 million children's building blocks, equal to the number of students who drop out of high school every year in the U.S." Embedded within this zoomed out re-representation was the quote, "Education is the most powerful weapon we can use to change the world—Nelson Mandela."

This led to a working group activity in which we played with the online application mosaically.com to create Chris Jordan-like images of data in motion. First, each person in the group chose a statistic that they wanted to re-present through motion. Then, we each identified a macro-level photo to be rendered from small photos of micro-images. Figure 1 depicts two images of a re-presentation of data created within our working group activity. The zoomed-out figure on the right depicts a representation of numeracy rates. This macro-image of numeracy rates is formed with many very small images of mathematics textbook covers (a zoomed in version on the left side of Figure 1). In the working group, we talked about using relevant images to represent data.

The use of the online application was cumbersome in some ways, which limited our choices of what to represent and how to represent it. We were limited by the photos available to us, which impacted the qualities of the overall image. We were limited by the application's inability to differentiate different parts of the photo. For example, we wondered if certain textbooks could be used for parts of the photo representing lack of numeracy and other textbooks for stronger numeracy. We were limited by shapes (rectangular photos, in the case of this application). We realized that Chris Jordan (or anyone else with more artistic expertise and time than we had in our working group) would be limited in similar ways, even with more time to create his representations. This highlighted the in-tensions among aesthetics, aesthetic vision, and data.

DATA IN CONTEXT

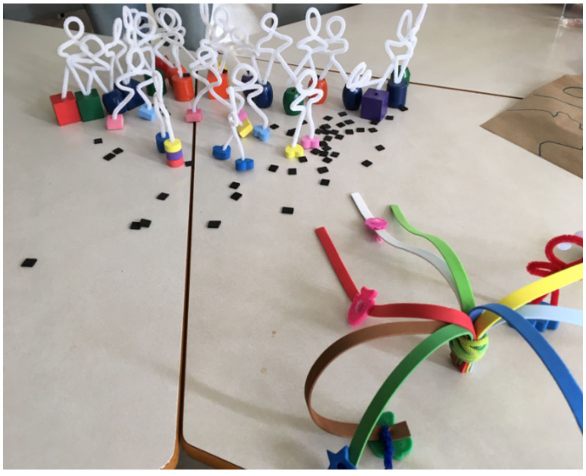

Figure 2.

But what of the situatedness of data? Data is never alone, it is always situated within a context that is itself re-presented. The context, often already reconstituted through re-interpretation, has the potential to both frame and provoke our understandings. In fact, our understanding of what comprises a context is dependent on the data (made) available to us. What do we make of the objects in Figure 2? What happens if we give the re-presentations (our) context? The next paragraph describes the representation photographed in Figure 2. Before reading it, consider what the representation suggests to you. Then, after reading the description, consider how the description framed your understanding, and how the description provoked your understanding.

One of the working group participants had recently read this in a National Geographic article: “Comprising less than 5% of the world’s population, indigenous people protect 80% of global biodiversity” (Raygorodetsky, 2018). Colleagues in the working group worked together to create this image of twenty people, one of whom stands guarding the biodiversity.

En explorant cette re-présentation, nous avons pris en compte les questions, défis ou limitations auxquels les présentateurs de données étaient confrontés dans la réalisation de leur nouvelle re-présentation. La différence de couleur des rubans a-t-elle une importance ? Ou la taille et la forme des « pieds » du peuple ? Ou encore la position des objets sur leurs rubans ? S’agit-il simplement de conséquences de contraintes de conception (matériaux disponibles, temps fourni) ? Et en tant qu’interprètes de ce modèle, sommes-nous enclins (à tort) à interpréter la signification de certains aspects lorsque ce n’est pas approprié ?

### DATA IN EXPECTATION

As noted above, in creating visual and tactile re-presentations of data, group members discovered that no matter their original intentions, re-presentations are inherently limited. The limitation could be because of tools, translation of thought to practice, abstraction, or other factors. Translating intensions to a tactile model was a cyclical process, often causing group members to compromise their original intentions. This cyclical and compromising process was different than expectations and led to discussions around how to convey the human within data.

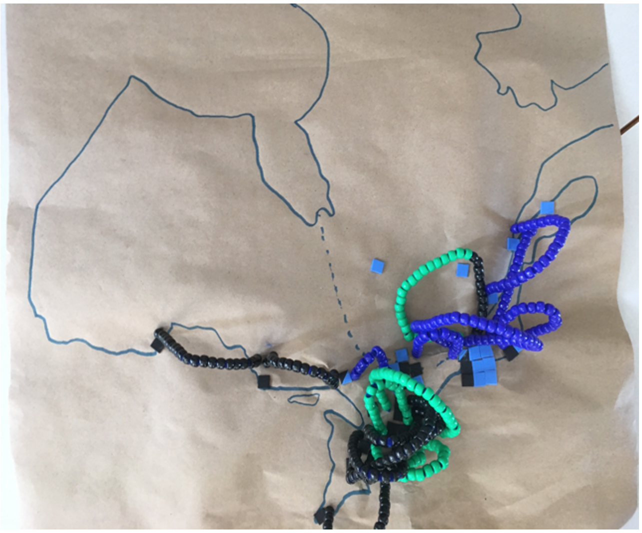

Figure 3. Representing language demographics in relation to university representation.

Figure 3 is a representation created by our working group participants to depict the universities in Ontario and Québec that operate in English, in French, and bilingually. The intention of the group members was to depict the proportions of language demographics in the provinces juxtaposed with the universities that support those languages. A problem/tension confronted in this design was how to represent the universities. Should each university be depicted with the same kind of representation even though some have many more students than others? Also, false binaries were a concern, relating to the English versus French universities—some universities claim to be bilingual but in fact favour one language significantly. The nuances of the data are necessarily lost in data representation. The work our participants did on this re-presentation opened up discussion of important questions in data representation.

- How is data "labelled" or "categorized," and what are the implications?
- Who does the categorizing?
- Does this impose binarity?
- What biases exist behind, or are revealed by, the categories?

While these questions are important to consider in the reading of data representations, we noted that the doing of representation—the act of making representations—alerts us to the kinds of category and binary problems that are present in data representations. The experience of making such representations then orients our reading of other data representations made by others.

In our discussion, we expressed concern for the damage we can do by drawing lines around demographic groups, not recognizing the range within apparent groups, and the intersectionality of groups. We also recognized the importance of the values behind a person's choices about which differences (which characteristics) to emphasize in any representation of data. On the other hand, we noted the potential value of representations for drawing attention to injustices. This tension is present in any representation of data.

### DATA IN PURPOSE

These questions are among those raised by the Working Group participants and discussed as we further explored and tried to humanize our physical re-presentations of data.

- Who collects the data?
- Why?
- For what purposes?
- Are the "purposes" for collecting the data the same as the uses of the data?
- How do we collect data for making decisions about learning, but de-personalize it?
- Why do Google and Facebook have access to what my students are doing in my classroom, and I do not?

At the outset of the working group we had delimited our focus on humanizing data to exclude discussion of the ethics of big data. However, our focus on raising attention to human choices in data and on using data for human interests led us to see that big data is central to these concerns. The data-hoarding practices of big data businesses are directed by humans who decide which data should be collected and who decide how the data is to be used.

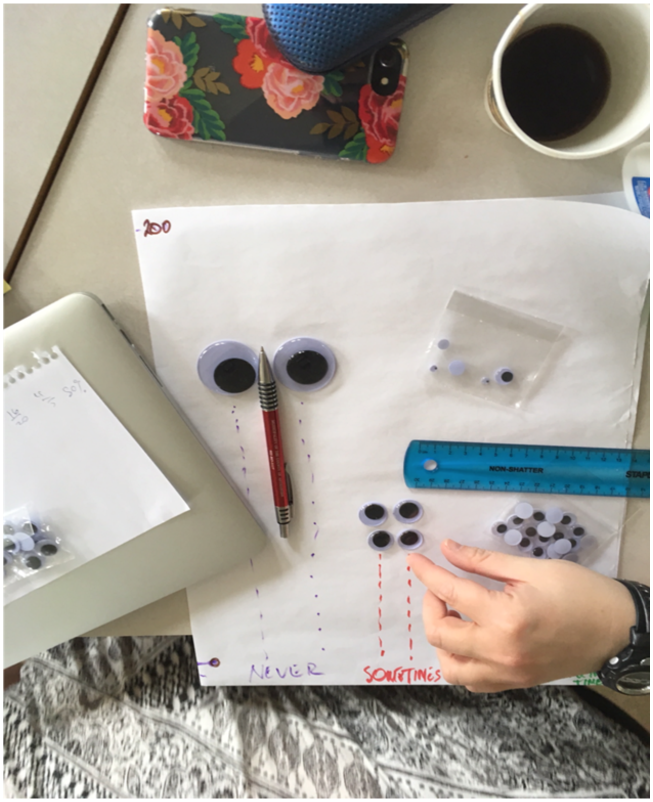


Figure 4. Human centric design.

Figure 4 is a representation of data one of the participants brought with them from a project they were working on. Arising from this re-presentation, we discussed assessment and assessment practices. Education practices are rife with assessment, which tends to be focused on data collection and representation. Teachers assess students. Students assess teachers (especially in universities where student assessment surveys are common). Jurisdictional authorities assess teachers and programs. We considered the impacts of the increasing ubiquity of surveillance that accompanies new technologies for recording data. For example, researchers and teachers can now have real-time data on physiological responses of students. The many ethical issues associated with our questions about data practices raised questions about whose responsibility it is to dialogue with students about these things. Is a mathematics educator's responsibility limited to facilitating practices of data handling (collection, interpretation, representation) or does it include the ethics?

## DATA FROM A HUMANIZING PERSPECTIVE

Our working group's construction of data representations and our discussions within and reflecting on these activities helped us recognize and grapple with the tensions of re-representing data—the in-tensions inherent to any re-presentation of any data. In preparing our summary presentation to the CMESG community, we became acutely aware that we were again experiencing the in-tensions associated with re-representation of data. We chose to summarize our discussions and learning with four statements:

- Data does not stand alone.
- En faisant, on crée le sens.
- Les catégories sont cachées.
- Nous modélisons les données. Les données nous modélisent.

A short elaboration of each statement was given by a working group member in the opposite language of the statement (French versus English and vice-versa). Our in-tensions in choosing these four statements and only including short elaborations was to continue the cyclical and interpretative process that is part of humanizing data. Although we introduced our statements to the wider group at the CMESG closing plenary, we knew that our re-presentation would be re-presented again as CMESG members interpreted the statements within their own-context, expectation, purpose, and motion.

We now use these statements to look across the tensions described above.

### DATA DOES NOT STAND ALONE

Data is always situated and interpreted within the realities of both the producer and receiver. There are no passive parties with data: data is constantly enacted and re-enacted within the in-tensions and constraints of all the parties. It is possible for data producers to obscure data, contexts, relationships, and purposes, as it is possible for the data receiver to obscure data, contexts, relationships, and purposes. This is inherent in any re-presenting. However, when we humanize data, we bring these possibilities to the forefront of our awareness.

Les données, telles qu'elles sont re-présentées, reflètent les intentions et les contraintes des producteurs. L'interprétation dépend de la sensibilisation des consommateurs. Les producteurs peuvent masquer des données, des contextes, des relations et des objectifs. Les consommateurs peuvent imposer des données, des contextes, des relations et des objectifs.

### MEANING IS CONSTRUCTED IN MAKING AND READING REPRESENTATIONS

Meaning making of data is an active in-the-moment process. This learning came as a result of our experiences with the tensions involved in creating physical representations of data. As our intentions for the re-presentations were thwarted by the construction parameters of the activities, we had to continuously renegotiate our understanding of the data and its re-presentation. The activities in re-presenting data using physical materials drew to the surface some of the tensions that even those who consider themselves very data-literate may remain blind to. In particular, relationships in data and complexities of data interpretations given physical constraints and limitations surfaced in new and unexpected ways for participants.

Notre travail de négociation vis-à-vis les contraintes matérielles a attiré notre attention sur les limites de nos modèles spécifiques, ainsi que sur l'idée générale que tout modèle mathématique a des limites. Ainsi, l'expérience de la création de modèles a influencé la manière dont nous avons donné un sens aux modèles en général. Si nous ne pouvions pas représenter fidèlement les données que nous voulions montrer, que pourrions-nous faire à la place ? Nous avons considéré les différentes représentations, ajusté et recentré, et parfois, nous avons simplement accepté que nous devions repenser ce qui doit être modélisé. Nous étions très conscients que nos décisions en ce qui concerne ce qui a été approximé, ce que nous avons ignoré, ou ce sur quoi nous nous sommes concentrés étaient probablement très différents de la re-présentation typique des données. Cela a renforcé notre conviction que les décisions sur quoi et comment représenter les données sont importantes et ont de multiples conséquences.

### CATEGORIES ARE MASKED IN DATA REPRESENTATIONS

This phrase could be iterated in different ways with different foci. Are categories hidden, masked, or obscured? These words mean different things by definition but also in connotation. And the connotations are not the same in French and in English. We subsequently spent some time choosing and discussing our intentions behind the wording. A number of times over the course of the working group we were reminded that any categories used in data representation are arbitrary in the sense that there are other possible categorizations, but not arbitrary in the sense that the categories are (intentional) expressions of the subjectivity of the people collecting, organizing, and representing the data.

Les catégories sont-elles cachées ou masquées ? Nous n’étions pas sûrs. Chaque mot a une connotation différente et le choix du mot le mieux adapté semble dépendre des intentions de ceux qui ont créé les catégories. La manière dont nous classons les données dépend de la vision des données que nous souhaitons représenter, de ce que nous permettons de voir ou de démasquer.

### DATA MODELS US. WE MODEL DATA

Much of our conceptualization of the world around us is dependent on our experiences with data that has been delivered to us, mediated through the hands of humans and their intentions. In this way we are constructed by data, our sense of self and our environment is constructed by data. However, we can have a hand in selecting data that we use to orient our experience of the world. As we generate, choose, and re-present data, we then mediate the experiences of the people around us. We help construct their worlds.

How much choice do we have in this process of re-presenting data? How much is imposed on us? How much do we impose?

Data models us. We model data. L’importance de cette relation réflexive est de plus en plus présente dans notre esprit compte tenu de l’expansion des applications d’intelligence artificielle qui entraîne la collection, la présentation et l’analyse de données dans de nouveaux domaines. Dans quelle mesure pouvons-nous vraiment choisir laquelle de nos données est modélisée ? Dans quelle mesure pouvons-nous être affectés par les choix de ceux qui re-présente les données qui nous modélisent ?

## LINGERING QUESTIONS

It has been a couple of months since our working group at CMESG. In exploring how to humanize data it seems we raised more questions than we answered. Maybe the very act of asking more questions than answering them **IS** what humanizing data is? What do you think?

As we reflect on our working group experiences, a number of questions still linger:

- Are we at risk of 'data burnout'? Que signifie même cette question si les données sont si omniprésentes ? Is it possible to 'burnout'?
- Considering different types of data, are some more human than others? Certains sont-ils plus réciproques que d'autres ? Are some really less in-tension-al (more objective) than others?
- How do we become more aware of the reciprocal relationship of data to the human? Comment la société devient-elle plus consciente ? Et comment pouvons-nous aider la société à prendre conscience ?
- What are the real boundaries of data? Are there boundaries? Qui définit les limites et dans quel but ?
- Avec notre reconnaissance de la subjectivité de la re-présentation des données, il peut être tentant de se méfier complètement des données. How can we support data-informed, science-informed decisions while still educating the subjectivity of data re-presentation?

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# RESEARCH AND PRACTICE: LEARNING THROUGH COLLABORATION

# RECHERCHE ET PRATIQUE : APPRENDRE EN COLLABORANT

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## PARTICIPANTS

| | | |
|---|---|---|
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| Lennie Comeau | Daniel Lumsden | Evan Throop Robinson |
| Beth Herbel-Eisenmann | Sarah Mathieu-Soucy | Annette Rouleau |
| Mariam Koozehkanani | Scosha Merovitz | |

## INTRODUCTION

In the last two decades, colleagues have been invited at CMESG/GCEDM to develop working groups or topic sessions around questions related to *teachers' practices* and the *relation between teachers and researchers*. Table 1 presents a list of the previous CMESG activities related to those themes.

In line with the work of these colleagues, the purpose of our working group was to consider the ways that research in mathematics education can acknowledge the reciprocal relationship between research and practice. Some have argued the importance of taking into account both the point of view of researchers and of teachers to address questions related to mathematics education (Bednarz, 2013; Goos, 2014). In our working group, we focused on the affordances of such collaboration as well as the challenges that this collaboration can raise at every step, from both the research community's and the education community's point of view. We invited the participants to explore not only the roles that teachers and researchers play when they collaborate, but also to examine how this collaboration between research and practice in mathematics education can further enhance mathematics education and teacher research.

| Year | Activity | (Co-)leader(s) | Theme |
|---|---|---|---|
| 2003 | WG-B | Louise Poirier<br>Florence Glanfield<br>Vicki Zack | Teacher Research: An Empowering Practice? |
| 2006 | WG-C | Chris Breen<br>Julie Long<br>Cynthia Nicol | Developing Respect and Trust When Working with Teachers of Mathematics<br>Le développement du respect et de la confiance dans le travail avec les enseignants de mathématiques |
| 2009 | WG-E | Jamie Pyper<br>Hassane Squalli<br>Laurent Theis | Studying teaching in practice<br>Étude des pratiques d'enseignement |
| 2012 | TS-A | Miroslav Lovric | Collaboration Between Research in Mathematics Education and Teaching Mathematics: Case Study of Teaching Infinity in Calculus<br>Collaboration entre la recherche dans l'éducation des mathématiques et les mathématiques de l'enseignement : Étude de cas d'enseignement au sujet de l'infini dans le calcul |

Table 1. Themes related to teachers' practices and the relation between teachers and researchers at CMESG/GCEDM.

In this report, we summarize the discussion that took place in the working group. The content of the report follows mainly a chronological order. Indeed, we had organized the three sessions following the work by Desgagné (1998) on collaborative research and its three moments of collaboration:

| | |
|---|---|
| Co-situation | ...renvoie à l'esprit collaboratif qui imprègne l'élaboration de la problématique. Elle consiste à faire en sorte que l'objet de recherche se construise à l'intersection des préoccupations du milieu de pratique et du champ de recherche concerné. |
| Co-opération | ...renvoie à l'esprit collaboratif qui imprègne la démarche de collecte de données. Elle consiste à faire en sorte que les données émergent de la structure d'interaction aménagée entre le chercheur et les praticiens. |
| Co-production | ...renvoie à l'esprit collaboratif qui imprègne la démarche d'analyse et de mise en forme des résultats. Elle consiste à faire en sorte que la production de connaissances prenne une forme qui soit utile aux praticiens et en même temps aux chercheurs. |

Thus, we invited the participants to think about the collaboration before the project has even begun (meeting 1—preparing the collaboration). We then worked in small groups on developing preliminary designs of a variety of collaborative projects (meeting 2—designing a collaboration). Finally, we addressed questions related to things that happen during and after the collaboration (meeting 3—what do we learn from collaboration? How do we share it? What dilemmas might be faced?).

## MEETING 1—PREPARING FOR THE COLLABORATION

We began the working session working on a scenario[1] about a Faculty of Mathematics Education (FME) that wishes to establish a partnership between university researchers in mathematics education and practitioners involved in the teaching field. We invited the participants to develop a University-School Protocol to guide research conducted in partnership with practitioner settings in mathematics education. Through a role play, the participants acted as researchers in mathematics education or mathematics teachers and had to brainstorm interests, concerns, expectations, needs, etc. Following this brainstorming, they continued in their roles to create guidelines for research-practice partnerships and therefore the elements of a collaboration protocol. Four participants acted as teachers, four participants acted as researchers, and three participants acted as analysts and observed the discussion between the researchers and the teachers.

First, in their respective sub-groups, those playing the role of researchers were asked to meet together to brainstorm about interests, concerns, expectations, and needs, in order to develop research-practice collaboration, as well as how they see their role and the role of the teachers. Those playing the role of teachers were asked to discuss the concerns they have in their practice that might lead them to be interested in working with researchers in mathematics education. The analysts were asked to work together to plan how they would observe a group of researchers and teachers who were collaborating to develop a research-practice protocol.

Then, we asked the teachers, the researchers, and the analysts to form two small focus groups (two teachers, two researchers and one or two analysts). The teachers and researchers were asked to discuss, from their role's perspective, why they think it would be interesting to work together, discussing for instance, what are the interests, hopes, expectations, needs…. From this 'brainstorming', they were asked to identify benchmarks for partnership research and thus the elements to consider in developing a protocol. The analysts had to take notes and observe the interactions in order to report later.

### THE ELEMENTS TO CONSIDER IN ESTABLISHING A PROTOCOL

This activity shed light on what appeared most important for both teachers and researchers and that should form the basis of a collaborative research project. Table 2 summarizes the main ideas and concerns about collaboration that arose from two groups' discussions.

| Group 1 | Group 2 |
|---|---|
| • Value teachers' perspectives, contributions, etc.<br>• Understand teachers' needs, difficulties, timing, etc.<br>• Tackle a project together, not top-down.<br>• Produce sharable products.<br>• Be aware of what the teacher feels comfortable with or not (filming, release time, etc.). | • Everyone must benefit the collaboration<br>• Determine what the goals are<br>• Determine the constraints and issues that need to be addressed<br>• Ask critical questions about why we do what we do<br>• Engage both teachers and researchers in all aspects of the project. |

Table 2. Elements to take into account to establish a protocol.

[1] Based on Desgagné (n.d.) and Bednarz, Corriveau, Saboya, Barry, et Maheux (2011).

The analysts observed the discussions and presented their observations of the two focus groups. Here are some of the key elements:

- There was no discussion from researchers about theoretical and methodological aspects of the research. There was no academic jargon used but rather the discussion was more of a conversation. The teachers and researchers seemed to **share the same language**.
- There was little discussion about the topics/themes that could bring researchers and teachers together or how a topic/theme might emerge from the collaboration.
- There were many more **discussions about technical aspects** of the collaboration:
    - timing
    - the meaning for the curriculum
    - the nature of the experimentation in classrooms: when? Which class? Would the researcher be willing to teach? Would the researcher be coaching, observing?
    - video recording: why would we want to film?
    - the public nature of data: what data? Why data?
- The analysts mentioned that researchers **need to clarify their methodological requests**. The discussion between teachers and researchers forced the researchers to be explicit and explain the rationale behind some methodological aspects of the research, such as long-term collaboration and the need to video record in the classroom.
    - Example: When co-designing a lesson, the timeline may be long (two or three years) because the researcher would like to test out the idea more than once and in different contexts in order to see what works and what does not work.
- The analysts pointed out that **teachers have a need to explain why there are limits of some requests in practice**. In doing so, teachers point out that researchers be aware of the complexity of school organization.
    - Example: A teacher might see two or three years of collaboration as too long because the teacher might not teach in the same school the next year, nor at the same level, etc.
- The discussion brought teachers and researchers to **negotiate research requests**.
    - Example: The researcher should not bring the idea of video recording the classroom too early, but rather wait until a safe atmosphere is created.
    - Example: Instead of the researcher coming with all his equipment and staying in the classroom to record the teacher, the researcher could give a camera to the teacher who chooses to record what she wants to share and is relevant to the project.
- A **shared goal** arose from discussion: the benefits for students. Both teachers and researchers want what is the best for students.

## OTHER OBSERVATIONS BROUGHT FORWARD IN THE WHOLE GROUP DISCUSSION

- **Teachers' and researchers' identity**. Teachers' and researchers' identities are interrelated, and not as separate as they seem or as we may argue they are. Perhaps collaboration begins with being able to jump into the other's shoes and thus, we can move from respective perspectives to a collective perspective. What might that mean?
- **Heterogeneity of teachers' preferences**. Teachers work in a variety of ways and have different preferences as to how they want to work with researchers. As an example, some teachers might want to meet during their working time because they see the collaboration as a part of their work (just as the researchers), but some prefer to meet outside of their work day so that they are available to students during the school day.

- **The institutional level**. Researchers have to be aware of the institutional dimension and demands and the ethical dimension in the collaboration: school board (and their approval), parents (and their consent), etc.
- **Different research perspectives**. There are different perspectives such as "introducing innovations" as opposed to "observing teachers teach." Why do we need to collaborate? Is it to implement researchers' new approaches or to better understand teachers' practices and their points of view? Are both collaborative perspectives? Is intervention a deficit-oriented approach? What would be the contribution of the teachers in this perspective? Should we not rather try to understand the complex practice of teaching and the actions of the teacher? What would be the researcher's contribution in this case?
- **Who needs who?** On the one hand, researchers would like to create a space where teachers could share their interest in working with researchers, an open avenue for teachers to come to researchers. On the other hand, teachers may feel they do not need the researcher. Are there misconceptions about this kind of research? What might need to change so that both researchers' work and teachers' work are valued?
- **Creating a collaborative space**. The collaboration has to be a safe space where people feel respected and valued, since only then will we be able to address important issues. Some collaborative space could be created before research begins. For instance, the researcher could create opportunities for teachers to know them such as by doing a workshop at their school, or co-planning or co-teaching with teachers.

### SPECIFIC QUESTIONS TO ASK AT THE BEGINNING OF A COLLABORATION

At the end of the first meeting, Beth shared questions she asks when she begins a new collaboration or when new collaborators join a team:

- What is a strength or two you would bring to our collaboration?
- What is something you have been wondering about in your practice/research?
- What is something you want to work on in the context of this partnership?
- What is a question (or two) that you have?
- What is a concern that you have going into this partnership?
- What is a hope you have for this partnership?

## MEETING 2—DESIGNING A COLLABORATION

The concept of *double relevance* (Desgagné, 1998; Dubet, 1994) was introduced to serve as a basis for thinking during the session. Double relevance means the researcher places himself or herself at the intersection of the research and practice worlds by addressing his or her research purpose to both communities (research and practice) throughout the entire research process. On one hand, the researcher cannot claim to construct knowledge with the practitioner without, fundamentally, considering the knowledge that, with or without the researcher's contribution, the practitioner constructs and develops throughout the course of his or her experience. On the other hand, the researcher is involved in a research process that must comply with the usual standards of research in mathematics education: he or she has to be credible and competent. In other words, the research needs to be relevant and meaningful to both the researchers and to the practitioners.

We put forward the idea that double relevance could be or needed to be considered in all phases of the research:

- the development of the research question and design;
- the way that data is collected;
- the way that data is analyzed; and
- the way that results of the research are shared.

Thus, we asked the group to consider: What methods of data collection, analysis, and reporting of results might respect the collaborative nature of research? What does double relevance mean for each of these phases?

### DOUBLE RELEVANCE IN DATA COLLECTION/CREATION/GENERATION

The participants first shared what they were already doing in their research-practice partnerships and then, they created new ways of thinking about data creation with a collaborative spirit.

#### About using reflexive journals

Evan shared a way of creating data (related to students) that was relevant for both research and practice. He uses a 'math journal' as a reflexive journal for students to write about the mathematics they are doing. The teacher is able to see the way students are feeling about mathematics throughout the mathematical activities, and the researcher uses the journals as data.

Minnie talked about a reflexive journal. She mentioned that asking the teachers to keep a reflexive journal may not be 'double relevant'. She suggested that journaling adds a lot of extra work for the teacher since it is not something that they already do in their practice. She proposed instead to record a short conversation (a debriefing) between the researcher and the teacher to have access to her thoughts, rather than asking the teacher to write in a journal.

Claudia related a collaborative research project (Barry, 2009) in which the researcher wrote a research journal and the teacher asked him if he could read it. The teacher read the journal and commented on it in the journal itself. It became, for them, another way of communicating. It helped to track what was happening in the research project and it also made the researcher re-think the way he was writing his notes in the journal, knowing it would be read by his collaborator.

*About the ownership of the data*

Some of the group argued that the data created should be shared among all collaborators since they are being part of their creation (e.g., videos, journals). However, this idea raised an ethical issue as some question whether the data can be owned by a group. Issues of preserving anonymity in collaboration also arose.

#### Double relevance in data analysis

*Co-analysis*

There was a discussion about researchers and teachers co-analyzing the data. One idea that was brought forward was that the teacher's lens in analysis might add layers and context to the data. Thus, the results of the co-analysis would not be generic but very contextualized. It was highlighted out that a teacher's expertise helps to point out details about students, which may or may not be represented in the data. Some in the group proposed analyzing data jointly but at different moments.

For example, a teacher could analyze what a student has done and then the researcher can analyze it starting with the teacher's analysis.

*About the language used in the collaboration*

The notion of language was discussed as being extremely important in collaborative research. Although one might think that the researcher's theoretical language might alienate practitioners, perhaps the researcher might be surprised. By bringing theoretical and methodological words and concepts to the collaborators, the researcher can see the ways that the teachers understand these ideas and make sense of them. In doing so, this might add interesting layers to the theory. The researcher might also build a 'new model' or a 'theoretical concept' from what emerges from the collaboration. In doing so, she stays very close to the language used in the collaboration.

Double relevance in dissemination

There was also discussion about ways that dissemination of the research project could benefit both research and practitioner worlds. It was suggested that the research be disseminated in a variety of ways to reach a variety of audiences. Several dissemination methods were suggested:

- development of materials for teacher education and teacher professional learning;
- presentation of the research or workshops to school board, teacher associations, and/or broader community;
- development of two types of articles—one type for research journals and another for practitioner journals;
- create videos about the research project—for dissemination through research project and university websites and for dissemination on professional learning or school board websites.

Other considerations in regards to double relevance

There were several other things discussed with respect to double relevance. For instance, Sarah wondered about what defines collaborative research since, as she sees it, any kind of research could be made relevant for both teachers and researchers. The discussion that arose from this question brought the group to argue that 'collaboration' in research-practice partnership does not only mean involving the teachers, but it also means that teachers are necessary in the understanding of a phenomena related to their practice, and the researcher could not do the research without them. We added that it implies considering teachers' analysis of that phenomena through meaningful ways for them (that may imply using other means of data collection than standard ways such as interviews or observations). We also put forward the idea that the collaboration has to be seen as an ongoing practice.

We then worked in small groups in designing collaborative research projects having those questions in mind (keeping in mind that all of the processes must respect the collaborative nature of this work—double relevance). We asked groups to consider

- What are your research questions?
- Who are your research partners?
- What might data collection look like?
- How will the data be analyzed?

- What are some methods of sharing the results?
- By what means can we make space for the experience of the collaborating practitioner? / If you are a practitioner, in what ways do you want to engage in this research collaboration?
- In what ways does the practitioner experience contribute to and support the research?
- How does this contribution and engagement in the research support the practitioner?

Each group shared their project with another small group so that they could be provided with critical and constructive feedback. The groups then shared the main elements of each project with the whole group.

## MEETING 3—WHAT DO WE LEARN FROM COLLABORATION? HOW DO WE SHARE IT? WHAT DILEMMAS MIGHT BE FACED?

The third session began with a Graffiti wall activity. We asked the participants to use the front board as a graffiti wall and jot down whatever came to their mind with respect to defining or describing important components of collaborative research in mathematics education. Following this we asked groups to use the ideas from the graffiti wall, their own experiences, relevant readings (we had given three papers—Corriveau & Bednarz, 2016; Goos, 2014; Goos & Geiger, 2006—about collaborative research), and their colleagues' ideas, to collectively create a definition for collaborative research in mathematics education. The following are the definitions that were created:

Collaborative research definition 1

> *The process of learning and working together in a trusting and supportive relationship toward a common goal or interest respectful and collaborative dissemination of the learning. Collaborative research is a symbiotic endeavor between teacher and researcher. It is a partnership that values each other's expertise with shared goal that may provide benefits (same, similar, or different) to both partners.*

Collaborative research definition 2

> *Collaborative research begins with mutual interest and motivation to explore a common problem related to mathematics education.*
>
> *It builds on a relationship based on respect and valuing complimentary knowledge in order to develop trust.*
>
> *It creates a synergy, and a symbiotic relationship.*
>
> *It capitalizes on what each party brings to the process (common goal, data collection, analysis, dissemination of results).*

### THREE SCENARIOS TO BE 'RESOLVED'

In order to examine dilemmas that might arise in collaborative research situations we asked the participants to work in one of three small groups. Each group received a scenario and they were tasked with answering the question or addressing the dilemma as well as to share their reflections. The following are the three scenarios that were provided.

Scénario 1 : *Ta tâche n'a aucun sens* (voir le scénario complet en Appendix 1) :

J'étais au doctorat et pour mon projet de recherche (Corriveau, 2013), j'organisais un groupe de réflexion sur la transition secondaire postsecondaire en mathématiques. L'objectif du travail de recherche était de mettre en lumière la manière dont on fait les mathématiques à chacun des ordres. En vue de la première rencontre, ma directrice de recherche et moi avions retenu une tâche mathématique prise du texte de Coppé et al. (2007). Il s'agissait, à mon avis, d'une tâche à la jonction entre le secondaire et le collégial, elle constituait une occasion en or pour amoindrir le fossé entre les deux ordres scolaires. J'ai donc décidé d'en faire la tâche d'ouverture. Cette tâche allait convaincre les enseignants de la pertinence du projet ! Lors de ma première séance, je présente la tâche aux enseignants en leur demandant s'il s'agit d'une tâche qu'ils pourraient faire avec leurs élèves ou leurs étudiants. À ma grande surprise, ils rejettent tous la tâche (voir Corriveau, 2017). J'étais sous le choc. Mon projet était-il vain ? Qu'allais-je bien pouvoir faire pour poursuivre les discussions ?

Scénario 2 : *Changement de point de vue : que peut-on apprendre au moment d'analyser ?*

Un projet de recherche collaborative s'organise autour de la question de transition interordres. La chercheure organise plusieurs rencontres avec un groupe d'enseignants de la fin du secondaire et du postsecondaire. Par le biais d'activités visant l'explicitation de leurs manières de faire des mathématiques à chacun des ordres, plusieurs enjeux de transition sont mis en lumière. C'est le cas lors de la première rencontre où le groupe met de l'avant des ruptures dans le passage de l'étude des fonctions à la fin du secondaire au premier cours de calcul différentiel au postsecondaire. Entre la première rencontre et la deuxième rencontre, la chercheure analyse la séance et dresse un premier portrait des enjeux soulevés dans les discussions. Ainsi, à la deuxième rencontre débute avec le portrait qu'en a fait la chercheure. Elle questionne les enseignants en leur demandant si ce qu'elle propose reflète ce qui s'était dit trois semaines auparavant. Or, à la lumière des ruptures mises en évidence, les enseignants semblent complètement changer d'idée. Ils établissent maintenant des liens entre les manières de faire aux deux ordres. La chercheure repart donc avec les données de la deuxième séance et en fait une analyse. Au fil des autres rencontres, ce même phénomène se reproduit. Les enseignants ne réfutent pas ou ne confirment pas les analyses proposées par la chercheure, mais « sautent du coq-à-l'âne » en repérant tantôt des ruptures et tantôt des ponts. Ils reviennent constamment en arrière et changent d'avis sur leurs propres analyses mises de l'avant. Que devrait conclure cette chercheure ?

Scenario 3: *Question of legitimacy* (see complete scenario in Appendix 2)

The project in question involved two researchers and primary mathematics teachers in a single group. All teachers held positions in early years of primary school. The project focused on the use of manipulatives in primary mathematics education. The purpose was to develop, with the help of teachers, a better understanding of doing mathematics with manipulatives and to develop classroom practices in using these materials. The following year, when the research was over and the researchers were no longer working with the group of teachers, the teachers submitted two requests to their school administration in light of the work done with the researchers: they asked for time release to engage in creating more manipulative materials, and they also asked for funds to buy new manipulatives. The requests were rejected. The teachers were very upset and contacted the researchers to ask for their support. They asked them to contact the school administration on their behalf. What should the researchers do?

Scenario 4: *Dilemmas in classroom research: Field notes on a Starbuck's napkin* (see Appendix 3 for the complete scenario)

This scenario involved two researchers and a Grade 9 teacher. The focus of the project was to describe teacher practice in implementing mathematics curriculum. The scenario described is fictional but based on particular aspects of the research project and represents dilemmas that might typically be faced in education research. The focus of the scenario is teacher and researcher reflections after a lesson. The described lesson focused on understanding some basic ideas about algebra such as operations with algebraic terms. The teacher wanted to facilitate this lesson by creating a math talk learning community (Hufferd-Ackles et al., 2004). The dilemma that arose considered the degree to which researchers might intervene or provide advice to a teacher about their lesson.

## FOLLOWUP

The group had a lively discussion about these scenarios and several issues were brought forth or reiterated in the discussion such as

- The researcher attitude: how to create a comfortable environment for the teacher where it is possible to disagree (with the researcher).
- Methodological insight (the breaching situation): how a fine line between familiarity and strangeness can give access to what is important for teachers.
- Analytical aspect: the importance of staying very close to what teachers say.
- Conceptualization: how to take into account the teachers' ways of thinking to conceptualize a phenomenon.
- Ethical issues: should the researchers take a position in an issue that concerns teachers? Should the researchers take a position and be involved in supporting a request teachers have for their school principal or policy makers? If the researchers get involved, would that mean that they have 'more power' than the teachers (and intervene in the legitimacy of the teachers in their request).
- Ethical issues: collaborative research may suggest changes in practice that shake up common ways of doing things or intervene with policies and prescriptions…
- Research-practice partnership: what about the other practitioners and the institution? Should the principals, policy makers, etc., be involved at some point… Should institutional questions/aspects be address within the group of teachers and researchers…

We concluded the three days and many members of the group commented that the group itself was extremely diverse in terms of their background, their experiences, and in their current roles. Some suggested that many of the aspects of collaborative work were shown in the way in which the group worked as everyone contributed to a safe and respectful space that valued the variety of perspectives that we all shared.

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## APPENDIX 1

### TA TÂCHE N'A AUCUN SENS, AUCUN LIEN !

Claudia Corriveau, *Université Laval*

J'étais au doctorat et pour mon projet de recherche, j'organisais un groupe de réflexion sur la transition secondaire postsecondaire en mathématiques. L'objectif du travail de recherche était de mettre en lumière la manière dont on fait les mathématiques à chacun des ordres. Quatre enseignants du secondaire et trois enseignants du collégial avaient accepté de participer à la recherche.

En vue de la première rencontre, ma directrice de recherche et moi avions retenu une tâche mathématique prise du texte de Coppé et al. (2007).

Soit une fonction *f* définie sur l'intervalle [-3; 3] dont on connait le tableau de valeurs suivant :

| *x* | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| *f(x)* | 2 | 1 | -1 | 0 | 0,5 | 1 | 2 |

a) Tracer une courbe compatible avec ce tableau de valeurs.
b) Peut-on en tracer d'autres ? Si oui, tracez-en une. Si non, expliquez.

Il s'agissait, à mon avis, d'une tâche à la jonction entre le secondaire et le collégial. En effet, elle était intéressante parce qu'elle ne se situait ni au secondaire, ni au collégial. C'était une tâche de transition, elle constituait une occasion en or pour amoindrir le fossé entre les deux ordres scolaires. J'étais comblée. J'ai donc décidé d'en faire la tâche d'ouverture. Cette tâche allait convaincre les enseignants de la pertinence du projet !

En janvier 2011, je tiens ma première rencontre avec les enseignants. Nous avons une journée complète pour discuter des façons de faire les mathématiques à chacun des ordres et pour mieux comprendre les difficultés liées à la transition. Je suis très nerveuse. J'espère que je serai à la hauteur. J'accueille les enseignants et je leur soumets la fameuse tâche en leur demandant s'il s'agit d'une tâche qu'ils pourraient faire avec leurs élèves ou leurs étudiants. Je ne m'attendais pas à des réactions comme celles-ci :

> Sam : *Ça, c'est quelque chose qu'on fait pas, ça a aucun sens, ça a aucun lien…*
> Colette : *C'est plutôt l'inverse j'imagine, vous donnez une fonction, vous demandez de la tracer…*
> Sam : *On peut en avoir des comme ça aussi mais ça va donner une fonction qu'on connaît déjà, une fonction à travailler, mais une fonction comme ça…*[soupir]
> Scott : *C'est une question complètement décontextualisée, ce qui n'est pas souhaitable dans le sens des attentes actuelles.*
> Serge : *Comme enseignant du secondaire, j'aurais pas été intéressé par cette question-là, ça vient d'où les points ?*
> Sam : *Il faudrait le relier à quelque chose de concret.*
> …

> Colette : *Travailler avec le tableau de valeurs pour représenter des fonctions, moi c'est pas quelque chose qui m'intéresse parce que c'est pas assez global.*

Ils rejettent tous la tâche. J'étais sous le choc. Mon projet était-il vain ? Qu'allais-je bien pouvoir faire pour poursuivre les discussions ?

## APPENDIX 2

## A QUESTION OF LEGITIMACY?

Claudia Corriveau, *Université Laval*
Doris Jeannotte, *Université du Québec à Montréal*

This case is inspired by a project that involved two researchers and primary mathematics teachers in a single group. All teachers held positions in early years of primary school.

The project focused on the use of manipulatives in primary mathematics education. The purpose was to develop, with the help of teachers, a better understanding of *what is doing mathematics with manipulatives*[2] and to develop classroom practices in using these materials.

The results of this work between the teachers and researchers led teachers to want to use such materials in their class, including non-commercial materials developed by the group. Indeed, at the time of data collection, there had been a great deal of material developed within the group or provided by the researchers. The joint work shed light on how to support students when using materials.

The following year, when the research was over and the researchers were no longer working with the group of teachers, the teachers submitted two requests to their school administration in light of the work done with the researchers:

1. They asked for time release to engage in creating more manipulative materials to use in their class;
2. They also asked for funds to buy new manipulatives.

The requests were rejected for the following reasons:

i. The teachers had already participated in a research project the following year and the cost of substitute teachers was supported by the research, but the work of finding substitute teachers to replace them had required a great deal of administrative work.
ii. The numerous cuts made in the school board did not make it possible to provide additional funds for the purchase of equipment (except for computer equipment which benefited from a special envelope).

The teachers were very upset and contacted the researchers to ask for their support. They asked them to contact the school administration on their behalf. What should the researchers do?

[2] Assuming it is something different from doing mathematics without manipulatives.

# APPENDIX 3

## SCENARIO 4: DILEMMAS IN CLASSROOM RESEARCH: FIELD NOTES ON A STARBUCK'S NAPKIN

Christine Suurtamm, *University of Ottawa*
Martha Koch, *University of Manitoba*

It was a cool March day as we loaded the video equipment back into the trunk of the rental car and headed to Starbucks. We had just left our third day of video-taped observations of Janet's Grade 9 mathematics class and as we walked with Janet down the hall after the class, she expressed her frustration and questioned what to do with the next day's lesson. This was the second day of promoting student discussion on the difference between adding and multiplying the two terms $4x^2$ and $2x^3$ and Janet was not sure that the students' understanding was progressing in the way that she had hoped.

Janet was one of several case study teachers that we had been observing during a large-scale research project. The purpose of the case studies was to be able to describe the experiences of teachers who are enacting classroom practices that reflect current thinking in mathematics teaching and learning (NCTM, 2000; Jacobs et al., 2006). These case studies not only offer us descriptions of engaging classroom practices, they also provide insights into how teachers enact these practices and reveal the dilemmas and challenges they face as they develop such practices (Suurtamm & Graves, 2009; Suurtamm et al., 2010). Janet described that she was focusing on developing a math talk learning community based on some of the reading she had done (e.g., Fosnot & Dolk, 2001), the professional development experiences she had, as well as her own teaching experiences.

As such, in our classroom observations, we were witnessing Janet's work on creating a math talk community (Hufferd-Ackles et al., 2004) where students grapple with mathematical ideas through discussion. The lessons that we observed were focused on developing algebraic concepts and the direction of the lesson followed the path of the students' thinking. In the first lesson that we observed, students worked with the notion of a like term and tried to find different explanations for when terms are alike. In the second lesson, a student suggested considering the difference between $4x^2 + 2x^3$ and $(4x^2)(2x^3)$ and how each of them can be simplified. Students worked in pairs to discuss the differences and several pairs went to the front of the room to share their explanations. One pair suggested that if the two terms were multiplied then the answer would be $8x^6$. Janet then asked the class if they had any questions for the pair. There were no questions and the students sat down. Another pair presented the idea that the solution to the multiplication would be $8x^8$ as you multiply the 4 with the 2, then the two exponents would be multiplied, giving you $x^6$, but then you had to multiply the $x$ by the $x$ to give you $x^8$. Janet again asked if there were any questions for the pair. Some students asked why you had to multiply the $x$ with the $x$, and the presenting students repeated their explanation and then sat down. The bell rang and the discussion was to continue into the third day. On the third day, Janet began by asking students to create a model of what is meant by $x^2$ and $x^3$. Students worked in pairs and a few pairs presented their models. The models included a diagram of a square and a cube and prompted a discussion about whether these squares and cubes could be added and/or multiplied. There was also a sharing of $x^2$ meaning $x$ times $x$ and $x^3$ meaning $x$ times $x$ times $x$. Janet then moved the discussion back into student pairs and suggested students use these models to further discuss the difference between $4x^2 + 2x^3$ and

$(4x^2)(2x^3)$ and how these expressions can be simplified. The bell rang and Janet stated that the next class would begin by making some conclusions about this problem.

As we packed up our video equipment, Janet packed up her teaching materials, and we began our walk down the hall. As we walked, Janet talked through her thoughts.

"*The one problem that I have is I get so drawn in to… to sort of where they're struggling and I want to keep going with it and then I look at the clock and I'm like, 'Okay, this is not my plan for today at all.' You know? At least, it was for the first 20 minutes, but then it kind of turns into a whole big discussion . . . I'm at a place right now where I'm a little bit worried about how am I going to get to the end, so I can't really afford to have days like today, but I think it's—I think I let it happen… But, like, the difference between 2x, for instance, meaning x plus x versus meaning, like, an area created by the multiplication of the number two times the number—whatever the value of x is. That, to me, sounds like it's ridiculous semantics, but it's part of the reason why they struggle with simplifying polynomials because when they look at 2x, well, Is that x plus x or is that…?' And I love the fact that it started to come out… but this changes the plan for tomorrow. I'm not sure where we are going next.*"

The discussion continued for a few more minutes and then Janet stopped in at her department office, dropped off her things, proceeded to hall duty, and we got in the car and, as we had the previous two days, drove to Starbucks to have a coffee and debrief our observations. As we sat down, the conversation began.

M: *It seemed as though Janet was looking for some guidance from us at the end of the lesson as to where to go next. Did you get that sense as well?*

C: *Sometimes I get that sense during the lesson too. Like, when a pair presents, or when Janet responds to student thinking she sometimes looks at me as I stand behind the camera. Sometimes it is a look that seems to be seeking approval, other times it's just a look of shared pleasure as a student provides a really sound response. Perhaps it's just a natural response to share a reaction to students' thinking between mathematics teachers.*

M: *Yes, but there are moments when I'm not sure where the thinking is going.*

C: *Yes, I wasn't sure what the students were going to use for models of x2 and x3. I know as we walked down the hall, I asked Janet whether they had previously used algebra tiles, but maybe I shouldn't have asked that question because she seemed to take it as a suggestion or criticism of her lesson. Remember, she said "I was kind of looking to see if anybody was going to go the algebra tile route, because we did use algebra tiles to model the x squareds. But I didn't put them out today, which is maybe a bit of my mistake because maybe if I put them on the table, they might have gone towards that in their representations. Um… yeah. Now that I think about it, that's a good point". But, you know, I didn't really mean it as a criticism of the lesson. I just wondered what models they had been exposed to. I don't know, perhaps I just should have kept quiet and not mentioned it.*

M: *Well, that is the hard part—we are just observing, but we also want to hear the teacher's reflections on the lesson, and it is only natural to have a conversation. And of course, they know we have some expertise in this area. It seems crazy not to tap into it. But at the same time, you don't want them to think that you are evaluating what they are doing. We're grateful that they provide us with the opportunity to observe the lessons. Teaching isn't easy.*

C: *Hmmm—I know what you mean.*

M: *But, doesn't it seem like this one discussion is going on for quite a long time and the students are not too sure where it's going? I think if I were a student in the class, I might be a bit frustrated. When students present, how do the other students assess whether the thinking is mathematically reasonable?*

C: *Well, I was quite confused about the discussion about like terms that they had on the first day. What is a like term is really just a definition—not something to mathematically argue about. Once we have the definition, we can then argue whether two terms satisfy the criteria, but the definition is just that. So, I wonder about whether teachers understand that there are definitions and conventions that we just agree on as a mathematical community and then there are things that we can reason about mathematically but we need to rely on those established norms. I'd love to have this conversation with Janet but I'm not sure it's appropriate. But yet, she also seems to be looking for guidance.*

M: *And I'm kind of worried about what the kids are thinking and feeling. There appears to be a lot of confusion. I wonder how they will sort through all those ideas?*

C: *I know.* (grabs a napkin to write) *How did that student come up with* $(4x^2)(2x^3) = 8x^8$?

M: *I know, I know, let's look at that again…*

We left that Starbucks discussion and many other similar discussions reflecting on the range of dilemmas that we face as researchers. What are our roles as researchers in a classroom? While these roles can be clearly determined at the start of the research, can they shift back and forth from merely observation to participant observer in different contexts? How does one observe a classroom without the teacher feeling evaluated? And does not the researcher slip into an evaluative mode to a certain extent even when trying not to? What does the teacher expect from the research? We were working with teachers who were developing new practices and many seemed to want to use us as sounding boards to discuss their ideas and receive feedback. While we nodded as they spoke to confirm their story, we were not providing the advice and dialogue that they may have been seeking.

We have spent a great deal of time discussing these dilemmas and challenges and while it is not clear that they are easily resolved, we look forward to continuing the discussion, over coffee, with others.

# INTERDISCIPLINARITY WITH MATHEMATICS: MIDDLE SCHOOL AND BEYOND

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## INTRODUCTION

In this Working Group (WG), we explored the opportunities and challenges of teaching and learning mathematics in an interdisciplinary (I-D) context. To follow Staats (2014) this means "drawing a context from another discipline authentically enough to support learning in both disciplines" (p. 7), mathematics plus the "other". While the notion of interdisciplinarity is not new, contemporary promotion of STEM/STEAM, new mathematics curricula in which the "focus is on real-life, relevant contexts" (BC Math Curriculum, 2018, para. 9), and ongoing efforts to foster mathematical literacy (OECD PISA, 2019) made this a timely focus for a CMESG working group.

The middle grades (6–8) were well suited as a starting point for the exploration of interdisciplinarity with mathematics because a) the mathematics in the curriculum begins to permit richer contexts in which students can be collaborators; b) classroom structures at this level typically still permit greater flexibility for the use of the learning space, and incorporation of what some may perceive as not-associated disciplines, allowing creative and interesting ways to understand a problem or issue; c) students have the personal persistence to sustain longer term inquiries; d) it is a time in students' learning where habits of non-integration and non-association of disciplines can be examined, explored, and changed; and e) these years are often positioned by teachers, school systems, parents, and society, as a transition phase—finishing, closing out, and consolidating one phase of learning (the elementary school years) and moving into a second phase of learning (the secondary school years).

Our Learning Goals for this WG were

- that participants learn about the evidence (literature, experience, etc.) that supports the praxis of interdisciplinarity in the middle school years.
- that participants appreciate the value of interdisciplinarity as a whole-school learning experience as an individual teacher/class learning experience.
- that participants are introduced to the principles of learning that underlie how the WG presents interdisciplinarity and begin to appreciate their application in thinking about and applying these principles to learning situations/contexts.

To work towards those Learning Goals, we engaged in a learning and experiential trajectory that included not only sharing and discussion but also participating in and attempting to create interdisciplinary activities. The three days were designed to, day 1, share and thus explore how we know and experience interdisciplinarity; day 1 and 2, engage in tasks that will help us understand some underlying principles to interdisciplinarity; day 3, dis-integrate and then integrate our thinking about interdisciplinarity with mathematics, and throughout all three days examine our assumptions about the benefits and risks of teaching mathematics in an interdisciplinary manner.

## DAY 1

Jamie opened our session on the first morning with a welcome and acknowledgement that our work and play during our time together would be taking place on the unceded and ancestral territories of the Mi'kmaw people and expressing our gratitude for this opportunity. Our intention was to approach the working group with an openness to the interplay of Western ways of knowing (i.e., the current curricular expression of mathematics and other subject areas) and Indigenous Ways of Knowing (e.g., relationship to place, respect, reciprocity, and responsibility (Porsanger, 2004), with an awareness that "the language that we use shapes the way we think" (Kovach, 2005, p. 25). As WG leaders, we understood our positioning as allies and that what we presented would naturally come from an ally understanding, but our WG participants were wonderfully well positioned to bring diverse perspectives and knowledge to our thinking and work.

We then spent some time talking together to get to know each other. A focus for the conversation was a diagnostic activity involving personal experiences and the creation of a visual graphic representing the aggregate of those personal experiences. Participants were invited to introduce themselves, describe where they were from, and describe an interdisciplinary activity that they had been a part of. As we went around the room, participants helped create an infographic on the whiteboard to represent their interdisciplinary experiences. The basic structure of the graphic was an $x - y$ axis, with

- the horizontal axis representing the extent to which the activity was about learning mathematics (on a scale of 0 to 100),
- the vertical axis representing the extent to which the activity was about learning (an)other discipline(s).

For each participant, once they had determined a location on the graph that represented their thinking, participants were invited to mark the location with a circle whose size reflected the duration of the activity. A rough scale was established with a small dot representing a single activity of short duration (e.g., an hour or less) and increasing up to a large circle to represent a full-year course. In addition, we colour-coded the circles to identify the age group of the participants in the interdisciplinary activity. On the margin we noted the 'other' disciplines involved. See Figure 1.

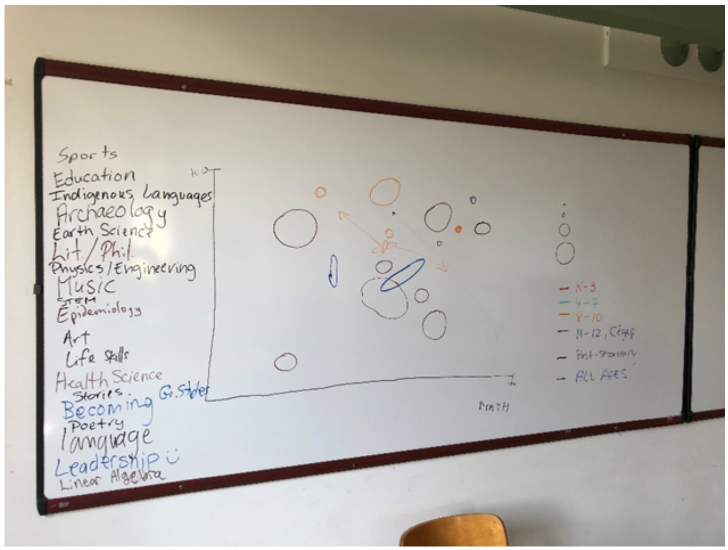


Figure 1. Interdisciplinary experiences of the WG members.

We acknowledged the imprecision in the schema and appreciated the extensions and creative adaptations from individuals as they considered the characteristics of the activities they offered. Our aim was to bring to light the scope of activities that can be considered as 'interdisciplinary', as well as to acknowledge the range of experiences and understandings of interdisciplinarity that participants were bringing into the working group.

Some of the participants' experiences, illustrated as circles or shapes on the graph, included whole courses where math consisted of the largest part; an epidemiology example at the college level; an integrated mathematics/literature/philosophy program; and a technology course in which math is explored as it fits 'under the hood' in projects. The intent was to use circles, but the ways in which interdisciplinarity has been experienced and considered, and the nature of the axes we designed for this conversation, meant the participants needed to be creative with their drawings on the graph. Circles became ovals, and lines were needed to connect smaller circle nodes of related experiences. Noticeable is that the prominent experiences are from Grades 8–10 and post-secondary learning. The durations of the interdisciplinary learning experiences shared were mostly the size of large units or whole courses. There were fewer experiences of short-term or quick injections of interdisciplinarity into learning.

Participants interpreted the design of the graph with math on the horizontal axis, and the other disciplines represented on the vertical axis as an important feature and valuable way of thinking about mathematics with respect to other disciplines in the context of interdisciplinarity. This placement of mathematics on the horizontal axis (low and acting like the ground) seemed to compensate for what the participants felt was the usual perception of placing mathematics 'high up' and with a perceived importance over other disciplines. On the horizontal axis, mathematics became understood in a grounded sense versus a bird's eye view of interdisciplinarity if mathematics had been placed on the vertical axis.

### GLOBAL WATER ISSUES—AN EXAMPLE

Armed with a better understanding of the wide scope of activities that could be considered under the umbrella of interdisciplinarity, we offered a particular example of a planned interdisciplinary unit on global water issues for a Grade 6 class (Lee & Tosic, 2018).

Vancouver teachers, Darryl Lee and Alice Tosic (2018), in response to a perceived "lack of connections in school mathematics" (p. 24) developed a mathematics unit with a water usage theme. As part of their project, they conducted research to examine whether this approach would enable their students to learn the targeted mathematical concepts from the British Columbia (BC) provincial grade six Mathematics curriculum, as well as about water issues, while at the same time encourage students to consider real-world factors when problem-solving and foster positive attitudes and beliefs toward mathematics (Lee & Tosic, 2018).

The unit was centred on the essential questions "How much water do we use?" and "How much water do we have?" The activities developed to help students answer these questions were cross-referenced to curriculum requirements and were taught over a six- to seven-week period. Implementation of the activities involved cycles of motivation (e.g., brainstorming, videos), application to personal situations, non-contextualised mathematics instruction on needed topics and skills and application of the skills to real-world data.

Thanks to Darryl and Alice, we were able to share some of the example activities that they used in their project with the working group members, including a Water Walk Activity (determining the time it would take to carry the water needed for one day from a source 5 km away), an assignment to find the approximate volume of a local lake, and an assignment to predict when a reservoir would run out of water.

Observations during the teaching of the unit and pre- and post-surveys of the students showed that students did learn the targeted mathematics, with some evidence of some deeper understanding gained with respect to the concept of volume and increased knowledge of water issues. Students also seemed to show an increased ability to consider real-world factors in problem-solving, in particular an increased awareness of assumptions. However, students did not generally perceive the topic of water usage to be relevant and their attitudes with respect to the relevance of mathematics to their lives did not change significantly. The unit did not seem to increase their enjoyment of solving mathematics problems or increase their interest in mathematics.

While the teachers themselves found that they were more enthusiastic about their teaching and felt good about teaching the mathematics in a meaningful context, Darryl Lee, in a conversation with Susan prior to the CMESG meeting, shared that the parents were not all keen on the approach. Some were concerned that curriculum objectives would not be met, while others were concerned because they felt ill-equipped to help their child with questions at home related to the activities.

As a group we discussed some of the issues around interdisciplinarity raised by this case. In particular, we considered

- Depending on the topic and the context, it may be challenging for teachers to design interdisciplinary activities that will accomplish defined learning objectives, particularly if the curriculum is very prescriptive.
- Affective goals may not be reached if the students do not see the chosen context as relevant or inherently interesting.
- Real-world contexts are not necessarily automatically relevant to a given audience.
- While the dispositions of the students with respect to the relevance of mathematics or of water usage measured immediately after the unit did not generally seem to have improved, we wondered if that might not change if students were surveyed again when older with respect to this experience.
- Students may have thought of these as word problems like they find in textbooks, rather than a rich learning context that has words for descriptive and informative context purposes.
- There appears to be a disconnect between fun mathematics and school mathematics.
- Third-world problems may have been perceived as depressing, therefore kids pushed those thoughts away.
- Students may be less excited about mathematics and science, even though those ways of knowing are integral to understanding social problems or issues.
- Teachers attempt to 'mix things up' with fun and exciting little things interspersed into their regular teaching. But if students have the notion they do not like mathematics but like the contextual experience, should teachers work to keep the mathematics hidden?

## INTRODUCTION TO ESCAPE ROOMS AS A VEHICLE TO EXPLORE AND EXPERIENCE INTERDISCIPLINARITY

To move the working group participants towards building their own interdisciplinary tasks, we presented some examples of escape room games. These scenarios, whether in an app online or paper-based, provide motivating stories that require players, either individually or in teams, to solve a variety of problems or tasks to complete an ultimate mission, often to "escape" from a difficult situation. While the realism of the contexts provides a natural platform for calling on application of knowledge from multiple disciplines, the scenarios are often set in fictional worlds that allow licence for creativity and adaptability to suit the intended audience.

We viewed a series of apps called "The Room", from *Fireproof Games*, recipient of numerous game-of-the-year awards, which present a 3-D tactile world reminiscent of the buildings and contraptions imagined and created around the 1800s through which a story develops and unfolds as one solves physical, mechanical, scientific, and mathematical puzzles. Another app was "Can you Escape?", by *MobiGrow*, an electronic escape room style of game. A paper escape room, by *Escape-Team.com*, was copied and made available for working group participants; this was the most popular version of an escape room with the working group participants. It included cryptographic codes, mazes, word and number puzzles, and multiple opportunities to cut artifacts out of paper to work with while solving the problems.

Other options were available such as

https://nowescape.com/blog/101-best-puzzle-ideas-for-escape-rooms/

https://www.pinterest.ca/peacock39/escape-room-activities/
https://www.60out.com/blog/solve-common-escape-room-puzzles
http://datenightwingman.com/diy-home-escape-room
https://www.circusbus.com/mobile-escape-room-toronto.html

Exploring these escape rooms offered an opportunity for the working group to consider how this type of activity applied the concepts of interdisciplinarity, and to experience the relationship of mathematics with other subject areas involved in the various interdisciplinary contexts. The working group members were then invited to form subgroups, with the task of developing their own interdisciplinary activity using an escape room metaphor over the next two days. These subgroups were based on the age-group that participants were interested in focussing on.

The memberships of the small groups remained static throughout the three WG days although a couple of WG participants were not able to stay for the full three days because of work demands.

The small groups were

- Beatrice, Liam and Melinda
- Edward, Taras, Wes, Andrew, and Lauren
- Phillip, Limin, Sarah, and Carrie-Ellen
- Eva, Ralph, France, and Osnat

To help focus the activity and to help the groups get started, the following considerations were posed:

- What is your context/story (Mars Mission, Zombie apocalypse, climate change, etc.)?
- Which disciplines would you like to incorporate?
- Is there specific mathematics content you would like to hit, or is there some obvious mathematics that arises from your context?
- With these considerations in mind:
    - What is your ultimate 'big problem'?
    - Develop some smaller problems that will contribute or lead up to solving the 'big problem'.

## THOUGHTS FOR THE DAY

At the end of the first working group day, participants were asked to critically reflect on their own thinking about interdisciplinarity and ask a question that they had with respect to interdisciplinarity at this time. The following are the exit cards for the twelve participants; we leave it for the reader to interpret.

> Experiences and understandings of interdisciplinarity differ—contention around definitions and intention of interdisciplinarity.
>
> How do we create interdisciplinarity beyond sticking together problems from different disciplines? What are wholistic approaches? How do we incorporate interdisciplinarity in the existing framework?

Interdisciplinarity is the natural organization of knowledge; disciplines are arbitrary constructs.

Which is the most important to guide the learning experience? Is it guiding questions or matching curriculum goals?

Process vs Product

Piaget is mostly misunderstood with the use of 'level of abstract reasoning' as an excuse to group students into 'ability groups' in mathematics.

How can technology (digital & non-digital) provide support to implementing interdisciplinary teaching and learning?

Interdisciplinarity has several pieces.

Successful practices in middle/junior high schools?

As it pertains to math ed, is interdisciplinarity better at teaching math or helping us (and students) use math?

The subjects are much more connected than we sometimes realize.

How can we convey this to our students?

I still have in mind this idea to see ID from two perspectives—see maths in 'other' disciplines or see other 'disciplines' in math.

What is a discipline? (Is it with content of the way we think or…?)

There are lots of thoughts.

What is the best way to bring interdisciplinarity into the middle school classroom?

Interdisciplinarity depends on the construction of disciplines. Some cultures may not split knowledge into disciplines (e.g., Indigenous cultures)

What does interdisciplinarity mean then?

Kathleen M. Thompson wrote a lovely article about preserving the integrity of each discipline when teaching in an integrated approach. "Maintaining integrity in an interdisciplinary setting."

> I think it is important for students to choose interdisciplinary topics meaningful to them…to give them freedom to explore what they're interested in.
>
> What are our goals when introducing course components with interdisciplinarity, and how do we help students see value in it? (i.e., the same theme as how students didn't see 'math' in the water activity)

## DAY 2

To ground the work of creating their own interdisciplinary activities, we chose to begin Day 2 with discussion of some theoretical underpinnings of interdisciplinarity, as well as considering the influence of curriculum on the implementation of interdisciplinarity in the classroom.

### THEORETICAL FOUNDATIONS (CURRENT CONCEPTUALIZED PRINCIPLES TO I-D)

While thinking about interdisciplinarity, we found it helpful to identify a few principles of learning we felt acted as a foundation to our conceptualization of interdisciplinarity: problem-solving; problem-based learning; inclusivity, diversity, and differentiation; and self-efficacy. These four principles created a network of ideas for us: each principle acting as a network node, and the connections between nodes became the focus of our inquiry for the working group. These four principles also represent a more humanist and interpersonal way of looking at teaching and learning through interdisciplinarity; it is not necessarily the focus on the subject areas and content that could drive learning, but the personal connection and relationship one has to another and to the subject matter and task.

Interdisciplinarity has a rich and long history in research efforts and the classroom experiences of teaching and learning mathematics. However, we decided to forego the usual introduction of definition and description at this moment by beginning with a personal anecdote of how we can think of this kind of learning and perhaps how we can increase the value and change the nature of the benefit we get from interdisciplinarity. Jamie related a story of a conversation he had with Grey Thunderbird (Tim Yearington) who is the Algonquin-Métis Knowledge Keeper in the Office of Indigenous Initiatives at Queen's University. Tim talked about a ceiling of knowledge which equates to complacency with one's knowledge. People may not be aware of their limitations of sight, and to metaphorically break the ceiling, to help them see new, it can be helpful to perform a small shift in something normal and usual. For example, if one always places a coffee cup in a particular place on the desk, move it to the other side of the desk, and consider it there, think, reflect; what has just happened and what is the impact? This kind of action "helps one to jumpstart one's re-integration" (T. Yearington, personal communication, January 22, 2019) of what they are thinking inside [their mind] with what is happening outside [their body]. Interdisciplinarity can be helped by 'moving the mug', giving a new perspective to consider.

Interdisciplinarity as a teaching strategy requires problems to solve, likely, relatively large, authentic problems that are relevant to the learner. Interdisciplinarity will also require creative thought and critical thinking to figure out what can be done to solve what looks like non-routine problems (Mayer & Wittrock, 2006), especially when data is not complete or readily available (Hollingworth & McLoughlin, 2005). Iterative, cyclical, and non-linear processes of problem-solving may require different habits of mind (Cuoco, Goldenberg, & Mark, 2010) to help one know what to do, when to do it, and why, as a lived experience of "knowing-to" (Mason & Spence, 1999).

As a way of acting and thinking when dealing with complex and world-like, relevant contexts, Problem-Based Learning (PBL) offers a holistic and structured framework to help with learning. Five goals that underlie Problem-Based Learning (PBL) include

1. Construct an extensive and flexible knowledge base;
2. Develop effective problem-solving skills;
3. Develop self-directed, life-long learning skills;
4. Become effective collaborators; and
5. Become intrinsically motivated to learn. (Hmelo-Silver, 2004, p. 240)

The underlining of words is ours, to help focus attention on the attributes of PBL that align with the characteristics many curricula identify in their front matter, describing the expected learners' outcomes and behaviours. The impacts of PBL as a structure and holistic process of problem-solving in authentic contexts also suggest those same attributes for learners: greater self-regulation and problem-solving skills (Seyhan, 2016); improved perceptions of problem-solving ability (Temel, 2014); increased creative-thinking ability, self-regulation learning skills, and proficiency in self-regulation (Yoon, Woo, Treagust, & Chandrasegaran, 2014); and an increase in academic achievement (Günter & Alpat, 2017). Again, the underlining is ours, to bring attention to the way the language used can be directly compared to curricula expectations.

Throughout all of this, in an interdisciplinary context, with complex problems requiring critical and creative thinking, and processes of problem-solving actions, the individual as a person remains the centre of the learning. In particular, four sources of self-efficacy (mastery, verbal persuasion, vicarious experience, and physiological/affective response) are valuable aspects to one's learning (Bandura, 1997). When one has mastery experiences, receives verbal persuasion in the form of encouragement and the next steps to be taken, can see others' complete work and achieve success with problems, and feel, possibly anxious as the task starts, but feel good about oneself at the conclusion of the task, one's belief in oneself as a person who can solve problems and complete complex contextual tasks increases; one's self-efficacy increases. Increased self-efficacy, supported by these four sources, has been shown to be correlated with increased achievement, risk-taking, resilience and persistence. "Perceived efficacy explains differences in school achievement after controlling for student characteristics, enrollment stability, teachers' experience, and prior school achievement" (Pajares & Urdan, 2006, p. 12).

These four principles were offered to the working group as four possible pillars to a conceptualization of Interdisciplinarity. Conversation and insight were interjected by participants as each principle was presented.

Interdisciplinarity itself as a pedagogical strategy has a long and successful history. A book by Humphreys, Post, and Ellis (1981) about Interdisciplinarity contains a number of philosophical and practical perspectives articulated in an effort to show how well positioned interdisciplinarity is as a learning strategy. To provide some insight into the way interdisciplinarity was understood and thought about historically, a number of ideas and quotes were presented by people such as

> The goal of education is not to increase the amount of knowledge but to create possibilities for a child to invent and discover. Teaching means creating situations where structure can be discovered.—Jean Piaget (p. 8)

> …education is to convey the ideas of value that allow people to discover meaning and purpose in their lives… If education does not serve to make us wise, to provide ideas that will enable us to choose between one thing and another, living will become a meaningless tragedy.—E. F. Schumacher (p. 4)
>
> Children's concepts basically evolve from direct interaction with the environment. Children need a large variety of enactive experiences.—Jerome Bruner (p. 16)
>
> …the coevolution of language and thought….social constructivism…zone of proximal development.—Lev Vygotsky (p. 31)

## A CURRICULUM FOCUS

While the above theoretical underpinnings provide ample motivation to implement interdisciplinary approaches in school, the traditional school curriculum is often perceived as a constraint, or even a barrier, to attempts to do so. To illustrate that this need not always be the case, we shared the example of the new British Columbia K–12 curriculum.

This 'new' curriculum, which has been in place at the K–9 level since 2016 and at the 10–12 level since 2018, takes a more holistic approach. While it still breaks teaching areas into traditional subjects (English Language Arts, Science, Mathematics, etc.), it is built around core competencies that transcend disciplines: Communication, Thinking, and Personal and Social (https://curriculum.gov.bc.ca/competencies). The curriculum for each subject within each grade level is very compact, consisting of a single page, organized around what students need to Know, what they need to be able to Do, and what they need to Understand. The "Know" comprises of content topics, the "Understand" outlines the big ideas, and the "Do" describes the specific skills linked back to the core competencies (https://curriculum.gov.bc.ca/curriculum/overview).

As an example, we took a look at the Grade 6 Mathematics curriculum (https://curriculum.gov.bc.ca/curriculum/mathematics/6). From this it was easy to see how Darryl and Alice had felt able to attempt to use an interdisciplinary approach in their grade 6 mathematics class. They were easily able to address at least two of the five "Big Ideas": "Mixed numbers and decimal numbers represent quantities that can be decomposed into parts and wholes" and "Properties of objects and shapes can be described, measured, and compared using volume, area, perimeter, and angles." With respect to content, they were able to address multiplication and division of decimals, volume and capacity, and line graphs. More significantly, their interdisciplinary approach gave a more natural context for supporting development of several of the curricular competencies, specifically

- using reasoning and logic to explore, analyze, and apply mathematical ideas;
- estimating reasonably;
- applying mathematical understanding through inquiry, and problem-solving;
- reflecting on mathematical thinking;
- using mathematical arguments to support personal choices.

These types of curricular competencies have always been part of the BC Mathematics curriculum but have been in the background. Now they have been brought to the forefront, and teachers have both the challenge and the opportunity to provide their students with experiences that allow them to develop and practice these skills. The 'one-page' curriculum for each subject within a grade-level also supports overlaying mathematics with other subjects to look for potential to integrate learning objectives and ideally create a richer learning experience.

This change in the format of the presentation of the BC curriculum is not the only change in British Columbia that supports a more interdisciplinary approach to teaching mathematics. The province has also introduced a Graduation Numeracy Assessment, which all students will take in grade 10 (https://curriculum.gov.bc.ca/assessment/grade-10-numeracy-assessment). It is very deliberately not tied to the specific curriculum of any particular grade level, rather it "focuses on the application of mathematical concepts learned across multiple subjects from kindergarten to Grade 10" (para. 2). Mathematics, more specifically, numeracy, is not intended to be taught only in mathematics class.

We took a few moments to explore the sample test online:
https://www.awinfosys.com/eassessment/eexams_sample.htm. The questions all represented using mathematics in context (e.g., water usage, pit houses once used by First Peoples, a partnership to form a video game company), requiring interpretations of graphs and varieties of representations of information and simple calculations based on data provided to make comparisons and decisions.

Consideration of these changes in BC provided the group with an opportunity to think about how curriculum and assessment can encourage or inhibit interdisciplinary approaches. For their escape room design, working group participants were encouraged to consider the curricular objectives that their activities would address. The BC curriculum pages were offered as a resource to them if they thought that choosing a particular grade-level might help them focus their efforts.

### TIME TO WORK ON ESCAPE ROOMS

After a short break, the subgroups were given time to work on developing their escape room activities. It was a time for discussion and debate as some groups struggled with the notion of an 'escape' room and with the many decisions that had to be made by the group: What context? Should we start with the mathematics or with the context? Should the activities be authentic or fantastical? Creating an interdisciplinary experience proved to be challenging, but the room was abuzz with conversation. One group immediately took advantage of the many manipulatives and resources available in the classroom, using these as inspiration for their project. Some of the struggles are identified in the Exit Cards completed at the end of the session.

Figure 2. Small groups in the WG working on Escape Room tasks.

EXIT CARDS

Three prompts were presented for Exit Cards at the end of day 2:

a. One question I have…
b. One thought that has occurred to me…
c. One anxiety I have…

The responses are collated below.

Questions:

- *How to move from contextualization to interdisciplinarity?*
- *How do we evaluate the suitability and/or success of interdisciplinary activities?*
- *How do I build activities that respect all disciplines and not only motivate mathematics but build mathematical knowledge?*
- *How to support teachers to do this work?*
- *How can we improve support our struggling learners? How is self-efficacy impacting academics?*
- *How can we design inter-disciplinary activities on our own? (i.e., How do we know what we don't know about the lenses of other disciplines?)*
- *What was meant by affective/physiological experience in terms of sources of self-efficacy?*
- *How can we ask interdisciplinary questions?*
- *When does integration not work? When do we need to stick to direct teaching? (balance scale of integration vs direct)*
- *How often is this done in real life?*
- *Why such a discrepancy in abstraction and complexity between the big ideas in science and those in math for the same grade level?*

Thoughts:

- *Collaboration between disciplines is necessary.*
- *'Story' is a fundamental concept—we need a linear thread to make it comprehensive, entertaining, and interesting.*
- *There are many different ways to enact an interdisciplinary approach.*
- *We need to consider how to involve specialized learning needs.*
- *Since teacher training and self-efficacy is so key to the success of an interdisciplinary and inquiry approach, this should be a mandatory aspect of teacher training and P.D. for existing teachers.*
- *I think it is important to not just do interdisciplinary tasks for the sake of doing them. The motivation should be made explicit to students to help them see value in the task.*
- *Focused curriculum (strict) vs open curriculum*
- *The goal is extremely important. It seems that once the goal of the interdisciplinary project/lesson is determined then it is easier to plan.*
- *It is hard to truly connect the subjects and not just use them to make math interesting.*
- *Experimenting with loose protocol makes us consider control of variables, calibration, new hypotheses, and refine our question.*

<u>Anxieties:</u>

- *Is this too 'ambitious' thinking about the system which is in place now…Bah, il faut rêver pour que les choses evolvent!*
- *Tension between 'closed' and 'open' problems. Latter is more like 'real world problems'—applicable and interesting, but former are more manageable and potentially satisfying.*
- *Are these experiences 'Meaningful' (whatever this means) to students?*
- *Time factor to implement properly.*
- *The artificial constraints of our disciplines and scheduling in the school system will not allow time for authentic inquiry and interdisciplinary thinking to be authentic and responsive. A new model of school is required—multi-age, multi-discipline, connected and issues-based with PBL.*
- *That teachers don't know enough about other disciplines, and collaboration can be difficult, given curriculum, restraints. Also, that 'fun' activities like this escape room don't teach effectively.*
- *Collaboration is a challenge—through practice and planning.*
- *The amount of time and/or expertise needed to do a good job of this.*
- *Trying to do this in a real classroom.*
- *That school is no longer a place for learning, for becoming. It is primarily a control valve for the flow of privilege.*
- *Is there loss of identity of math epistemology (we need to learn both to appreciate math internal consistency <u>and</u> be able to model*
- *external' (real-world) situations).*

## DAY 3

The *Wordle* (using www.wordle.net) (Figure 3), shown below was presented to the WG at the start of Day 3. It is a creation using all the words from the participants' responses to the three exit-card prompts from Day 2.

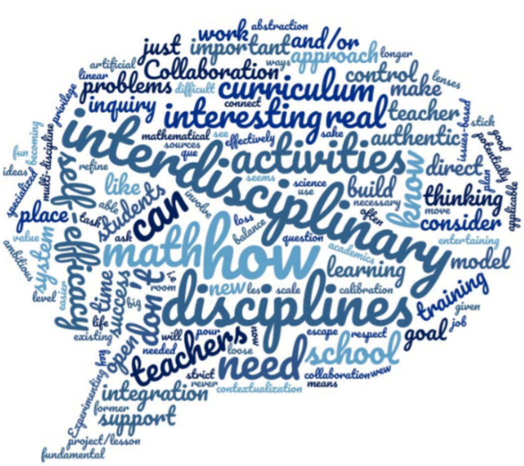


Figure 3. *Wordle* of participants' responses to the three prompts at the end of Day 2.

WG participants knew of their own responses to the prompts from Day 2 but did not have access to the responses from the other participants. They had an opportunity at the start of Day 3 to consider the *Wordle* as a compilation of all their responses. Participants noticed that from their responses to the three prompts from the previous day, there were a lot of 'how' statements and not a lot of 'why'. The words 'math' and 'interdisciplinary' were the largest words in the *Wordle*, not surprising given that *interdisciplinarity* was the theme of the WG, and the participants were predominantly mathematicians and mathematics educators. Other words noticed were 'interesting', 'real', and 'authentic'. This presented an interesting backdrop to the WG's reflections as each small group presented their thinking and ideas that emerged from the Escape room inspiration.

### TIME TO FINALISE SMALL GROUP WORK

The small groups then had about thirty minutes to complete some work on their tasks, close their discussions, and finalise potential products.

We observed that over the three days of the WG in which the participants worked on their 'Escape Rooms', the small groups appeared to keep much of their work in their heads; they did not take advantage of the internet or try to map rooms on paper or storyboarding. Some groups spent much of the allotted time wrestling with both the restrictions imposed by the escape room theme and with the vast freedom of being able to choose *any* context, *any* discipline, *any* problems. In order to bring in the mathematics meaningfully into the context of another discipline, the designers needed to settle on, and know something about, the other discipline(s). The small groups did not appear to overtly use the four points to consider while creating an escape room that was displayed each day (see Day 1) to direct their work, thinking, and action. This last day, with our time running out, was the most visibly productive, as the small groups made final decisions as best they could and made notes on chart paper in preparation for their presentations.

### PRESENTATIONS

Each of the groups led the WG through an introduction and explanation of their escape room ideas. For some small groups, they had manipulatives and demonstration, for other small groups, their explanations were supported with details written on chart paper. The following are summaries and images (where appropriate) of each small group's thinking at this time.

#### <u>Small group: France, Eva, Ralph, and Osnat</u>

This group presented a set of activities that centred around the use of dominoes. Questions included If a seesaw is created using a metre stick, and a domino balanced at one end is projected into the air by pushing down at the other end, how is the height of the projectile affected by the position of the fulcrum? Arrange the dominoes on edge in various patterns, knock one over and observe the consequences. What aspects of the arrangement affect the outcome? When a chain of dominoes are aligned on edge and the first is knocked down to topple the rest, how is the speed of the reaction affected by the spacing of the dominoes, or by the slope of the surface they are arranged on?

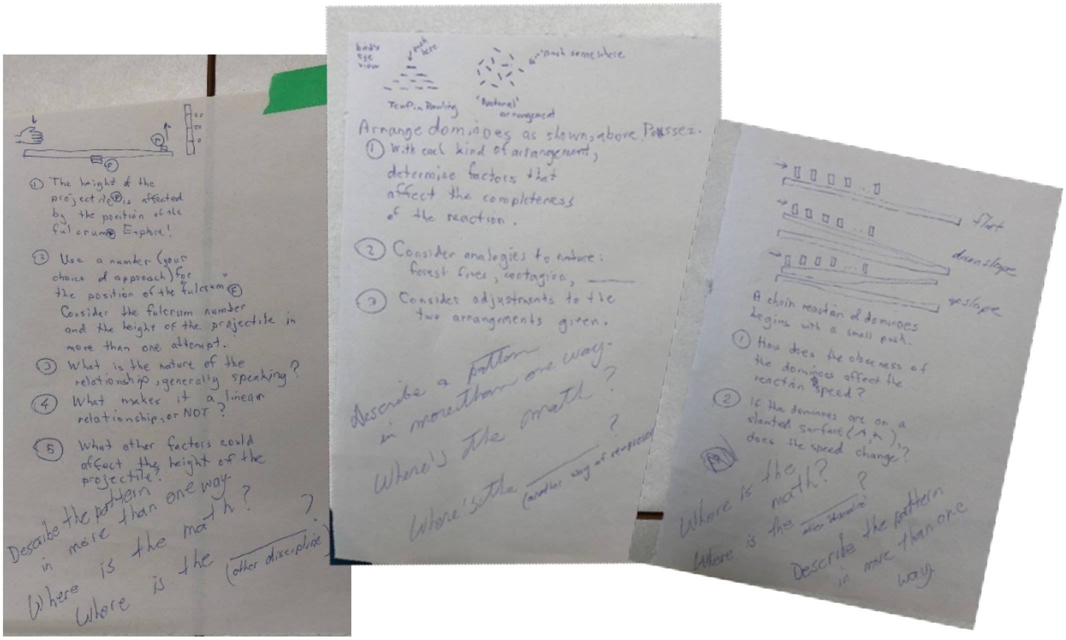


Figure 4. Small Group dominoes activities.

> *It all started with dominoes. Playing with dominoes focused our work into intense discussions.*
>
> *We were never off task. Playing with a concrete manipulative itself stopped us from drifting off and losing track of the purpose of the learning. We considered this 'real-world' because the dominoes ARE real objects.*
>
> *As teachers we were thinking about questions such as "how does the activity bring up questions to ask about it?" And we thought about the nature of open and closed tasks; how is playing like this open, in what way is it open, and to what degree is it open?*
>
> *Interdisciplinary tasks could be great ways to introduce statistics, for example, considering the issue of "multiple tries, or forestry and tree placement—is random tree placement beneficial and for what benefit? Perhaps strategic tree placement can reduce forest fires?"*

While the group appreciated the context of an escape room to give some kind of space to think about interdisciplinarity, they said, "*We escaped the escape room, pretty much right off the bat*." The group decided to focus on the mathematics and problems that would be interdisciplinary rather than the overarching theme of an escape room. They started with the domino problems, but then asked, Where is the mathematics in these activities? What other disciplines or areas of knowledge were involved? Connections were seen with physics (friction and force), theory of forest fires, and even art.

<u>Small Group: Beatrice, Liam, and Melinda</u>

In this escape room activity, aimed at the Grade 7/8 level, students are given the mission to hatch a dinosaur egg and care for a baby dinosaur in modern times. This would require both mathematics and science, including information about temperature, climate, health, and nutrition.

*We think we ultimately were getting away from the escape room situation! A kind of pivot from one focus to another. We were thinking about the "Math first, and let that thinking lead to interdisciplinary considerations next.*

*The Grade 8 curriculum of science had lots of possibilities to fit our tasks.*

*It was easier to connect to math competencies than to the context!!*

*The art curriculum offered big ideas of perspectives.*

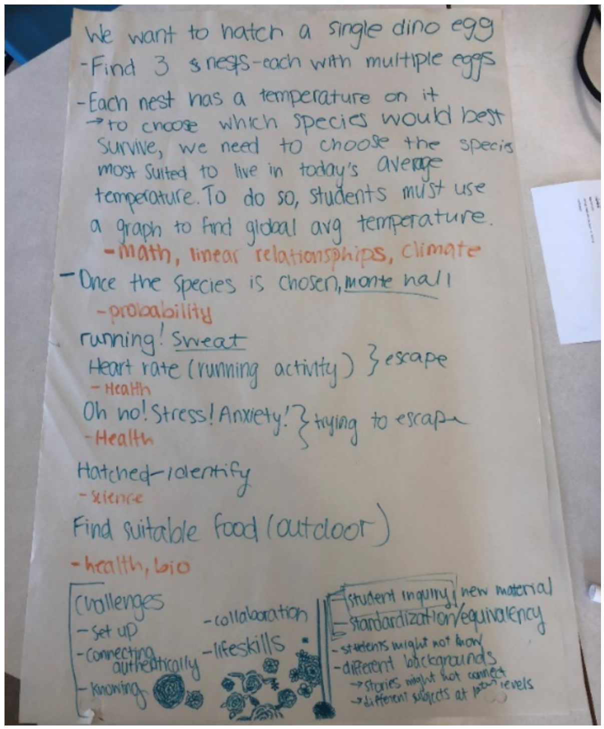


Figure 5. Small group dinosaur task.

<u>Small Group: Phillip, Limin, Sarah, and Carrie-Ellen</u>

This group chose middle school students as their target audience and chose obesity as their main theme. They considered this as an opportunity to intertwine mathematics objectives with objectives in healthy living, food technology, sciences and life skills curricula. They struggled with whether to use open questions that are accessible to all learners, allowing students to work at their own level, or to use more closed questions to ensure more consistent outcomes. They considered providing teams of students with specific characters, having particular ages and health conditions/behaviours, and having them consider possible outcomes for those characters. They appreciated that within this context there were many opportunities to introduce increased complexity as desired.

*How does the teacher prepare for open exploration of topics? We may be experts in math, but how deep can we go as non-science "specialists"?*

> *To learn something about other disciplines, for it not to be fake, we need to collaborate to know the goals of the other discipline.*
>
> *Collaboration takes time!*

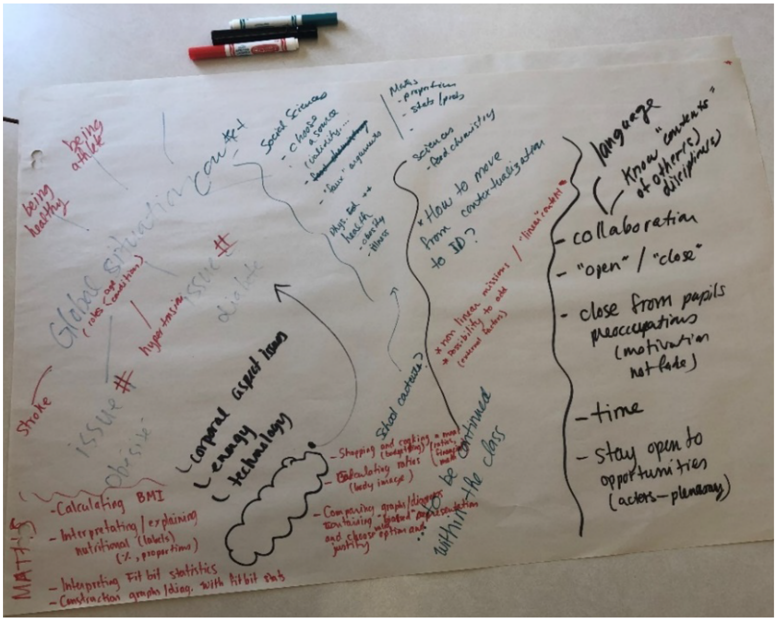


Figure 6. Small group notes on 'Obesity' theme.

<u>Small Group: Edward, Taras, Wes, Andrew, and Lauren</u>

This group focussed on post-secondary students and chose the theme of a time machine for their activity. The specific tasks included determining the day of the week of Shakespeare's birthday (modular arithmetic); de-crypting a code using the Bacon cipher; and using this code to travel to a time when negative numbers were not being used in Europe, but were in use in India, and solving a problem both with and without the concept of negatives. They envisioned being able to extend this to branch to other calendars and cultures, connecting to history and astronomy.

They did not conceive of this as a classic escape room but rather as more like a big video game in which students could move from room to room. Similar to the previous group, they struggled with whether to use open or closed questions.

The post-secondary context provides some additional challenges to interdisciplinarity. In particular, the disciplines are typically very segregated with no over-arching curriculum. The group wrestled with many philosophical issues, including:

> *We were thinking of 'un-disciplinary'. Interdisciplinarity is a moving away from disciplines and maybe 'un-disciplinary' is helping to move farther…*
>
> *To move away from some socio-cultural and political and lived experiences such as the English discipline enforced in Residential schools.*

> *Word beginnings such as 'trans-' and 'multi-' instead of 'inter-' disciplinary allowed us to think of creating "a place to open privilege."*

## CLOSING DISCUSSION

Following the presentations, we scrambled groups and spent some time reflecting on the activities and discussions of the last few days, focussing our attention on what we had learned about both the benefits and the challenges of taking an interdisciplinary approach. Benefits included the potential for creativity, collaboration between teachers and other experts, motivation of students, breaking down of traditional barriers, and providing students with relevant, applicable learning experiences. Challenges included traditional curriculum structures, potential lack of teacher expertise in the 'other' discipline, creation of authentic experiences, expectations of parents and students, and the time demands.

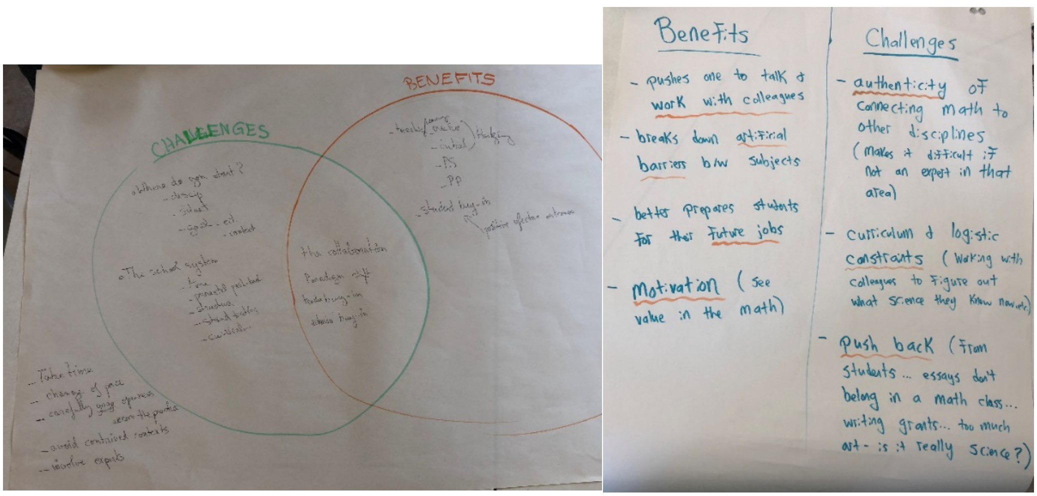


Figure 7. Benefits and challenges examples from small group discussions.

Some additional thoughts that emerged:

> *Does math disappear as a discipline if it is appreciated in each of the other disciplines, for example in physics, in chemistry, in accounting…? Is this good or bad? Or is this a risk? Is it the dismantling of disciplines and the reduction of mathematics to just those calculations that help complete a task or find a solution to a problem?*
>
> *Interdisciplinary learning is complicated, worthwhile if done carefully and thoughtfully, that is, not forced. Interdisciplinarity has potential as a research possibility: to learn about math and about context.*
>
> *There might be a change to how learning goals are designed and written because of the recognized work involved in interdisciplinarity.*
>
> *Being able to play with objects because of interdisciplinary problems may be beneficial to understanding interdisciplinarity.*
>
> *Interdisciplinarity also requires a re-questioning of the word discipline. Disciplines are ways of knowing and within knowing there is discipline. Disciplines as constraint provides a lens in which to work.*

## CONCLUSION

The theme of our Working Group (WG) group was interdisciplinarity. The following are some of the insights of the members of our WG and what is presented are some variations on a theme.

*Nous avons passé du temps à partager des expériences, à examiner des théories, puis à travailler en petits groupes dans le but d'essayer de créer une activité interdisciplinaire pour les étudiants.*

There are many ways to approach interdisciplinarity—this emerged as the groups worked. Some approached starting with a problem, a story, the mathematics, the curriculum, playing with dominoes. In doing so, they all experienced and explored the subverting of learning, the diverting of an initial purpose of the learning, and the complex layering of interdisciplinary learning.

*Nous avons eu quelques difficultés avec le mot « discipline ». Il envoque des barrières qui ont été contestées à la fois comme arbitraires, artificielles et contraignantes, même oppressives.* That led to the re-questioning of the word 'discipline'. *En même temps, il a été observé que « les disciplines » sont des manières de savoir et que, dans les moyens de savoir, il y a de la discipline. Nous pouvons voir les disciplines comme une lentille, mais aussi comme une forme de contrainte, mais elles peuvent être nécessaires.* We can see disciplines as a lens, but also a form of constraint, but these can be necessary.

Interdisciplinarity can be challenging in the face of school structures, and the collaboration and commitment needed, however if done carefully and thoughtfully, not just for its own sake, it can be worthwhile. It can motivate, inspire, and show connections that are often overlooked.

In closing, our relationship to the world around us has an important influence on how Interdisciplinarity is integrated and then implemented. Opportunities for interdisciplinarity naturally bring us into learning spaces where respect for one another, each other's knowledge and abilities, and a responsibility to each other, the disciplines, and curricula is authentically and honestly present. Problems of practice and problems of life and learning are natural contexts and situations in which interdisciplinarity can flourish. However, traditional and historical approaches may provide barriers and challenges, existing as a habit of what teaching and learning is, rather than what it could be. An interdisciplinarity approach requires becoming aware of habits of practice, knowing, and learning. Interdisciplinarity could be considered a distinct break from the traditional, Western learning approach. "To modify a habit, you must decide to change it" (Duhigg, 2012, p. 270), "and once you know a habit exists, you have the responsibility to change it" (p. 271). A cyclical and/or non-linear process of experimentation and failure, with small moments of success, is to be expected and necessary for long-term habit change.

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# CAPTURING CHAOS? WAYS INTO THE MATHEMATICS CLASSROOM

# CAPTURER LE CHAOS ? ENTRÉES SUR LA CLASSE DE MATHÉMATIQUES

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## INTRODUCTION

There is a general consensus that the classroom is a multifaceted and often chaotic environment, which is experienced differently by each individual. This complexity makes authentically 'capturing' teachers' and students' experiences a difficult undertaking. It is impossible to attend to everything, and as such any research method is to some extent limited (e.g., in the scope of what it can measure). Moreover, observations are never neutral: tools and stance always create their own 'realities', bringing forth phenomena and issues in their own language.

La classe de mathématique est un environnement multiforme et souvent chaotique. Cette complexité rend difficile le projet de « capturer » (authentiquement) les expériences des enseignants et des étudiants. De même, il est impossible de tout contrôler et, en tant que telle, toute méthode de recherche est limitée dans une certaine mesure (par exemple concernant la portée de ce qu'elle peut mesurer).

Si l'enseignement des mathématiques est considéré comme une « discipline nécessairement, et pour le mieux, éclectique » (Rowland, 1999, p. 5), il est raisonnable de penser que le domaine de l'enseignement des mathématiques s'enrichit de l'utilisation de méthodes de recherche variées et multiples. La théorie du chaos elle-même a parfois été sollicitée pour conceptualiser des méthodologies de recherche conçues pour aborder le chaos en classe.

If mathematics education is viewed as a "necessarily and beneficially an eclectic discipline" (Rowland, 1999, p. 5), then it is reasonable to infer that the field of mathematics education is enriched through the use of varied and multiple research methods. Chaos theory itself was occasionally solicited to conceptualize educational research methodologies designed to embrace classroom chaos.

At the outset of this meeting, in light of the preceding comments, the goal was to cultivate an understanding of what takes place in the mathematics classroom. We were guided by the following questions:

- How do teachers and students experience chaos within the math classroom, and how should these experiences be of concern for mathematics education research?
- In the face of chaos, how do we deal with ideas such as subjective versus objective measures of classroom activity, the validity, credibility, affordances and limitations of teaching and research methods?
- Can we 'capture' or make use of the complexity of what is happening in the math classroom? How can we work with/around the technical and theoretical constraints?
- How does it affect our way of thinking about doing mathematics in a teaching/learning context? How does it affect how we think about research?

Comment le chaos est-il vécu en classe de mathématiques ? Quelles sont les implications de nos limites et possibilités pour comprendre et communiquer le chaos de la classe ? Quels liens faire avec la « vie mathématique » des élèves et des enseignants ?

Nous présentons ici quelques traces de ces trois journées fructueuses de questionnement, de jeu, de réflexion et de discussion de ce thème.

What actually transpired was three fruitful days of questioning, playing, thinking and discourse, in pursuit of this significant endeavour. What follows is an account of the activities and events that occurred and the questions that were raised over the three days this group met.

## DAY 1: EXPERIENCING CHAOS / L'EXPÉRIENCE DU CHAOS.

Nous avons demandé aux participants ce que le chaos en classe signifie pour eux. On parle de l'inattendu et du chaos en nous même, souvent présent derrière l'ordre apparent du monde, de la classe, des élèves…

Our first day involved getting to know each other and our topic. As part of our early introductions and greetings, we asked WG participants to share what chaos in the classroom means to them. A few individual contributions, and some shared questions, provide a small window into our experience. One participant shared that, for her, chaos occurs when "*things are happening that are unexpected or unanticipated…in a sense, the chaos is in me*." Another shared a photo (Figure 1) of a classroom with students sitting in neat rows, their bags sitting neatly in the aisle next to each chair. For her, the chaos in the photo lived in the students and the psychological baggage each child brings to the classroom.

Figure 1. Chaos in the classroom: image shared by one of the participants.

A few questions emerged as part of the sharing and ensuing discussion. These included

- Is the input 'error' (that instigates the departure from the intended conception) based in students' interpretation of the problem?
- If we look at student misconceptions through the lens of chaos theory, is it then possible to start with the conception the student has conveyed and 'rewind' to determine the initial input (i.e., the initial misconception)?

Then we introduced a task that allowed participants to experience 'messiness' and create some data for later analysis. We intentionally used the term 'messiness' as it was recognized that participants come with their own interpretations of chaos and complexity. Each group of four or five was asked to document their work and their thinking through multiple media.

### ACTIVITY 1: THE COINS PROBLEM

**The Coins Problem**: A collection of coins is spread out on a table, and you are told the number of heads that are face-up. Is it possible (without looking at the coins) for you to divide the collection into two groups, with each group having the same number of heads visible face-up? Write up your solution so you can share it.

Figure 2. The 'coins'.

Nous avons ensuite expérimenté une tâche propice au « désordre » : Une collection de pièces est étalée sur une table, et on vous indique le nombre de « faces ». Sans regarder les pièces, pouvez-vous divisez la collection en deux groupes, chacun ayant le même nombre de faces? Les équipes se mettent rapidement mis au travail, empruntant différents voies. Nous en profitons pour documenter un peu ce travail (par photos et vidéos) afin d'y revenir le lendemain.

**Instructions:** In groups of 3 or 4. Work on the problem, and collect data (to be analysed next session). Data needs to be collected at 3 short moments (3-5 minutes) during the activity:

a) in the first few minutes, at the initial stage of the solving;
b) somewhere in the middle, e.g. when some insights seems to emerge;
c) by the end, when you wrap-up, and prepare to share your finding.

Participants take turns to collect these data in one of the following ways, while others work on the problem:

- **Video**: Simply film the team, trying to get a good recording of how the group works, and the current stage of the work (e.g. what have been found, current hypothesis, etc.)
- **Notes**: Step back for a few minutes, observe what is going on, and try to make a short list or a description of what is taking place in terms of: (a) Understanding of the situation ; (b) Use of concepts ; (c) Forms of reasoning.

Figure 3. The coins problem and one group working on it.

The groups quickly selected roles and began working at the problem (Figure 3), each with a different approach. As an observer, it was interesting to see the different interpretations of task; some groups saw it as a probabilistic task and others saw it as a "generality". Of course, some groups also played with the counters and engaged in pattern making (Figure 4).

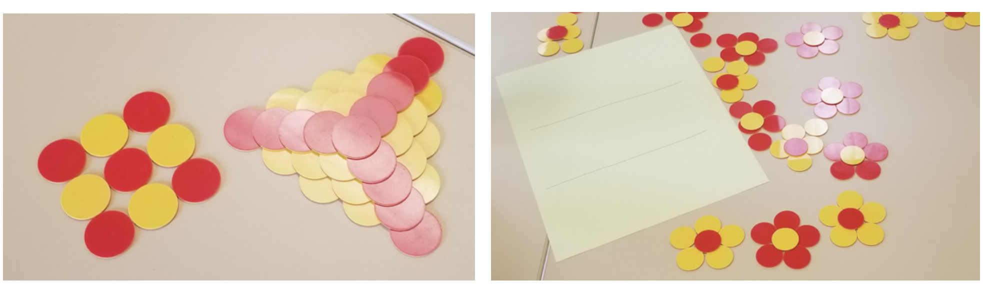

Figure 4. Playing with the counters.

DISCUSSION

After the morning break, participants returned to discuss how they approached the problem, and how they connected with it. Some of the general themes included

- How each group or individual approached the problem and their connection with it. More than one person commented that because they saw the problem as probabilistic, they felt a sense of disconnect and 'pulled back' from the problem due to lack of interest.

- How some participants grappled with the challenges around documenting, and how that role affected their participation in the problem. For example, it was felt by some that the videographer could not fully participate in the solving.

After this debrief, Jean-François provided a brief history and summary of chaos theory to ensure group members had some common ground upon which to base a discussion.

### CHAOS THEORY / LA THÉORIE DU CHAOS.

It is hard to talk about chaos in the classroom without saying a few things about chaos theory. We thus decided to sketch a quick picture, hoping at least to provide us with some background, ideas, words, and so on. Chaos theory is a branch of mathematics that looks at systems in which the smallest changes can have a very large impact. The story goes this way: In 1961, Edward Lorenz was simulating weather patterns with 12 variables (for temperature, wind speed, etc.) when one day he decided to look at an old sequence of data again. To save time, he decided to start the simulation in the middle, feeding the computer with a printout of calculated data. The computer used 6-digit precision, but the printout rounded variables off to 3-digits: 0.506127 was printed as 0.506…but consensus at the time was that this should not have any significant practical effect. The simulation, of course, ended up producing very different results, and Lorenz eventually showed that even detailed atmospheric modeling cannot, in general, make precise long-term weather predictions. Chaos theory was taking shape.

> On peut difficilement se passer de parler un peu de la théorie de chaos, son origine dans les travaux de Lorentz, et ses nombreuses « applications » aujourd'hui, y compris en éducation. Cette rapide ouverture nous conduit à réfléchir à nos théories, par exemple. Qu'ont-elles à dire par rapport au chaos de la classe ? Qu'est-ce qu'une théorie du « chaos dans la classe de mathématiques » devrait faire ? Devrait ou pourrait-elle parler du « genre » de mathématique qu'on y fait ?

Since then, we know that chaotic behaviors are at the heart of many other phenomena. Chaos theory has applications in many fields such as economics (stock market), sociology (groups, populations), computer science (cryptography), psychology (groups behaviors), biology, philosophy…. It connects to notions such as randomness, feedback loops, repetition, self-similarity, self-organization, complexity, and so on. Many found inspiration in chaos theory to think about education (Figure 5). Some invite us to embrace complexity instead of dissecting teaching or learning into isolated analyzable components. Others challenge our views on errors, divergences, bifurcations, diversity, collectivities, repetitions, control, constraints, interactions…and more! Some seem to suggest that even research in mathematics education is rather chaotic considering how, for example, theoretical frameworks used in research often depend not only on the phenomenon of interest, but on researchers' beliefs.

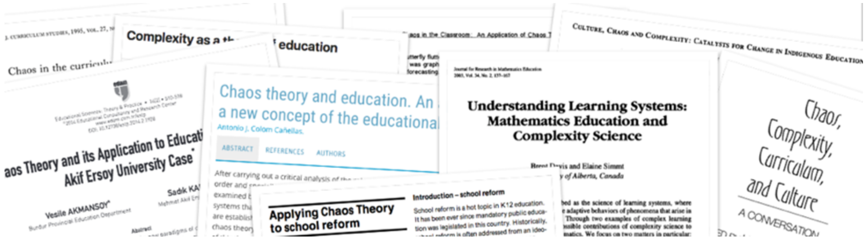


Figure 5. Images of articles on Chaos and Complexity Theory in Education.

Speaking of theory, we took the occasion to ask ourselves how different approaches to understanding teaching and learning (e.g., Constructivism, la théorie de situations didactiques, Sociocultural Approaches) might have to say about chaos. And we asked, what would a 'theory of mathematics classroom chaos' need to do? Should or could it talk about the 'kind' of mathematics that is done?

Concluding that first day, participants were invited to keep thinking about their own experience of chaos in relation with mathematics education or research, and either write a few paragraphs, or make a drawing.

## DAY 2: MAKING SENSE? OF CHAOS / FAIRE SENS ? DU CHAOS.

Day 2 predominantly involved grappling with qualifying the characteristics of complex classrooms. First, we looked holistically at the bigger picture, using variation to help us. Then we changed the scale of the data and tried to extract from the particular, shifting the lens to look more closely at data that was captured in various settings.

The plan for the day was to delve deeper into making sense of chaos. To ease into this, we began by revisiting the tasks from Day 1. We began by asking participants to observe and compare two video clips: one taken the previous day depicting participants working on the coins problem (filmed at normal speed), and one contributed by a participant, showing high school students working on vertical whiteboards (filmed on fast-forward). Almost immediately, questions were raised regarding the working group leaders' intentions in their choice of the two videos.

> Le deuxième jour, il a été question de qualifier une salle de classe dite « complexe » : le sont-elles toutes autant ? De la même manière ? Nous avons revisité les tâches du premier jour…et puis nous nous sommes rapidement « écartés » du chemin prévu, faisant un détour du côté d'une vidéo proposée par Peter. La discussion prit aussi un détour pour remettre en question les intentions poursuivies par le groupe de travail : devrait-on resserrer notre attention sur certains contextes cherchant par exemple à faire place au chaos, et aller plus loin dans nos compréhensions de ce qui s'y passe ?

Questions about the leaders' intended use of the participants' videos followed shortly. Concerns were allayed, as video's intended use was only to show within the working group. The video of the participants that was chosen was selected because it showed all groups working, and the 'fast forward' video was shown because it illustrated a different kind of 'messiness'. The end result of the activity was a fruitful discussion about the immense impact the decisions we, as researchers, make about what data we capture and how we capture it has on our results, not only due to what is 'seen' but what is missed. We began to question what we see, and think we see, in our data. Other salient points included questioning how we think about planning, and a raised awareness of our innate aim to reduce complexity.

> Le visionnement d'enregistrement nous a entrainés du côté des considérations éthiques et méthodologies, nous permettant de réfléchir sur les impacts des décisions que nous prenons à propos des données que nous recueillons et sur la manière dont nous les capturons sur nos résultats. Que dire de tout ce que l'on manque, de tout ce que l'on ignore ? Que dire de notre tendance à réduire la complexité ? Ou encore : Pourquoi le chaos dans la classe de mathématiques est-il « difficile » ?

ANALYZING DATA / L'ANALYSE DE DONNÉES.

Following the aforementioned discussion, we dug deeper into the data. Participants were asked to separate themselves from the members they had worked with on Day 1. Leaders distributed photocopies of data collected from each groups' work on the coins problem, consisting of pictures of notes, written work, as well as journal entries collected previously from Math 10 students given the same task. The effect of splitting up the previous day's groups was that each new table grouping had one member who had knowledge of each data artefact. Participants were tasked with using their own experience and frame of the problem to understand others' work. Although it was requested that they not contribute to analysis of their data item, it was noted that some participants could not help talking about their personal experiences with the tasks. Others resisted talking about their artefact, allowing their new group mates to construct their own interpretation.

> Les participants ont ensuite été invités à se séparer des membres avec lesquels ils avaient travaillé le premier jour pour examiner avec d'autres des traces du travail de chaque groupe (sur le problème des pièces). De nombreux points sont soulevés : ce qui est communiqué ou non et comment, le travail écrit par rapport au travail produit, les interprétations variées du problème, les règles, les hypothèses (parfois auto-imposées), les différentes questions mathématiques qui peuvent émerger (recherche d'une solution générale ou de cas), les rôles des traces dans le travail, et ainsi de suite.

The prescribed task was for each group to produce an "accurate and interesting description of what (mathematically) takes place in the data." Table groups were asked to produce a poster and to be prepared to report out on their area(s) of focus and what they found/determined/noticed in the data.

A rich and varied discussion followed, as each table interacted with their data and their companions. Some groups dissected the artefacts, while others had a more general discussion.

REPRESENTING AND PRESENTING DATA / PRÉSENTER ET REPRÉSENTER DES DONNÉES.

Following the break, each group was asked to present their 'poster' and discuss their focus and findings. The posters took several forms, as one can see in Figure 6.

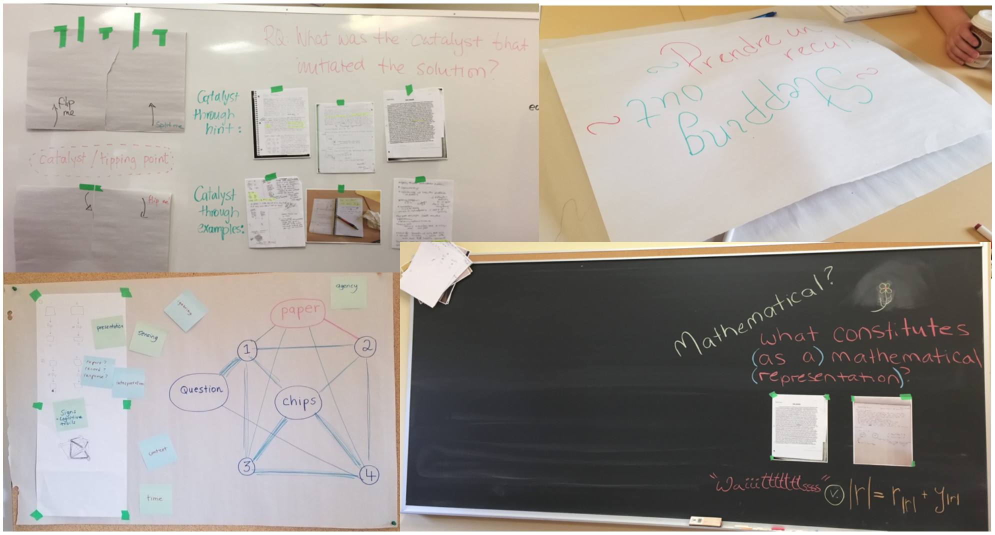


Figure 6. Group representations of discussions about the data from the coin activity.

Un travail de « bricolage » autour de la question de la représentation des données nous a occupés un bon moment, et nourri une conversation très fine et très riche « à travers » les données plutôt que simplement « à propos » des données. Nous avons été fascinés par notre habileté à donner sens au chaos sans pourtant qu'il ne cesse, jamais, de nous poser des défis, de nous échapper.

There was some excellent dialogue 'through' the data provided, as well as discussions 'about' the data. Participants toured the posters in a gallery walk (Figure 7) and afterward came together to share thinking as a larger group. Discussions revolved largely around themes of how groups interpreted and engaged with the problem; the format and nature of the data given; and decisions made by the leaders. In each of these themes can be found important considerations regarding the limitations on the conclusions we can make when analyzing data.

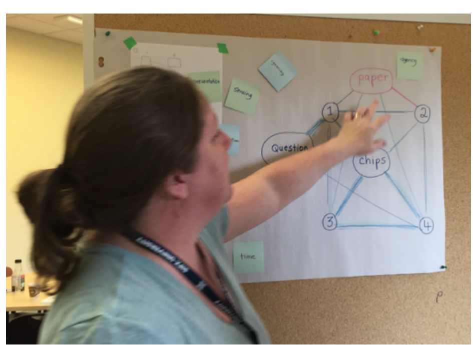


Figure 7. Ann explaining her group's representation of the data.

During the activity on Day 1, there was a question about whether or not the coins could be 'flipped'. One point of discussion was the impact of that particular query. Some groups overhead the comment about flipping, but it did not resonate because it did not align with their interpretation of the problem. This prompted the question: "If a group interpreted the problem differently than the intention of the leaders, does that make it wrong?" A short conversation followed, regarding the interest lying in which ideas are taken up, and which are not. And despite the different interpretations, all somehow resulted in order.

On the topic of the format of the data, Peter described the artefact (the writings) as a "*dead note*," when looked at it "*after the fact*." Some participants said that during the activity they used notes/signs for the purpose of thinking with and/or thinking through. The feeling of some members of the WG was that after the fact, these artefacts are different. Others argued the artefact is never dead while it exists, as there are signs to be sensed; that when the artefact is looked at, it is 'lived/living', and it is evolving as we interact with it. Shifting the path slightly, one participant raised the distinction between work created 'in the moment' (i.e., the writings produced by the participants) as compared to work that is 'produced' (i.e., the student journals). Further, there was talk about the difference between the 'intentionally produced writing' done by the group, as opposed to individuals' 'private writing', intended only for personal use. Florence, for example, commented that at one point during the task, her notes became 'quiet'; a personal recording while others were working on the task orally.

This conversation flowed into a gentle questioning and critique of decisions and comments made by the leaders. For example, some commented on the decision to provide only the paper artefacts and not the video, as some students work better orally and thus would not have produced any of the paper artefacts chosen. There was talk about the comments and guidance given by the leaders during the activity, such as asking groups if they had thought about flipping, potentially implying that the group was approaching the task incorrectly. We returned to this discussion on day 3.

The main points to be drawn from these conversations are the awareness we must have as researchers and interpreters of data. In our role as collectors of data, we influence and impact what is done and what is represented (for example, full immersion in the activity versus having to video the activity at intervals). We agreed that we must be aware of our biases and our perspectives, as we cannot lay them aside. We also questioned: "What are we actually able to draw out of the data?", and "What can we know for sure?"

Day 2 concluded with the assignment of two small homework activities around 'chaos', listed below (full details at: www.tiny.cc/CMESGhomework). Participants were asked to reflect on what the activities bring to mind with respect to mathematics teaching and learning.

La deuxième journée s'est terminée par l'attribution de deux petits devoirs autour du « chaos » (www.tiny.cc/CMESGhomework). Les participants ont été invités à réfléchir aux activités en pensant à l'enseignement et l'apprentissage des mathématiques.

- Activity 1–Cosine Convergence: Randomly pick 5 real numbers and recursively calculate the cosine (with a calculator or computer) of the result. What's happening?
- Activity 2–Recursive Sine: In the first figure on the back of this page, use a ruler to start at $x = 0.4$ and meet the curve going vertically. From that point, move horizontally until you touch the diagonal line. Still using your ruler, now move vertically so you find a

second point on the curve. Find 5 points in this manner. Then use the second figure and try it again. What is happening?

## DAY 3: QUESTIONING OUR A PRIORI / ÉCHAPPER AU CHAOS.

Day 3 can be broken into three segments: finishing up activities; bringing home the ideas WG participants grappled with over the previous two days; and, after the break we reflected on our experiences in the group and shared some thoughts, feelings, and realizations that arose during our time together.

### CHECKING HOMEWORK / LES DEVOIRS.

We began by revisiting the two activities assigned for homework at the end of Day 2 (Figure 8). Participants were asked to consider how the activities and their experience with them might feel like something related to mathematics teaching, learning, or researching.

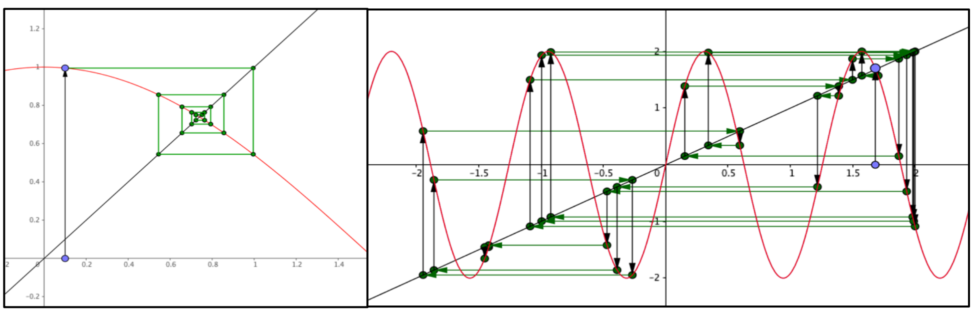

Figure 8. Cosine convergence and recursive sine homework results.

Speaking from personal experience, David supposed it (a chaotic classroom) would be okay, although the "*turbulent phase*" at the start of the year would be initially messy until the students got used to it. Continuing on this vein, the teacher would have to create the 'space' within the classroom for the students to evolve, and this would be messy, but "*not too messy*." Referring to the photo of the classroom shared earlier (Figure 1), the comment was made that "*it does not have to be visually chaotic to be chaotic*." For example, the social norms of the classroom can be ritualized (such as the structure seen in Japanese classrooms), yet the mathematics and socio-mathematical norms do not have to be so.

### LOOKING INWARD: QUESTIONING OURSELVES / UN REGARD VERS L'INTÉRIEUR : SE QUESTIONNER.

The second segment of Day 3 tasked each table group to consider the following questions, keeping in mind the idea of new/different 'ways' of researching the mathematics classroom.

1. Why is it so difficult to think outside the box, to make something different, to create something radically new, to move students or teacher out of their comfort zone?
2. Why do we struggle so much to overcome our own biases, as researchers? Can we really be open to something 'new'? Would we even recognize it?

Nous avons ensuite proposé aux groupes de discuter les questions suivantes :

Pourquoi est-il si difficile de sortir des sentiers battus, de faire quelque chose de différent, de créer quelque chose de radicalement nouveau, de sortir les étudiants ou les enseignants de leur zone de confort ?

Pourquoi avons-nous tant de mal à surmonter nos propres préjugés en tant que chercheurs ? Pouvons-nous vraiment être ouverts à quelque chose de « nouveau » ? Le reconnaîtrions-nous ?

Groups were told they would have five minutes to present their most interesting/ compelling/ informative interpretation of the questions. They were given time to prepare their presentation. A summary of each groups' responses follows.

The key idea contributed by Group 1 revolved around chaos and order. Wes stated, "*By not reducing chaos we are denying our humanity*," explaining that it is human nature to reduce, and that what living things do is organize the environment (Figure 9). Further, this organization occurs "*with*," not "*to*," the environment, in relation and supporting it because nothing is isolated. Yet the order we produce eventually turns back into chaos. David pointed out the counters from the coins activity, which some groups had used to create patterns (Figure 4), now residing jumbled together in their plastic bag.

Figure 9. Unsolicited order…maybe not reducing chaos is denying our humanity?

The responses shared by Group 2 referred to our biases, particularly the fact that we can never put them aside. They wondered, if it is given that our biases direct our attention, and that we each bring different lenses to our research (in collecting and viewing data), is there significance in working with others? They proposed, "*Through the intersection of multiple biases, maybe we can come to something.*"

Group 3 focused on the 'boxes' that prevent learning from happening, asserting that there is a reason why we have boxes, and that we are not trying to get "*outside the box*." They suggested that we privilege some boxes over others in that we attend to only certain things, such as only some ways of learning, and not all. One particular point shared is that we should "*avoid doing the things that cause learning not to happen*," adding the example that the superficial physical set-up of the classroom (e.g., vertical non-permanent surfaces) does not guarantee the moving of ideas, but NOT doing it might help inhibit learning.

The fourth table group considered the role of time and the movement of ideas in research. In particular, they noted the difficulty of tracking the synchronicity of ideas, asking the following questions:

- "*How can we preserve the time across different things we are looking at through space and time?*"
- "[Given that] *it is impossible to look at everything at once, how do we synchronize what we look at in separation? How do we capture or track back that movement of ideas?*"

A group discussion followed the individual table summaries, so others could comment on what they had heard. Joyce suggested we give up on 'capturing', and instead think about how to learn more about it/from chaos in the classroom. Jo added, "*The idea of 'capturing' points to the thing being outside us, chasing something separate from us. The 'chase' IS the thing; 'capture' implies something that exists. We are observing something and wanting to communicate these observations*." Jean-François noted, "*Just because I observe something does not mean that I can describe or communicate it.*"

## SUMMING UP: WHAT HAVE WE LEARNED? / FAIRE LE POINT: QU'AVONS-NOUS APPRIS ?

> Enfin, bon an mal an, nous avons tenté de saisir en quelques mots ce qui se dégage de tout ça…. Et nous avons encore une fois constaté nos difficultés à répondre à nos questions. Nos mots et nos interprétations peuvent-ils suffire ? Nous trahissent-ils trop ? Nos biais sont toujours présent, et c'est peut-être pourquoi il est si important de collaborer.

After the morning break WG participants returned for one final meeting to engage in a general sharing in/sharing out session. Participants were asked to take two minutes to reflect on the past two and a half days, and consider sharing their response to the following:

- Is there something that (for you) was most relevant/compelling/important/memorable?
- Is there something important that you saw/did/heard/learned that you feel you can share?

Though varied, a few themes emerged in what was shared. In some cases, a thought shared by one group member was taken up and added to by others. A small representation follows.

One such thought concerns the power of words (or a word) and how our interpretations of them/it drive or direct the flow or direction of the conversation and activity. For example, the word 'chaos' has an informal definition, a Greek mythology definition, and a mathematical definition. The word 'mistake' implies that there is a predetermined route. If there is an implication (intended or not) that there is a 'right' approach to a problem, then alternate approaches may be perceived as wrong, causing individuals to withdraw and feel it is an unsafe environment to take risks. How we use words, and how people pick them up and interpret them, can have great impact on others. Florence shared a quote from her elders, "*the blood remembers*." We are shaped by our experiences, and we interpret all we encounter based on our experiences. Often we have a bodily response, such as physical withdrawal, all of which ties to the next thought, regarding biases.

A second participant noted, "*we cannot park them* [our biases]. *We have them and our knowing is always undergoing; we always bring forward our own interpretation*." In sum, we need to be aware of how great an influence we have, and how powerfully our actions, even if insignificant or unnoticed by us, may affect others.

With these thoughts resonating, we concluded our time together. We learned from each other, and we made some progress, even if it was just to take a deeper look at what we think we know and can say about our research. We acknowledged that regardless of how and what we 'capture' or 'chase' what is happening in the mathematics classroom, our data and analysis will be coloured by our biases. Given this fact, perhaps the best thing we can do is to seek collaboration with others. If, in collaboration, we talk about what each of us sees, and acknowledge our biases, it may invite opportunity. It is hoped that participation in these conversations sparks additional collaborative research into what is going on in the secondary mathematics classroom.

## REFERENCES

Rowland, T. (1999). *The pragmatics of mathematics education: Vagueness and mathematical discourse*. Routledge.

Panel

Panel

# ÉTABLIR ET MAINTENIR DES COLLABORATIONS ENTRE DIDACTICIENS ET MATHÉMATICIENS

# INITIATING AND NURTURING COLLABORATIONS BETWEEN MATHEMATICIANS AND MATHEMATICS EDUCATORS

## ANIMATEUR / CHAIR

Frédéric Gourdeau, *Université Laval*

## PANÉLISTES / PANELISTS

Paul Deguire & Manon LeBlanc, *Université de Moncton*
Ed Doolittle, *First Nations University* & Kathy Nolan, *University of Regina*
Ryan Gibara & Sarah Mathieu-Soucy, *Concordia University*

## PRÉSENTATION (EN FRANÇAIS)

Fidèles à nos traditions, le panel nous a réunis autour d'enjeux communs, chacun s'exprimant en français ou en anglais, à partir de son bagage et de sa perspective. Nous avions préparé notre panel en évoquant plusieurs questions, et les fils conducteurs y sont aussi nombreux que les nuances qui s'imposent, permettant de présenter une partie de la réalité complexe que nous avons abordée.

Le thème du panel est lié à notre spécificité. Le GCEDM a été fondé par des mathématiciens et des didacticiens et, chaque année, il rend possibles des conversations qui ne pourraient sans doute pas avoir lieu sans lui. Malgré cette forte tradition, il y a encore relativement peu de collaborations entre des mathématiciens et des didacticiens. Nos trois paires de présentateurs sont de ceux qui ont établi des collaborations de cette nature. Lors du panel, nous avons abordé le potentiel de telles collaborations, la manière dont elles ont été établies, maintenues, ainsi que certaines des difficultés rencontrées.

## PRESENTATION (IN ENGLISH)

True to our tradition, we gathered around common issues, each of us contributing in French or English, from his or her own perspective, based in our personal experience. We prepared the panel with many questions in mind. There were common treads in our discussions, as well as nuances, which helped to have a significant conversation about a complex reality.

The theme of our panel is connected to our specificity. Mathematicians and mathematics educators founded CMESG, and each year, our group enables conversations, which might not otherwise take place. In spite of this strong tradition, there are still relatively few collaborations between mathematicians and mathematics educators. Our three pairs of presenters have already forged a working relationship. In the panel, we addressed the affordances of such collaborations, how they were initiated, nurtured, and some of the difficulties faced in doing so.

## LES QUESTIONS ABORDÉES

Dans notre préparation, nous avons discuté de certains aspects qui nous paraissaient importants. Les questions qui suivent étaient autant de pistes pour préparer le panel, et permettent de mettre en perspective les contributions des panélistes.

- Étant donné que vous avez travaillé ensemble, qu'est-ce que vous avez trouvé le plus intéressant ?
- Comment avez-vous réconcilié des objectifs différents (s'ils étaient différents) ou identifié des objectifs communs ?
- Comment avez-vous pu vous engager dans ce projet étant donné les autres contraintes liées au travail (recherche, enseignement, thèse, etc.) ?
- Avez-vous commencé par les objectifs, les projets, ou les projets sont-ils nés alors que vous vous connaissiez ? Qu'est-ce qui est venu en premier ?
- Comment cette collaboration est-elle perçue ou reconnue dans votre département / université / contexte (étudiant dans un cas) ?
    - Pour les mathématiciens, est-ce reconnu au même titre que de la recherche en maths ?
    - Pour les éducateurs, est-ce reconnu comme de la recherche ?
    - En fait, comment est-ce reconnu ?
- Sur le plan personnel, il existe des obstacles à ce type de collaboration. Nous sommes des experts dans un domaine, mais pas dans l'autre (vraisemblablement). On peut avoir peur du jugement de l'autre, qui peut penser que ce qu'on dit ou fait est inapproprié, inexact, incorrect. Cela s'est-il présenté pour vous ? Si cela ne s'est pas produit, comment la confiance de base a-t-elle été établie ?
- Dans nos communautés respectives (incluant les conférences et les résultats des recherches), il y a des normes culturelles.
    - En mathématiques, les concepts sont simples, les mots bien définis. On prouve.
    - En éducation, la réalité est complexe, le langage chargé. On ne prouve pas, au sens du matheux.
    - La question : *Comment avez-vous vécu les différences culturelles entre les communautés de mathématiques et d'enseignement des mathématiques, si cela a été le cas ?*

## OUR INITIAL QUESTIONS

In our preparation for the panel, we listed some aspects which seemed important. The list of questions, which follow, gives some background to help to understand how the panelists prepared their contributions.

- Given that you have been working together, what has been most interesting for you?
- How did you reconcile different goals (if they were different) or identify common goals?
- How did you manage to devote time and energy to this project given your other responsibilities at work (research, teaching, thesis work, etc.)?
- Did you start from the goals, the projects, or did the projects came stem from an existing relationship? What came first?
- How is that collaboration considered or recognised in your Department / University / context?
  - For mathematicians, is this recognised at all, as it is not maths research?
  - For educators, is this recognised as research?
  - How is it recognised?
- On a personal level, there are barriers to this type of collaboration. We are experts in one area but not in the other (presumably). We can be afraid of being judged or afraid that the other might think that we are saying or doing something which is incorrect, inappropriate, or inaccurate. Did you feel this? If not, how was the basic trust established?
- In mathematics and mathematics education communities, conferences, research, there are cultural norms.
  - In maths, concepts are simple; the meaning of words is precise. We prove.
  - In education, reality is complex, language often loaded. We do not prove, at least not in a mathematical sense.
  - The question: *How did you experience the cultural differences between mathematics and mathematics education communities, if that has been the case?*

## NOS PANÉLISTES, DANS LEURS PROPRES MOTS / OUR PANELISTS, IN THEIR OWN WORDS

### RYAN AND SARAH

In this panel, we shared about a research project that we initiated alongside Laura Broley. We shared our unique experience initiating a research project as three doctoral students from two different disciplines, mathematics (Ryan) and mathematics education (Sarah and Laura). To put it simply, we started collaborating following the initiation, in Fall 2016, of a mentorship program at Concordia University for mathematics graduate students who are asked to be involved in teaching activities. Laura and Ryan were asked to take charge of the project and develop it. Informal discussions between Sarah and Laura regarding the mentorship program led to all three of us wanting to look at the program from a research perspective. Sarah, therefore, came into the project as a new eye. Our work was meant to be aligned with the goal mentioned in multiple papers (Speer et al., 2017; Speer et al., 2009) to use research to inform the design, improvement, and efficacy of professional development programs for graduate students involved in teaching.

A few factors facilitated our worlds coming together. In our university, mathematics and (post-secondary) mathematics education graduate students are in the same department, which allowed us not to feel like outsiders in this project. We were also lucky that our project was recognized by our university. Indeed, we were able to get funding from the university to go and share our project nationally (Broley, Mathieu-Soucy, & Gibara, 2018) and internationally (Broley, Mathieu-Soucy, Gibara, & Hardy, 2018).

On a more personal level, it was nice to collaborate with peers and try to do something ourselves. It was also something to take our mind off our respective thesis work. While the university required us to have a professor overlooking our work because we were all still graduate students (Professor Nadia Hardy agreed to be part of our project), we were still able to lead the direction of the project.

Another aspect that we believe allowed our collaboration to go smoothly is that Sarah and Laura were both originally trained in mathematics before transitioning into mathematics education and remember their experiences of that transition. It allowed them to be somewhat comfortable in both worlds. Also, they remember well their recent transition into mathematics education, and how one can easily feel bombarded by names of theories and people, which allowed them to be especially sensitive to Ryan's experiences.

Being students brought its share of challenges to our research project, though. While Laura and Sarah were initiating Ryan to mathematics education, they were still learning about it themselves as doctoral students. In this sense, it was an asset, given our limited experience, to have two people in mathematics education to conduct this research. At least, speaking the same language, we had an idea of what was to be done and what would be culturally appropriate, as well as knowledge of potential conferences where to present our work. Also, another challenge was finding the balance between conducting this project, about which we were excited and passionate, and finishing our degrees. We each had a thesis to write regarding other, independent, research that is unaffiliated to the mentorship program on top of Curricula Vitae to build and pressure to publish in our respective fields.

When it comes to the aspect of collaboration between researchers of two disciplines, we worked in a mindset where we were aware that our different knowledge and background gave us different strengths. In that sense, some parts of the work were better performed by one member of the collaboration as per their personal strengths and, importantly, this did not lessen the value of the collaboration. In that sense, no one wanted to bring the group down and slow our progress. It was worth it to benefit from everyone's strengths, given the limited time we each had to put into this project. We did, however, make sure to challenge one another and take the time to talk in order to understand our individual views.

One topic we addressed that got some reactions after the panel was when we mentioned that, as doctoral students, we had in mind that what we do during our graduate studies has an impact on our future and on our ability to find work afterwards, especially an academic position. We mentioned how there were a lot of voices present in our lives, a lot of voices that gave us a lot of different (and sometimes contradicting) advice and how it was a challenge for us to navigate these recommendations, while navigating our personal motivations for doing research. This is particularly true when our interests are slightly outside our main research topic or slightly outside what is generally valued in our respective field. For example, Ryan shared how he is sometimes discouraged from pursuing projects related to mathematics education as a mathematics graduate student. With many postdoctoral opportunities being very competitive, he is presented with the advice that he should dedicate as much of his time as possible to his mathematics research. Should Ryan then spend his time on a project that will, according to what he hears, not be valued on his résumé, even if it is something he enjoys and believes in, or should he invest his time into something that will add more value to his résumé? While we heard after the panel multiple people telling us to avoid thinking about résumés and positions, to do what we love and to be ourselves, and that this should be enough to find work after we graduate, we were left wondering how to consolidate these comforting words with other opinions we have encountered so many times in the past.

### MANON ET PAUL

Au fil des ans, nous avons eu la chance de collaborer à deux niveaux, soit au sein d'un comité de révision des cours de mathématiques pour les étudiantes et étudiants inscrits au baccalauréat en éducation primaire et en tant que président et vice-présidente du Groupe d'action pour les mathématiques en Acadie (GAMA).

#### Cours de mathématiques pour les étudiantes et étudiants inscrits au baccalauréat en éducation primaire

Le processus de révision du baccalauréat en enseignement primaire a commencé en 2012. Dès le début, les discussions réunissaient des gens de différents milieux, incluant des mathématiciens, des didacticiens et des agents pédagogiques. En 2015, un comité interfacultaire formé de professeures et professeurs en mathématiques et en éducation des trois campus de l'Université de Moncton a été créé. Son mandat était de revisiter le format et le contenu des cours de mathématiques destinés aux étudiantes et étudiants inscrits au baccalauréat en enseignement primaire. Les mathématiciennes et mathématiciens sont responsables du contenu, alors que les didacticiennes et didacticiens s'assurent que les principes didactiques présentés dans le cadre théorique des programmes d'études sont respectés. Cette reconfiguration de programme a servi de tremplin pour la collaboration qui s'est établie entre le département de mathématiques et statistiques et la Faculté des sciences de l'éducation. Les cinq cours de mathématiques préexistants ont été modifiés et un laboratoire se donnant cinq fois par semestre a été ajouté à la formation, afin de permettre aux étudiantes et étudiants de mettre en pratique leurs habiletés en résolution de problème en appliquant et enrichissant ce qui a été vu en salle de classe. De plus, un nouveau cours de modélisation a été ajouté à la formation. Ce cours a comme objectif d'approfondir la compréhension des étudiantes et étudiants à travers l'utilisation d'exemples concrets et d'amener les gens à prendre conscience de l'utilité des mathématiques en travaillant sur des problèmes faisant appel aux contenus vus dans les quatre cours précédents de mathématiques et dans le cours de statistique descriptive. Avec ce cours, nous tentons donc de créer une expérience positive avec la résolution de problèmes pour les étudiantes et étudiants, avant qu'ils se retrouvent en salle de classe. Nous souhaitons les préparer pour qu'ils soient mieux équipés pour répondre à la question « Pourquoi fait-on des mathématiques ? ».

#### GAMA

Pour des raisons historiques, il y a peu de tradition mathématique dans les écoles francophones du Nouveau-Brunswick (N.-B.). Ce constat nous a amenés à vouloir développer la vie mathématique dans les écoles francophones de la province. Le tout a débuté en 2009 avec une « Journée mathématique », qui se déroulait une fois par année et qui permettait à des gens travaillant au

Ministère de l'Éducation et du Développement de la petite enfance (MEDPE) du N.-B., à des agentes et agents pédagogiques associés aux différents districts scolaires francophones de la province, à des enseignantes et enseignants des Collèges communautaires du N.-B. et des écoles, ainsi qu'à des professeures et professeurs de l'Université de Moncton de se rencontrer pendant une journée pour discuter de mathématiques. Or, bien que ces journées étaient très intéressantes, il demeure que les gens qui y assistaient étaient déjà convaincus de la beauté et de l'utilité des mathématiques. L'impact de ces rencontres au-delà de la pièce où elles prenaient place n'était pas important (pour ne pas dire inexistant). Le nombre de participantes et participants a grandement diminué au fil des années, ce qui nous a poussés à repenser notre stratégie. C'est alors que GAMA vit le jour en 2012. Essentiellement, ce groupe vise à a) développer ou améliorer la vie mathématique dans les écoles et b) amener les gens à développer une attitude plus positive envers les mathématiques dans le système scolaire. GAMA est un petit groupe, son défi principal est de trouver dans les écoles suffisamment de gens intéressés à apporter leur contribution à l'enrichissement de la vie mathématique en milieu scolaire.

Actuellement, quelques membres du GAMA sont actifs dans les écoles (par exemple, en menant un club de mathématiques). Le groupe est également présent dans les écoles grâce à deux activités, soit le Concours GAMA et la Fête des maths. Le Concours GAMA est une compétition amicale qui s'adresse à tous les élèves du système scolaire francophone et qui a lieu chaque année à la période automnale. Dans le cadre des différents concours, les élèves ont eu à créer un logo pour le GAMA (Figure 1), à prendre des photos des mathématiques autour d'eux, à expliquer les mathématiques présentes dans différents emplois, etc. Ce concours n'a donc pas comme objectif d'évaluer les habiletés procédurales ou de résolution de problèmes des élèves, mais vise plutôt à leur faire voir et à parler des mathématiques autrement.


Figure 1. Logo du GAMA créé par Geneviève Gauthier, une élève de l'École secondaire Népisiquit de Bathurst.

La Fête des maths a lieu au mois de février chaque année (depuis 2013). Un thème est ciblé pour chaque édition et des activités portant sur ce thème sont envoyées dans les écoles. Voici quelques exemples de thématiques exploitées durant les différentes Fêtes des maths : Jeux olympiques (2014, pour les Jeux de Sochi), Cryptographie (2015, pour coïncider avec la sortie au théâtre du film « Imitation Game »), Arts et mathématiques (2020).

Les couleurs de notre collaboration

*Étant donné que vous avez travaillé ensemble, qu'est-ce que vous avez trouvé le plus intéressant ?*

- La capacité qu'on a d'identifier des sujets d'intérêt commun et la facilité avec laquelle nous pouvons travailler ensemble sur ces sujets quand il n'y a pas de structure qui donne plus d'importance à un groupe qu'à un autre.

*Comment cette collaboration est-elle perçue ou reconnue dans votre département / université / contexte (étudiant dans un cas) ?*

- Le travail effectué au sein du GAMA n'est pas vraiment reconnu, sinon avec un peu de gratitude. Concrètement, cela influence peu notre carrière, car on considère ce travail comme un service à la collectivité. Nos collègues sont probablement heureux que le travail soit fait… et que ce soit d'autres qui le fassent ! (Car cela ne nuit pas à leur carrière).

*Comment avez-vous vécu les différences culturelles ?*

- La perception que la didacticienne avait des mathématiciennes et mathématiciens a évolué au fur et à mesure que la collaboration s'est développée. Une vision restrictive et stéréotypée s'est vue transformée, alors que le discours de plusieurs mathématiciennes et mathématiciens s'est avéré plus didactique que ce qui était attendu. Du côté du mathématicien, la plus grande prise de conscience fut dans la réalisation que ce n'est pas parce qu'il y a beaucoup d'information en surface qu'il n'y a pas de profondeur dans le message émis. Notre amour commun des mathématiques a eu raison des différences culturelles qui sont, au fond, un peu artificielles.

*Sur le plan personnel, il existe des obstacles à ce type de collaboration. Ces obstacles se sont-ils présentés pour vous ?*

- Le fait de connaitre les gens avec qui nous travaillons, avant de commencer à collaborer, a eu un double effet, soit de contribuer à être moins intimidés par l'autre et de contribuer à être plus intimidés par l'autre ! D'un côté, étant donné que l'on connait l'autre, les preuves de sa compétence n'ont pas à être faites. Chacun arrive avec son expertise et ces expertises se complètent pour former un comité à la fois riche et diversifié. D'un autre côté, lorsque la relation initiale est hiérarchique (dans ce cas-ci, le mathématicien a été le professeur de la didacticienne lorsqu'elle était inscrite au baccalauréat en éducation secondaire), il est difficile de briser la relation préétablie pour en développer une d'égal à égal. Passer du « nous » au « tu » est plus difficile qu'on pourrait le croire, même après dix ans !

## EDWARD AND KATHLEEN

Our plenary panel presentation began by displaying a visual timeline to reflect the overlapping, intersecting, and multiple ways in which we collaborated over many years while working at our respective universities (Kathy at the University of Regina and Ed at First Nations University of Canada).

### Kathy

Over the past few months, Ed and I met several times to discuss our various collaborations over many (more than 15!) years. We thought we might best portray the diversity and duration of these collaborations/conversations through a timeline, shown in Figure 2 (not to scale!).

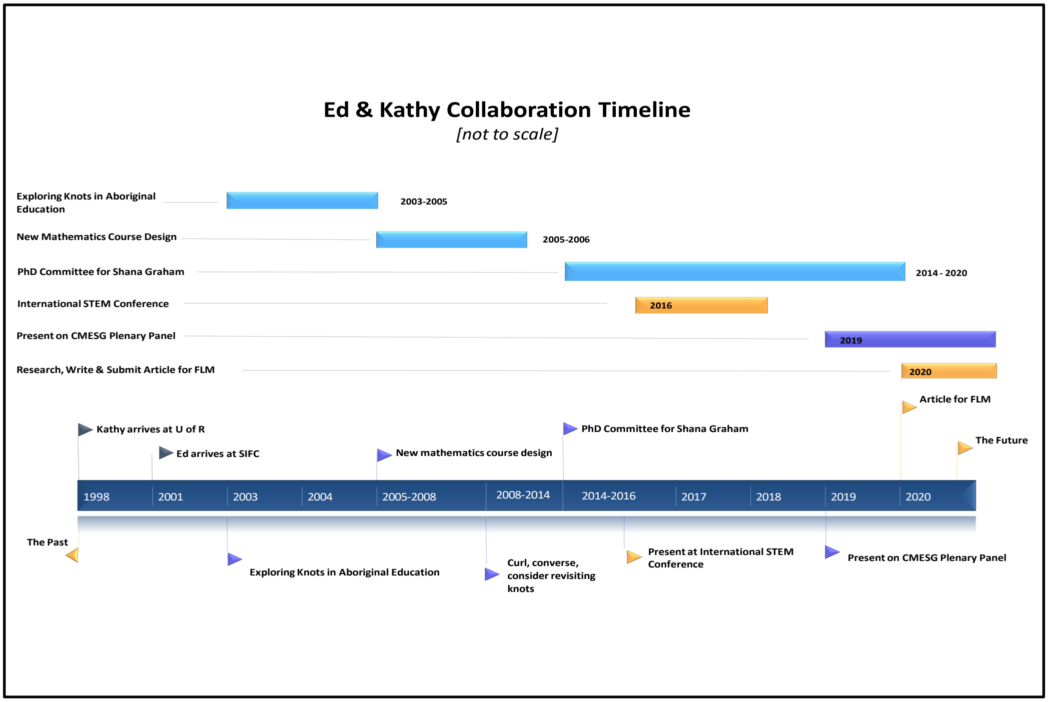


Figure 2. Timeline of Ed and Kathy's collaboration.

Highlighted/listed on the top left are the key areas or projects of collaboration that we recalled during those meeting to construct the timeline, beginning around 2003 until the present (actually, even into the future with our plans for extending our work together). It is the first project named in the list—a research project on Exploring Knots in Aboriginal Education—that brought us together for the first time (and Ed will speak about this project shortly).

As you can hopefully see from the timeline on the bottom of the figure:

- We have a past, as well as a future, for our collaborative work
- We have research projects, as well as course design, conference presentations and PhD student committees and commitments
- We have 'pauses' (I prefer not to call them 'gaps'), where we just curled together and conversed about "starting up that knot project again…"

Ed

After I completed my PhD in pure mathematics, I went to work for Queen's University's Aboriginal Teacher Education Program (ATEP), mostly helping with program administration. After time, however, I was asked to teach the math methods course for the community-based program on reserves in Northern Ontario such as the remote fly-in community of Kasabonika Lake. I was reluctant to teach the courses because I knew so little about math education, but because ATEP had so much difficulty finding instructors, I agreed to try. I consulted with colleagues at

Queen's, in particular Bill Higginson and Lynda Colgan, and I attended a math education conference that took place at Queen's.

It was at one of the sessions at that conference that I experienced one of the most transformative experiences of my career. I cannot remember who was speaking at the session, unfortunately. But one of the topics was division by fractions. The speaker showed how $6 \div 2 = 3$ could be realized on a number line as three jumps of two steps at a time. Then he showed how $6 \div \frac{1}{2} = 12$ could be realized in exactly the same way, as twelve jumps of half a step. It sounds like such a simple thing, and no doubt to most math educators it is one of many simple models for division that they know and teach. But for me, it was extraordinary.

I went to school for 26 years straight (with breaks only in the summer), completed three degrees in mathematics, and graduated with a PhD in pure mathematics (partial differential equations) in 1997. However, even with all that education, I must say that I never really understood division until that moment at the conference at Queen's. To me, division by a fraction was 'invert and multiply'. That was the way I was taught, and it had always worked for me, so I had no motivation to explore any further.

I have always been interested in the difference between knowledge on the one hand and deeper understanding on the other. I could feel the difference between knowing a theorem in mathematics and really understanding it. Deeper understanding involved not just the statements of the theorem and its proof but also knowledge of the context and history of the theorem, examples and counterexamples, special cases and extensions, alternative approaches, and the obstacles and the open paths either blocking or leading to generalization, all of which can lead to a more visceral, deep understanding of the subject at hand. My successes in higher mathematics were often tied to situations in which I had achieved a deep understanding.

What I had not realized until that moment was that deep understanding is also possible with elementary mathematics. When I was young, first learning about theorems in number theory, I wondered whether it was possible to understand arithmetic in the same way, to prove it correct and understand why it worked, but no one and no book I read seemed able to answer my questions. Arithmetic remained for me a set of instructions until that moment at the conference.

Since then I have come to appreciate mathematics education a great deal. I learn more whenever I get a chance, and I have had the privilege of being able to seek out research opportunities with colleagues like Kathy Nolan.

<u>Kathy</u>

I call this "my journey to Ed." Beginning with my experiences as a learner of mathematics—while quite 'successful' in terms of grades and in conveying a message that I was 'good at' math, I recognized that direct teaching with a focus on mastering and executing procedures dominated in those mathematics classrooms so much so that when I became a mathematics teacher, I struggled to do anything different in the way of pedagogy in my own classrooms.

Then I became a mathematics teacher educator. Shaped and influenced by these previous direct teaching experiences, but also strongly influenced by newly discovered research literature, I wanted to do things which engaged students in conversation, in movement, in collaboration—actions that I was convinced would lead to a different kind of learning, one with deeper conceptual understanding of mathematics.

Enter inquiry—not just a pedagogical tool but a philosophy of teaching and learning that really challenged and disrupted the philosophy of teacher-directed approaches. However, in the first few years, my pre-service teachers just were not 'buying into it'. It was too different from what they knew and experienced themselves as learners. And, let's be honest, in those early days they were not experiencing much in the way of inquiry modeling from me! I kept pondering how I (along with my pre-service teachers) could model inquiry teaching without having ever experienced it as a learner. This dilemma is a common feature of inquiry-based pedagogy in the research literature.

As I set about to make my push for inquiry in my teacher education classes, I recognized that I wanted the process of linking the mathematics to curriculum to be more organic/authentic but for this I needed a deeper conceptual understanding of mathematics, especially more context-embedded mathematics. In other words, I was looking for mathematical understanding that could reflect meaningful contexts, especially those contexts and versions of mathematics that invited marginalized and/or neglected learners of mathematics.

So, I guess the summary of this brief story is that I recognized a need to 1) know mathematics at a deeper level, 2) understand this mathematics in meaningful contexts, and 3) make strong and clear connections to curriculum so that there would be 'buy in' from pre-service teachers. I began conversations with Ed. And the rest is history.

<u>Ed</u>

I have something to admit: I failed kindergarten. I hope that does not mean that the rest of my 26 years of education are invalid and I must start over again.

In my kindergarten class, there was a chart on the wall with six tasks that we were all supposed to complete by the end of the year. I cannot remember what they all were, but they were things like "can spell first name" and "can count to 10" (or whatever the highest number was). Most of my classmates could do all six of the tasks. I could only do five. By the end of kindergarten, I still could not tie my own shoes. I remember being mildly shamed by the public nature of the reporting on my status in the class.

One of the ironies of my situation is that about half of all people, adults and children, tie their shoes incorrectly. 'Correctly' may be a matter of judgment, but I suggest that it means that one ties one's shoes in a reef knot rather than a granny knot. You can do a quick test just by looking at someone's shoes. If the knot goes across the shoe, it is probably a reef knot. If the knot goes up and down the shoe, it is probably a granny, which tends to come loose faster than a reef knot and also tends to jam so that it does not come undone when you want it to come undone.

I found tying my shoelaces boring, so I tended not to do it, which caused my mother to worry about safety and cleanliness: I might trip over my shoelaces or drag them through mud. She responded by buying me slip on shoes, PF Flyers if I recall correctly. When it became clear that not tying the shoelace knot was going to be a problem, my mother and numerous other relatives all gave me conflicting advice on how to tie my shoes which did not help me much as far passing kindergarten was concerned. Eventually, too late to earn that sixth star, I chose to learn the simplest method my relatives showed me and I still do it that way now, almost fifty years later, although I have consciously modified the method to ensure that I am tying a reef knot, and I tighten my shoelaces before tying the knot so that they do not come undone as easily. (Why shoelaces become undone is a subject for active research: see, e.g., Daily-Diamond et al, 2017.)

That early experience with knots, and experiences in Scouts Canada and with sailing led me to an interest in the mathematical theory of knots. As a graduate student in mathematics at the University of Toronto, I was fortunate to be able to study knot theory (a branch of topology) with Kunio Murasugi, one of the foremost experts on the subject. Professor Murasugi kindly supported my travel to the first academic conference I ever attended, at the Santa Fe Institute, on knots and DNA (which does get tied into knots) where I met Lou Kauffman and Vaughan Jones and many other interesting mathematicians and scientists. I recall asking Lou Kauffman about something in a book he had written, the 'rope trick', and on the spot, near the coffee urns at break time, he whipped the belt out of his pants so he could demonstrate it to me.

Later, when I became interested in Indigenous math education when I was at Queen's, I came up with the idea of approaching mathematics through knots and braids, which are certainly a part of Indigenous culture. I put together a presentation for the students in the Aboriginal Teacher Education Program (a community-based program which leads to a Bachelor of Education degree and a teacher's certificate). We decided to bring the students from their communities to the Queen's campus for a few weeks in the summer, because it was easier to teach courses requiring facilities like labs, art materials, and sports gear on campus as opposed to in the communities.

The session we did on knots and mathematics was a great success in some ways, I thought. It seemed to engage that students in a way that the standard curriculum did not. One recollection of the event I have is that the students were enthusiastically following along with me when I was leading them in learning to tie a figure eight knot. I held the rope over my head so that they could all see the steps better, and they imitated me to such an extent that they all tied their figure eight knots with their arms held above their heads too.

I used a tub of water to show them how most ropes sink in water, which could be a problem for maritime applications, but that some modern (and even some ancient) rope materials float. I then tied a floating rope into a heavy knot called a monkey's fist and asked the students whether the knot would still float or would it sink. Based on the weight of the knot, most of the students predicted it would sink, and then I threw it in the water to show them it would float, which is a demonstration of the concept of density versus mass, a concept in the science curriculum.

I felt that my presentation was less of a success in its connection to the mathematics curriculum. The connection between knots and mathematics was clear to me, after having taken graduate courses in the subject, but I had difficulty finding connections with the K-12 curriculum. One attempt I made was to talk about the concept of prime and composite knots (reef knots and granny knots are composite, being formed of two overhand knots; overhand knots and figure eight knots are prime, not being composed of simpler knots). But I felt that that was not enough to encourage the teacher candidates to try teaching about prime and composite knots alongside prime and composite numbers. Other interesting parts of knot theory had no connection to the K-12 curriculum that I could imagine.

I felt that I needed help from an expert in math education to connect my knot ideas to the K-12 math curriculum so that (I felt) there would be a chance that they would actually use some of the knot material when they were teaching in their own classrooms. So when I started in my first Assistant Professor position at First Nations University (then called the Saskatchewan Indian Federated College) in 2001, I sought out a math educator who might be able to help me in that endeavour, which is how I started to work with Kathy on our Knots and Aboriginal Math Education project.

We have still not made great progress on that project, but at least it provided the spark which has led to a fruitful collaboration which is nearly twenty years long at this point and is still going. I feel blessed at having Kathy and so many other generous colleagues in math education who have helped me understand that there is so much more about math education than just curriculum. I would like to take this opportunity to thank you all.

## CONCLUSION

Ce panel a permis de mettre en lumière des collaborations existantes, d'y puiser une source d'inspiration et de partager autour des enjeux auxquels font face les membres de notre communauté face au manque de reconnaissance de la valeur du travail fait en collaboration, lequel n'est généralement pas reconnu d'emblée dans les communautés de recherche en mathématique et en didactique. Dans une intervention, Florence Glanfield a remarqué que nous étions plusieurs à occuper des postes dans lesquels nous pouvions travailler afin que ce travail soit davantage reconnu, par exemple lors d'engagement ou de promotion.

## CONCLUSION

This panel enabled us to highlight some existing collaborations, which can be a source of inspiration. We discussed some of the challenges faced by members of our community as they engage in collaborative work, given the lack of recognition of such work has in both the mathematics and the mathematics education research communities. In one of the last interventions, Florence Glanfield drew our attention to the fact that many of us are in positions where they can work towards a recognition of the value of such work, for instance in hiring and promotions.

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# Topic Sessions

# Séances thématiques

# CULTURALLY SUSTAINING MATHEMATICS EDUCATION: CONNECTING INDIGENOUS KNOWLEDGE AND WESTERN MATHEMATICAL WAYS OF KNOWING

Ruth Beatty, *Lakehead University*
Colinda Clyne, *Upper Grand District School Board*

In our session we presented an overview of our multi-site research study. Since 2012 nine teams across Ontario comprising Indigenous leaders, artists, educators and non-Indigenous educators worked together to explore the mathematics inherent within Anishinaabe or Métis cultural practices. The projects took place in different community settings that varied in terms of contexts and participants. Each individual project was at the local grassroots level and driven by the views, opinions, resources, and interests of participating community. We engaged decolonizing research ethics with a commitment to making the research meaningful for community partners by establishing consensual collaborations, where different people brought different expertise and experiences to the work (Whetung & Wakefield, 2019). Initially our goal was to demonstrate the efficacy of this approach in order to transform Canadian mathematics education into a form that respects both Western and Indigenous traditions. However, through the processes of co-planning, co-teaching and co-reflecting, we realized that the work was primarily about lifting up Indigenous knowledge and shifting research practices towards an emphasis on ethical relationality (Donald, 2012; Wiseman et al., 2020).

In order to make math meaningful and relevant for First Nations and Métis students, we made explicit connections to the mathematics inherent in Anishinaabe or Métis culture and provided an opportunity for all students to experience culturally sustaining mathematics instruction. We brought together two cultural knowledge systems: the mathematical knowledge inherent in traditional Indigenous technology, design and artistry and the mathematics found in the Ontario curriculum. We looked to ethnomathematics as a framework, which views school mathematics as one of many diverse mathematical practices that is no more or less important than mathematical practices that have originated in other cultures and societies (Wagner & Lunny Borden, 2012; Mukhopadhyay et al., 2009; D'Ambrosio, 2006).

To make these cultural-mathematical connections, we identified relational protocols from Indigenous knowledge systems to build long-term relationships with community that are grounded in respect, relevance, reciprocity, and responsibility (Archibald et al., 2019; Kirkness & Barnhardt, 1991). Placing relationships at the heart of this work aligns with Indigenous research methodologies that ask not whether results are valid or reliable, but whether those entering into the research process have fulfilled their roles and responsibilities within the relationship (Wilson, 2001). We were guided by the people we worked with and co-created meaningful approaches to both mathematical instruction and to research with respect to methodology, analysis, and the dissemination of findings. This commitment to ethical relationality allowed us to "build bridges, to render consensus emerging among our own conversations intelligible to a wider mainstream audience in dire and urgent need of alternative, complementary ways of knowing and being (aka epistemologies and ontologies)" (Darnell, 2018, p. 231).

Project sites included two federal schools each within a First Nation and three provincially funded public schools each with ties to a specific First Nation. Additionally, we worked in four classrooms in urban settings in which community research team members were invited via the board's Indigenous Education Advisory Committee and so did not represent one specific First Nation, but rather urban Indigenous communities that created themselves as "intimate, human, and self-defined spaces" (Smith, 2012, p. 127).

We followed a cyclical process that began in each community with an initial consultation to ensure the work was grounded in local circumstances and responsive to the community's educational goals. We were the "holders of space… creat[ing] the space to put Nishnaabeg [and Métis] intelligence at the center and to use its energy to drive the project[s]" (Simpson, 2017 p. 15). Throughout the projects, team members engaged in ongoing consultation with community over the course of months or years. After initial consultations, project team members then co-planned and co-taught a particular form of Indigenous technology and/or artistry chosen by the community either based on their own priorities for cultural revitalization or based on the funds of knowledge of participating artists. The activities included different kinds of beading (loom, medallion, and peyote stitch), birch bark basket making, moccasin making, and Métis finger weaving. Teams then co-analyzed video recordings of the lessons and reflected on the experience in terms of cultural and mathematical understanding.

## STUDENT EXPERIENCES

Centering instruction within Indigenous knowledge created experiential connections to mathematics as students engaged in the cultural activities the community partners brought to the classroom. The learning began with cultural teachings and the significance of activity in Anishinaabe or Métis culture. Students then participated in the activity, through which the mathematics emerged.

One example we showed was the mathematics of looming (Beatty & Ruddy, 2019). As the name implies, looming is a type of beading that is done on a loom and involves stringing beads onto weft threads and weaving them through warp threads (Figure 1).

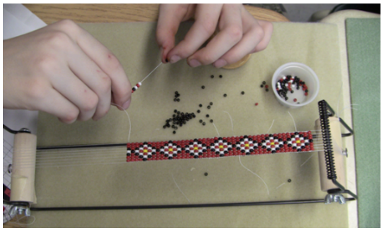

Figure 1. Creating beadwork using a loom.

The first step in creating a design is to define the space to be used by using a template (Figure 2). The columns represent the weft threads, and the number of columns corresponds to the horizontal length of the beadwork. The rows represent the spaces between the warp threads, and the number of rows corresponds to the width of the beadwork. The horizontal and vertical grid and the two-dimensional patterns created within the grid offered the potential for exploring number sense, and patterning and algebraic reasoning.

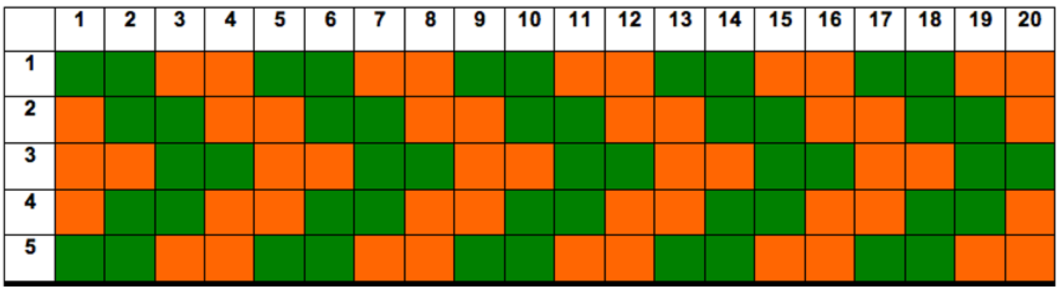


Figure 2. A completed bead looming template.

One pattern introduced to the students was a chevron pattern. In order to make identifying the unit of repeat easier, we put 'occluders' on the board. These could be dragged to hide different parts of the pattern in order to help students discern which columns were repeating in order to identify the unit of repeat, or 'pattern core'. The classroom teacher moved the occluders to highlight the succession of cores (1–4, 5–8 etc.) and students agreed that each was an iteration of the original unit of repeat. She then took a sample of the unit of repeat and pulled it over the completed template to demonstrate how the unit repeated along the template (Figure 3).

She asked the students how many times the core repeated.

> Teacher*: How many of the core did I find in my 20 columns?*
> Liam: *Oh now it makes sense! Five times 4!*
> Jonathan: *Five times of 4.*
> Anne: *Let's write that thinking down.* [Writes 5x4]. *Five groups of 4…*
> Students: *Equals 20.*

Figure 3. Identifying the unit of repeat, or 'pattern core', of a chevron pattern.

Through these investigations, the students co-constructed an understanding of multiplicative thinking, and the language of multiplication was introduced. When determining the unit of repeat, students identified groups of columns that made up the unit or pattern core, then focused on how many iterations of the unit was possible on the 20 column template, and progressed from skip counting by fours, to repeated addition, to thinking about 'groups of'. In addition, students also explored the number of beads in each pattern core and used that to determine the total number of beads in one template through skip counting by units of 20 to determine the number of total beads.

Another example we showed was exploring the mathematics of circular medallions in Grade 10 (Figure 5).

Figure 5. Beaded circular medallions.

Using the planning template (Figure 6), the students discovered that if they started with a centre bead, each successive ring held one bead per section (8 beads for ring one, 16 beads for ring two, etc.).

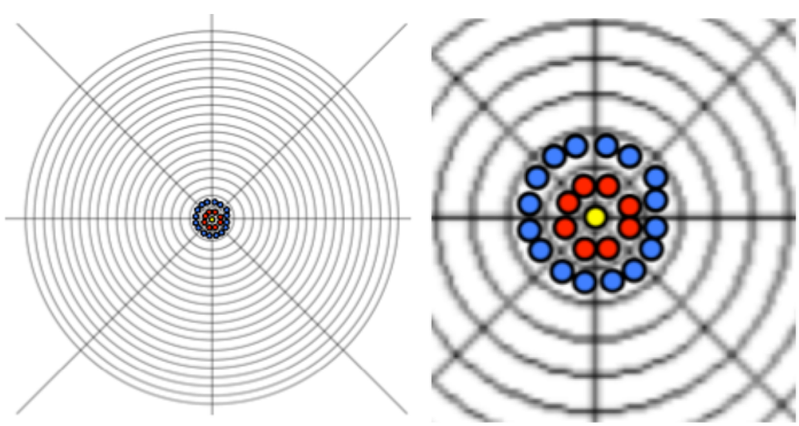

Figure 6. Medallion template illustrating the placement of beads.

Given this information, students were asked to find how many beads they would need for a 20-ring medallion and to find a rule (generalization) that would allow them to accurately predict the number of beads needed for any size medallion. The students noticed a pattern of consecutive odd square numbers, and most students identified the generalization $(2n+1)^2$, where $n$ represented ring number. Some students also came up with the expression $4x^2+4x+1$. The students initially were unsure how the two expressions represented the growth of beads in the medallion or how the expressions related to each other. They explored the expressions by creating vertical lines of circular counters to compare the total number of beads of different sizes of medallions and used these to construct a graphical representation, which demonstrated that the growth was exponential because the rate of growth accelerated with each ring (Cooney et al, 2010). They also created consecutive odd squares using tiles in order to decompose the squares to see the different parts of both expressions and find connections between the two (Figure 7).

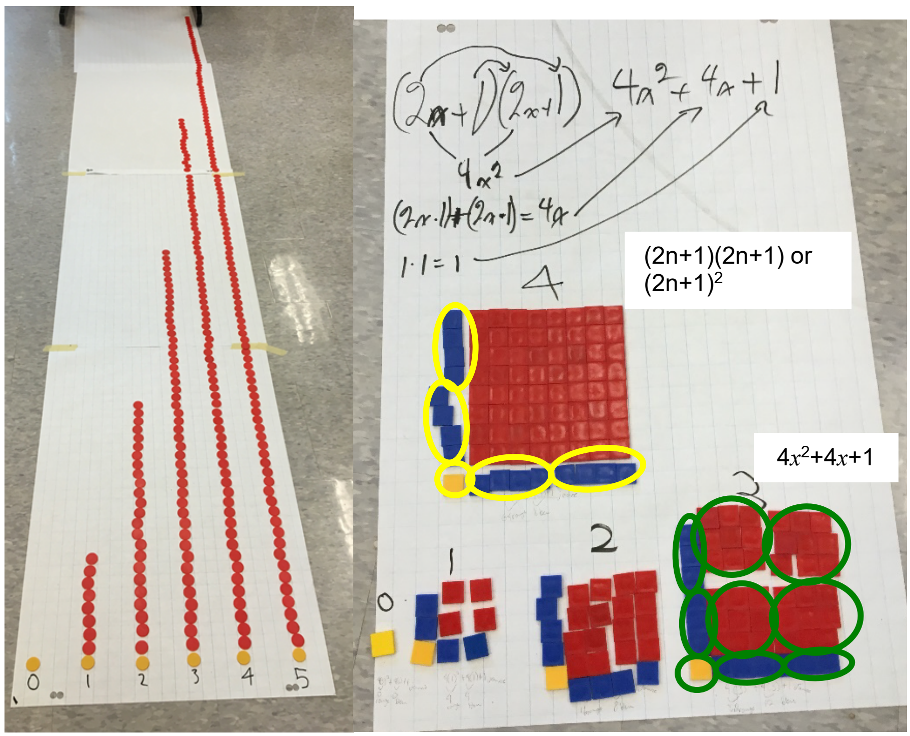


Figure 7. Two concrete representations of the mathematics of beaded circular medallions.

Through these kinds of explorations, the students formed strong relationships with community partners, and their reflections highlighted the importance of these relationships in their learning.

> *This project was the best days of the school year. The beading project was a better way to learn math than normal math worksheets because it's hands-on and has better results, and it's a bit of a history lesson too because you learn about how Métis people would learn and practice math. My favourite element over all was the people. Everyone was always smiling and laughing. They never left our sides without giving some sort of help. I really enjoyed the attitudes of the grown ups in the room—they were interesting people to be around and to hear stories from. I would much rather continue this project for the*

> *rest of the year, not only because of the beading, but mostly because of the people who made some of the best days of my life possible.*
>
> Grade 6 student written reflection, April 2018

Students also expressed an appreciation of learning math in context with community, which they stated helped them (some for the first time) to understand concepts they had learned through rote memorization and symbolic manipulation but that they had not understood at a conceptual level.

> *The Elders talked to us and got to know us individually—it was more one-on-one. And the math—it's a lot easier to understand, and a lot more engaging when we're able to feel the beads. It's a lot more hands-on and it's a lot more visual. It's not just equations being written on the board and us taking notes of the answers, it's trying to find the answer with your hands, and seeing it, and playing with the beads to get to the answer.*
>
> Grade 10 student interview, May 2019

## COMMUNITY PARTNERS EXPERIENCES

Attending to the mathematics of the processes of designing and creating artifacts, and the mathematics of the finished products, led artists to recognize that they had already been thinking mathematically within their artistry. This was a reciprocal revelation in that it transformed how we see Indigenous artistry but also transformed how we see mathematics. Community partners developed deep understanding of complex mathematical concepts, which strengthened their awareness of the place mathematics has in Indigenous culture and strengthened their own cultural and mathematical self-identities. Two examples are presented below.

C.R. is an Algonquin beader and has been part of the study for eight years. In that time, she has gone from developing a comfort with math, an example of reciprocity, to becoming a trailblazer in this work by mentoring other Indigenous artists in communities across the province. She has presented at research conferences, won a University award for her role in the study, been co-applicant on a number of federal research grants, and co-authored a book chapter in a resource on mathematics instruction. Collaborating in this work has been transformative for C.R., who now refers to herself as an ethnomathematician, and contrasts this to the messaging she received throughout her childhood.

> *When I was in school, I was told that Native people can't do math. I was told that my entire education in elementary school and in high school. When I went to college I was part of the first cohort of Native Community and Social Development, and there was a math component. And the students stopped going to that program because of the math, so the college dropped the math component. So, it's not just me. We've been told our whole lives we can't do math. That's why I do this work, that's what motivates me every day. I'm not going to let another kid grow up believing that they can't do math because that's awful. I didn't know I could do math until I was over 40 years old. And now I teach it.*
>
> Conference presentation, May 2019

Another example is L.M., a Métis artist and expert beader, who had trepidation about participating in a study focused on making mathematical connections. During an interview after the first two projects she was involved with, L.M. was asked what she would be taking away from the experience.

> *Math, definitely. I see it in a whole different light now. Well before, I just did beading, that's the way I was taught. I wasn't taught by anybody that it included math. And when they were breaking it down into the math components, even designing, I'm thinking I do that, but I never knew I did it. I just didn't know the names. So then I got really excited. Now I'm looking at my designing and everything else more so on the math side of it, rather than just designing. Like, it's amazing! At first I didn't understand the math, but I was able to still help out, and then when I did get the math it was…ok…a whole new door's opened! I like beading more now!*

Interview, June 2018

Since this interview, L.M. has co-taught both the skill and the mathematics of beadwork in a number of elementary classrooms. She has also presented to teachers at a school board-level professional development learning fair.

In each project, the artists we worked with did not initially perceive themselves as mathematical thinkers, however, through the process of co-planning and co-teaching and particularly from working and learning with students, they came to understand that they had always understood the mathematics of their artistry but had not made a formal connection to 'school math'. This project provided an opportunity for artists to recognize the culturally embedded mathematical competencies with which they had already been engaging.

## SETTLER EDUCATORS EXPERIENCES

Non-Indigenous educators came to recognize and appreciate the complex mathematics that emerged during the projects. "*This project has opened my eyes. When we bring Algonquin culture into our school and show that we value it, the learning that comes from it is phenomenal. It's like mathematics that I've never dreamed of!*" They also unpacked their own positions within the work and their responsibilities, including an appreciation of the expertise of community artists, and their commitment to continuing the work in a good way.

> *The community team members opened up so many pathways for us with the rest of the community. Now we have meetings with Elders and ask for permission for the work we're going to do. And having community working with us in the classroom means we know we're doing the work in a good way, because before, we were never sure. We weren't sure if we were doing a service or a disservice to our First Nations students, or to the community. These connections have been so beneficial. It's really phenomenal to be part of this change. We're building bridges, we're talking, we're creating dialogue and we're moving forward together.*

Grade 6/7 teacher interview

## CONCLUSION

Decolonizing education through reconciliation is a focus of the final report of Canada's Truth and Reconciliation Commission *Calls to Action* (2015). We believe that ethical relationality may be a powerful step towards reconciliation, and therefore decolonization in the domain of mathematics education. In our work, decolonization influenced both the teaching of mathematics and also the participatory action research process. The importance of relationships, and their connection to reconciliation, was summarized by an Anishinaabe research team member:

> *When we bring community into the classroom, we need to ask, what are you giving back? And for me, that's where I see the beauty of this project—that giving back to community partners. Watching their relationships developing with the students, and the children remembering that relationship. For example, in one classroom we were in the children were talking about how it was the best thing they'd done all year because of the relationships with the people in the room. Giving the community partners an opportunity to build relationships with youth, having their skills honoured, that's reciprocity. That's reconciliation.*
>
> Interview, December, 2018

Educational researchers have a responsibility to work with community to prioritize Indigenous knowledge and ways of knowing. Although the content of the work presented here is mathematics, what emerged more significantly in all of the projects was the centrality of reciprocal relationships.

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# RELATIONSHIPS WITH/IN/AROUND MATHEMATICS

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*University of Alberta*

> *In a fractured age,*
> *when cynicism is god,*
> *here is a possible heresy:*
> *we live by stories,*
> *we also live in them.*
> *One way or another we are living the stories*
> *planted in us early or along the way,*
> *we are also living the stories we planted –*
> *knowingly or unknowingly - in ourselves.*
> *We live the stories that either give our lives meaning,*
> *or negate it with meaninglessness.*
> *If we change the stories we live by,*
> *quite possibly we change or lives.* (Okri, 1997, p. 46)

## INSPIRATION

I was inspired by some writing that my friends and I have done over the last two years. In 2018 Jennifer Thom and I explored the notions of mathematics classrooms as living landscapes (as "distinctive features of a sphere of activity" ("Landscape, n.d.) and the ways mathematical ideas might be considered a topography or "writing of place" (Topography, n.d.) of the classroom (Thom & Glanfield, 2018). In 2019, Jennifer and I were joined by Métis Elder and scholar, Elmer Ghostkeeper, as we drew upon the notions of living landscapes and topography to re-conceptualize science, technology, engineering, and mathematics (i.e., STEM) as a "cultures' reading, experiencing, and building or writing of place" (Glanfield et al., 2019, p. 11).

In this writing, then, I will attempt to share with you the stories that were planted in me and the stories I planted in myself, knowingly or unknowingly, to articulate the ways in which relationships with/in/around mathematics could be conceived as a topography of the storied landscape (i.e., a distinctive feature of a sphere of activity) of mathematics education that I have experienced as a learner, teacher, teacher educator, and researcher.

## ACKNOWLEDGEMENTS

I start this writing by first acknowledging land and ancestors. I acknowledge that I gave this talk, and was visiting, the territories of Indigenous relatives, who have lived for thousands of years on Turtle Island and in the place now called Antigonish and Nova Scotia. I express my gratitude to those ancestors who took care of the land, lived with the land, for all of those years so that the CMESG/GCEDM community could gather together.

Secondly, I also want to acknowledge my ancestors, those that have taught me and walked alongside me as I have grown up and those ancestors that are in the Spirit World. These ancestors have taken care of me, touched my life and living, and have called me to live true to my spirit, to be honest to myself, and to accept myself the way the Creator made me. These ancestors include my family, great grandparents, grandparents, parents, aunties and uncles; include the children, teachers, teacher candidates, graduate students, and CMESG/GCEDM friends and colleagues that have walked alongside me in mathematics education; many Indigenous knowledge keepers and Elders, Dr. Cecil King, Dr. Eber Hampton, Dr. Edward Doolittle, late Elder Reverend Dayyn Umperville, late Elder Ken Goodwill, late Corinne Mount Pleasant Jétte, Dr. Lorna Williams, late Elder Marge Friedel, late Elder Narcisse Blood, Elder Elmer Ghostkeeper, Elder Isabelle Kootenay, Elder Mary Cardinal Collins, Elder Dr. Francis Whiskeyjack, Elder Francis Alexis, and Elder Alsena White; my teachers Sister Nativity, Mrs. Meardi, Dr. Tom Kieren, Dr. Jean Clandinin, and Dr. Margaret Haughey.

Next, I need to acknowledge where I am from, the land that shaped my experiences and life. I am from the part of Turtle Island that is now known as Northeastern Alberta. I was born in Lac La Biche and spent my growing up years on a ranger station situated on the edge of the boreal forest in a place called Wandering River. From Wandering River our family moved to Lac La Biche and then to Fort McMurray where I finished my K–12 education.

I grew up surrounded by trees such as black poplar, white poplar, birch, and black spruce and in a loving family. My father, the son of English immigrants, was a forest officer who had moved to northeastern Alberta in 1956. My mother was a descendant of Scottish and French fur traders who had fallen in love and had children with First Nations and Métis women. Our ancestors lived mostly in an area now known as Fort Chipewyan in the far northeastern corner of Alberta, Fort McMurray, and in the Edmonton area. While I was growing up, my maternal grandparents lived in Fort McMurray (where my mother was born and my parents were married); my maternal great grandparents lived in Fort Chipewyan; and my paternal grandparents had gone to the Spirit World before I was born. Our family spent a lot of time with my mother's family.

## STORIES I TELL

I believe that the stories that we tell about our relationships with mathematics and what it means to be mathematical shapes our journeys with learning and teaching mathematics. The stories that we tell about our relationships with mathematics are shaped by the experiences (Dewey, 1938) that we live from our early years and become stories that are planted in us, knowingly or unknowingly.

## STORIES I TELL OF BEING MATHEMATICAL IN RELATIONSHIP WITH FAMILY

As I mentioned earlier, my father was a forest officer. His office was a room connected to our living space, which meant that I grew up listening to the Forestry radio and paying attention to maps. Maps were very much a part of my father's work; there was a large map of the province of Alberta on the wall of his office and we would often hear radio messages related to locations on the map. For example, a lightning strike, or a smoke, might have been spotted in a particular location and the radio would share the suspected location. For example, section 32, township 20, range 14, west of the $4^{th}$ meridian. My dad would then plot the point on the large map. Dad would also have topographical maps as he would need to understand the topography of a particular region of the district if the Forestry were planning work related to timber management. From a very early age, Dad showed me how to read the locations on the large map and read those topographical maps with the 3-D glasses. Dad would talk about what could be learned about the topography of the region by looking at those maps.

My mother regularly engaged me in counting and reading. She would count up and count down when she was doing her morning exercises. I would mimic that counting with her—for example we would count up to 20 and then back to zero when she was doing her leg exercises. I remember that she would always count out loud when she was casting stitches onto her knitting needles. Over sustained periods of time I recognized the importance of quantity, and as I continued to grow up, I learned about one-to-one correspondence as I was put in charge of setting the table for meals. One of mom's monthly activities was balancing the checking account. I would sit next to her as she performed that task at the kitchen table. She would teach me about the concepts of adding, subtracting, and balance during these times.

My maternal grandparents were very much a part of my growing up. We regularly spent time with them, not only for special occasions but time each summer in Fort McMurray. My grandfather had had several different jobs in his lifetime, but I mostly remember him working for a company in the oil sands. His father and grandfather were master carpenters and had been responsible for building boats for the fur trade and the river trade in Northeastern Alberta. My grandfather had learned from his father and grandfather about carpentry, and I can recall being with my grandfather in his workshop where he would always have one or two projects in the works. He was constantly measuring with different tools. I learned the importance of measuring from him. My grandfather also grew up on the land as his mother and maternal grandfather were trappers and fur traders. Grandpa was very good at noticing changes with the land. Another favorite memory that I have of time with my grandfather is walking on the land around the ranger station when they would visit us. Grandpa would often ask me to notice how the trees were growing and would ask me why I thought that they would be growing in that way. We would look at the fauna around the tress and he would talk about the importance of the relations between the trees and their surroundings. He would ask me to notice the different colours of the leaves—varying shades of green and ask me why I thought the leaves could be different shades. He encouraged me to observe and pay attention to seemingly inconsequential things, like the different shades of green of leaves, ask me to think about these differences in relation to being with the land, and then would teach me why these observations might be in place.

My grandmother loved to play card games of any kind, crochet, and cook. She had grown up as the daughter of a Hudson Bay manager and in a family where there were many women who beaded. Although my grandmother did not have the patience required to teach me how to bead or crochet, she did love to play cards, and my earliest memories of time with grandmother was watching her during card games. As I grew up, I was increasingly intrigued by the different types of card games and was eventually invited to play. I have vivid memories of playing crib, and my 'job' was to count each person's hand and then score. During these experiences I learned about breaking apart a number and that there are many different ways of representing a number. I also was learning a great deal about probability as I would listen to the card games.

### A STORY I TELL OF BEING MATHEMATICAL IN RELATIONSHIP WITH SISTER NATIVITY

I started first grade in a small Francophone school in rural Alberta. There were three grades per room in the three-room school. In my classroom, I remember there being two rows of first grade, two rows of second grade, and one row of third grade. My earliest memory of school and mathematics was that Sister Nativity would give each child in the classroom a pile of popsicle sticks to count as our mathematics lesson. We would all bundle the sticks in groups of five. Following the bundle activity each of us would then report about the number of bundles (and loose sticks not in a bundle) to the whole class. While we were reporting, Sister Nativity would record our reports. If a child reported three bundles and two loose sticks, Sister Nativity would write on the board:

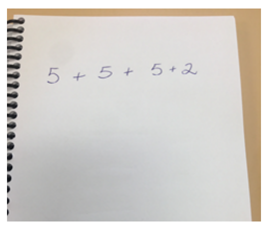


Figure 1.

Then, in unison, we would 'count' the number of sticks together, "Five, ten, fifteen, sixteen, seventeen." The child with this number of bundles and sticks on their desk would have to point to each bundle and stick as it was counted.

If a child had one bundle and two loose sticks on their desk, then Sister Nativity would add to the board the new sentence and would write:

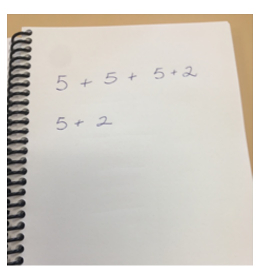


Figure 2.

Then, in unison, we would 'count' the number of sticks together, "Five, six, seven." And, the child at the desk would point to each as we collectively 'counted.'

Through each of the grade levels, Sister Nativity would introduce different ways of writing the sentences. For example, a Grade 3 grade level representation of Figure 1 would be written as in Figure 3.

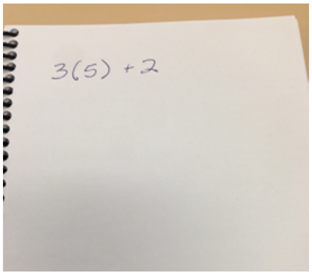


Figure 3.

Our chant would then be "Three groups of five plus two, fifteen, sixteen, seventeen." I do not remember how the third-grade children would point to their sticks when we counted.

Each child in each grade level would have their own number on the board, written in different ways, and we were all involved in the reading of those numbers. I do not think I really knew what I was learning at the time, but I remember Sister Nativity using this activity repeatedly in our lessons. I eventually learned that I could write two groups of five as ten; or two groups of five as $5 + 5$; or two groups of five as $2(5)$; and I began to learn equivalency of representations of quantity. I loved to pay attention to how the second and third graders were writing numbers and loved the patterns that I began to notice. I was known as being mathematical and as someone who was 'good' at 'mathematics' in Sister Nativity's classroom. These stories stayed with me throughout my K–12 schooling and when I finished high school, I pursued a Science degree, with a focus in mathematics, and then an Education degree, with a major in secondary mathematics.

## A STORY I TELL OF BEING A SECONDARY MATHEMATICS TEACHER

When I first started working as a teacher, I was very curriculum focused. I was very strict in following the program of studies and the textbook that was assigned. After the first six months of teaching I felt lost and felt like I was not complete, that something was not right for me as a human being, as a teacher, and as someone who loved mathematics. As I was driving the highways in central BC that summer, I realized that I was not teaching mathematics the way that I saw mathematics, as relationships. I returned to school to begin my second-year teaching and knew I wanted to teach in a different way, a way that focused on the relationships that I saw within mathematics. I started to love teaching as I began sharing my understanding of the relationships that existed in mathematics. As I gained in confidence, I became interested in how students saw relationships within mathematics. When I learned about how my students saw the relationships and realized that some of my students did not see any relationships, they just saw a group of discrete topics, and I noticed that they would forget those topics. I became fascinated with learning more about the ways in which mathematical concepts unfolded throughout the K–12 school curriculum, attending professional development sessions about elementary school mathematics, and always starting the beginning of each topic of study or unit of study with attempting to explicate the ways that I saw the new topic or unit in relation to previous units and topics that were covered in the school curriculum. I would plan for opportunities near the end of the topic or unit to ensure that my students had a chance to reflect on, and share, the relationships that they were finding as they studied the topic or unit.

## STORIES I TELL OF BEING A MATHEMATICS CURRICULUM LEADER

This fascination with the relationships within school mathematics developed further when I worked for a provincial ministry of education in designing and implementing mathematics programs of study and then developing the Grade 12 Diploma Examination in Alberta. As I reflect on the experiences of working with the ministry of education, I realized the importance of being in spaces and having conversations with people over sustained periods of time about our understandings of mathematical concepts, and about how we think about teaching mathematics helped me to further develop the ways in which I was able to articulate my understanding of relationships that I saw in mathematics.

It was while I was engaged in this work with the ministry of education that I had opportunities to work with teachers and Indigenous knowledge and language keepers across Northern Canada. I spent two weeks in a community in the Northwest Territories (NWT) one spring and learned a bit about the Dogrib language. The primary teachers in the school were working closely with Indigenous language holders as a way to engage the children with the Dogrib language. The language holders introduced me to the ways that the Dogrib language was different than English and that the ways that numbers were said and represented were different than base 10. It was while I was in this community that I learned that Indigenous languages were verb-based and not noun-based, like I was used to, in English. This was really made clear to me when the primary school teachers were focusing on geometric shapes. The language holders were teaching us the ways those shapes would be described in Dogrib. It was also made clear that, traditionally, in Dogrib, you would not have words for numbers that you would not need. For example, you would never 'need' a number like one thousand as that was bigger than any community. You would just use the word 'many.' This visit left me fascinated, but I did not yet have enough experience to do much with this information.

I then had a chance to pursue doctoral work in mathematics education and was greatly influenced by professors in my program. I learned about an enactivist (Varela et al., 1996; Maturana & Varela, 1987) view of cognition alongside Dr. Kieren. Enactivism suggests that an individual brings forth meaning through the interaction of a physical world and the individual. Proulx (2008) writes,

> The world of meaning is not in us, nor in the physical world, it is in the interaction of both in a mutually affective relationship. With my structure I make sense and give meaning to that physical world and bring forth a world of significance. It is a world of significance that is enabled by my structure, and also by the environment that I interact with. It is my structure that allows me to "see" or perceive things in the physical world, and so my structure allows me to give meaning to the attributes of the physical world. (p. 21)

I also learned about narrative inquiry (Clandinin & Connelly, 2000) as a research methodology alongside Dr. Clandinin.

> Narrative inquiry is a way of understanding experience. It is a collaboration between researchers and participants over time, in a place or series of places, and in social interaction with milieus. An inquirer enters this matrix in the midst and progresses in the same spirit, concluding the inquiry still in the midst of living and telling, reliving and retelling, the stories of the experiences that made up people's lives, both individual and social. (Clandinin & Connelly, 2000, p. 20)

These two theoretical framings helped me to understand the stories that were planted in me or that I planted in myself in relation to mathematics and mathematics education. And, it was these two theoretical framings that helped me to transition into moving to a new place and working in at a University that I had never before visited.

### STORIES OF LOCATING MYSELF AS AN INDIGENOUS MATHEMATICS EDUCATOR

When I first moved to the new place, I met Dr. Cecil King and the late Elder Reverend Danny Umperville. These two Indigenous knowledge holders and teachers taught me about the differences between an Indigenous worldview and a 'Western worldview'. In listening to the teachings of these two knowledge holders, I came to further make sense of the experiences that I had had with my family. I also had the chance to meet other Indigenous knowledge holders through a network of people who were supporting mathematics and science teachers of Indigenous youth, Corinne Mount Pleasant Jétte and Métis scholar Elmer Ghostkeeper.

I was so surprised when I met Elmer, he was another Métis person who liked mathematics. I had held a story with me from my first year of university, when one person, when learning that I was Métis, said, "I didn't realize that You people can do mathematics." I walked away and was silent. I did not know how to respond. It was the first time in my life that my identity as a Métis person would somehow contribute to something I could not do. I stayed silent about being Métis throughout the rest of my undergraduate degrees and my master's degree because I did not know how to respond to that comment, or any comment related to the ways in which my identity was related to what I could or could not study. I had not yet learned of the ways to reply to comments related to my identity.

I also met Dr. Edward Doolittle, who is Mohawk and a mathematician. Edward was the second 'native' person that I had met who liked mathematics. Not only did he like mathematics, he also had taken time to study the Mohawk language. Edward and I became friends and colleagues as we engaged in conversations with Indigenous and non-Indigenous teachers and knowledge holders.

Each of these Indigenous knowledge holders taught me and planted in me, stories of the ways that an I as a Métis mathematics educator and researcher could 'be' in an academy. They taught me about the complexity of residential school experiences, of the differences between what it meant to live with the land versus living off of the land, they taught me that numbers are alive, and most importantly they taught me to embrace Indigenous knowledge and wisdom traditions to further develop my understandings of relationships within and around mathematics. They also provided me with the possibility of inquiring into my experiences in the NWT working with Dogrib language holders.

## THE TRUTH ABOUT STORIES

I continue to be inspired by a 'landscape' filled with many diverse colleagues and friends who all walk alongside me, metaphorically and physically, including those whom are now in the Spirit World. These teachers generate a sphere of activity where I bring forth the meanings that I make of my experiences in my living as an Indigenous mathematics educator and researcher.

I started this writing with Ben Okri's words about the stories planted in us…and the way that we live in those stories. I have shared those stories that I've lived with/in/around mathematics as I have come to know myself as an Indigenous mathematics educator and researcher, who works at a university. I leave this writing with Thomas King's (2003) words, "The truth about stories is that that's all we are" (p. 153) and Clandinin and Connelly's (2000) words, "…all of us, lead storied lives on storied landscapes" (p. 8).

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# RENEWING SCHOOL MATHEMATICS THROUGH STUDENTS' WAYS OF DOING MENTAL MATHEMATICS[1]

# RENOUVELER LES MATHÉMATIQUES SCOLAIRES EN S'INSPIRANT DES PROCESSUS DE CALCUL MENTAL DES ÉLÈVES[1]

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> *What I have discovered is that through this process not only have the students learned, but also I have learned much about geometry from them. At first I was surprised—How could I, an expert in geometry, learn from students? But this learning has continued for twenty years and I now expect its occurrence. In fact, as I expect it more and more and learn to listen more effectively to them, I find that a larger portion of the students in the class are showing me something about geometry that I have never seen before.* (Henderson, 1996, p. 46)

## INTRODUCTION

The work presented here is about, of and for mathematics. I have quoted Henderson for two reasons. First, his recent departure from the field of mathematics and mathematics education is an occasion to pay tribute to the innovative nature of his thoughts and ideas on mathematics. Second, and more important for the work presented here, his papers continually remind us that mathematics is not the sole possession of mathematicians and that anybody who does mathematics can have a say in, and offer inspirational ideas for, its development: mathematics is not an epistemological absolute, but mostly a collective epistemological endeavour[2]. What Henderson points out is that students, and not only university students, can produce *new* mathematics, that is, mathematics that can be added to our current body of the discipline. This is what I discuss here.

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[1] La Séance Thématique était bilingue. C'est pour cette raison que le texte pour les actes lui aussi maille le français et l'anglais. This Topic Session was bilingual. It is for this reason that the paper in the proceedings interweaves both English and French.

[2] See, for that matter, a recent paper of mine and colleagues on (questioning) the place of mathematicians as the unique referent for school mathematics (Maheux, Proulx, L'Italien-Bruneau, & Lavallée-Lamarche, 2019).

Lorsque Peter Liljedahl, le président du GCEDM, m'a invité de la part du comité exécutif à offrir une Séance Thématique, il m'a en somme expliqué que mes travaux faisaient intervenir beaucoup de mathématiques et que ceci serait d'intérêt pour les membres de la communauté. Mes travaux étant centrés sur le calcul mental et les stratégies de résolution, ceci m'apparaissait une belle façon de les décrire. J'ai donc accepté avec enthousiasme, surtout que Peter est un vrai *gentleman*. À vrai dire, cette invitation était tout à fait à propos. Récemment, dans notre *Laboratoire Épistémologie et Activité Mathématique* (www.leam.uqam.ca), nous avons commencé à porter une attention particulière aux dimensions mathématiques des travaux que nous conduisons, au point d'en faire un objet concret d'investigation. Cette Séance Thématique se présentait alors comme une occasion de creuser plus en profondeur cette dimension mathématique.

As mentioned, my research program is centered on mental mathematics. The topics explored through the mental mathematics work we conduct are however not restricted to numbers and arithmetic, as we delve into algebra, statistics, geometry, functions, systems of equations, measurement, for example. We work in regular elementary, secondary and university classrooms, following teachers' invitations, where we give students a variety of tasks to solve. In these sessions, the participating students are usually given relatively short amounts of time to answer their tasks (from 15 to 20 seconds), posed orally and/or on the board, for which they cannot use paper and pencil or any other material aids. When time is up, in a plenary manner, students are asked to share and explain their strategies for solving the tasks given. In these contexts, because of time constraints, participants can barely refer to known procedures because they are often too time-consuming and hence non-economical in this context. This leads them to engage with particular strategies, often quite different from usual ones, which bring forth innovative mathematical ideas.

De façon plus précise, nos recherches se centrent sur l'étude et l'analyse des stratégies déployées par les élèves dans ces contextes de calcul mental, soit leurs richesses sur le plan mathématique pour résoudre les tâches, leurs caractéristiques spécifiques (vs les stratégies en contexte de papier et crayon), leur potentiel pour stimuler l'avancée des mathématiques en classe, etc. Ces idées forment le cœur de notre travail de recherche et ont été, et continuent d'être, diffusées dans la communauté scientifique (voir p. ex. Proulx, 2013, 2015, 2017, 2019; Proulx et al., 2017). Toutefois, dans cette Séance Thématique, j'aborde une dimension complémentaire qui émerge de ce travail de recherche. En lien avec les propos d'Henderson préalablement cités, je veux parler des mathématiques produites par les élèves dans ces contextes de calcul mental. En particulier, je m'intéresse ici à ce qui peut être soutiré au niveau mathématique des façons de faire des élèves pour résoudre les tâches de calcul mental, c'est-à-dire à ce que les façons de résoudre des élèves peuvent nous apporter mathématiquement.

## SETTING UP THE TOPIC SESSION / CONTEXTUALISER LA SÉANCE THÉMATIQUE

As reported above, in the mental mathematics studies we conduct, the students often offer strategies, ways of doing and thinking about concepts, that are mathematically interesting and insightful. At times, these mathematical ideas differ quite significantly from what one could consider 'standard' mathematical understandings, methods or ways of doing. This leads us to be quite interested in the nature of what they do and what it can mean mathematically. We began investigating some of the students' ways of solving and doing for drawing out mathematical ideas from them and to inspire ourselves mathematically from their work. In other words, we wanted to see what we could gain, mathematically, from their strategies and solving processes in mental mathematics contexts.

Habituellement, en didactique des mathématiques, nous tentons de comprendre le sens que les élèves donnent aux mathématiques en le comparant (implicitement ou explicitement) avec le corps de connaissances mathématiques que nous connaissons, utilisé alors comme référent. Comme communauté, cette approche nous a permis de développer des compréhensions importantes sur les façons de comprendre les mathématiques des élèves, et d'imaginer des façons de les aider à mieux comprendre ces mathématiques en retour. Toutefois, ici dans cette Séance Thématique, je propose de faire le contraire. Je propose d'analyser les façons de faire des élèves pour informer en retour les mathématiques scolaires elles-mêmes.

This can be seen as an occasion, as Steffe (Moore, 2017) puts it, for school mathematical research. The exploration of students' mathematical ideas, concepts or methods can help decipher elements of mathematical interest. These mathematical elements might be ones that are implicit in our ways of doing and may be kept hidden, or again ones that we are not necessarily aware anymore of using when we do mathematics. As well, these elements can potentially cast new light on our usual mathematical ideas, concepts and methods to understand this mathematics on new grounds. Informally, I call this the renewing of school mathematics, for lack of a better word. This is, obviously, work in progress. I present here some initial examples of the nature of these mathematical investigations.

## EXAMPLES OF/FOR RENEWING SCHOOL MATHEMATICS / EXEMPLE DE RENOUVELLEMENT DES MATHÉMATIQUES SCOLAIRES

### EXEMPLE 1: SUR LES OPÉRATIONS DE FONCTIONS (TIRÉ DE PROULX, 2015)

Dans une de nos études, des élèves de 5<sup>e</sup> Secondaire (15–16 ans) devaient résoudre mentalement des tâches graphiques reliées aux opérations de fonctions. Un exemple typique de tâche qui leur était donné leur demandait d'opérer mentalement (+, − , ×,÷) sur des fonctions représentées graphiquement dans le même plan cartésien, sans avoir recours à aucun papier ni crayon. Plusieurs stratégies, selon les tâches, ont été produites. Par exemple, pour la tâche suivante (Figure 1), les élèves devaient retrouver la fonction $g$ qui avait été additionnée à la fonction $f$ si la fonction résultante $f + g$ était donnée.

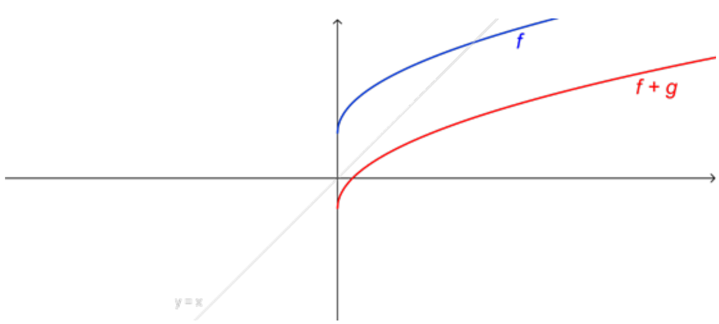


Figure 1. Tâche graphique demandant de trouver la fonction $g$ additionnée à la fonction $f$.

In order to decipher which function $g$ was, some students expressed that "*each function was parallel to the other*," that $g$ had to be a constant function "*for the curve to be translated down*," and that this constant function was "*negative for bringing the curve lower*" as a result of its addition. Initiated from a geometrical understanding of the graphical representation, this strategy brings forth an interesting mathematical tool, that is, the parallelism of curves in order to understand translations in a graphical environment, through operations, etc. This view of operations in terms of parallelism of curves contrasts with algebraic strategies usually used that add numerically a value to each image-length in the graph: here, it is a property of the curve taken as a whole that is affected, translated in parallel through being added to a constant function. Somehow, this leads to understanding the effect of adding a constant value, through the constant function, to another function. In this instance, for example, $f(x) = \sqrt{x} + 5$ to which $g(x) = -7$ is added produces a parallel function $(f + g)(x) = \sqrt{x} - 2$.

Toutefois, il y a plus à réfléchir mathématiquement au niveau de la représentation *algébrique* d'une fonction. Considérer la fonction comme un tout qui est translaté, qui est parallèle à une autre, amène à réfléchir sur ce qui se passe algébriquement lors de l'addition de deux fonctions. Il est possible de penser à l'addition de $f(x) = \sqrt{x} + 5$ et $g(x) = -7$ comme étant uniquement un exercice de manipulations algébriques qui se réduit à $\sqrt{x} + 5 - 7 = \sqrt{x} - 2$. Toutefois, prendre la fonction comme un tout permet de considérer l'ensemble des longueurs d'images de la fonction $f + g$ et de la considérer parallèle à l'autre ensemble des longueurs d'images de la fonction $f$. Sous cet angle, l'addition de la fonction $f$ à la fonction g à travers $\sqrt{x} + 5$ et $-7$ ne signifie plus en effet la même chose qu'une manipulation algébrique qu'un élève de 2e secondaire ferait. En s'appuyant sur ce que les élèves de 5e secondaire ont fait, le $\sqrt{x} + 5$ représente ici l'ensemble complet des longueurs d'images de $f$, auquel l'ensemble complet des longueurs d'images de $g$, soit $-7$, est additionné. Ceci montre que bien que nous puissions penser faire de simples manipulations algébriques, en fait, nous pouvons imaginer travailler avec des ensembles complets de longueurs d'images pour chacune des fonctions « algébriques » que nous additionnons. En d'autres mots, les élèves offrent un sens aux représentations algébriques en termes de longueurs d'images, et non simplement en termes de manipulations d'expressions algébriques, ce qui représente une dimension mathématique fort intéressante.

In another case, still in this same study, students had to find the function resulting from the addition of $f$ and $g$ in the graphical representation of Figure 2.

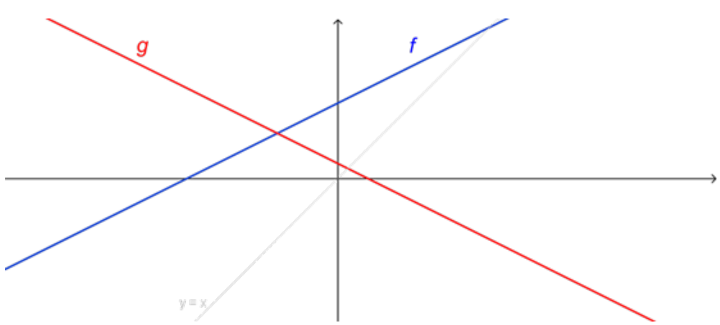


Figure 2. Task requiring the addition of two linear functions.

Dans ce cas, plusieurs élèves ont porté leur attention sur des points spécifiques des fonctions $f$ et $g$, tel que le montre la Figure 3 : (1) où $f$ croise l'axe des $x$ (abscisse à l'origine); (2) où $f$ et $g$ se croisent; (3) où $f$ et $g$ croisent l'axe des $y$ (ordonnée à l'origine); (4) où $g$ croise l'axe des $x$ (abscisse à l'origine).

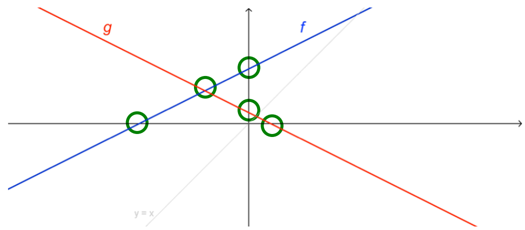


Figure 3. Points soulignés par les élèves pour additionner les deux fonctions.

This strategy, focused on significant points that enable one to determine where the function is, leads to insights in relation to what happens if we were to multiply these two functions (instead of adding them here). Paying attention to points 1 and 4 permits one to evaluate the general shape of image-length for an $x$ smaller than that at point 1: with negative values of $f$ multiplied with positive images of $g$ giving negative values in its multiplication. The same is true for images with an $x$ bigger than the one of point 4. For values in $x$ situated between points 1 and 4, the multiplication of images would give a positive value, leading one to see the quadratic function ($2^{nd}$ degree) created by the multiplication of two linear functions ($1^{st}$ degree). Albeit one can make sense of it algebraically, this graphically explains why the resulting curve would be a parabola and thus why the resulting function of a multiplication is a curve of $2^{nd}$ degree (and can also be linked to the study of inflexion points and zeros, obtaining in this case the following analytical table $[+|-|+]$). The students' entry in the task through points of significance opens toward an added sensitivity, this time for envisioning multiplication; a renewal of graphical elements related to these functions and operations on them.

EXEMPLE 2: SUR LA RÉSOLUTION D'ÉQUATIONS (TIRÉ DE PROULX, 2013)

Dans une autre étude, des étudiants universitaires en mathématiques avaient à résoudre des équations algébriques en contexte de calcul mental, encore sans papier ni crayon. Certaines des stratégies déployées par les étudiants étaient habituelles, alors que d'autres offraient des façons différentes de concevoir ce que signifie résoudre une équation algébrique.

One of these strategies was suggested when students were given the following task: solve $\frac{2}{5}x = \frac{1}{2}$ for $x$. One student explained first to aim at getting rid of the $\frac{1}{2}$ through doubling the entire equation (multiplying by 2). This gave him "*4 fifths of x equals 1*," which he then multiplied by the inverse of $\frac{4}{5}$ to obtain the value of $x$ in this equation. This procedure to find the value of $x$ can lead to rethink what we do when we solve algebraic equations. Somehow, the usual discourse is that we attempt to isolate the $x$ in order to know what it is worth, and this is attempted in potentially the least number of steps efficiently possible. However, in this case, the first step did not aim to work on the $x$ and focused on the equation itself, its form, its structure, to make it simpler. This strategy of transforming the equation by doubling it, albeit maybe local and working only for this specific algebraic equation, underlines the fact that solving an equation for $x$, what we refer to as isolating

$x$, becomes mostly a matter of establishing equivalent algebraic equations. Somehow, through the various steps we go through to solve for $x$, we produce a family of equations that are all equivalent and that all have the same solution for $x$ that makes them an algebraic equality.

Isoler le $x$, alors, ne se résume pas tant à isoler le $x$ qu'à produire une famille d'équations équivalentes (et il y en a une infinité) et d'arriver à une équation qui offre plus facilement la valeur de $x$ satisfaisant l'équation initiale (et toute la famille d'équations équivalentes en fait). Il arrive souvent que cette équation finale soit de la forme « $x = quelque\ chose$ ». Celle-ci est souvent perçue comme l'étape finale qui permet de trouver la valeur de x, mais elle n'est en fait qu'une seule équation possible de la famille entière d'équations algébriques, souvent considérée comme étant la plus simple de toute la famille pour déchiffrer la valeur possible de x (voir Arcavi, 1994, sur ce point). Ainsi, résoudre une équation pour x ne revient pas à isoler le x, mais plutôt à transformer l'équation initiale pour obtenir d'autres équations qui seraient plus faciles à lire pour soutirer la valeur possible de $x$. En effet, lorsque confrontés à l'équation $2x = 2$, plusieurs seront satisfaits de cette étape et pourront affirmer directement que $x = 1$ sans avoir besoin de continuer et d'isoler le $x$ (en divisant par 2 de chaque côté, par exemple). Ce type de travail de l'équation sous l'angle des équations équivalentes et de la famille d'équation qui en découle permet de réfléchir en retour de façon importante sur le sens de la résolution d'équations.

### EXAMPLE 3: ON ARITHMETIC MEAN (SEE PROULX, 2017)

In one study, mental mathematics work was done on statistics with Grade 9 students. Here again, numerous strategies were deployed. For example, the following task was given to students (inspired from *À vos maths! Chenelière éducation*):

> Here are four distributions:
> a) 41, 42, 44, 45;
> b) 41, 42, 43, 44, 45;
> c) 41, 42, 47;
> d) 40, 42, 44, 45.
> Which distributions have the same mean?

Some students explained having paid attention only to the units in each distribution, subtracting for each datum the value of 40. They then added the units and divided them by the number of data, obtaining the same mean for A and B, in this case 3, which became a 43.

Cette stratégie mène à considérer que ce qui est commun dans une distribution peut être « mis de côté » pour analyser d'autres éléments de celle-ci, sans changer l'essence même de la distribution et ses données. De plus, ceci montre qu'il existe une famille de distributions qui n'ont pas tout à fait les mêmes données, mais qui partagent la même structure : même distance à la moyenne, même écart-type, etc. Cette stratégie montre que les distributions « 41, 42, 44, 45 » et « 1, 2, 4, 5 » partagent la même structure, uniquement translatée de 40. La Figure 4 illustre ce principe pour deux distributions continues.

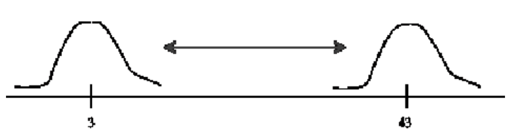


Figure 4. Translation de droite à gauche du graphique d'une distribution

These students' strategy evokes an entire family of distributions that share the same structure, measures, and relationships between data, without having the same values for these data. This also leads to reflect on the fact that there might be, in this entire family of distributions, one distribution that is easier to make computations with for evaluating its mode, average, standard deviation, and so forth, and which could be used to infer back information on the initial distribution. All one has to do in this case is to adjust the values of the measures obtained, for example, for the mean or median, with the distance value between both distributions used (here, in this case, one would have to add 40 to the measure obtained, as it is the 'distance' between both distributions). This strategy leads here to cast some interesting light on some fundamental properties of statistic distributions, ones worth paying attention to.

EXEMPLE 4 : SUR LA RÉSOLUTION D'ÉQUATIONS (TIRÉ DE PROULX, 2013)

Il n'y a pas que les stratégies et réponses dites adéquates qui peuvent être source d'inspiration au niveau mathématique. Un certain nombre de stratégies de calcul mental dites erronées ou incomplètes peuvent aussi contenir des éléments intéressants. À titre d'exemple, lors de l'étude avec des étudiants universitaires sur la résolution d'équations mentionnée auparavant, une des tâches était de résoudre pour $x$ l'équation suivante $\frac{6}{x}=\frac{3}{5}$. Parmi les stratégies diverses offertes, certains étudiants ont proposé de lire l'équation « à l'envers », comme si elle était $\frac{x}{6}=\frac{5}{3}$, expliquant que « Si mon nombre est 6 fois plus gros que $\frac{x}{6}$, alors il est 6 fois plus gros que $\frac{5}{3}$ », donc $\frac{30}{3}$ qui donne 10 comme réponse.

Of interest in this strategy is that this reading of the equation, done 'from below', raises the question as to if both algebraic equations referred to are equivalent. Even if both equations are algebraically valid, and give the same solution of 10, it is not clear how these two equations are equivalent. In effect, in going from one to the other, from $\frac{6}{x}=\frac{3}{5}$ to $\frac{x}{6}=\frac{5}{3}$, this transformation implicates a change in the conditions for which each equation is true. The first equation $\frac{6}{x}=\frac{3}{5}$ has a condition that is absent in the second one $\frac{x}{6}=\frac{5}{3}$, that is, $x=0$ needs to be excluded from one of the possible solutions of $\frac{6}{x}=\frac{3}{5}$ before looking for which values of $x$ can verify the equality. In other words, the conditions for which $\frac{6}{x}=\frac{3}{5}$ is true are double: the value of $x$ needs to be different from 0 and also needs to be such that if one divides 6 by $x$ one gets $\frac{3}{5}$. In the second equation $\frac{x}{6}=\frac{5}{3}$, only the second condition is present and $x=0$ does not need to be excluded from the solution set (even though it is not a solution in this particular instance). In short, neither $\frac{6}{x}=\frac{3}{5}$ nor $\frac{x}{6}=\frac{5}{3}$ are equivalent equations. One can refer to a similar, but extreme case to help see how both equations are not equivalent: $\frac{6}{x}=0$ and $\frac{x}{6}=0$. These two equations do not share the same solutions, even if both equations are the result of the same transformation from one to the other (by reading it 'from below'). This leads to reflections about what it means to produce equivalent algebraic equations: not only does it need to maintain the solution to the equation, but the domain of possibilities for $x$ needs also to remain unchanged. Here the domain was transformed, and thus the pair of equations are no longer equivalent, even if the solution is maintained the same.

## FINAL REMARKS / REMARQUES FINALES

L'analyse mathématique des stratégies d'élèves en contexte de calcul mental présentées dans cette Séance Thématique se veut une occasion de réfléchir, voire de questionner, différents concepts des mathématiques scolaires pour potentiellement les approfondir et/ou en raffiner notre compréhension. Cette analyse veut montrer, d'une certaine façon, que l'étude des (stratégies) mathématiques que les élèves déploient peuvent nous inspirer mathématiquement et même nous informer sur des dimensions mathématiques que nous tenons pour acquises ou que nous avons même parfois oubliées. Dans notre laboratoire, nous croyons que cette avenue a beaucoup de potentiel pour notre champ de recherche, si ce n'est que pour mieux comprendre les mathématiques que nous tentons de faire comprendre aux élèves en retour. Bien qu'intéressantes, les analyses initiales présentées dans cette Séance Thématique représentent toutefois qu'un bref aperçu de ce qui pourrait être fait. Un travail beaucoup plus systématique serait à réaliser pour explorer le potentiel de cette approche, soit de cette utilisation des mathématiques des élèves comme source d'inspiration pour mieux comprendre les mathématiques scolaires.

Let me conclude with an anecdote. I was once in a thesis defense as external examiner, and the overarching research question of the thesis was something like: "How can we help middle school students learn algebra?" At first sight, this question made sense as many of us probably want that. Somehow, however, I felt that this question was horribly one-sided and could have been complemented by another: "How can we help algebra to be learned by middle school students?" When I asked it orally, grins and smiles were offered by the rest of the panel—accompanied by the stupefaction of the unfortunate candidate. I admit that it was a difficult question, and one that I could not even address at the time. The examples and analyses offered above on students' mathematical strategies might be seen as attempts to address this question more directly. This time, about more than only 'helping' algebra!

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# THE REAL NUMBERS: FROM KINDERGARTEN TO SIGNAL PROCESSING

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*Dalhousie University*

The set of real numbers is endowed with both geometric and algebraic structures that make it an extraordinarily useful tool, the most basic mathematical tool, in understanding and modeling physical reality. My purpose in this talk is to discuss strategies for the development of an adequate image of the real number line in the minds of students who want to pursue a degree and career in some area that requires strong quantitative reasoning.

There are two aspects of the real numbers where the level of understanding of many students is lacking, which dampens their potential for achievement in a wide range of areas. These two areas of concern are the *geometric nature of the continuum* and the meaning of *multiplication*. In this brief note, we focus on multiplication. Before presenting strategies to address the meaning of multiplication, we will present some of the reasons why it is important to make a significant effort to improve student outcomes in these areas.

As the information age matures, it carries with it incredible advances in areas such as medical technology, artificial intelligence (AI), and e-commerce. What is referred to as the knowledge economy represents an increasing portion of the overall economic activity in many countries of the world, including Canada. So-called STEM (Science, Technology, Engineering, and Mathematics) careers take up an increasing portion of the workforce, and most STEM positions are in upper income ranges. Mathematics, in particular, is at the core of many of these emerging specialties. Online commerce depends on secure encryption based on recent advances in number theory. Artificial intelligence and deep learning algorithms are built with neural networks, which are essentially very large systems of simultaneous linear equations in many variables. The Page rank algorithm running every Google search is based on theorems from linear algebra. The stunning advances in medical imaging could not have occurred without recent discoveries in the mathematical area of harmonic analysis, the theory of which is based on deep properties of the real numbers.

Those in command of the mathematical maturity necessary to be thought leaders and technical developers in novel applications of technology, such as those mentioned above, form a new elite in modern society. Unlike the aristocracy of the past, one does not need to be born into this elite class. The key to entry is a good command of mathematics, and a good understanding of the real numbers provides an ideal foundation for developing a command of more advanced mathematics.

## THE NUMBER LINE

Continuous exposure leads to familiarity, and familiarity morphs into deeper understanding over time. A number line is present in some form in many classrooms from K to 12. There should be one in every mathematics classroom—of course, it will take different forms as the concepts to be illustrated become more advanced. It may even be in virtual form such as can be represented in the Desmos (https://www.desmos.com/calculator) interactive environment. One of the strands through the curriculum should have the outcome goal of creating the concept of 'the' real number line, which is independent of any physical representation of it, in each student's mind. They should be able to imagine it as a geometric object that is a perfect, or ideal, line with no numbers marked on it. Two fundamental features of this ideal line are that it looks exactly the same if you translate it by any amount, and it looks exactly the same if you zoom in or out by any amount. Continuous exposure to physical representations of the number line and the language used by teachers and textbooks should gradually build familiarity and eventually this deeper understanding.

Addition and subtraction are typically introduced with manipulatives, such as blocks, that the students can count. As soon as is convenient, addition and subtraction should also be illustrated on a number line in the classroom. This ties counting and measurement conceptually together from the beginning and helps make negative numbers a natural concept for the students.

As students are introduced to new numbers, the location of a particular number on the number line should be a priority for every learner. Besides cutting a pie to illustrate the fraction $\frac{3}{8}$, students should be visualizing where $\frac{3}{8}$ sits in the interval between 0 and 1. When irrational numbers are introduced and particular ones, such as the $\sqrt{2}$ or $pi$, are introduced, the issue of locating them on the number line becomes a compelling learning opportunity. When trying to locate an irrational number on the line, one must naturally deal with choosing a level of precision and then zooming in on the line enough to resolve points with that level of precision. Exposure of the students to zooming in or out and translating on the number line, over several years and in many different contexts, will build a mental image of the line as a homogeneous continuum. The process of zooming is just as fundamental as that of translating.

## MULTIPLICATION

The point of view we are taking is that the operation of multiplication is a consequence of zooming in or out on the number line. If we represent the number line with a very stretchable elastic cord with 0 and 1 marked and if we have a rigid line just under it with 0 and some point $a$ (larger than 1 to start) marked, we can stretch the elastic line so that the mark at 0 stays over the rigid 0 and the mark of 1 is over the rigid $a$. The advantage of this method is that students can actually do this for values of $a$ that are not much larger than 1. For any point $b$ on the elastic line, its position after stretching is sitting over $ab$ on the rigid line. We can use this to explore the meaning of multiplication and relate it to the concept of repeated addition for positive integers. There are limits to how much you can explore with an elastic strip, so we attempt to illustrate the concepts via geometric diagrams.

In Figure 1, there are two copies of the number line, one is labeled $L_1$ and the other is labeled $L_a$, where $a$ is some number larger than 1. The line $L_a$ represents $L_1$ uniformly dilated so that the unit interval, from 0 to 1, becomes the interval from 0 to $a$. It is drawn below $L_1$ so we can illustrate the effect of this dilation on other points in $L_1$. The point $P$ on the vertical line is a perspective point. The straight line from $P$ to $a$ on $L_a$ passes through the point 1 on $L_1$. If $b$ is any other number, we mark its location on $L_1$. The dashed straight line passing through $P$ and $b$, located on $L_1$, intersects $L_a$ at a point which we call $ab$. Indeed, one can take this to be the definition of the product of $a$ with $b$. Another point $c$ is also illustrated as well as the resulting point $ac$. Ideally, one would mark the location of all points involved on both copies of the line, but the picture becomes much too cluttered.

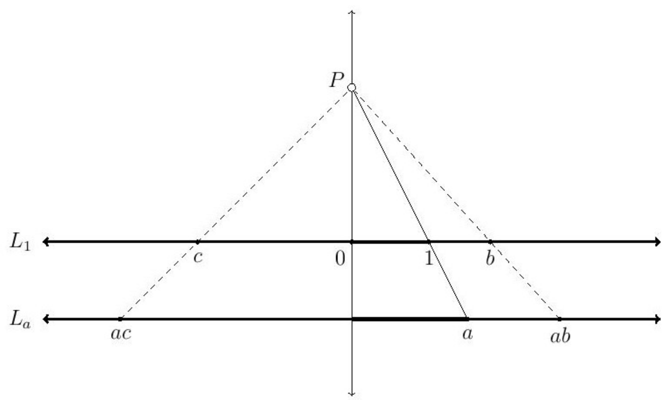


Figure 1. Multiplication by $a$ with $a > 1$.

With appropriate graphing software, one can create a dynamic version of this diagram where the point $a$ can be moved along the line. As $a$ approaches 1 from the right, $P$ travels upward. At $a = 1$, the perspective point is at infinity and the dashed lines become vertical. With $0 < a < 1$, $P$ is positioned below the lower copy of the line as illustrated in Figure 2. Students find it quite fascinating to play with a dynamic version and watch the changes as $a$ slides past 1.

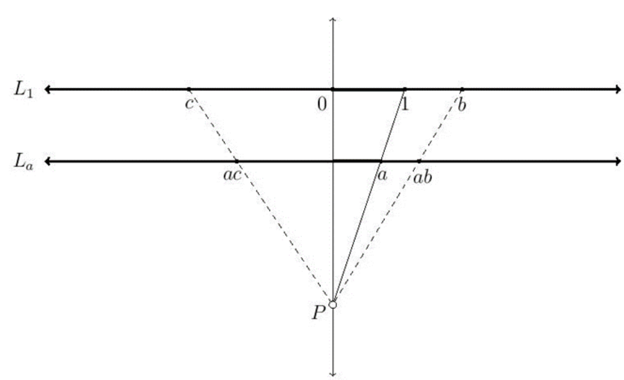


Figure 2. Multiplication by $a$ with $0 < a < 1$.

When $a$ approaches 0, $P$ comes up to 0 and all dashed lines point into 0 on the lower copy of the line, making it clear why multiplying 0 by any other number must result in 0. As $a$ passes to the left of the origin, $P$ moves up to sit between the two copies of the number line and the resulting situation is illustrated in Figure 3. This provides a compelling visual response to the often asked question of why the product of two negative numbers is positive. It is particularly effective with dynamic software where the students can slide the numbers along and watch the resulting motion of the product.

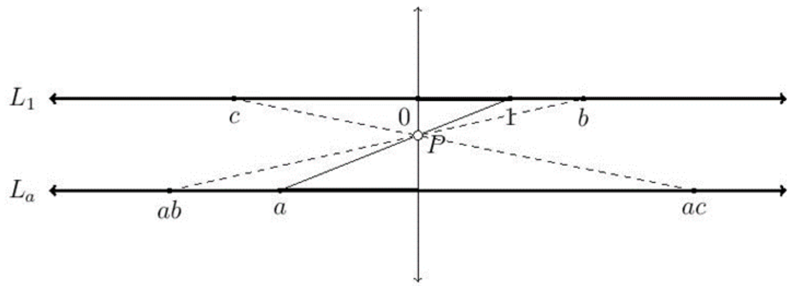


Figure 3. Multiplication by $a$ with $a < 0$.

Algebraic properties of arithmetic such as $ab = ba$ or $a(b + c) = ab + ac$ can be illustrated with similar diagrams for various locations of the respective points on the line. In fact, all the field axioms can be illustrated. Exploring with such diagrams would make nice exercises for students in their last high school algebra course. The message should be that these rules of symbol manipulation are not arbitrary but are simply consequences of translation and dilation on the number line.

## CONCLUDING REMARKS

We make no claim of originality of the ideas discussed in this note. We do not know of anyone who has presented diagrams like those in the figures to provide an interpretation of what multiplication means for real numbers, but we have not done a careful literature review.

# New PhD Reports

# Présentations de thèses de doctorat

# EXPLORING COLLECTIVE CREATIVITY IN ELEMENTARY MATHEMATICS CLASSROOM SETTINGS

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## INTRODUCTION

According to the National Council of Teachers of Mathematics *Principles and Standards* document (NCTM, 2000), "in this changing world, those who understand and can do mathematics will have significantly enhanced opportunities and options for shaping their futures" (p. 5). At the same time, researchers in creativity see it as an essential life skill and recommend that it should be fostered by the education system (Burnard & White, 2008; Craft, 2000; Sawyer, 2011). In the field of mathematics education, doing and understanding mathematics are often described as creative processes. For example, according to King (1992), doing mathematics means creating mathematics that is new and significant to at least the doer(s)—an assertion that should be considered at both the individual and the collective levels (Martin et al., 2006). Moreover, Davis (2005) argued that the classroom should be understood as a collective learner, and Sawyer (2011) saw that effective creative learning needs the collaboration of teachers and students while they are improvising together within the structures provided by the curriculum and the teachers. Such arguments suggest a shift in our focus as mathematics educators from individual to collective creativity in mathematics learning environments, or at least consideration of levels other than the individual at which mathematical creativity may emerge.

Research indicates that a positive collaborative educational environment is essential in enabling creative development (Kelly, 2012). However, a review of the literature suggests a dearth of published empirical research that deals with creativity as a collective activity (Levenson, 2011). Moreover, although scholars in pedagogy, mathematics education, and teacher education have generated a rich literature base promoting learning for, fostering, and characterizing mathematical creativity (e.g., Cropley, 1992; Haylock, 1997; Sriraman, 2009), only a few of the current definitions of creativity are suited to the distributed and collective enterprise of the classroom. My doctoral study brought together theories related to creativity and theories related to collectivity in order to investigate the nature of collective creativity in mathematics learning environments, how it can be fostered, and the teachers' role in this endeavor.

Due to the paucity of space in this paper, I focus solely on characterizing and describing collective creative acts of a group of sixth grade students while they were working on a mathematical task.

## THEORETICAL FRAMEWORK

Based on an extensive review of the literature on creativity, I initially framed the study by a number of themes to encompass a variety of paths and visions of creativity, which I used in the analysis of data collected in a Grade 6 mathematics learning environment. Through analysis, I then refined and (re)developed the themes to distill four metaphors to describe the experience of creativity with(in) the collective: summing forces, expanding possibilities, divergent thinking, and assembling things in new ways. These were embodied in, and representative of, varied, emergent, interwoven and recursive learning acts I observed in the learning environment, which I combined in an overarching framework that I refer to as collaborative emergence.

### METAPHORS OF CREATIVITY

1. Summing forces: This metaphor encompasses the ways in which learners co-act and sum their efforts to enable productive steering (Aljarrah, 2018) "that is not simply located in the actions of any one individual but in the collective engagement with the task posed" (Martin et al., 2006, p. 157). In this case, productive steering of a group of learners, working as a collective, cannot be attributed to any single learner. Instead, it should be attributed to the summing of their forces through coherent coactions as described by Martin et al. (2006).
2. Expanding possibilities: Expanding might be understood as broadening the learners' horizon by gaining new insights based on previous insights. It is a kind of stretching the space of the possible as a result of the evolving and the growth of the learners' basic insights. In this case, any new insight is dependent and contingent upon previous insights (Aljarrah & Towers, 2019).
3. Divergent thinking: Divergent thinking requires students to diverge outside their known content-universe, outside their safe zone of acting and thinking, outside the problem's clearly given conditions and information, and even outside the content of the planned curriculum; it is a metaphor that is based on considering many potential pathways, looking in many directions, journeying outside a known content universe, and thinking outside-the-box (Aljarrah & Towers, 2019).
4. Assembling (things in new ways): This metaphor implies looking for associations and making connections. This vision of creativity is based on an assumption that many educative things are with(in) the reach of learners in their learning environment (Aljarrah, 2018). This idea of making connections between seemingly unrelated ideas and concepts was well-established as one of the five mathematical processes that were suggested by the National Council of Teachers of Mathematics (NCTM, 2000).

### IMPROVISATION AND COLLABORATIVE EMERGENCE

Herein, I use collaborative emergence as an overarching theoretical framework for the four metaphors: summing forces, expanding possibilities, divergent thinking, and assembling things in new ways.

Sawyer has done extensive work in the field of creativity (e.g., Sawyer, 2001, 2003, 2011), in particular identifying creativity as a collaborative emergent phenomenon. According to Sawyer (2011), collaborative emergence is not a final end product; rather, "it is a constantly changing ephemeral property of the interaction, which in turn influences the emergent processes that are generating it" (p. 465).

Sawyer (2003) concentrated on the improvised and collective nature of group creativity. According to him, group creativity is (1) unpredictable, in that each moment emerges from preceding flow of the performance; (2) collective, in that members of the group influence each other from moment to moment; and (3) emergent, in that the group demonstrates properties greater than the sum of its individuals. Sawyer (2003) argued that creativity emerges from the collective activity of the group, and "the group itself becomes a creative agent, originating novel creative ideas that can only be thought of as a group property" (p. 62).

In the field of mathematics education, Sinclair et al. (2013) proposed a new way of framing creativity in the mathematics classroom, which "emphasize[d] the social and material nature of creative acts" (p. 239). According to them, creative acts occur in the confluence of material agency, the agencies of the people in the classroom, and the agency of the mathematical discipline. "Creative acts collectively engender… a new space, which enable[s] new forms of arguments to emerge" (p. 251).

## METHODOLOGY

In my study I aimed to explore and describe collective creativity in elementary mathematics learning environments. To fulfil the objectives of the study, I adopted a design-based research methodology (DBR) (Design-Based Research Collective, 2003). DBR is specifically intended to aid the development of theory through shaping particular forms of learning and systematically studying those forms within context. An advantage of design-based research is the reintegration it brings between practice and theory and between researchers and practitioners (Anderson, 2005).

I explored mathematical creativity with(in) a collaborative problem-solving environment. Two mathematics teachers and 25 of their sixth-grade students in a Canadian school setting participated in my research study. In collaboration with the teachers, students were grouped into seven groups, each consisting of three or four students. Students participated in collective problem-solving sessions inside and outside their classrooms. The video-recordings of these group activities formed the core of the data. The sessions outside the classroom can be described as task-based interviews with the groups (Aljarrah, 2018). During the problem-solving sessions, my attention focused on the group as a whole and not simply what each of the participating individuals was contributing.

The processes of analysis followed Powell et al.'s (2003) analytical model for studying the development of mathematical thinking, which consists of seven interacting, non-linear phases: (1) viewing attentively the video data, (2) describing the video data, (3) identifying critical events, (4) transcribing, (5) coding, (6) constructing storyline, and (7) composing narrative (p. 413).

Following Powell et al.'s (2003) analytical model, the first step of analysis was to watch every video in order to familiarize myself with its content. The next step was to choose the most significant video records and to prepare written, time coded descriptions of their contents. After that, I started reviewing these written descriptions to identify *critical events* that are related to my research questions. Following Flanagan's (1954) description of critical incidents "as extreme behavior, either outstandingly effective or ineffective with respect to attaining the general aims of the activity" (p. 338), I considered an event to be critical if it was helpful in triggering and/or explaining the nature of collective creativity in elementary mathematics learning environments. I transcribed the critical events and used my initial list of themes gleaned from the literature review to code the different kinds of collective creativity I observed.

## FINDINGS

Here, I articulate my understanding of the nature of collective creativity in mathematics learning settings based on excerpts from a 25-minute problem-solving session / interview that features a group of three sixth grade students: Zaid, Mark, and Kyle (pseudonyms). The students had been described by their teacher as high-ability students who usually work individually in class. The session took place in a quiet room in the Student Centre in the participant school. I introduced the following task, designed based on examples from Empson and Levi (2011), to the group and asked them to work on it together:

> *Three children, Alex, Zac, and John, shared a chocolate bar. Explain in as many ways as you can how those children may divide the chocolate bar into three pieces such that Alex will get twice what John got, and John's part is no more than one-fourth of the original bar and no less than one-tenth of it.*

Here, I use brief accounts from the excerpts to exemplify the four metaphors, which I use to describe the experience of creativity with(in) the collective. (Note: In my transcript I use dashes to show an interruption of one speaker by another).

### METAPHOR #1: SUMMING FORCES

The group tried to develop an initial collective understanding of the problem and to bring its obvious and non-obvious conditions within the reach of their eyes and hands. After a brief conversation that was based on trial-and-error and guessing acts (e.g., when Zaid asked, "*How much does Zac get?*" Mark suggested to just give him a third, and Kyle suggested to give him whatever was left); Mark also suggested, "*Draw the chocolate bar*," and on a shared piece of paper, he drew a rectangle and split it into quarters. Kyle commented that the diagram did not need to be "*that big*." Mark responded, "*The bigger it is, the better it is to work with.*" The three students engaged in a conversation while they were working collectively on their shared representation of the chocolate bar. Zaid was taking a close look at the diagram while saying, "*I guess, um, sooooo, um.*" His initial actions, as well as the actions of the others in the group, reflected kind of a hesitating as they whiled over the problem. This was followed by initiating potential pathways to approach the problem. Kyle noted: "*It does not say what Zac gets* [Pause 2 seconds] *because we can do like John and Zac get 25%* [each], *and then, um, or sorry, no* [Pause 2 seconds]." Mark supported Kyle's suggestion by restating the given (conditions) of the problem: "*Okay, Alex has to get twice what John got, and John's part cannot be more than one-fourth.*" Based on Mark's statement, Kyle refined his suggestion and stated with confidence, "*Yes, like if John and Zac will get one-fourth* [each]*, and then Alex gets, um, two-fourth—.*" Mark summarized the group's suggestions and expressed their collective agreement to proceed to next steps. He interrupted Kyle's statement and stated, "*Oh, yeah, it would work, yeah because if John—.*" Kyle interrupted and completed Mark's statement: "[because if John] *gets 25%, Alex gets 50%, then there is 25% left from the bar, we just give that to Zac.*" If we merge Mark's and Kyle's statements, we get "*Oh, yeah, it would work, yeah because if John gets 25%, Alex gets 50%, then there is 25% left from the bar, we just give that to Zac.*" Zaid was also part of this collective aha moment; he summarized the group's initial plan: "*I guess we just have to work with, um, Alex and John because Zac does not* *matter.*" This comment emerged as a natural conclusion of their conversation, and the whole group agreed with it.

To this point, the group members had been collectively engaged in 'summing forces' activities. They tried to understand the problem and to consider the conditions of it. They listened respectfully to each other and responded thoughtfully to the wonderings and suggestions that emerged through the conversation. Borrowing from Martin et al. (2006), "although [students] bring identifiable personal contributions to the collective action, it is through the coacting of these that [productive creative acts emerge and grow]" (p. 157). Students' productive steering toward 'better' mathematical ideas and thoughts—that appear to be helpful for the group in developing a solution to the problem— "[was] not determined by all the individuals having the same understanding" (p. 157). Instead, it was a result of summing of their diverse contributions.

### METAPHOR #2: EXPANDING POSSIBILITIES

Here, I share an excerpt from the problem-solving session with this group of students that can be described as one of "expanding combinatorics" (Sawyer, 2003, p. 7). Here are the three students' voices while they were discussing, explaining and expanding their ideas and thoughts:

> Zaid: *Technically, if you just kept on zooming in, slicing like into three, then zooming in to the last section* [Pause 2 seconds] *depressing into three that goes on forever then, it goes on for ever.*
>
> Mark: *Yeah* [Pause 2 seconds] *yeah, forever—*
>
> Zaid: *Because as long as one section always belongs to Zac—*
>
> Mark: *Or then we can do—*
>
> Zaid: *No, wait, wait, wait, we have to divide it to* [Pause 2 seconds] *like once you use the entire bar, you have to start dividing into four* [Pause 2 seconds] *okay? If you know what I mean, so like you know like how it goes… John, Alex, and then John Alex, and then John Alex, and then so—*
>
> Mark: *No, but—*
>
> Zaid: *It is one-third two-third, one-third two-third, one-third two-third* [Pause 2 seconds] *and then this one (the fourth part) we have to divide it into four because from here on out, Zac has to get—*
>
> Kyle: *Has to get one—*
>
> Zaid: *Has to get part of it, they cannot just say oh Zac you don't get any, right? So the last one* [of Zac's portion] *you have to divide it into—*
>
> Kyle: *Four—*
>
> Zaid: *Four—*
>
> Mark: *Yes, four—*
>
> Zaid: *Into four and then the first three* [Pause 2 seconds] *like the first three quarters is one and two and then Zac gets the last one. And the last one that Zac got you have to divide into four again.*

For example, the group used their *realistic* options (strategies) to divide the chocolate bar to generate and explain their *mathematical (technical)* ones, which according to them *go on forever*.

They considered tenths, ninths, eighths, sevenths, sixths, and fifths as their *realistic* options for dividing the chocolate bar. As is evident in this excerpt, the students expanded their realistic options based on imagination and playing activities (e.g., the interplay between the realistic and the mathematical and the use of the 'zooming in' expression to imagine and to justify the possibility of going on for ever). Those expanding activities were prompted earlier by Kyle's suggestion, "*As long as John always gets half of what Alex gets, Zac does not really matter.*" According to them, since Zac's portion could be anything, they could take one of his parts, divide it into four equal-sized sections, and give one to John and two to Alex. They could then divide *that* last section (i.e., Zac's) into four, give one to John and two to Alex, and so on; they could keep *zooming in* to the last section and do the same thing an infinite number of times. According to them, this idea can be applied to all *realistic* options (i.e., the tenths, ninths, eighths, sevenths, sixths, and fifths).

### METAPHOR #3: DIVERGENT THINKING

The group were making every effort to find a more sophisticated (for them) explanation of their claim that the dividing process could go on forever, hence the reason why they were willing to look into multiple directions (for example, the use of number line and decimals). Mark initiated a new conversation about using the number line and decimals to explain why, '*mathematically*' the process of dividing the chocolate bar could "*go on forever*":

> Mark: *You could, you could also put it in decimals.*
>
> Zaid: *But, I mean if you did decimal fractions then you can do anywhere from here* [indicating 0.1 on the number line] *to all the way to* [indicating 0.25]—
>
> Mark: *Look, if you did zero point two five, to zero point one*—[drawing a line segment to represent a part of the number line with 0.1 and 0.25 as its endpoints]—
>
> Zaid: *And, twenty-five hundredths, and umm* [while he was writing on a piece of paper].
>
> Mark: *Yeah.*
>
> Zaid: *If you used the decimal fractions there will be a loooooot.*
>
> Mark: *If you did this that is endless. You could do*—
>
> Zaid: *Oh, yeah endless, because you could just keep adding like…point one, point one, point one, point one, one, one, one, one, one, one, one, one, one…*[he wrote 0.1000000000000001 on one of their shared pieces of paper]—
>
> Kyle: *You could keep going zeros, um, all the way.*
>
> Mark: *That is technically more, um*—
>
> Zaid: *Or you could just keep going one, one, one, one, one, ... (0.111111111...), because if you do point one, if there is two ones it is a bigger number by, umm, one-hundredths.*
>
> Mark: *Exactly, for decimals it is endless.*

### METAPHOR #4: ASSEMBLING THINGS IN NEW WAYS

While the three students (Mark, Kyle, and Zaid) were collaborating to solve the chocolate bar problem, they recalled and put forward lots of ideas and many pieces of information and tried to look for associations and make connections between those pieces of information. Such creative actions "collectively engendered a new space, which enabled new forms of arguments to emerge" (Sinclair et al., 2013, p. 251).

Students in this group were actively engaging in connecting and reconnecting activities. During the session, they moved back and forth between different representations and their varied suggestions. While doing this, they tried to find connections between those representations and suggestions that might help them to find new strategies for the problem. For example, Zaid suggested, "*Maybe you can just keep going like* [Pause 2 seconds], *this thing you can split into three* [while he was working on a one-fifth on a diagram], *John gets this* [in addition to his fifth], *and Alex gets two more of those* [in addition to his two-fifths], *and then the last*—". Kyle interrupted Zaid and expressed his concern that doing "*this* [splitting] *will be kind of tough.*" Mark returned to the diagram that represented the tenths and stated, "*This is the smallest piece you can get, um, if we erase a couple of these ones that might work?*" Zaid suggested to work on a new diagram, and Kyle suggested taking all main possibilities (i.e., the tenths, ninths, eighths, sevenths, sixths, fifths, and quarters) and trying to find connections between them.

During the session, no one in the group asked others for further explanation of their comments, gestures, and representations because those emerged as a result of their interactional conversation. Even when the group was working on diagrams, they worked together (adjusting, adding, deleting) as if it were a one-person action. In addition to their gestures and representations, students used some new words that emerged during the conversation (e.g., *zooming in, expanding, mathematical, technical, realistic*), which were then sustained and enacted through repetition. They used what they have creatively by finding connections, combining ideas and information, and assembling and re-assembling or disassembling things for the favor of looking for a better construct.

## DISCUSSION AND CONCLUDING THOUGHTS

Based on data analysis and interpretation, I argued that collective creative acts are *the (co)actions, and interactions, of a group of curious learners, while they are working collaboratively on an engaging problematic situation. Such acts, which may include (1) summing forces, (2) expanding possibilities, (3) divergent thinking, and (4) assembling things in new ways, trigger the new and the crucial to emerge and evolve* (Aljarrah, 2017). I believe that such a description offers a framework to explain both individual and collective creative acts in mathematics classrooms, and may be considered as a base for promising pedagogical initiatives to develop and sustain learning environments that "occasion deep and powerful understandings of mathematical concepts—the kind of classrooms in which students have opportunities to construct their own knowledge through a creative process of inquiry" (Martin & Towers, 2011, p. 253). Thus, a noteworthy question at this point is What do learning and teaching look like with(in) each metaphor of creativity?

In the following chart, I present some entailments of the four metaphors of creativity (see Table 1). Such entailments are significant to understanding the underpinnings of effective creative learning in mathematics classroom settings.

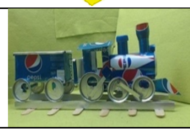
| **Metaphors of Creativity** | **Visualizing** | **Grounding Metaphor** | **Learning** | **Teaching** |
|---|---|---|---|---|
| Summing Forces |  | Sum of Forces | Productive Steering | Nudging |
| Expanding Possibilities |  | Inflating | Growth | Extending |
| Divergent Thinking |  | Multidirectionality | Widening Perception | Re-orienting Attention |
| Assembling Things in New Ways |  | (De)construction | Assembling | Triggering Disintegration |

Table 1: The entailments chart of the four metaphors of creativity

Here, I will discuss briefly the logical implications of two metaphors of creativity: expanding possibilities and divergent thinking. As is indicated in Table 1, expanding possibilities draws on a grounding metaphor of inflation. Hence, students seemed to be inflating their initial ideas and thoughts (they restricted themselves to the use of fractions) to show that the dividing process could go on forever. To assist students in expanding possibilities, the teacher should be ready to extend possibilities by stretching the space of presented possibilities. Divergent thinking is grounded by the notion of multidirectionality. Hence, students seemed to be looking into multiple directions (i.e., the use of number line and decimals) to find a more sophisticated (for them) explanation of their claim. To assist students in divergent thinking, the teacher should be ready to re-orient attention to aspects of the problem that are ripe for alternative interpretations. Of course, both of these teaching actions demand that the teacher understand the relevant mathematics deeply and have the flexibility of practice to be able to see new possibilities and alternative interpretations and be able to act in the moment to cultivate them. Moreover, teachers' attendance to the teaching-learning acts entailed by the metaphors of creativity would support them in their effort to "trigger an ongoing mode of engagement with mathematics that [would serve] to create ever-more contested mathematical understanding, which in turn would further occasion the emergence of new insight" (Metz et al., 2014, p. 46). Such new insights are more likely to emerge when the classroom proceeds in a collaborative, improvisational fashion within the structures provided by the curriculum and the teachers, where students are "allowed to experiment, interact, and participate in the collaborative construction of their own knowledge" (Sawyer, 2004, p. 14).

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# MODÈLES DE PROCESSUS DE COMPRÉHENSION DE LA DÉRIVÉE SOUS L'ANGLE DES REPRÉSENTATIONS

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## RÉSUMÉ

*Ma thèse présente des modèles de processus de compréhension d'étudiants autour du concept de dérivée. Ces modèles sont décrits en termes de types de représentations utilisées et des actions posées sur ces représentations. Afin d'y arriver, un* teaching experiment *a été mis en place permettant ainsi de faire une observation fine de l'activité mathématique d'étudiants en action.*

## INTRODUCTION

Le cours de calcul différentiel m'intéresse depuis plusieurs années. En fait, dès que j'ai suivi ce cours, je me suis questionné sur ma réelle compréhension des concepts que nous voyions. Or, bien que j'aie très bien réussi, j'avais toujours l'impression que l'essence même des concepts du cours m'échappait.

Ce questionnement, très personnel, s'est avéré partagé lorsque j'ai consulté les travaux sur le calcul différentiel en didactique des mathématiques. En effet, un constat récurrent au cours de mes lectures, que ce soit au Québec ou ailleurs dans le monde, est que les étudiants semblent avoir de la difficulté à mobiliser et en mettre en lien les concepts du cours de calcul différentiel nécessaire pour résoudre des problèmes. Le problème ne serait pas nécessairement dans l'aspect procédural du calcul d'une dérivée en un point par exemple, mais plutôt dans la reconnaissance de l'utilité du concept afin de résoudre un problème (Selden, Mason, et Selden, 1989, entre autres).

Plusieurs chercheurs en didactique des mathématiques ont étudié ces difficultés en entrant par différents angles : les conceptions des étudiants (par exemple le travail de Biza, Christou, et Zachariades (2008) autour du concept de tangente), les obstacles épistémologiques (par exemple, les travaux de Sierpinska (1985) sur la limite et Hitt (2003a) sur l'infini) ou l'enseignement (par exemple, les travaux de Dufour (2011), Hardy (2009) et Odierna (2004)). Tous ces travaux ont grandement contribué à notre connaissance de l'enseignement et de l'apprentissage du calcul différentiel.

À la lumière de ces études, il y a lieu de se demander comment on peut caractériser et favoriser une compréhension plus conceptuelle dans les cours de calcul différentiel. Afin d'approfondir cette question, certains auteurs suggèrent des environnements à mettre en place pour atteindre ces objectifs (Zerr, 2010, entre autres). Or, la plupart de ces études adoptent un point de vue sur l'appréciation des étudiants pour ce type d'environnement plutôt que sur la nature des environnements eux-mêmes. D'autres auteurs ont modélisé la compréhension d'étudiants des concepts en calcul différentiel. Par exemple, Zandieh (2000) propose un schéma théorique pour analyser cette compréhension. Ce schéma sous forme de tableau à double entrée présente trois niveaux de compréhension du concept de dérivée soit taux, limite et fonction, qui sont jumelés à des types de représentations soit graphique, verbal, physique et symbolique (Zandieh, 2000).

Bien que ces études apportent un éclairage important sur l'apprentissage, la compréhension et l'enseignement des concepts du calcul différentiel, des questions demeurent. Comment cette compréhension évolue-t-elle ? Comment et quand les étudiants rencontrent-ils les difficultés répertoriées précédemment ? Comment les surmontent-ils ? L'objectif général de ma thèse est donc d'observer le processus de compréhension de la dérivée, plus particulièrement la construction d'une compréhension conceptuelle, dans un contexte d'enseignement conçu pour favoriser une telle compréhension.

## POSITION SUR LA COMPRÉHENSION

Afin de pouvoir décrire le processus de compréhension, je me suis inspirée de deux cadres théoriques sur les représentations en mathématiques. Il s'agit de la théorie des registres de représentations sémiotiques de Duval (1995, 2006) et des représentations fonctionnelles définies par Hitt (2003b).

La théorie des registres de représentations sémiotiques de Duval repose sur le principe qu'un concept mathématique est représenté par des représentations appartenant à différents registres. Duval défini les registres comme des systèmes qui sont régis par des règles de conformité établies. Ainsi, dans le domaine de l'analyse par exemple, les registres de représentations retenus sont : algébrique, verbal (oral ou écrit), graphique, numérique, tabulaire. Au-delà du registre auquel les représentations sont associées, Duval insiste également sur le fait que la compréhension réside dans l'articulation des représentations provenant de différents registres. Or, les représentations décrites par Duval sont plutôt « institutionnelles ». En effet, en les définissant à partir des registres qui se doivent de respecter des règles précises, elles n'admettent pas les représentations produites au fil de l'apprentissage, voire lors d'une résolution de problème. Pensons par exemple, à une esquisse graphique dessinée rapidement lors d'une résolution de problème. Ces représentations ne respectent pas toujours les règles des registres auxquels elles sont associées, mais ne sont pas moins importantes pour autant dans le processus de compréhension des étudiants. Ce type de représentation est décrit par Hitt (2003b). Ils les nomment « représentations fonctionnelles ».

Le processus de compréhension relève, selon la position adoptée pour ce projet, des **types de représentation** (algébrique, graphique, verbale, tabulaire, numérique, schématique, gestuelle) utilisés ou produits par les étudiants, de la **nature de ces représentations** (institutionnelles ou fonctionnelles), et des **actions posées sur celles-ci** (reconnaissance, production, traitement, conversion, coordination).

L'objectif de recherche peut donc être précisé et prendre la couleur de la position théorique adoptée :

- Construire un ou des modèles du processus de compréhension de la dérivée des étudiant.e.s dans un contexte particulier.
    - Identifier les principales représentations sollicitées ou produites par les étudiant.e.s, qu'elles soient fonctionnelles ou institutionnelles.
    - Décrire et analyser la façon dont les étudiant.e.s ont recours à ces représentations en termes de reconnaissance, production, traitement, conversion ou coordination.
- Examiner de quelle manière le contexte mis en place (tâches, approche, interactions, etc.) a pu contribuer à faire ressortir, évoluer et articuler ou non différentes représentations produites ou manipulées par les étudiant.e.s.

## MÉTHODOLOGIE

Afin d'atteindre ces objectifs, un *teaching experiment* (TE) (Steffe et Thompson, 2000) a été mis en place. Il s'agit d'une méthodologie qui permet l'observation d'étudiants en pleine activité mathématique, au cœur de leur processus de compréhension. De plus, les chercheurs qui utilisent cette méthodologie ont un double objectif. Celui d'observer l'activité mathématique d'étudiants, mais également d'agir sur cette activité. En effet, le *teaching experiment* nécessite la planification et la tenue de séances d'enseignement qui ont des objectifs d'apprentissage précis (de contenu mathématique en particulier et de compétences). De plus, cette méthodologie a pour but de construire des modèles de l'activité mathématique observée afin de contribuer à une meilleure compréhension de cette activité. Ainsi, il faut préciser que le but n'est pas de produire des modèles qui pourront être reproduits ou généralisés. L'idée est d'utiliser ces modèles afin qu'ils soient réinvestis pour mieux guider l'activité mathématique autour des concepts ciblés.

Pour cette thèse, cinq séances d'enseignement d'environ 1h30 chacune ont eu lieu. Six étudiants y ont participé. Les séances d'enseignement étaient planifiées au fur et à mesure de la prise de données. C'est-à-dire qu'une première analyse sommaire était faite après chaque séance afin de s'y appuyer pour préparer la séance suivante. L'approche utilisée pour ces séances s'inspire de la méthode ACODESA proposée par Hitt et Morasse (2009). Il s'agit de privilégier un apprentissage collaboratif, des débats scientifiques et des périodes d'autoréflexion. Il y a donc eu beaucoup de travail en équipe lors des séances. De plus, une attention particulière a été portée aux représentations. D'abord pour la préparation des séances, un soin a été mis à proposer des problèmes assez ouverts et avec des représentations données appartenant à différents registres de représentations. De plus, pendant les séances, les représentations proposées par les étudiants ont été privilégiées et réinvesties autant que possible. Cette façon de faire avait pour but de rester le plus fidèle à leur processus de compréhension.

## MODÈLES DE PROCESSUS DE COMPRÉHENSION : UN EXEMPLE

Après une analyse approfondie du travail des étudiants lors des séances d'enseignement, deux modèles collectifs et un individuel de processus de compréhension ont été construits. Il est apparu rapidement évident, étant donné l'approche utilisée mettant de l'avant le travail d'équipe, que les processus de compréhension seraient collectifs. En effet, l'activité mathématique observée était partagée par les membres d'une équipe. Un modèle individuel a tout de même été produit à partir du travail d'un des participants qui avait parfois de la difficulté à prendre part au travail d'équipe.

Ainsi, plusieurs moments individuels ou en duo avec la chercheure-enseignante ont permis de produire ce modèle.

Les modèles construits dans la thèse sont de type descriptif et sont présentés sous la forme de textes suivis, format souvent privilégié lors de recherches utilisant le *TE*. Les modèles mettent en lumière les principales représentations et actions posées sur ces dernières lors des séances d'enseignement par les étudiants et relèvent des extraits de discussion entre les étudiants ainsi que certaines de leurs productions. Il est donc difficile de présenter les modèles dans leur entièreté dans ce rapport de recherche. En conséquence, un exemple d'une partie d'un modèle est présenté dans ce qui suit. Cet exemple reflète la façon dont les modèles sont présentés dans la thèse, mais surtout la façon dont l'aspect processuel de la compréhension des étudiants a été mis en lumière. Ainsi, si vous désirez plus de détails quant aux modèles des processus de compréhension construits, je vous invite à consulter la thèse directement (Dufour, 2019).

Les modèles ont été construits à partir de l'analyse de la résolution de trois problèmes proposés aux étudiants lors des séances d'enseignement. Chaque équipe de travail, composée de trois étudiants, a suivi des étapes de résolution différentes. Voici les principales étapes de résolution d'une équipe pour le premier problème analysé qui travaille le concept de dérivée en un point en contexte de croissance d'une population de bactéries : s'approprier la tâche (représenter graphiquement une situation décrite en mots, identifier ce qui est recherché), travailler sur la table de valeurs (représentation proposée dans le problème), identifier un intervalle du domaine qui contient un changement de croissance, proposer le concept de dérivée et identifier son utilité, identifier algébriquement la valeur recherchée. Ces étapes servent de contexte afin de « structurer » le modèle du processus de compréhension, mais elles ne sont pas le processus de compréhension en tant que tel.

À partir de ces étapes, les grandes lignes des actions des étudiants posées à l'aide de différentes représentations ont pu être énoncées. Voici un schéma de ces grandes lignes toujours pour le premier problème proposé.

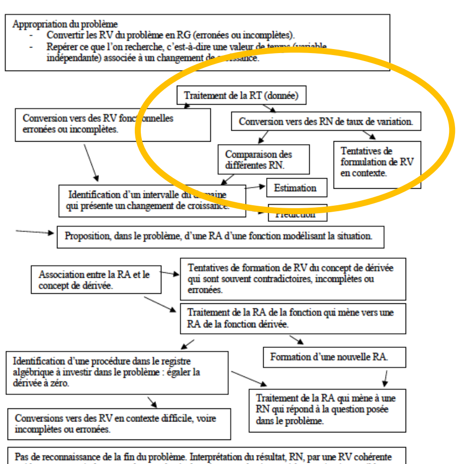



Figure 1. Les grandes lignes de la résolution du premier problème par une équipe (Dufour, 2019, p. 118).

Ces grandes lignes informent déjà davantage que les étapes de résolution sur le processus de compréhension des étudiants du concept de dérivée en un point. Or, l'aspect processuel, c'est-à-dire les liens entre les grandes lignes énoncées dans la Figure 1 ou ce que les flèches de la Figure 1 représentent, n'est pas illustré par ce schéma. Les modèles construits dans la thèse détaillent cet aspect en insistant sur les passages entre les grandes lignes du schéma. Prenons la partie mise en évidence dans la Figure 1 afin de donner un exemple des détails fournis par les modèles.

Pour ce problème, une table de valeurs (représentation tabulaire, RT) a été proposée aux étudiants. L'équipe concernée a rapidement pu traiter cette représentation en mettant en évidence les écarts entre les données du tableau (voir ligne A Figure 2) ce qui représente des variations de temps et des variations du nombre de bactéries dans la population. Elles ont également pu réinvestir cette information en mettant ces variations en relation pour obtenir des représentations numériques (RNs) de taux de croissance moyen sur différents intervalles de temps selon les données du tableau (voir ligne B de la Figure 2). Il s'agit donc d'une conversion de la RT vers des RNs. Une participante reconnaît même que la variation du nombre de bactéries pour un intervalle en particulier est plus grande que les variations du nombre de bactéries pour d'autres intervalles, mais que la variation de temps est aussi plus grande. Le groupe s'intéresse alors à comparer les différentes RNs obtenues et savent identifier un changement de croissance de la population de bactéries dans l'intervalle [12, 14]. Cela leur permettra, plus tard, de se concentrer sur cet intervalle pour répondre à une question du problème. De plus, elles cherchent à interpréter ces taux de variation en produisant des représentations verbales (RVs) en contexte (voir ligne C de la Figure 2). Dans cet extrait, l'étudiante peut produire une RV fonctionnelle qui lui permet de mettre en évidence les idées de variation et de « moyenne ». En effet, l'étudiante prend soin de parler du nombre de bactéries **qui fait augmenter** le nombre de bactéries dans la population. Elle ne se limite pas au nombre de bactéries dans la population, mais bien à sa variation en s'intéressant aux différents écarts obtenus entre les données du tableau. De plus, elle précise que c'est « en moyenne » ce qui indique qu'elle considère la comparaison, par la représentation d'un taux, des deux écarts correspondants dans la table de valeurs. Enfin, bien qu'elle n'utilise pas la RV plutôt institutionnelle de « taux de croissance moyen », sa RV fonctionnelle laisse croire que cette participante a entamé un processus de conversion (interprétation) des RT et RNs vers une RV en contexte.

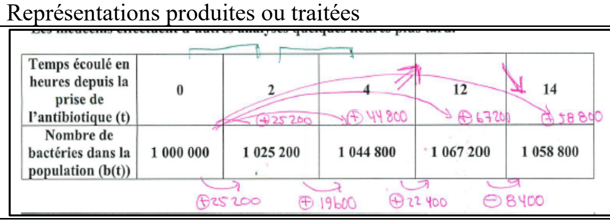
<table>
<tr><th></th><th>Actions posées par les étudiants</th><th>Représentations produites ou traitées</th></tr>
<tr><td>A</td><td>Traitement de la représentation tabulaire (donnée)</td><td>
<table>
<tr><td>Temps écoulé en heures depuis la prise de l'antibiotique (t)</td><td>0</td><td>2</td><td>4</td><td>12</td><td>14</td></tr>
<tr><td>Nombre de bactéries dans la population (b(t))</td><td>1 000 000</td><td>1 025 200</td><td>1 044 800</td><td>1 067 200</td><td>1 058 800</td></tr>
</table>
</td></tr>
<tr><td>B</td><td>Conversion vers des représentations numériques de taux de variation</td><td>$$\frac{1067200 - 1044800}{12 - 4} = \frac{22400}{8} = 2800/h$$</td></tr>
<tr><td>C</td><td>Tentatives de formulation de représentations verbales en contexte</td><td>Karine : […] le nombre de bactéries qui fait augmenter le nombre de bactéries de ma population, elle est comme vraiment petite là ! Tu sais comme dans l'espace de 8 heures, je pense que ça faisait, en moyenne, comme 2800 bactéries par heure ?!</td></tr>
</table>

Figure 2. Actions sur les représentations posées par les étudiants d'une équipe lors de la résolution du premier problème.

## ÉLÉMENTS PLUS GÉNÉRAUX QUI RESSORTENT DES MODÈLES

Les modèles de processus de compréhension construits dans la thèse ont également permis de ressortir des éléments plus généraux, communs aux modèles, qui peuvent contribuer à une meilleure connaissance des processus de compréhension pour le concept de dérivée en particulier. Trois éléments seront discutés ci-dessous : le statut particulier du registre algébrique, l'apport du registre verbal pour la recherche de sens, l'importance des concepts collatéraux.

Tel qu'il est conclu dans Dufour (2011), le registre algébrique a une importance particulière dans l'enseignement ainsi que dans les manuels scolaires. En effet, ce registre semble souvent privilégié dans l'enseignement et même désigné comme étant « ce qui est mathématique ». Nous avons remarqué que cette tendance semble gagner les étudiants également. En effet, la représentation algébrique d'une fonction semble souvent un déclencheur pour leur permettre de reconnaître le concept de dérivée comme « utile » pour la résolution d'un problème. Il n'est pas de même si les étudiants sont face à un graphique ou une table de valeur par exemple. De plus, certains étudiants donnaient un statut de rigueur et de validité important à ce registre. Dans certains cas, le problème ne pouvait pas être terminé tant qu'une solution algébrique n'avait pas été proposée. Les solutions dans les registres graphique ou tabulaire semblaient être, pour eux, de l'ordre de l'exploration. On voit bien ressortir ce regard sur le registre algébrique chez Antoine en particulier. Par exemple, à un certain moment, Antoine compare sa démarche de résolution, faisant intervenir des représentations algébriques incomplètes et des représentations graphiques, à celle de Jérémie qui présente une « impressionnante », mais erronée, démarche uniquement dans le registre algébrique. Antoine dit « Wow ! À comparer à mon *brouillon*, c'est vraiment… good ! » (Dufour, 2019, p. 210). Avant même de comprendre la démarche de Jérémie, Antoine met de côté la sienne et semble accepter d'emblée celle de Jérémie. Plusieurs exemples de ce type ont été repérés au cours des séances d'enseignement.

Une approche privilégiant le travail d'équipe a permis d'observer plusieurs représentations verbales produites par les étudiants. En effet, à plusieurs reprises, la nécessité pour eux de produire une représentation verbale de leurs idées associées à différents concepts (verbaliser ou mettre en mots) les amène à devoir réfléchir sur les représentations en jeu et à en dégager une cohérence. Souvent, c'est la production de ce type de représentation qui leur permet de prendre conscience d'une coordination ou cohérence entre les représentations en jeu ou encore de la présence d'éléments contradictoires entre ces représentations. Ce qui les amène par la suite à modifier ou revoir les représentations produites en les mettant, parfois pour la première fois, en comparaison. La tentative de produire une représentation verbale des taux de croissance moyens calculés lors de la résolution d'un problème dont il a été question préalablement dans cet article (voir p. 189 et Figure 2) est un exemple de l'élément pointé. Après avoir formulé la RV donnée à la ligne C de la Figure 2, la participante poursuit sa tentative en ajoutant « nombre de bactéries… eh moyen là, je ne sais pas trop quoi, la différence entre les deux [nombres de bactéries], elle commence à diminuer ». Cet extrait montre la difficulté qu'elle a à produire une RV en cohérence avec ses RNs. L'étudiante est par la suite troublée par cette difficulté. Elle réalise qu'elle ne sait trop identifier à quel concept doivent être associés ses calculs (RNs) et cherche donc à clarifier cette situation. Cet exemple montre que la production d'une RV peut déclencher une réflexion prometteuse autour des concepts en jeu.

Le concept de dérivée s'appuie évidemment sur plusieurs autres concepts. Les études sur le calcul différentiel abordent d'ailleurs les difficultés liées aux concepts de tangente, de limite, d'infini par exemple. Il est indéniable que ces concepts sont primordiaux dans le processus de compréhension du concept de dérivée. Dans cette thèse, le concept de taux de variation est apparu comme un élément particulièrement clé dans ce processus (tel que souligné également par Carlson (1998) entre autres). À plusieurs reprises, ce concept a représenté un élément déclencheur d'un blocage pour les participants. En effet, bien que les participants réussissent parfois à identifier le concept de dérivée comme utile pour arriver à des solutions aux problèmes proposés, c'est dans l'opérationnalisation de cette idée qu'ils identifiaient mal, dans la représentation algébrique d'une fonction linéaire par exemple, le taux de variation. Cela menait ainsi à une mauvaise représentation du concept de dérivée et donc, à une solution erronée du problème.

## CONCLUSION

Enfin, les modèles construits pour cette thèse ainsi que les éléments généraux ressortis contribuent aux connaissances à la fois pour l'enseignement et pour l'apprentissage du concept de dérivée en particulier et aussi au processus de compréhension de façon plus générale. En effet, les cadres théoriques et d'analyse mis en place dans la thèse se sont avérés très aidants afin de mieux comprendre et décrire des processus de compréhension d'étudiants.

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# TEENAGERS AT A CROSSROAD: EXPLORING TEENAGE NEWCOMERS' IDENTITY AS LEARNERS OF MATHEMATICS AND OF ENGLISH AS AN ADDITIONAL LANGUAGE

# ADOLESCENTS À LA CROISÉE DES CHEMINS : EXPLORATION DE L'IDENTITÉ DE NOUVEAUX ARRIVANTS À L'ADOLESCENCE EN TANT QU'APPRENANTS DE MATHÉMATIQUES ET DE L'ANGLAIS COMME LANGUE ADDITIONNELLE

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## ABSTRACT

*This PhD thesis was set to explore teenage newcomers' identity making as learners of mathematics in the context of their transition to the English-speaking Canadian educational system. Drawing on Ricoeur's (1992) etymological distinction between* idem *and* ipse *(identity as sameness, and identity as selfhood), I use Ivanič's (1998) theoretical framework that allows for the investigation of identity as a dynamic concept, which includes four identity-related interconnected dimensions: autobiographical, discoursal, authorial, and socioculturally available identities. For ease of reference, I use the acronym ADAS to discuss teenage newcomers' mathematical identity as they transition into a new educational system while learning English as an additional language. The research design comprised three sets of data collection that yielded six families, 16 individual interviews, and four focus group discussions.*

## RÉSUMÉ

*Cette thèse de doctorat visait à explorer la façon dont était façonnée l'identité des adolescents nouveaux arrivants qui apprenaient les mathématiques lors d'une transition vers le système d'éducation canadienne anglophone. Prenant comme assise conceptuelle la distinction étymologique établie par Ricœur entre* idem *et* ipse *(identité en tant que ressemblance et identité en tant qu'individualité), j'utilise le cadre théorique d'Ivanič (1998) pour examiner l'identité en tant que concept dynamique qui comprend quatre dimensions interconnectées : la dimension autobiographique, la dimension discursive, la* *dimension d'auteur, et la dimension des identités socioculturelles disponibles. L'étude incluait trois séries de collecte de données à l'aide de six entretiens familiaux, 16 entretiens individuels et quatre groupes de discussion.*

## INTRODUCTION

Transitioning into a new country during high school years—where mathematics and English function as stepping-stones in career and education trajectories—compels the investigation of teenage newcomers' identities as learners of mathematics and of English as an additional language (EAL[1]). The transition into a new country is no longer only about the mastery of new linguistic patterns; there is growing evidence that shows identity-making plays a crucial role in one's engagement with, and commitment to, learning not only EAL but also mathematics. In this research, I explored the experience of teenage newcomers learning mathematics and EAL through the lens of identity making.

## THEORETICAL UNDERPINNINGS: SHIFTING EPISTEMOLOGIES

In spite of it being a word everybody—everywhere—uses, identity seems to generate different meanings that heavily depend on context, purpose, and user (Brubaker & Cooper, 2000). My entry point to framing identity work draws on Ricoeur (1992) who distinguishes between the meaning of *selfhood* and the meaning of *sameness* that are equivalent to "the Latin *ipse* or *idem*" (p. 2) respectively. The former implies continual development of sense of self, the latter of static representation over time; the former is oriented toward the future, the latter toward the past; the former sees identity as a dependent variable, the latter as an independent variable; the former brings forth the diversity between individuals, the latter elides individuals to create a unified façade (see Fellus & Biton, 2017; Norton, 2000).

We may qualify the concept of ipse identity within the sociocultural framework as assigned, interpretive, collective, and negotiated. Within this framework, identity is perceived as "an idea that a person constructs" rather than "an underlying substance to be discovered" (James, 1890, as cited in Polkinghorne, 1988, p. 149). This perception of identity allows us to make the transition from treating identity as "a flattened reality that consists of physical objects in time and space along with their relations" (Polkinghorne, 1988, p. 149) to seeing it as a "construction built on other people's responses and attitude toward a person and is subject to change as these responses, inherently variable and inconsistent, change in their character" (p. 145). Drawing on Sfard and Prusak (2005) and Fellus (2019), the term identity is ontologically understood as storied self. This resonates with Bauman's (1996) definition of identity as actions of "being looked at by others, being framed and moulded by their scrutiny, demands and expectations" (p. 18). Thinking of identity in these terms turns our attention to the possibility of exploring the concept as a co-

[1] There are two main reasons why I chose the term EAL over other widely used terms such as ESL or ELL. First, many times, learners' English is not their second language. In fact, in cases of multilingual speakers, English may be their third or fourth language, which justifies the use of EAL over ESL. Second, in order to foreground an asset-oriented discourse rather than a deficit-oriented discourse that is embedded in terms such as ELL, I use the term EAL throughout for consistency and for the purpose of solidifying this transition in discourse.

constructed, negotiated—however scripted and constrained—phenomenon (Ivanič, 1998). I thus turned to research on identity in EAL and in mathematics education.

### IDENTITY IN EAL

Norton (2000) not only demonstrates how learners' EAL autobiographical narrations and positioning of self shape choices they make in relation to learning EAL but also convincingly describes how the use of one-dimensional terms for the purpose of identifying students are counterproductive and reductive. She suggests, for example, the use of the construct *investment* rather than *motivation* to understanding learners' engagement in learning (Norton, 2000). More specifically, Norton (1997) rationalizes that rather than asking, "Is the learner motivated to learn the target language?" and "What kind of *personality* does the learner have?" we should ask, "What is the learner's *investment* in the target language? How is the learner's *relationship* to the target language socially and historically constructed?" (Norton, 1997, p. 411). The term has blazed a new research trajectory that substantiates the agentive, self-regulated aspect of learning EAL (see Angelil-Carter, 1997; Babino & Stewart, 2017) that learners use to fulfill their future aspirations and visions of belonging to a specific professional group (Norton, 2000).

### IDENTITY IN MATHEMATICS EDUCATION

The concept of identity in mathematics education has gained increased attention in the last two decades. Sfard and Prusak (2005) discuss storied selves as a tool to examine learners' actual and designated identities in the form of constructed, ascribed, or self-assigned "narratives about individuals that are reifying, endorsable, and significant" (p. 16). McFeetors and Mason's (2005) work on constructing identity narratives of success through interactive journal writing, students' portfolios, and teacher's field notes reveals that "succeeding in school mathematics is less a matter of learning mathematics content than it is a quest for a more positive sense of identity in mathematics class" (p. 17). Through the "amplification"—to use their term—of students' voices, students developed identities as successful learners of mathematics who trust their mathematical thinking thus contributing to our understanding of identity work through stories of positive personal experiences and discourse.

### TOWARD A NETWORK FRAMEWORK OF IDENTITY

Considering the context-specific identity work that was the focus of this study, and in order to reflect the complexity of identity work and at the same time bring together what we already know, I imported Ivanič's (1998) four-part identity model from the field of academic writing. The model includes autobiographical identity, discoursal identity, authorial identity, and socioculturally available identities, or in short ADAS (Fellus & Glanfield, 2019). Conducting a theory-based literature review in EAL and in mathematics education, I compiled identity-related research studies. Forty-five mathematics education studies (Fellus, 2019) and 24 EAL studies (Fellus, 2018) were identified. Of the 69 papers that focus on identity as a dynamic construct, 58 per cent were published between 2008 and 2018—perhaps reflecting a growing interest in identity shortly after its introduction into the respective fields of mathematics education and EAL. None treated identity making in mathematics among teenage newcomers. To better reflect the complexity of Ivanič's (1998) identity aspects, I use the word *dimensions* when referring to any of them because this word more readily connotes the breadth and depth of identity work than the word *aspects*. I now turn to a short description of each of these dimensions, which will buttress the framework of the present study.

Autobiographical Identity in EAL and in Mathematics

McAdams and Olson (2010) contend, "Narrative identity is the storied understanding that a person develops regarding how he or she came to be and where he or she is going in life" (p. 527). That is, the interpretation of lived experiences may change across time and space (Polkinghorne, 1988). It is through the investigation of autobiographical identities in EAL and mathematics not "as a quality or attribute" (Ivanič, 1998, p. 16) but rather as a medium through which one's identity is continually constructed that we see how learners' engagement is driven by their interpretation of past learning experiences (Norton 2001). The scarcity of autobiographical identity in mathematics education (Towers et al., 2016) depicts a vast terrain of uncharted paths in research on K-12 autobiographical identity.

Discoursal Identity in EAL and in Mathematics

Toward the end of the twentieth century, the idea that discourse plays a pivotal role in one's identity construction has taken center stage in scholarly work (Harré & Gillett, 1994). Ivanič (2006) demonstrates that one's discoursal identity is simultaneously constructed by address (how individuals are talked to by others); by attribution (how individuals are talked about by others); and by alignment (how individuals sound like others). Because interaction is the main mode of teaching and learning (Vygotsky, 1978), we must understand how learners' identities are co-constructed in and through interaction. To better grasp these discoursal identities, Sfard and Prusak (2005) offer a tripart model for identity stories that "are reifying, endorsable, and significant" (p. 16). These stories can be represented by a configuration of $B^{A}C$, where A is the object of the story told, B is the speaker, and C is the recipient (p. 17).

In the context of EAL and mathematics, system-wide labeling of EAL learners within a deficit discourse became institutional markers (Ortmeier-Hooper, 2008) thus precluding the use of a language other-than-English as "a resource that provides sets of meaningful choices" (Fleming et al., 2012, p. 42). While the use of labels is often practiced with good intentions, often policies and practices that unintentionally treat EAL learners as "uninvited guests" (Yoon, 2008) and learners of mathematics as either mathematically abled or disabled (Damarin, 2000) get stuck in ruts, and it is attending to how learners' identity is constructed in and through this deficit discourse that can help break out of these ruts.

Authorial Identity in EAL and in Mathematics

I draw on Bakhtin's (1981) notion of authorship that positions speakers as 'owning' their words when they populate them with their own accents and intentions. Within the context of mathematics education and EAL, authorial identity is understood as a process of identity making where learners have "legitimacy to contribute to, take responsibility for, and shape meanings that matter" (Wenger, 1998, p. 197). This process is made possible by acts of appropriation of and ownership over ideas. In the context of teenage newcomers learning mathematics, the following questions are paramount: who has the right to claim ownership over their English and thus see themselves as able to contribute as full and equal participants in conversations in and about mathematics? How do learners of English author their identities as owners of mathematical ideas? And how do they perceive the affordances and obstacles of learning the target language in the development of authoring their selves as learners of mathematics? In the context of learning a language, Norton (2000) calls on teachers to "help learners claim the right to speak" (p. 142) as a function of the legitimacy attributed to the learner to use the language.

The strong association between authoring and ownership and learning mathematics was substantiated in research. Consider, for example, the work of Wirtz and Kahn (1982) who had three groups of elementary school children work on word problems. One group worked on word problems by closely following the textbook. The second group did not do any textbook word problems at all. The third group was asked to author their own word problem for a whole year. Wirtz and Kahn (1982) found that the third group attained the highest achievements on standardized tests thus highlighting the relationship between authorship and learning. Langer-Osuna and Esmonde (2017) dub this the practice of "conceptual agency" and "intellectual authority" (p. 640). This is because when learners reason, argue, provide explanations, and defend their respective stance, they, in actual fact, engage in work on authorial identity. Operationally speaking, the unit of analysis in authorial identity is "the speaker's position, opinions, and beliefs" (Ivanič, 1998, p. 26). It is this notion of "students' opportunities to author, justify, and debate mathematical ideas [that positions] them with mathematical authority" (Langer-Osuna & Esmonde, 2017, p. 645) and where they can experience "knowledge ownership" (Hogan, 2008, p. 110).

Socioculturally Available Selfhoods in EAL and in Mathematics

Available selfhoods are co-constructed by cultural, historical, and sociological frameworks (Ivanič, 1998). These may include cultural heritage, family stories, metanarratives, or literary characters that take forms and shapes in fiction, literature, movies, and social and cultural narratives. All these function as repositories for identities that are contextually and socioculturally available (Ivanič, 1998). The unique affordances of socioculturally available identities are that they are open for selection and appropriation.

Socioculturally available selfhoods can increase or limit one's possible life trajectories (Bauman, 2004/2012). It is this perception of identity as socioculturally available for the taking that this dimension offers. Some sociocultural contexts allow individuals "to choose what [they] desire and renounce what [they] resent," (p. 38) others are stapled with identities imposed on them. Conceptualizing identity as such, "The job of an identity-constructor is, as Claude Lévi-Strauss would say, that of a *bricoleur*, conjuring up all sorts of things out of the material at hand" (p. 49).

In EAL, Fang (2016), and others, found that such available identities take the form of sounding like a native speaker. EAL learners wish to be deemed members of a particular group of speakers of the target language. This, however, is complicated when languages are stratified and marginalized (Barwell, 2016; Marx, 2002). Within mathematics education, research has found that learners of mathematics align with or reject socioculturally available narratives and scripted roles about doing mathematics that are propagated through literature and popular media (Epstein et al., 2010), metanarratives (Sfard & Prusak, 2005), and pedagogical approaches (Hogan, 2008).

## RESEARCH QUESTIONS AND METHODS

In order to capture the complex nature of identity making in the context of transition, the research design was set to include three different modes of data collection: family interviews with at least one parent and their teenage children, one-to-one interviews with the teenagers and their parents; and all-parent and all-teenager focus groups. The purpose for the research design was to maximize the breadth and depth of the data collected so that the following overarching research question can be answered: How do teenage newcomers craft their mathematics-related and EAL autobiographical, discoursal, authorial, identities, and how do they draw on socioculturally available selfhoods?

In order to preclude problems associated with language differences between the researcher and research participants (see McCay & Wong, 1996), it was decided to recruit participants who speak Hebrew, as I am a native speaker of Hebrew. Six families reached out following a Call for Participation. Altogether, I completed six family interviews, 16 individual interviews (eight teenagers and eight parents), and four focus group discussions.

The interview questions were designed to elicit identity-related stories. Several rounds of analyses were conducted to first examine how identity is formulated in and through each of the four dimensions and later explore the inter-relationship between the four dimensions. Data were scrutinized employing a deductive analysis (using the ADAS framework), an inductive analysis (Corbin & Strauss, 2008), and a horizontal analysis by dimension and case (LeCompte, 2000).

## KEY FINDINGS

The inductive vertical and horizontal analyses generated 12 categories that were collapsed into the three following overarching themes through an examination of the similarities and differences between and among the quotes: *At a Crossroad*, *At the Nadir of Transition*, and *Making it Against All Odds*. Findings reveal that teenage newcomers' identity making as learners of mathematics and EAL is multifarious, multidirectional, and inter-animated. While the teenage newcomers are continually engaged with identity making as speakers of EAL and learners of mathematics in a new educational system, their collective identity as Israelis, who make it against all odds, and the socioculturally available selfhoods, draw a complex picture that depicts the intricate relationship between and among the four identity dimensions. The answer to the overarching research question collectively demonstrated that the teenage newcomers' identity making as learners of EAL and mathematics was directed by their autobiographical narratives (how they chose to reconstruct their past experience in a timeline that constituted their current way of being a student of EAL and of mathematics); their discoursal identity (how they were spoken to and about by their parents and siblings as learners of mathematics and how they identified themselves as current users of EAL); authorial identity (the positions, statement, beliefs, and perceptions they expressed about what it means to learn EAL and mathematics); and socioculturally available selfhoods (how they draw on prototypical possibilities for selfhoods from stories their parents share with them about using EAL and doing mathematics).

Teenage newcomers already have identities as learners and doers of mathematics, but, in the context of transitioning into a new country, this mathematics-related identity is now reshaped by their identities as learners of EAL. Consider, for example, Loki Port Noy (teenager), who did not see himself as a legitimate speaker of English (discoursal identity) and thus chose not to engage in interaction of any sort. Early on, he realized that his previous discoursal identity as mathematically abled is stripped away as his knowledge of mathematics can no longer be of service to his authorial identity because, as he put it, "two plus two is not *arba*!" (Loki Port Noy, teenager, Personal Interview). For him, the fact that he needs now to do mathematics in English—a language of instruction he is not feeling comfortable using—precludes him from doing very simple operations in mathematics such as adding two and two, which no longer sum up to "Arba" (four in Hebrew).

Bar, another teenager, constructs her autobiographical, discoursal, authorial, and socioculturally available narratives to overcome her fear of sounding like one who does not belong (discoursal identity) and to develop a can-do mindset in learning mathematics as an EAL student. She manages to do well at school through developing and building on her authorial identity as she ventriloquates her mother's stance about what it means to do well in mathematics. It is only by examining Bar's four identity dimensions that we can see how her socioculturally available selfhoods of Israelis who can make it against all odds contribute to her identity making as a learner of EAL and mathematics.

Yoking together two research areas on identity in EAL and mathematics education creates a synthesis that reflects a real-life context for many teenage newcomers. It is thus not a temporary compound but rather an area of research that is most compelling. Using a unified model that brings together the four dimensions discussed in this research may shed much needed light on the intersection of experiences of learning high school mathematics while learning an additional language among teenage newcomers as they integrate into school at a key point in their life.

## PEDAGOGICAL IMPLICATIONS

Being aware of identity making as a determining factor in students' academic success calls for a paradigm shift in mathematics education that will allow for practitioners to adopt a "knowledge-making epistemology" so that "theorizing the identities of learners will help advance mathematical learning for all students" (Jorgensen, 2014, p. 314) including teenage newcomers. When system-wide labeling identifies learners as "linguistically deficient and disregards their proficiencies in languages other than English" (Shapiro, 2014, p. 387), it unintentionally constructs a counterproductive discoursal identity that shuts down opportunities for authorial identities to develop. This study showed how the context of being at a crossroad was consequential to the teenage newcomers' developing identities as learners of EAL and mathematics. How identity is associated with learning trajectories is not a trivial question. Rather, it is related to the larger question of what schooling is about.

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# FLIPPING THE SECONDARY MATHEMATICS CLASSROOM: TEACHERS' PERCEPTIONS ON THE USE OF VIDEO INSTRUCTION

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## INTRODUCTION

The flipped classroom is an instructional strategy that reverses the traditional classroom that is more teacher-directed learning, to an environment where it is more student-centred, and the instructor acts a facilitator. This is achieved through technology, where students can access teacher instructional videos online through different media sources.

Four secondary school mathematics teachers took part in the initial stages of the research study, where I conducted face-to-face semi-structured interviews. Out of the four participants, one participant went further with this research study that included observing his class, viewing his videos, post interviews after each class, and perceptions of student surveys once the observation period was completed.

## THEORETICAL FRAMEWORK

The theoretical framework used for flipped classroom often focuses on the reasons not to use class time to deliver a lesson (Bishop & Verleger, 2013). A research team, along with experienced educators at the Flipped Learning Network (Hamdan et al., 2013), identified four key pillars: "flexible environment, learning culture, intentional content, and [a] professional educator" (p. 5) as a theoretical framework for the flipped classroom.

## THE FLIPPED CLASSROOM

The flipped classroom model is a pedagogical approach in which the teacher flips the sequence of instruction and homework. In a traditional classroom, the teacher typically introduces a topic or concept using lecture-based instruction. The purpose of these lectures is for students to gain knowledge and understanding of the topic. Teachers disseminate new knowledge while students play a more passive role in their learning by listening and taking notes. Students are then tasked with applying this knowledge through assignments done outside the classroom. In contrast, the flipped classroom allows students to learn new content at home via video instruction. During classroom time, students apply and develop their understanding by collaborating with others.

The way in which teachers flip classrooms has changed over the years. One early method was to give students material to review in preparation for classroom activities (Strayer, 2012). The use of technology in a flipped classroom allows teachers to reverse the order of traditional instruction and to emphasize collaborative learning (Hong et al., 2011). Bergmann and Sams (2012) recommend using technology as a standard tool for information sharing. Teachers can, for example, share articles, videos and presentations for students to view prior to class.

A second element of the flipped classroom promotes the construction of knowledge through collaboration (Bergmann & Sams, 2012). During class time, teachers may offer a brief re-cap of the information shared in the pre-class material; however, spend the majority of class time on collaborative work projects (Bergmann & Sams 2012). To put it in more familiar terms, the classroom time is used for what is typically known as 'homework' in the traditional model.

Lampert (2001) suggests that group discussion and work projects during class time allow students to deepen their understanding of the content by collaboratively constructing new knowledge. The activities focus on the construction of knowledge through social engagement and the application of that knowledge and understanding (Bergman & Sams, 2012). During class, students receive immediate feedback from peers and from their teacher because they are actively engaging with the content in their group work activities and discussions (Bergman & Sams, 2012; Lampert, 2001).

A benefit of the flipped classroom is that it provides the opportunity for student engagement and for students to work collaboratively. For this to work effectively, teachers must put particular structures in place in order to facilitate teamwork among students. Tuckman and Jensen (1977) describe small-group development as occurring in four stages. The first stage in Tuckman's model is *forming*, in which group members learn from one another and speak about the opportunities and problems that may arise (Tuckman & Jensen, 1977). This stage is important as it allows all group members, including those who are more introverted in nature, to outline their objectives and have a voice in a small group setting.

The second stage in Tuckman's model is *storming* (Page & Donelan, 2003). Tuckman describes storming as the stage during which individuals try to position themselves within their group and, as a result, potentially experience power struggles until compromises are established (Tuckman & Jensen, 1977). *Norming* is the third stage in Tuckman's model. At this stage, roles within the group are established and accepted, and group members begin making decisions how to approach problems to meet the objectives that have been set out (Tuckman & Jensen, 1977).

In the fourth stage, *performing*, the team is clear on the processes and has a clear vision of how to work together toward their shared goals. While disagreements may arise, they are quickly resolved at this stage (Tuckman & Jensen, 1977). The final stage in Tuckman's (1977) model was added after the original model, in 1977, is known as *adjourning*. This is the stage in which the group disbands, hopefully with its objectives fulfilled and its members having developed mastery of the concepts that were set out by the teacher.

The flipped classroom allows teachers to observe the learning in real time as students openly discuss the content, ask questions, apply the new knowledge, and get immediate feedback from both their instructor and their peers (Ozdamli & Asiksoy, 2016).

FLEXIBLE ENVIRONMENT

One method of creating a flipped classroom environment is the Harkness method. It is named after philanthropist Edward Harkness who, in 1930, made a donation to Phillips Exeter Academy. The donation allowed the school to reduce class sizes to eight students and to restructure the physical environment from the traditional row of desks to a large oval table that accommodated everyone. The oval table enabled the instructor to see everyone's eyes during the group discussion, raising the instructor's awareness of student engagement while ensuring minimal interruption by the instructor and promoting a student-centred, open-minded environment. Harkness teaching evolved as a result, and this method is used in many countries today (Courchesne, 2005; Jones, 2014).

The Harkness pedagogy allows students and their teacher to sit around an oval table and discuss concepts related to the subject at hand (Williams, 2014) with the teacher acting as a facilitator rather than as a lecturer, or interrogator (Courchesne, 2005). The learning process is guided through effective questioning and by encouraging students to share their thoughts and possible solutions (Jones, 2014). This form of instruction accommodates students to a higher degree and provides the opportunity for students to increase their critical thinking skills through independent learning. This environment is designed for students to actively engage with one another and therefore differs from a traditional classroom in which students are expected to play more of a passive role in their learning (Hamden et al., 2013).

Another area where the flipped classroom is flexible is it allows time for small group tutorials (Bergmann & Sams, 2015). For those students struggling with concepts, the flipped classroom allows teachers to clear up any concept difficulties that students may have (Bergmann & Sams, 2015). Furthermore, teachers have the ability to bring those students having difficulty together, while others in the class can now work independently (Bergmann & Sams, 2015).

When a teacher works with a group of students who are experiencing difficulty regarding the subject matter, teachers can create a digital whiteboard. This involves a group of people that come together and share ideas, and content, where the lesson being explained can now be recorded (Bergmann & Sams, 2015). Additionally, small group tutorials also benefit from the amount of time that a teacher would normally spend with one student. In the flipped classroom, teachers have the opportunity to work with every student, thus making great use of their time (Bergmann & Sams, 2015).

This pedagogy has different touchpoints between student and teacher. Students can now connect with their teacher by watching online instruction and have the ability to playback and listen to concepts that they may not have understood the first time. With the use of technology, a student can now access peers through learning platforms if they need assistance on a concept, or have a touchpoint with their teacher, if they may have a question.

The flipped classroom allows teachers to monitor students' learning. Depending on the learning management system, teachers can access how long a student may have watched a video. One has to caution, as this might not be accurate due to a student playing a video but doing another task. Teachers can also embed questions within their videos, to see which students attempted the questions, but also get a sense on what concepts they need to stress in class.

### LEARNING CULTURE

In many traditional classrooms, the teacher is seen as the 'sage on the stage', where students simply memorize the concepts and facts and later replicate that knowledge on an assessment, without thinking what they are replicating (King, 1993; Jones, 2014). King (1993) refers to this as the transmittal model, where a student's intellect is seen as an empty container, that instructors pour knowledge into. The instructor is seen as the expert with all the knowledge and is responsible for imparting that knowledge to the passive students. This approach does not recognize student autonomy as students are generally not active in their learning. Constructivist theory would disagree with this approach, as it is assumed that students use their present knowledge and past experience to aid them in comprehending the material (King, 1993).

The learning culture of the flipped classroom takes a different approach in which the teacher is seen as the 'guide on the side' (Hamden et al., 2013; Jones, 2014). This should not be misinterpreted to believe that the teacher does not have an end goal. The role of the teacher does have learning goals and objectives but acts more as a guide, eliciting the learning from the students rather than directing it.

In this approach, the learning culture shifts from teacher-centered to student-centered learning by providing video lectures outside of the classroom and providing further collaborative opportunities in class (Lambert, 2012). Moreover, concepts are presented in a way, where students take that information and interact with peers and share ideas from what they already know (King, 1993). The culture of the flipped classroom is for instructors to provide more face-time with their students, by moving around the room to provide instant feedback to students, providing one-on-one instruction (Bergmann & Sams, 2012).

### INTENTIONAL CONTENT

In flipped classrooms, intentional content is about selecting the best available content that can be delivered both inside and outside the classroom. Educators in flipped classrooms therefore deliberately select and explore content that affords students the chance to gain a better conceptual understanding of real-life situations (Hamden et al., 2013; McDonald & Smith, 2013).

Bloom's (1968) taxonomy on learning has six levels of increasing difficulty. The six levels in Bloom's taxonomy are remembering, understanding, applying, analyzing, evaluating, and creating. In a non-flipped classroom, instruction is focused primarily on developing the first two levels of Bloom's taxonomy: remembering and understanding, which are lower ordered tiers of Bloom's taxonomy (Bergmann & Sams, 2012). In a flipped classroom approach, Bloom's taxonomy is flipped where the lower ordered tiers are now conducted outside of the classroom, and the higher ordered tiers of applying, analyzing, evaluating, and creating are now done in the classroom with the teacher as the facilitator (Bergmann & Sams, 2012). Teachers typically use their instruction time to introduce concepts using relevant examples to foster a degree of understanding.

In the flipped classroom setting, the order of Bloom's taxonomy is reversed, with remembering and understanding taking place outside the classroom through the use of stimulating video lectures that permit students to review content multiple times in order to understand the concepts being presented (McDonald & Smith, 2013). Inside the classroom, students can collaborate with one another on different aspects of their learning based on the video instruction. Applying, analyzing, evaluating, and creating take place in the classroom itself based on students' understanding and remembering of the material reviewed outside the classroom environment. It is in these higher-order skill developments that can simultaneously deepen the students' understanding and knowing of the topic.

### PROFESSIONAL EDUCATORS

In the flipped classroom setting, educators act as facilitators while monitoring the progress of their students (Jones, 2006; King, 1993). While this teaching pedagogy shifts teachers away from the traditional classroom role, they must nonetheless possess the skills to assess students' work and facilitate class discussions. To be committed to students and their learning, teachers in the modern flipped classroom must also embrace technology for this pedagogy to be successful (Fulton, 2012).

In addition, student-teacher relationships are established in the flipped classrooms, due to the flexibility of this learning environment. Teachers have access to students one-on-one, for those who are struggling with concepts. As a result, it is essential that student-teacher relationships in early grades have a great effect on their current and future academic and behavior needs (Hughes & Kwok, 2007; Meehan, Hughes, & Cavell, 2003).

## RESEARCH PARTICIPANTS

A purposive sample was used (Creswell, 2012) with specific conditions to better determine who could be asked to participate. Participants were selected using the following criteria: (1) be a certified mathematics teacher; (2) teach in an independent school; (3) have experience using a flipped classroom; and (4) have experience using video instruction. The recruitment process of the possible participants was deliberate in nature. To recruit possible participants, an email was sent to department heads of independent schools in the province of Ontario. Once a list of possible participants was compiled, I reached out to these prospective participants to ask if they would be interested in participating in a qualitative study that focused on teacher perceptions with the use of video instruction in the flipped classroom. It was important that participants were made aware of the details of the research study and the scale of their voluntary involvement.

As a result, four participants were selected for this research. Each participant received an information letter that described the nature of the research study. The information letter included an informed consent document that explained the role of the participant and their rights as a voluntary participant in the study. Those who agreed to be part of the study signed the consent form acknowledging that their participation was voluntary and that they could decline and withdraw from the study at any time. All four participants took part in the initial interviews, regarding video instruction in the flipped classroom. Out of the four participants, only Isaac agreed to continue with the research study.

Isaac is a colleague of mine, who is a grade nine and ten mathematics teacher. He has been teaching for eleven years and is an advocate for the use of technology in the classroom

## DATA ANALYSIS

Case study qualitative analysis was utilized to explore the challenges and benefits of the flipped classroom as they pertain to the use of video instruction. Each respondent was assigned an alias name that linked the participant to their respective responses. This helped to maintain a high degree of anonymity in the responses from each participant. It was important and ethically necessary to ensure the protection of each participant and their given responses.

The participant responses were then coded and labelled by using key words from the objectives. Themes were then identified based on common responses to each code that was determined during my research. Creswell (2012) suggests that further analysis of the coded transcripts are needed to reduce to four to six main themes. The purpose of this theming was to capture similarities while identifying any significant differences in the responses given.

One participant agreed to further be observed through classroom observation and viewing of their video instruction. Additionally, after each observation, this respondent was asked further questions and was provided a verbal summary of what was said in order to improve accuracy of statements provided by the participant. According to Creswell (2012), member checking is an effective way to determine accuracy of the findings collected. Isaac had the opportunity to provide feedback on transcripts of my interviews with him and to verify if my interpretation of the data collected from Isaac is accurate.

## MAJOR FINDINGS

The major findings were drawn from the initial interviews with four participants, daily observations in Isaac's classroom, face-to-face talks with Isaac, and Isaac's perceptions from a student online survey. There are three major findings identified from this research study relating to teacher's perceptions on video instruction:

1. Students are in favour of shorter videos that were five to eight minutes in length and preferred not to have more than two concepts in a video. Isaac's students felt, with shorter videos, they were able to stay on task and found the videos more engaging. Students felt better prepared to complete their homework and begin to reuse the videos in preparation for upcoming tests and exams.
2. Teachers should not use the flipped classroom method all of the time. Several students in Isaac's class preferred teacher-directed instruction. Despite several students still wanting a non-flipped classroom approach, they did prefer to have access to videos to re-engage with lesson delivery and to act as a study guide for unit tests and midterm and final examinations. Isaac also found some concepts were more effective when taught in a non-flipped format, as explanation of a new concept took some time for students to comprehend. Additionally, he felt that implementing a flipped classroom depended on the dynamics of the classroom and the students.
3. Not all students enjoyed the flipped classroom. Some of the high-achieving students preferred to work alone and did not appreciate having to work at home as well.

## CONCLUSION

Educators have to remember that the flipped classroom is not designed to replace the non-flipped classroom, as the learning environment may not be conducive to all learners. However, if used properly, it can help improve a student's self-regulation because it holds the students accountable for their education, therefore increasing the likelihood that the student will feel more engaged in their own learning.

One obstacle that stands in the way of implementing the flipped classroom is time (Rosenberg, 2013). From a teacher standpoint, teachers need time to construct videos, especially if a teacher is initially adopting a flipped classroom model in their classroom. Bergmann and Sams (2012) cautions teachers to plan for their classroom time now that lectures are provided as individual work and typically done at home. Secondly, the flipped classroom pedagogy supports the development of self-regulating learnings, and teachers need to be aware that in addition to teaching the curriculum, they also play a significant role in coaching students to become self-regulating learners and more time is needed to effectively apply this theory.

As seen in this research study, technology has a played a significant role in the flipped classroom. It is important that teachers do not put the emphasis solely on using the technology in their teaching. If they choose to incorporate technology, it is more important that they consider how the technology used best applies in their respective classrooms in order to support student learning and self-regulated learning practices. Technology should not be a distraction to teaching and learning, it is better used to augment the learning experience and when possible, create access to learning.

Finally, the flipped classroom and the use of video instruction in the flipped classroom have many benefits for student learning. It helps with students' anxiety through the use of classroom collaboration and students having the ability to work one-on-one with a teacher during class. Students have the ability to ask questions and get immediate feedback from their peers or teacher, which often creates a space for active learning (Hamden et al., 2013). Additionally, when a student misses class, they now have the opportunity to view a video on what concepts they missed.

The focus of this research study was on teacher perceptions on the use of video instruction in the flipped classroom. The benefit of this instruction is threefold. As teachers were now able to 1) give more attention to students who are struggling in mathematics; 2) students now have the class lecture, literally in their hands, as these videos can now be accessed anywhere; and 3) the flipped classroom is very transparent when it comes to parental involvement. Parents can have access to what their child is studying and therefore, communication between the teacher and parent is now greatly enhanced (Bergmann & Sams, 2015).

Lastly, the importance of the flipped classroom is the relationships this pedagogy creates between student and teacher. When students feel comfortable with a teacher, the opportunity for the learning environment to be positive is enhanced, and students feel more open to communicate their struggles or successes with their teacher or peers (Chen & Chang, 2014). Students are more prone to be more engaged in the subject, which in helps with their critical thinking skills which is a major facet in education.

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# GROWTH OF MATHEMATICAL UNDERSTANDING FOR LEARNERS WHO EXPERIENCE DIFFICULTIES IN MATHEMATICS

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I begin with a story about David and his teacher. David's story leads to questions around the assumptions we make about who can understand mathematics and the barriers we place on those we assume are not able to understand mathematics. David's story sets the stage for the description and findings of my subsequent exploration of the growth of understanding for learners who experience difficulties in mathematics.

## DAVID

David was a student in a self-contained Grade 4/5 split special education classroom. Before the start of the year the teacher met with the school principal to discuss the children she was about to teach. All the children in the class were officially labelled with some label. Most of the children in the class were either diagnosed with a learning disability or ADHD or both. Then there was David. David had a label of 'slow learner'. The label of slow learner was meant to convey low intelligence, certainly below average, and falling somewhere between 75 and 90 on a normal curve of intelligence (Williamson & Paul, 2012). Those with educational authority around David had little hope for him. They believed that teaching would not matter for him, especially in mathematics. This was the explicit description of David conveyed to the teacher by the principal before she began the school year. The teacher was told that it was her responsibility to focus on the other students who may possibly have potential to learn mathematics but not to 'waste' too much time and effort on David. David just could not understand mathematics.

The teacher taught mathematics to her class as usual. Then at one point during the year, the teacher was exploring student generated algorithms for multiplication with her students. Student generated algorithms is an especially abstract reasoning to develop and use (Barmby, Harries, Higgins, & Suggate, 2009). Through constructing their own algorithms, learners develop multiplicative reasoning, specifically reasoning with sets of numbers. And, at the same time, this reasoning is important and necessary for more advanced mathematics. The teacher gave the class the equation to solve, $16 \times 4$, with the instruction that they could use anything in the room as a tool to help them solve the problem. The students began working away, and the teacher began moving around the classroom to discuss how the students were solving the problem. The teacher had lost track of David, even though she wondered how David would interact with the problem.

David's teacher peripherally noticed that David was manipulating beads on an abacus. However, her gaze did not last long as a child's remark called her attention elsewhere. Then, while the teacher's attention was elsewhere, David called out in a proud voice "*64. The answer is 64*." The teacher looked up and put her full attention on David. The teacher immediately went to discover how David was thinking about the question and how he arrived at his answer. "*David, why do you think the answer is 64? Can you show me your reasoning?*" The teacher asked David. David pushed all the beads on the abacus to one side. He said, "*You see, 16 is 10 and 6*." The teacher nodded to show she was listening. David continued, "*Well, 10 times 4 is this*." David pushed 4 rows of beads aside on the abacus, counting 1,2,3,4 as he pushed each row. "*And, then you have 4 sixes*." David counted out six beads on 4 rows and pushed them aside to align with the rows of ten. "*Then you count them all together, I know that these are 10, so 10, 20, 30, 40, that's 40. Then this is 6 so 46*." Then David counted each bead, one-by-one from 46 to 64 to get the answer.

The teacher had just witnessed David do something he was not supposed to do-David showed abstract reasoning and mental flexibility with numbers, two important foundations to number sense and higher order mathematics (Fosnot & Dolk, 2001). The teacher noticed the important mathematical moment for David and went to share the news with her principal. However, the teacher received a response she was not expecting from the principal: "*David really can't do mathematics. He is really low. What you think you saw is not really what David can do-he is a slow learner*."

From David's story we can wonder: "Why was David's mathematical insight dismissed?" and "Why did David's label of inability overshadow his ability?"

This study is an exploration of abilities and the growth of knowing of the concept of zero for learners experiencing difficulties in mathematics. Instead of answering why a label can be so powerful as to overshadow ability, I explore abilities and pathways of growth through making developing mathematical understandings explicit. In the following sections, I first situate mathematics difficulties within the broader literature, I then discuss the Pirie-Kieren Theory of Mathematical Understanding (PK) as the theoretical framework and the methodology for the project. Finally, I introduce Angela and her relationship with mathematics, as well as her specific pathway of understanding of zero. I conclude with findings about growth of understanding for learners experiencing difficulties learning mathematics.

## MATHEMATICS DIFFICULTIES

There are many different terms for students who experience difficulties in mathematics due to a learning disability or mathematics difficulty (MD). These terms have included acalculia, dyscalculia, mathematics disability, mathematics difficulties, arithmetic disorders, and mathematics disorders. There has been no real consensus as to the definitions of each term-the meanings can vary by researcher, context or field. Additionally, these definitions of learning disabilities (LD) that are utilized tend to have two problems. First, the definitions tend to be too general and expound little on exactly what is an LD. Second, the definitions tend not to answer why one child may be diagnosed and another child not diagnosed (Kavale, Holdnack, & Mostert, 2005). However, there are some commonalities across the definitions in use that may be helpful in understanding the label. One commonality is that of the relationship between the labeling of an LD and perceived classroom achievement (Gersten, Clarke, & Mazzocco, 2007).

Mathematics difficulties can be experienced inconsistently across mathematical domains and even within the same mathematical concept (Houssart, 2004). A child may experience success with an algebraic concept one day and experience difficulty with the very same concept the next day, and vice versa.

## THE PIRIE-KIEREN THEORY OF MATHEMATICAL UNDERSTANDING (PK)

First in 1989 and again in 1994, Pirie and Kieren outlined a theory of growth in understanding that represented understanding as a dynamic recursive process. Pirie and Kieren developed a model as a methodological tool for their theory (Figure 1 is a depiction of a participant's mapping of her growth of understanding). The model displays a series of eight nested circles, each circle representing a mode of growth. While each circle is labeled a different mode of growth—Primitive Knowing, Image Making, Image Having, Property Noticing, Formalising, Observing, Structuring and Inventising—the image of nesting represents the recursiveness of each circle.

For this research project, I utilize only the first three modes of the PK theory: Primitive Knowing, Image Making, and Image Having. Because knowledge can only come from knowledge, Primitive Knowing is a starting point; it is the repository for all the previous understandings that a learner has accumulated and is required for the new growth (Martin & Towers, 2015). Image Making is when learners are actively engaged in learning and begin to create mathematical mental objects. When the learner is able to use their mental object, or mental image, without relying on the activity that initiated the mental image, they are at the third mode of Image Having. Image Having is an abstraction of Image Making (Pirie & Kieran, 1994).

## METHODOLOGY

The data shared in this paper comes from one part of a four-part study-an approximately 1.5-hour mathematical task-based individual clinical interview of one of the participants: Angela. Angela was 11 years old at the time of the study and experiencing mathematics difficulties. I videoed our interactions as Angela progressed through seven tasks I either designed or modified from another source for the study. Each task revolved around a specific understanding of zero. In this paper I share analysis and findings from one interaction around the idea of zero as a placeholder.

## ANGELA

> *Like sometimes when we're doing a boring lesson I'll hate it* [referring to math]. *Sometimes when it's an easy lesson, then I'll like it. Sometimes we have a test coming up…and…I study. Sometimes I get it wrong, and I'm just like, I hate math…Usually I hate math.*

Angela holds the belief that the high amount of effort she expends attempting to do well in mathematics should be proportionate to a high achievement level. At the same time as receiving support and expending a lot of energy trying to do well, Angela's achievement in school has been labored and inconsistent. However, the mathematics that Angela says she hates is not actually mathematics. Angela hates the structures of schooling (McDermott, 1996) that surround the teaching of school mathematics: testing/grades and learning procedures devoid of connections.

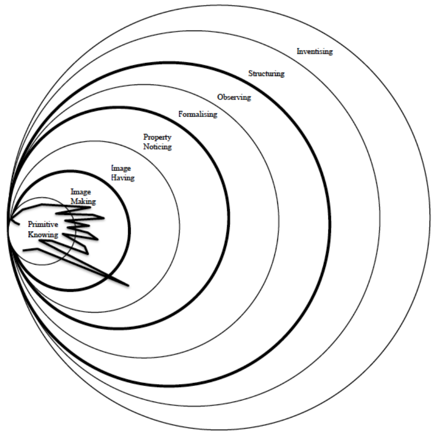


Figure 1. Angela's mapping.

Throughout Angela's journey of growth in understanding zero, she hovers in, out and around Primitive Knowing. Each time Angela moves into Image Making, and once into Image Having, she does not stay there for long; she quickly folds back to her previous understandings of zero. One reason for this constant recursive movement is that Angela does not usually trust her own ideas. And if developed further, often these ideas could lead to more growth. After proposing an idea that could potentially demonstrate growth toward creating images of zero, Angela will often look to my face to gauge the accuracy of her statement. Relying on adult input, such as Angela's gauging the correctness of her answer through my non-verbal cues, is a typical response by children who experience mathematical difficulties (Shih, Speer, & Babbitt, 2011).

## ZOOMING IN ON ANGELA'S UNDERSTANDING

I presented Angela with two number cards, one with the number "105" and one with the number "150." I then asked Angela, "Is the digit 0 worth more in 105 or 150?" (Cockburn & Parslow-Williams, 2008, p. 19). Angela began answering the question, and as she did, she came to more and more realizations about the concept of zero:

> Angela: *I don't know. 'Cuz this number is 50* [Pointing to the 50 in 150] *so, it's not like…and in this number* [points to the 150].
> Angela: *Oh* [with certainty]*! This one* [points to the zero in 150] *because like, because then it makes it 50. If you didn't have the zero because then it would be 15* [uses hand to cover the zero in 150].
> Angela: *if you didn't have the…* [Uses hand to cover the zero in the 105—looks confused],
> Angela: *but also…this number is just like putting the number in the middle 'cuz there's nothing there. It's just 5*. [points to the zero in 150]. *150, zero.*
> Angela: *Yeah but this number* [points to 105] *would also be 15.*
> Angela: *[*covers the zeros on both cards]*…I don't know maybe not…*
> Angela: *it would be one and five because there has to be something in the middle.* [pointing to the 150]. *This is at the end.*

Here, in order to decide which zero is worth more, Angela strategizes using her primitive knowing to explore isolating the zero, and then, removing the zero. Angela's interactions with this exploration lead her to Image Making, making connections around zero as a placeholder that (a) zero has a relationship with the surrounding numbers, and (b) zero depends on its relationship with the surrounding numbers for zero's identity.

Angela's first strategy is to use her primitive knowing of decomposing numbers to isolate the '50' from 150. Angela is utilizing her primitive knowing of place value zero until the tens place. This strategy of dipping back into Primitive Knowing to retrieve the place value zero up until the tens place, leads Angela to begin Image Making about place value zero's relationship with the surrounding numbers. At the same time, because Angela has not yet thickened her knowing to include place value zero in other place value places, another cognitive conflict arises. Angela then removes the explicit zero first from 150. Angela is left with the two numerals, 1 and 5 to make fifteen. This satisfies Angela until she realizes that removing the zero from 105, leaves her with the same numerals, 1 and 5 to make fifteen. The two identical numbers of 15 cause a cognitive conflict (Movshovitz-Hadar & Hadass, 1990) for Angela. This conflict then propels Angela to immediately dip back into Primitive Knowing. She gathers the same understandings as before, place value zeroes for the tens place and the ones place, but now uses her primitive knowings in a different way. This time, Angela focuses on the 105. Different than the 150, the 105 has a zero in the middle, in the tens place. This slight difference affords Angela to reason that even with the removal of the zero, there is still something there: "*it would be one and five because there has to be something in the middle*." The one and five could not join together because there is still something in the tens place. She explains that even if the zero is removed, something has to be positioned between the one and five. The zero to Angela in 105 is different than in 150. Thus, she reasons that the numerals "*1*" (one hundred) and "*5*" (fifty) in 150, can be 15 because they are beside each other, but the one (hundred) and five in 105 cannot be fifteen because something has to be placed in the middle between them. As Angela plays and moves between the different zeroes in 105 and 150, she is on the cusp of Image Making. Angela is making images of the relationships between the place value zero and zero.

## CONCLUSION

Angela demonstrated growth on a small scale in the previous depiction of her learning. Acknowledging that growth also happens on a small scale (Siegler, 1996) is important for repositioning learners with mathematics difficulties from 'unable' to 'able' in specifically understanding abstract mathematical concepts. Angela's journey was not a linear one—her mapping began and ended with Primitive Knowing. Even though Angela moved in and out of different types of knowings, sometimes recursively moving backwards, accessing previous knowings to help her push her growth forward, she still ended in the same place as she began. If truth be told, Angela did not really start in Primitive Knowing nor did she really end in Primitive Knowing despite what the model conveys. Beginning and end points on a model of growth are discretionary. Growth is not absolute because it is continuous and not discrete. Pathways of growth came before the first marker and pathways of growth will come after the last marker. I chose where on the continuous model I would start and end the tracking of growth. The point at the beginning and the point at the end of my analysis indicates a choice by me, the researcher; it does not indicate a beginning nor an end of the whole pathway. When Angela began her journey in this study, she had a concept of zero as a number side by side with a concept of zero as nothing (not a number)—both in her Primitive Knowing. Angela ended her journey in this study in Primitive knowing, again revisiting zero as a number and zero as nothing. However, Angela's understandings around zero and nothing are now more robust then they were to begin with. Now, she has in her Primitive Knowing that zero as nothing and zero as a number can coincide beside each other.

Understanding that learning is messy and that movement backwards is also a movement of growth is an important idea that will help us to see that real and true mathematical growth can occur for David, for Angela, and any learner who typically experiences difficulties learning mathematics.

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# Ad Hoc Sessions

## Ad hoc sessions

# MATHEMATICS AS A LIBERATING ART

Derek Postnikoff
*University of Saskatchewan*

"When am I going to use this?"—the instantly recognizable cry of the frustrated mathematics student. How should the mathematics educator answer? Suppressing the challenge through an authoritarian threat of discipline was once a popular response that is now recognized as completely unacceptable. Ignoring the question altogether or conceding the uselessness of the lesson would be tantamount to surrender and would all but guarantee student disengagement and a loss of respect. It is no wonder then that contemporary math educators feel pressure to focus on the practical uses of mathematics in an attempt to make the material relevant to their students and improve engagement. A lot of impressive work is being put in to developing realistic, relatable mathematics lessons, and I applaud those who are contributing to this important effort.

However, when a math student asks, "When am I going to use this?", I suggest that it often constitutes an instance of the fallacy of complex question, similar to "Did you make a lot of money from selling those stolen exam answers?" Both answers dictated by the internal logic of the latter question ("yes" or "no") count as a confession of wrongdoing; a better response is to reject the logic of the exchange and respond, "I did not steal or sell any exam answers." Similarly, by consistently explaining precisely and explicitly how students will use every piece of mathematics they are taught, an educator implicitly reinforces the troublingly restrictive presupposition behind the complex question—that the sole value of mathematical study lies in its directly foreseeable practical application.

I contend that it is incumbent upon math educators to help their students appreciate that the value of mathematics extends beyond its immediate practical utility and, in particular, to understand that the study of mathematics is valuable insofar as it fosters mental liberation. Since the time of Socrates, mathematics has been considered a foundational liberal art; arithmetic, geometry, and logic comprise three of the traditional seven *ars liberalis*. Mathematics frees us from error by helping us overcome our faulty numerical instincts and the inexactness and uncertainty of the concrete world. Practicing abstract mathematical reasoning helps a student develop the capacity for hypothetical and counterfactual thinking and allows them to see a greater plurality of possible paths; a student competent in this way of thinking would be unlikely to ask "When am I going to use this?" Computational competency frees one from depending on technological devices for their calculational needs. By learning to write and read proofs, a student's intellectual autonomy is greatly enhanced—their reasoning abilities allow them to see the truth freely without requiring expert input, helping the student become an independent mathematical authority. Mathematics can even liberate us from the bonds of finitude!

The next time one of your math students asks, "When am I going to use this?", embrace the teachable moment and give a response that aims to liberate!

These are not new ideas, but they bear repeating until they are widespread.

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# IF THE BINOMIAL EXPANSION THEOREM CAN BE VISUALLY DEVELOPED THROUGH PERMUTATIONS, THEN WHY DOES THE COMBINATION USE FORMULAS?

Gale Russell
*University of Regina*

This ad hoc question explored a problem that emerged from a series of tasks I have been using in all levels of the EMTH (math methods) courses that I have taught over the past five years. It began as an introduction to permutations and combinations, but then beautifully developed into Pascal's triangle and the binomial expansion theorem.

The first question I ask students is "*How many different 3-cube high towers are possible if you can only make use of two colours in their construction?*" Typically, 90%+ of the students come to the answer eight (permutations) based on constructions similar to that in Figure 1, and the remainder provide four (combinations) based on constructions similar to that in Figure 2.

Figure 1. Permutations of two-colours.

Figure 2. Combinations of two-colours.

This difference in opinion is quickly resolved through an impromptu debate amongst the students who decide 'order matters' (a whole other ad hoc session possibility) and go with permutations.

Next, I tell the students, "*Linking cubes can be purchased in sets of 1000, with 100 of each of 10 colours. If you started with 1-cube high tower possibilities and kept going building successively taller tower possibilities, what would be the tallest full set of towers you could build? How many extra linking cubes of each colour would you need to build the next set of towers?*" Figure 3 shows one way that the students begin work on solving this set of questions

Image 3: Building of increasing height towers

At this point, there are always a handful of students that say, "*Hey, I've seen this before… it's that triangle thing*"—Pascal's Triangle. Another outcome checked off, and I now see a way to take them even further by asking the students to describe each row of towers for me to write out.

> Students: *1 red and red and red or 1 red and red and yellow or 1 red and yellow and red …*
> Me: *I'm getting tired of all this writing… isn't there something easier I could write*
> Students: *1 r and r and r or 1 r and r and y or 1 r and y and r…*
> Me: *Well, that's a bit better, and I know I can use & for 'and', but what about all the 'or's'?*
> Students: *"And" is multiplication and 'or' is addition (we did stop to make sure that this was a correct conclusion in this case)… so* $1rrr + 1rry + 1ryr + 1yrr$*…*
> Me: *Make sure I put all the rs and ys in the right order*
> *Students: Well* $rrr$ *is just* $r^3$ *and rry is the same as* $ryr$ *and* $yrr$*, so you could just use* $r^2$
> Me: $1r^3 + 1r^2y + 1r^2y + 1r^2y + \cdots$
> Students: *Why don't you just write* $1r^3 + 3r^2y + \cdots$

And, voila, the binomial expansion theorem. EXCEPT…we started by clearly stating that this was a permutation situation, but the binomial expansion theorem involves combinations.

The ad hoc session was to discuss where this change in combinatorics happens and why. We think we narrowed down where it happened…the step where $ryr$ and $rry$ are identified as the same, but still cannot justify the actual changing of combinatoric because the 3 in $3r^2y$ makes up for saying they are the same. The ad hoc session started again during the car ride to the Halifax airport with new attendees, and now I am also wondering if the problem occurs because the students are selecting from colours rather than selecting colours for positions. If you too want to join this ongoing and rewarding discussion, I would love to hear from you: gale.russell@uregina.ca.

## Mathematics Gallery

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## Gallérie Mathématique

# MOVING ACHIEVEMENT TOGETHER HOLISTICALLY (M.A.T.H.): AN INDIGENOUS APPROACH TO MATHEMATICS EDUCATION

Ellen Carter, Evan Throop-Robinson & Lisa Lunney Borden
*St. Francis Xavier University*

Moving Achievement Together Holistically (M.A.T.H.) is a research program rooted in Indigenous approaches to mathematics teaching and learning, as well as Indigenous methodology. This project draws upon L'nui'ta'simk (Mi'kmaw ways of knowing or Mi'kmaw epistemology) to transform pedagogical practice in both Mi'kmaw Kina'matnewey (MK) and public schools. The program is based on lessons learned from Lunney Borden's (2010) doctoral research which generated a model for decolonizing mathematics education for Mi'kmaw students.

We visit various K to 8 classrooms to work alongside teachers as they implement lessons based on the ideas of verbifying and spatializing number and operation. Through the gallery walk, we shared the results of a lesson titled "sets of, rows of, jumps of," which highlights the concept of multiplication in various contexts without yet teaching the word. We designed a series of four learning centres where students would use dice to obtain two numbers, which they then would use to build multiplication facts as sets, rows, or lengths. For example, at a *set* centre, if a student rolled three and five, they could choose to build three sets of five, or five sets of three. We provided two variations of the set model—with and without a ten frame. The *rows of* centre provided students with square colour tiles to build area models and a blank grid to shade what they built. The *jumps of* centre used number lines. A researcher (Lisa) and the teacher circulated around the room engaging students in discussion about what they were building and how they were getting the totals. The teacher continued working with these centres for a few days to deepen students' understanding of building multiplication conceptually. We returned the next week with story problems that each involved these concepts, still having not introduced terminology.

As we were engaging in discussion about the problems, one student explained that she was building things a certain number of times: "*I built 3 four times and I built 2 five times…*" Then she paused and said, "*Wait a minute, this is just times!*" She then told everyone in the room that this was "*just times!*" She declared, "*I know what times is now, it's just groups!*"

Our goal with these lessons was to move from process to concept. We wanted students to informally build multiplication before we introduced the name of this concept and the symbols for the concept. We believe this was done successfully, and the teacher also explained that she later used a similar approach when introducing division as fair sharing. She remarked that she found this to be a very effective approach.

The activities themselves allowed us to see how students were thinking about quantities and working with repeated quantities. Students were able to move easily from one model to another depending on the context. Set models and area models were favoured more than the number line model. While we continue to analyse this data, we believe it was an effective way of teaching the students what multiplication means which, in turn, made the problem-solving easier. This approach moves students through a process and allows them to name the process when they figure it out, just like mathematicians do.

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# STUDENT PROBLEM-SOLVING IN THE TRANSITION FROM ELEMENTARY TO SECONDARY MATHEMATICS

Alexandre Cavalcante[1], Olga O. Fellus[2], Elena Polotskaia[3], Louis-Philippe Turineck[1], & Annie Savard[1]
*McGill University[1], University of Ottawa[2], Univérsité du Québec en Outaouais[3]*

One area of particular difficulty in learning mathematics is the transition from arithmetic to algebra. This difficulty is associated with students' skills in solving word problems. In elementary school, problems are solved using concrete numbers and operations on them. In secondary school, however, problems are solved using equations, inequalities, and functions through knowledge of structures and relationships (Cai & Knuth, 2011; Kieran, 1989; Schmidt & Bednarz, 2002). The leading perception in our project is that holistic relational thinking is an effective problem-solving tool both in arithmetic and in algebra. Working with middle-school students who are struggling with mathematics, our objective, therefore, was to produce new relevant instructional materials and pedagogical strategies.

During a period of a two-year project, we worked closely with two teacher-participants to co-design a collection of word problems, the easiest of which requiring one arithmetic operation and the most difficult requiring one algebraic equation with multiple operations for their solution. We used the problem-solving cycle model (Savard, 2008; Polotskaia, 2015) to develop specific teaching strategies. The first year of the research project was devoted to the composition and testing of various relational problem-solving activities. This period also allowed the teacher-participants to appropriate the approach of teaching relationally—an approach which was new to them by their admission. We conducted interviews with students before and after the implementation of the new teaching approach where they used the relational approach to solve word problems with missing numerical data, which were provided upon request.

Preliminary results show changes in students' mathematical sense making while solving problems. The following were observed: (a) Students requested less numerical data from the interviewer after the project intervention. This indicates that they are more likely to think about relationships than about numerical operations. (b) Students used more strategies to understand the problem including more frequent use of the graphical representations of the relationships. (c) The success rate for the more complex problems was higher after the intervention.

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# EXPERIMENTAL MATH SPACE

Amenda Chow
*York University*

https://amchow.info.yorku.ca/experimental-math-space/

Figure 1. Logo designed by Giuseppe Sellaroli.

At university, in comparison to other science disciplines such as biology, chemistry, and physics, students in mathematics do not have a laboratory component to complement their mathematics lectures. Housed in the Department of Mathematics and Statistics at York University, I am developing an experimental math space that aims to fill this gap and consequently, enhance experiential education in the mathematics discipline.

Students learning in this space are surrounded by tactile objects that support and reinforce the mathematical ideas taught in lectures. They may use these objects to construct mathematical experiments that model practical applications such as feedback loops, resonance, optimal path finding, and random walks. This provides an indelible experience for undergraduate mathematics students. Students are also working in teams and learning mathematical software that are needed to operate the mathematically-inspired pieces of equipment. Portable pieces from this space are also able to enter classrooms during lectures for demonstrations that enhance mathematical concepts that are being taught.

Figure 2. Students in MATH 4090 (Mathematical Modelling) with their professor using robots to describe random walks in discrete time. This photo was taken on January 11, 2019 in the experimental math space.

A catalogue of the mathematically-inspired equipment we have can be found here: http://amchow.info.yorku.ca/experimental-math-space/items-we-have-available/.

A photo gallery can be found here: http://amchow.info.yorku.ca/experimental-math-space/experimental-math-space-photo-gallery/.

The learning outcomes for students using this experimental math space are:

1. designing experiments that are modelled by mathematical equations,
2. using mathematical software and tools commonly used in industry, which will aid in the design and collection of the results of the experiment,
3. writing scientific and technical reports, and
4. working and communicating in a team-oriented environment with peers and faculty.

This experimental math space is supported by York University's Academic Innovation Fund (http://avptl.info.yorku.ca/academic-innovation-fund/).

# MATH OUTREACH ACTIVITIES AT THE UNIVERSITY OF CALGARY

Lauren DeDieu
*University of Calgary*

At the Galley Walk, I shared information about the enrichment and outreach initiatives I have been involved with since joining the Department of Mathematics and Statistics at the University of Calgary. The poster I created was meant to spark a discussion about best practices in sharing mathematics with the wider community and how to reach underrepresented groups. Below I have outlined several of these outreach activities. Information about our other programs can be found here: https://math.ucalgary.ca/community_outreach.

## GIRLS EXCEL IN MATH (GEM) CALGARY

Girls Excel in Math (GEM) Calgary is an enrichment program for junior high school girls who are interested in having fun exploring mathematical topics that they would not normally see in the classroom. Topics include voting methods, graph colouring, fractals, and cryptology.

GEM sessions run four times a year from 10am–12pm on Saturday mornings at the University of Calgary. Students are recruited and mentored by excellent teachers at their own schools. Saturday sessions are led by these junior high school teachers with support from undergraduate student volunteers. In 2019, approximately 90 students and 10 teachers participated. For more information see https://science.ucalgary.ca/mathematics-statistics/engagement/educational-outreach/girls-excel-math.

## SONIA KOVALEVSKY (SK) DAY

Sonia Kovalevsky (SK) Day is a daylong event for high school girls organized by the University of Calgary Student Chapter of the Association for Women in Mathematics (AWM). This event was first held in March 2018 and had approximately 80 participants.

Students began their day together as a group. They worked collaboratively on mathematical communication activities and brainstormed about where math is found in the world. Students then broke into smaller teams and participated in a workshop led by faculty members and graduate students. At the end of the day, students created trifolds on their mathematical exploration and partook in a mini-research fair where they presented their findings to their parents and peers.

## CMS REGIONAL MATH CAMP (ALBERTA)

The CMS Regional Math Camp (Alberta) is a weeklong overnight summer camp for mathematically inclined Grades 7–10 students. The camp is held at the University of Calgary during odd years and at the University of Alberta during even years. Approximately thirty students are invited each year and participate in mathematics workshops, problem-solving sessions, friendly math contests, group activities, and discussions. Students also participate in field trips, physical activities and free time for friendship building. For more information see https://math.ucalgary.ca/community_outreach/cms_regional_camp.

# PROBLEM-SOLVING ONLINE ACTIVITY

Elena Polotskaia[1], Olga O. Fellus[2], & Viktor Freiman[3]
*Univérsité du Québec en Outaouais[1], University of Ottawa[2], Université de Moncton[3]*

We used the Equilibrated Development Approach to problem-solving to design an online environment (available in French and English is underway) offering special problem-solving activities for students of all ages. The main purpose of the environment (https://elenapolotskaia.com/mathematical-reasoning-development-games/) is to engage students in a self-directed learning of mathematics. However, the role of the teacher in this learning is crucial because the software does not teach problem-solving per se. Rather, it creates conditions for students to test their knowledge, identify gaps, and evaluate their progress. This means that the student takes on most of the responsibilities traditionally assigned to the teacher.

The environment offers four categories of problems: problems based on additive relationships; problems based on multiplicative relationships; problems based on combinations of additive and multiplicative relationships; and problems with fractional expressions. Numerical values of the problems are initially hidden behind letters. The solver can compose a sequence of arithmetic operations to solve the problem by using letters and/or numbers.

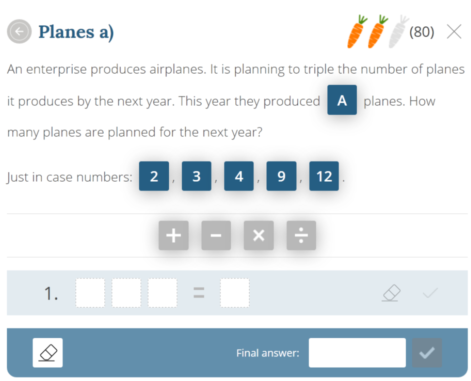


Figure 1.

Usually at the beginning, all students use numbers to solve computer task problems, but after training, the majority opts to use letters as they learn how to analyse and express relationships between quantities in a general form thus performing algebraic reasoning.

## FOR MORE INFORMATION

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# STUDENTS' PERCEPTIONS OF TEACHER EXPECTATIONS AND BELIEFS OF MATH LEARNING ABILITY

Andrew Vorhies
*University of Calgary*

In this poster, I presented and analyzed examples of student responses to vignette questions showing the female participants' constructed understanding of teacher beliefs and expectations of their math learning ability. I drew on data collected within a project investigating the role a male teacher plays in the relationship female learners build with mathematics in the early elementary years. Elementary education is important as students are developing their relationship with math. Strong math understanding is vital as many occupations are not considered math-centric but require significant math fluency. Female elementary teachers with a poor math relationship have been shown to pass on their relationship to female students (Beilock, Gunderson, Ramirez, Levine, & Smith, 2010). Gender role modelling appears to happen in elementary school for female students by their female teacher as males do not seem to be as affected (Boaler, 2015). In my study, I further explored female elementary students' experiences through one-on-one interviewing, and I also presented two vignettes to the participants to allow for an inquiry in a less direct manner. The first vignette eluded to a teacher, Mr. Smith, who believed both boys and girls can learn math equally and was followed by asking the participants if they thought there are teachers like this one. Some responses were: "*Yes. Because Mr. Smith thinks they're good at math. Everybody's good at math.*" "*Yes. Because everyone knows that girls and boys can be good at math, not just the, just the boys or just the girls.*" The second vignette eluded to a teacher, Mr. Jones, who believed boys are better math learners than girls and was followed by asking the participants if they wanted to be in this class. Some responses were: "*No, 'cause I'm a girl, and he doesn't think I'm good at math.*" "*No. Because he thinks boys are better than girls in math.*" I conducted theme-based analysis of the responses. One recurring theme was the participants all wanted to be good at math. The teacher's beliefs and expectations that females could learn math just as well as the male students was a major influence in the participants' desire to be good at math. Most participants did not answer why they wanted to be good at math when directly asked, but the vignettes provided considerable information beyond standard interview questions; such as, participants want the teacher to believe in their math learning ability and have high expectations, the vast majority of participants believed their teacher was similar to Mr. Smith, participants do not want to be in a class where the teacher does not believe in their ability or has low expectations, and the participants are very aware of teacher expectations and beliefs in learning ability in early elementary.

The teacher's beliefs and expectations of students' math learning ability matters greatly to these female participants. While the participants gave other reasons for why they wanted to be good at math; such as, peer or home influence, for the future, and challenge or reward, all participants indicated that teachers had the greatest influence on their relationship with math through expectations and beliefs. Findings of this project support other studies (Boaler, 2015) that found teacher expectations are a key influence on one's math relationship and achievement.

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# Appendices

# Annexes

## Appendix A / Annexe A

# WORKING GROUPS AT EACH ANNUAL MEETING / GROUPES DE TRAVAIL DES RENCONTRES ANNUELLES

1977 *Queen's University, Kingston, Ontario*

· Teacher education programmes
· Undergraduate mathematics programmes and prospective teachers
· Research and mathematics education
· Learning and teaching mathematics

1978 *Queen's University, Kingston, Ontario*

· Mathematics courses for prospective elementary teachers
· Mathematization
· Research in mathematics education

1979 *Queen's University, Kingston, Ontario*

· Ratio and proportion: a study of a mathematical concept
· Minicalculators in the mathematics classroom
· Is there a mathematical method?
· Topics suitable for mathematics courses for elementary teachers

1980 *Université Laval, Québec, Québec*

· The teaching of calculus and analysis
· Applications of mathematics for high school students
· Geometry in the elementary and junior high school curriculum
· The diagnosis and remediation of common mathematical errors

1981 *University of Alberta, Edmonton, Alberta*

- Research and the classroom
- Computer education for teachers
- Issues in the teaching of calculus
- Revitalising mathematics in teacher education courses

1982 *Queen's University, Kingston, Ontario*

- The influence of computer science on undergraduate mathematics education
- Applications of research in mathematics education to teacher training programmes
- Problem solving in the curriculum

1983 *University of British Columbia, Vancouver, British Columbia*

- Developing statistical thinking
- Training in diagnosis and remediation of teachers
- Mathematics and language
- The influence of computer science on the mathematics curriculum

1984 *University of Waterloo, Waterloo, Ontario*

- Logo and the mathematics curriculum
- The impact of research and technology on school algebra
- Epistemology and mathematics
- Visual thinking in mathematics

1985 *Université Laval, Québec, Québec*

- Lessons from research about students' errors
- Logo activities for the high school
- Impact of symbolic manipulation software on the teaching of calculus

1986 *Memorial University of Newfoundland, St. John's, Newfoundland*

- The role of feelings in mathematics
- The problem of rigour in mathematics teaching
- Microcomputers in teacher education
- The role of microcomputers in developing statistical thinking

1987 *Queen's University, Kingston, Ontario*

- Methods courses for secondary teacher education
- The problem of formal reasoning in undergraduate programmes
- Small group work in the mathematics classroom

1988 *University of Manitoba, Winnipeg, Manitoba*

· Teacher education: what could it be?
· Natural learning and mathematics
· Using software for geometrical investigations
· A study of the remedial teaching of mathematics

1989 *Brock University, St. Catharines, Ontario*

· Using computers to investigate work with teachers
· Computers in the undergraduate mathematics curriculum
· Natural language and mathematical language
· Research strategies for pupils' conceptions in mathematics

1990 *Simon Fraser University, Vancouver, British Columbia*

· Reading and writing in the mathematics classroom
· The NCTM "Standards" and Canadian reality
· Explanatory models of children's mathematics
· Chaos and fractal geometry for high school students

1991 *University of New Brunswick, Fredericton, New Brunswick*

· Fractal geometry in the curriculum
· Socio-cultural aspects of mathematics
· Technology and understanding mathematics
· Constructivism: implications for teacher education in mathematics

1992 *ICME–7, Université Laval, Québec, Québec*

1993 *York University, Toronto, Ontario*

· Research in undergraduate teaching and learning of mathematics
· New ideas in assessment
· Computers in the classroom: mathematical and social implications
· Gender and mathematics
· Training pre-service teachers for creating mathematical communities in the classroom

1994 *University of Regina, Regina, Saskatchewan*

· Theories of mathematics education
· Pre-service mathematics teachers as purposeful learners: issues of enculturation
· Popularizing mathematics

1995 *University of Western Ontario, London, Ontario*

- Autonomy and authority in the design and conduct of learning activity
- Expanding the conversation: trying to talk about what our theories don't talk about
- Factors affecting the transition from high school to university mathematics
- Geometric proofs and knowledge without axioms

1996 *Mount Saint Vincent University, Halifax, Nova Scotia*

- Teacher education: challenges, opportunities and innovations
- Formation à l'enseignement des mathématiques au secondaire: nouvelles perspectives et défis
- What is dynamic algebra?
- The role of proof in post-secondary education

1997 *Lakehead University, Thunder Bay, Ontario*

- Awareness and expression of generality in teaching mathematics
- Communicating mathematics
- The crisis in school mathematics content

1998 *University of British Columbia, Vancouver, British Columbia*

- Assessing mathematical thinking
- From theory to observational data (and back again)
- Bringing Ethnomathematics into the classroom in a meaningful way
- Mathematical software for the undergraduate curriculum

1999 *Brock University, St. Catharines, Ontario*

- Information technology and mathematics education: What's out there and how can we use it?
- Applied mathematics in the secondary school curriculum
- Elementary mathematics
- Teaching practices and teacher education

2000 *Université du Québec à Montréal, Montréal, Québec*

- Des cours de mathématiques pour les futurs enseignants et enseignantes du primaire/Mathematics courses for prospective elementary teachers
- Crafting an algebraic mind: Intersections from history and the contemporary mathematics classroom
- Mathematics education et didactique des mathématiques : y a-t-il une raison pour vivre des vies séparées?/Mathematics education et didactique des mathématiques: Is there a reason for living separate lives?
- Teachers, technologies, and productive pedagogy

2001 *University of Alberta, Edmonton, Alberta*

- Considering how linear algebra is taught and learned
- Children's proving
- Inservice mathematics teacher education
- Where is the mathematics?

2002 *Queen's University, Kingston, Ontario*

- Mathematics and the arts
- Philosophy for children on mathematics
- The arithmetic/algebra interface: Implications for primary and secondary mathematics / Articulation arithmétique/algèbre: Implications pour l'enseignement des mathématiques au primaire et au secondaire
- Mathematics, the written and the drawn
- Des cours de mathématiques pour les futurs (et actuels) maîtres au secondaire / Types and characteristics desired of courses in mathematics programs for future (and in-service) teachers

2003 *Acadia University, Wolfville, Nova Scotia*

- L'histoire des mathématiques en tant que levier pédagogique au primaire et au secondaire / The history of mathematics as a pedagogic tool in Grades K–12
- Teacher research: An empowering practice?
- Images of undergraduate mathematics
- A mathematics curriculum manifesto

2004 *Université Laval, Québec, Québec*

- Learner generated examples as space for mathematical learning
- Transition to university mathematics
- Integrating applications and modeling in secondary and post secondary mathematics
- Elementary teacher education – Defining the crucial experiences
- A critical look at the language and practice of mathematics education technology

2005 *University of Ottawa, Ottawa, Ontario*

- Mathematics, education, society, and peace
- Learning mathematics in the early years (pre-K – 3)
- Discrete mathematics in secondary school curriculum
- Socio-cultural dimensions of mathematics learning

2006 *University of Calgary, Calgary, Alberta*

· Secondary mathematics teacher development
· Developing links between statistical and probabilistic thinking in school mathematics education
· Developing trust and respect when working with teachers of mathematics
· The body, the sense, and mathematics learning

2007 *University of New Brunswick, New Brunswick*

· Outreach in mathematics – Activities, engagement, & reflection
· Geometry, space, and technology: challenges for teachers and students
· The design and implementation of learning situations
· The multifaceted role of feedback in the teaching and learning of mathematics

2008 *Université de Sherbrooke, Sherbrooke, Québec*

· Mathematical reasoning of young children
· Mathematics-in-and-for-teaching (MifT): the case of algebra
· Mathematics and human alienation
· Communication and mathematical technology use throughout the post-secondary curriculum / Utilisation de technologies dans l'enseignement mathématique postsecondaire
· Cultures of generality and their associated pedagogies

2009 *York University, Toronto, Ontario*

· Mathematically gifted students / Les élèves doués et talentueux en mathématiques
· Mathematics and the life sciences
· Les méthodologies de recherches actuelles et émergentes en didactique des mathématiques / Contemporary and emergent research methodologies in mathematics education
· Reframing learning (mathematics) as collective action
· Étude des pratiques d'enseignement
· Mathematics as social (in)justice / Mathématiques citoyennes face à l'(in)justice sociale

2010 *Simon Fraser University, Burnaby, British Columbia*

· Teaching mathematics to special needs students: Who is at-risk?
· Attending to data analysis and visualizing data
· Recruitment, attrition, and retention in post-secondary mathematics
Can we be thankful for mathematics? Mathematical thinking and aboriginal peoples
· Beauty in applied mathematics
· Noticing and engaging the mathematicians in our classrooms

2011 *Memorial University of Newfoundland, St. John's, Newfoundland*

- Mathematics teaching and climate change
- Meaningful procedural knowledge in mathematics learning
- Emergent methods for mathematics education research: Using data to develop theory / Méthodes émergentes pour les recherches en didactique des mathématiques: partir des données pour développer des théories
- Using simulation to develop students' mathematical competencies – Post secondary and teacher education
- Making art, doing mathematics / Créer de l'art; faire des maths
- Selecting tasks for future teachers in mathematics education

2012 *Université Laval, Québec City, Québec*

- Numeracy: Goals, affordances, and challenges
- Diversities in mathematics and their relation to equity
- Technology and mathematics teachers (K-16) / La technologie et l'enseignant mathématique (K-16)
- La preuve en mathématiques et en classe / Proof in mathematics and in schools
- The role of text/books in the mathematics classroom / Le rôle des manuels scolaires dans la classe de mathématiques
- Preparing teachers for the development of algebraic thinking at elementary and secondary levels / Préparer les enseignants au développement de la pensée algébrique au primaire et au secondaire

2013 *Brock University, St. Catharines, Ontario*

- MOOCs and online mathematics teaching and learning
- Exploring creativity: From the mathematics classroom to the mathematicians' mind / Explorer la créativité : de la classe de mathématiques à l'esprit des mathématiciens
- Mathematics of Planet Earth 2013: Education and communication / Mathématiques de la planète Terre 2013 : formation et communication (K-16)
- What does it mean to understand multiplicative ideas and processes? Designing strategies for teaching and learning
- Mathematics curriculum re-conceptualisation

2014 *University of Alberta, Edmonton, Alberta*

- Mathematical habits of mind / Modes de pensée mathématiques
- Formative assessment in mathematics: Developing understandings, sharing practice, and confronting dilemmas
- Texter mathematique / Texting mathematics
- Complex dynamical systems
- Role-playing and script-writing in mathematics education practice and research

2015 *Université de Moncton, Moncton, New Brunswick*

- Task design and problem posing
- Indigenous ways of knowing in mathematics
- Theoretical frameworks in mathematics education research / Les cadres théoriques dans la recherche en didactictique des mathématiques
- Early years teaching, learning and research: Tensions in adult-child interactions around mathematics
- Innovations in tertiary mathematics teaching, learning and research / Innovations au post-secondaire pour l'enseignement, l'apprentissage et la recherche

2016 Queen's University, Kingston, Ontario

- Computational thinking and mathematics curriculum
- Mathematics in teacher education: What, how… and why / Les mathématiques dans la formation des enseignants : quoi, comment… et pourquoi
- Problem solving: Definition, role, and pedagogy / Résolution de problèmes : définition, rôle, et pédagogie associée
- Mathematics education and social justice: Learning to meet the others in the classroom / Éducation mathématique et justice sociale : apprendre à rencontrer les autres dans las classe
- Role of spatial reasoning in mathematics
- The public discourse about mathematics and mathematics education / Le discours public sur les mathématiques et l'enseignement des mathématiques

2017 McGill University, Montréal, Québec

- Teaching first year mathematics courses in transition from secondary to tertiary
- L'anxiété mathématique chez les futurs enseignants du primaire : à la recherche de nouvelles réponses à des enjeux qui perdurent / Elementary preservice teachers and mathematics anxiety: Searching for new responses to enduring issues
- Social media and mathematics education
- Quantitative reasoning in the early years / Le raisonnement quantitatif dans les premières années du parcours scolaire
- Social, cultural, historical and philosophical perspectives on tools for mathematics
- Compréhension approfondie des mathématiques scolaires / Deep understanding of school mathematics

2018 Quest University, Squamish, British Columbia

- The 21$^{st}$ century secondary school mathematics classroom
- Confronting colonialism / Affronter le Colonialisme
- Playing with mathematics / Learning mathematics through play
- Robotics in mathematics education
- Relation, ritual and romance: Rethinking interest in mathematics learning

2019 St. Francis Xavier University, Antigonish, Nova Scotia

- Problem-based learning in postsecondary mathematics / L'apprentissage par problems en mathématiques au niveau postsecondaire
- Teaching primary school mathematics…what mathematics? What avenues for teacher training? / Enseigner les premiers concepts mathématiques à l'école primaire…quelles mathématiques? Quelles avenues pour la formation à l'enseighnement?
- Humanizing data / Humaniser les données
- Research and practice: Learning through collaboration / Recherche et pratique : apprendre en collaborant
- Interdisciplinarity with mathematics: Middle school and beyond
  Capturing chaos? Ways into the mathematics classroom / Capturer le chaos ? Entrées sur la classe de mathématiques

## Appendix B / Annexe B

## PLENARY LECTURES AT EACH ANNUAL MEETING / CONFÉRENCES PLÉNIÈRES DES RENCONTRES ANNUELLES

| | | |
|---|---|---|
| 1977 | A.J. COLEMAN | The objectives of mathematics education |
| | C. GAULIN | Innovations in teacher education programmes |
| | T.E. KIEREN | The state of research in mathematics education |
| 1978 | G.R. RISING | The mathematician's contribution to curriculum development |
| | A.I. WEINZWEIG | The mathematician's contribution to pedagogy |
| 1979 | J. AGASSI | The Lakatosian revolution |
| | J.A. EASLEY | Formal and informal research methods and the cultural status of school mathematics |
| 1980 | C. GATTEGNO | Reflections on forty years of thinking about the teaching of mathematics |
| | D. HAWKINS | Understanding understanding mathematics |
| 1981 | K. IVERSON | Mathematics and computers |
| | J. KILPATRICK | The reasonable effectiveness of research in mathematics education |
| 1982 | P.J. DAVIS | Towards a philosophy of computation |
| | G. VERGNAUD | Cognitive and developmental psychology and research in mathematics education |
| 1983 | S.I. BROWN | The nature of problem generation and the mathematics curriculum |
| | P.J. HILTON | The nature of mathematics today and implications for mathematics teaching |

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|---|---|---|
| 1984 | A.J. BISHOP | The social construction of meaning: A significant development for mathematics education? |
| | L. HENKIN | Linguistic aspects of mathematics and mathematics instruction |
| 1985 | H. BAUERSFELD | Contributions to a fundamental theory of mathematics learning and teaching |
| | H.O. POLLAK | On the relation between the applications of mathematics and the teaching of mathematics |
| 1986 | R. FINNEY | Professional applications of undergraduate mathematics |
| | A.H. SCHOENFELD | Confessions of an accidental theorist |
| 1987 | P. NESHER | Formulating instructional theory: the role of students' misconceptions |
| | H.S. WILF | The calculator with a college education |
| 1988 | C. KEITEL | Mathematics education and technology |
| | L.A. STEEN | All one system |
| 1989 | N. BALACHEFF | Teaching mathematical proof: The relevance and complexity of a social approach |
| | D. SCHATTSNEIDER | Geometry is alive and well |
| 1990 | U. D'AMBROSIO | Values in mathematics education |
| | A. SIERPINSKA | On understanding mathematics |
| 1991 | J .J. KAPUT | Mathematics and technology: Multiple visions of multiple futures |
| | C. LABORDE | Approches théoriques et méthodologiques des recherches françaises en didactique des mathématiques |
| 1992 | ICME-7 | |
| 1993 | G.G. JOSEPH | What is a square root? A study of geometrical representation in different mathematical traditions |
| | J CONFREY | Forging a revised theory of intellectual development: Piaget, Vygotsky and beyond |
| 1994 | A. SFARD | Understanding = Doing + Seeing ? |
| | K. DEVLIN | Mathematics for the twenty-first century |
| 1995 | M. ARTIGUE | The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching |
| | K. MILLETT | Teaching and making certain it counts |

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|---|---|---|
| 1996 | C. HOYLES | Beyond the classroom: The curriculum as a key factor in students' approaches to proof |
| | D. HENDERSON | Alive mathematical reasoning |
| 1997 | R. BORASSI | What does it really mean to teach mathematics through inquiry? |
| | P. TAYLOR | The high school math curriculum |
| | T. KIEREN | Triple embodiment: Studies of mathematical understanding-in-interaction in my work and in the work of CMESG/GCEDM |
| 1998 | J. MASON | Structure of attention in teaching mathematics |
| | K. HEINRICH | Communicating mathematics or mathematics storytelling |
| 1999 | J. BORWEIN | The impact of technology on the doing of mathematics |
| | W. WHITELEY | The decline and rise of geometry in 20th century North America |
| | W. LANGFORD | Industrial mathematics for the 21st century |
| | J. ADLER | Learning to understand mathematics teacher development and change: Researching resource availability and use in the context of formalised INSET in South Africa |
| | B. BARTON | An archaeology of mathematical concepts: Sifting languages for mathematical meanings |
| 2000 | G. LABELLE | Manipulating combinatorial structures |
| | M. B. BUSSI | The theoretical dimension of mathematics: A challenge for didacticians |
| 2001 | O. SKOVSMOSE | Mathematics in action: A challenge for social theorising |
| | C. ROUSSEAU | Mathematics, a living discipline within science and technology |
| 2002 | D. BALL & H. BASS | Toward a practice-based theory of mathematical knowledge for teaching |
| | J. BORWEIN | The experimental mathematician: The pleasure of discovery and the role of proof |
| 2003 | T. ARCHIBALD | Using history of mathematics in the classroom: Prospects and problems |
| | A. SIERPINSKA | Research in mathematics education through a keyhole |
| 2004 | C. MARGOLINAS | La situation du professeur et les connaissances en jeu au cours de l'activité mathématique en classe |
| | N. BOULEAU | La personnalité d'Evariste Galois: le contexte psychologique d'un goût prononcé pour les mathématique abstraites |

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| 2005 | S. LERMAN | Learning as developing identity in the mathematics classroom |
| | J. TAYLOR | Soap bubbles and crystals |
| 2006 | B. JAWORSKI | Developmental research in mathematics teaching and learning: Developing learning communities based on inquiry and design |
| | E. DOOLITTLE | Mathematics as medicine |
| 2007 | R. NÚÑEZ | Understanding abstraction in mathematics education: Meaning, language, gesture, and the human brain |
| | T. C. STEVENS | Mathematics departments, new faculty, and the future of collegiate mathematics |
| 2008 | A. DJEBBAR | Art, culture et mathématiques en pays d'Islam (IXe-XVe s.) |
| | A. WATSON | Adolescent learning and secondary mathematics |
| 2009 | M. BORBA | Humans-with-media and the production of mathematical knowledge in online environments |
| | G. de VRIES | Mathematical biology: A case study in interdisciplinarity |
| 2010 | W. BYERS | Ambiguity and mathematical thinking |
| | M. CIVIL | Learning from and with parents: Resources for equity in mathematics education |
| | B. HODGSON | Collaboration et échanges internationaux en éduction mathématique dans le cadre de la CIEM : regards selon une perspective canadienne / ICMI as a space for international collaboration and exchange in mathematics education: Some views from a Canadian perspective |
| | S. DAWSON | My journey across, through, over, and around academia: "...a path laid while walking..." |
| 2011 | C. K. PALMER | Pattern composition: Beyond the basics |
| | P. TSAMIR & D. TIROSH | The Pair-Dialogue approach in mathematics teacher education |
| 2012 | P. GERDES | Old and new mathematical ideas from Africa: Challenges for reflection |
| | M. WALSHAW | Towards an understanding of ethical practical action in mathematics education: Insights from contemporary inquiries |
| | W. HIGGINSON | Cooda, wooda, didda, shooda: Time series reflections on CMESG/GCEDM |
| 2013 | R. LEIKIN | On the relationships between mathematical creativity, excellence and giftedness |
| | B. RALPH | Are we teaching Roman numerals in a digital age? |
| | E. MULLER | Through a CMESG looking glass |

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|---|---|---|
| 2014 | D. HEWITT | The economic use of time and effort in the teaching and learning of mathematics |
| | N. NIGAM | Mathematics in industry, mathematics in the classroom: Analogy and metaphor |
| 2015 | É. RODITI | Diversité, variabilité et convergence des pratiques enseignantes / Diversity, variability, and commonalities among teaching practices |
| | D. HUGHES HALLET | Connections: Mathematical, interdisciplinary, electronic, and personal |
| 2016 | B. R. HODGSON | Apport des mathématiciens à la formation des enseignants du primaire : regards sur le « modèle Laval » |
| | C. KIERAN | Task design in mathematics education: Frameworks and exemplars |
| | E. MULLER | A third pillar of scientific inquiry of complex systems—Some implications for mathematics education in Canada |
| | P. TAYLOR | Structure—An allegory |
| 2017 | Y. SAINT-AUBIN | The most unglamorous job of all: Writing exercises |
| | A. SELDEN | 40+ years of teaching and thinking about university mathematics students, proofs, and proving: An abbreviated academic memoir |
| 2018 | D. VIOLETTE | Et si on enseignait la passion? |
| | M. GOOS | Making connections across disciplinary boundaries in preservice mathematics teacher education |
| 2019 | J-M. DE KONINCK | Découvrir les mathématiques ensemble avec les étudiants |
| | R. GUTIERREZ | Mathematics as dispossession: Reclaiming mental sovereignty by living mathematx |

## Appendix C / Annexe C

# PROCEEDINGS OF ANNUAL MEETINGS / ACTES DES RENCONTRES ANNUELLES

Past proceedings of CMESG/GCEDM annual meetings have been deposited in the ERIC documentation system with call numbers as follows:

*Proceedings of the 1980 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 204120

*Proceedings of the 1981 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 234988

*Proceedings of the 1982 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 234989

*Proceedings of the 1983 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 243653

*Proceedings of the 1984 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 257640

*Proceedings of the 1985 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 277573

*Proceedings of the 1986 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 297966

*Proceedings of the 1987 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 295842

*Proceedings of the 1988 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 306259

*Proceedings of the 1989 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 319606

*Proceedings of the 1990 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 344746

*Proceedings of the 1991 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 350161

*Proceedings of the 1993 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 407243

*Proceedings of the 1994 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 407242

*Proceedings of the 1995 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 407241

*Proceedings of the 1996 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 425054

*Proceedings of the 1997 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 423116

*Proceedings of the 1998 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 431624

*Proceedings of the 1999 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 445894

*Proceedings of the 2000 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 472094

*Proceedings of the 2001 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 472091

*Proceedings of the 2002 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 529557

*Proceedings of the 2003 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 529558

*Proceedings of the 2004 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 529563

*Proceedings of the 2005 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 529560

*Proceedings of the 2006 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 529562

*Proceedings of the 2007 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 529556

*Proceedings of the 2008 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 529561

*Proceedings of the 2009 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 529559

*Proceedings of the 2010 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 529564

*Proceedings of the 2011 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 547245

*Proceedings of the 2012 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 547246

*Proceedings of the 2013 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 547247

*Proceedings of the 2014 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 581042

*Proceedings of the 2015 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 581044

*Proceedings of the 2016 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 581045

*Proceedings of the 2017 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 589990

*Proceedings of the 2018 Annual Meeting* . . . . . . . . . . . . . . . . . . . ED 595075

## NOTE

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.