MEANINGFUL MATHEMATICS: NETWORKING THEORIES ON MULTIPLE REPRESENTATIONS AND QUANTITATIVE REASONING

Nicole L. Fonger
Syracuse University
nfonger@syr.edu

Nigar Altindis
Syracuse University
naltindi@syr.edu

This paper articulates a stance on the study of students' meaningful mathematics understanding with multiple representations. We integrate Thompson's theory of quantitative reasoning and Dreyfus' theory of multiple representations in our approach to frame and conduct empirical investigations of the study of meaningful understanding of function. We provide empirical data to support our approach to examining representational fluency and functional thinking from this networked stance. Our research articulates how the coordination of theories can be productive in informing the design, conduct, and analysis of contexts aimed to understand students' meaningful math learning with a focus on functional thinking.

Keywords: Functional thinking, Representational Fluency, meaning, Quantitative Reasoning, Multiple representations, Networking theories

Motivation and Aim

The complexity of studying and bolstering students' conceptual understanding of mathematical ideas remains a key challenge in mathematics education (Drijvers, 2019). In studying such a complex phenomena, Sfard (1998) articulates a tension of relying on one metaphor or theory of learning to the detriment of another. Consistent with this view on a need for multiple theories, in this work we aim to coordinate two theoretical orientations on understanding the mathematical idea of function—multiple representations and quantitative reasoning. Networking theories is one approach to theory development to advance knowledge in the field of mathematics education that may help to address such key challenges (BiknerAshbahs & Prediger, 2010; Cobb, 2007; Johnson & McClintock, 2018; Johnson, McClintock, Gardner, 2019). As such, this theoretical report is driven by an aim to coordinate theories for the purpose of developing new theory (cf. Cobb, 2007), not to supplant or replace existing theories.

We aim to network theories to better articulate a local theory for the purpose of addressing an enduring challenge of characterizing meaningful learning of mathematics with multiple representations. This theoretical report seeks to extend prior studies by offering both (a) a step toward the articulation of theory development focused on meaningful learning vis-à-vis the networking of theories, and (b) additional empirical investigation of the relationship between students' conceptions of functions and representational fluency. This aim emerged from both a review of empirical studies in the domain of scholarship on functions, a functions approach to mathematics, and functional thinking (e.g., Cai et al., 2010; Stephens et al., 2017a; Stephens et al., 2017b), and a discernment for the utility of networking of theories to expand possibilities for framing problems and understandings (e.g., Cobb, 2007).

In this report we also enact some of the recommendations and lessons learned from participating in a recent international conference working group focused on theoretical perspectives, networking theories, and methodological implications (Bikner-Ashbahs, Bakker, Johnson, Chan, 2019). We focus on the following recommendations of the working group as a structure for this report: (a) elaborate how the networking of theories helps to address a research problem, (b) take care in elaborating not only theoretical constructs being coordinated but background theories and underlying assumptions, and (c) communicate how the networking of theories and methodologies are in symbiotic exchange in all phases of research. We draw on empirical findings to bring these issues to life.

Theoretical Orientation on Meaningful Learning

Meaning is a creative act. Our current framing of meaningful understanding involves a critical focus on students conceptual or relational understanding (Skemp, 1976). We follow Lobato's (2013) articulation of concept to include meanings to include: (a) "*meanings*, which refer to one's interpretation of situations, conversations, symbols, and operations (Thompson & Saldanha, 2003; Voigt, 1994); and (b) *connections*, which include the specific links or ways of integrating representations, ideas, objects, and/or situations (Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986)" (emphasis in original, p. 2-3). From this stance, a person (e.g., student) is viewed as active in their meaning-making processes through the activities of creating and interpreting invariant features among representations.

Multiple Representations

We conceptualize representations as different "faces" of the same mathematical object; looking at an object through only one representation cannot reveal all features of the object (Kutzler, 2010). Representational fluency (RF) is "the ability to create, interpret, translate between, and connect multiple representations" in doing and communicating about mathematics (Fonger, 2019, p. 1). The notion that "using multiple representations" supports students' conceptual understanding of mathematics is prevalent in mathematics education research, practice, and policy in the United States (e.g., NGO & CCSSO, 2010; NCTM, 2000, 2014). Some argue that creating and interpreting representations are "important cognitive processes that lead students to develop robust mathematical understandings" (Huntley, Marcus, Kahan, & Miller, 2007, p. 117). This work draws predominantly on a theory of multiple representations (e.g., Dreyfus, 1991; Lesh, Post, Behr, 1987).

From this theoretical stance, scholarship contributes to an understanding of how the practices of representing and the processes of learning are complexly intertwined and emerge over time in symbiotic relation (Fonger, 2019; Selling, 2016). However, a lens on characterizing sophistication in representational fluency alone is often insufficient for garnering evidence of the nature of students' meanings of mathematical objects. Thus, to attend to the goal of studying meaningful learning with multiple representations, a coordination of lenses is needed (as we have argued in earlier work Fonger, 2019; Fonger, Ellis, Dogan, 2016). In our work, we aim to grow our theoretical orientation on meaningful learning to include not only students' representational activity in creating and connecting representations, but also hypotheses of the *meanings* students hold of the mathematical objects being represented.

Quantitative Reasoning and Functional Thinking

We narrow our focus in this report on the domain of scholarship on supporting and characterizing students' understanding of students' functional thinking (FT)—the ability to generalize and represent functional relationships (Stephens et al., 2017a; Kaput, Blanton, & Moreno, 2008). There is a growing body of work in this domain that draws on Thompson's theory of quantitative reasoning (Thompson, 1994, 2011). Much of this work is grounded in constructivism a background theory of learning (Glasersfeld, 1995; Piaget, 2001). From this perspective, an individual constructs knowledge through processes of assimilation and accommodation, creating mental models that are viable according to their interactions. A researcher then aims to build models of the mathematics of students (Steffe & Olive, 2010).

From a quantitative reasoning orientation, attention to students' conceptions of attributes as measurable and that vary are key (cf. Johnson, McClintock, & Gardiner, 2019). We see an overlap in literature that focuses on characterizations of students' functional thinking, and students' quantitative reasoning, that tends to fall along three types of reasoning: covariation, correspondence, and recursive.

As Johnson et al. (2019) have reported, investigations of students' covariational reasoning are intertwined with students' creation and interpretations of representations such as Cartesian graphs. Along a related line of investigation, Moore et al. (2013) found that pre-service secondary math teachers' meanings of graphs and functions were in some cases constrained by their attachment to canonical forms (e.g., horizontal axis is x, vertical axis is y), limiting their ability to draw meaning about a graphical representation of x as a function of y. In another study, Fonger, Ellis, & Dogan (2016) found students engaged in covariational and correspondence reasoning to support their sense-making of symbolic function rules as generalizations of relationships among quantities. Consistent with this study, students' reasoning about relationships among quantities can support students' meaningful use of multiple representations (Moore, 2014). Moreover, the meanings students hold of representations and representational conventions may constrain their meanings of mathematical ideas such as angles or linear functions (Moore, 2012; Moore et al., 2013). In the domain of research on functional thinking, we see a need to more explicitly integrate attention to how students are creating and connecting representations to shed light on the creative aspects of doing mathematics with representations as important tools that may support or constrain students' activity and meanings of and for mathematics (cf. Brownell, 1947).

Methodology

In this section we focus on how the networking of theories and methodologies are in symbiotic exchange in all phases of research.

Data Sources and Mode of Inquiry

We employed a case study methodology (Stake, 1995) in our focus on meaningful understanding of quadratic function. The participant of this case study was a secondary preservice teacher in her second semester of a two-semester methods sequence. The authors conducted a 60-minute audio-recorded semi-structured task-based interview with the PT. We prompted the PTs to think aloud and clarify the meanings of her thoughts, such as: Can you tell me what connections do you see?; and Did you see the acceleration in your table/graph? In this task, we intended to observe the PT's functional thinking and representational fluency while they interpreted a diagram and table to make sense of a quadratic relationship. We created enhanced transcripts be embedding participant's written artifacts into verbatim transcripts. This method improved our ability to code students' activity.

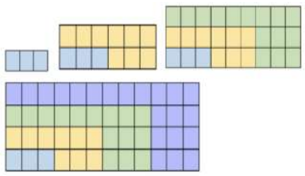
Task. Consider the following series of diagrams illustrating how a rectangle grows in several iterations.  In general, what is the relationship between height, h, and area, A, in the growing rectangle context? Why?	• **Structured Interview Probes**. • How did the diagram help you to notice the pattern you observed? • Can you describe any connections across the methods you used? Does it make sense? What is the connection between the symbolic generalizations you wrote? What about the rules and the diagram, or tables? • How would you sequence representations to teach this task? Why?

Figure 1: Task Design (adapted from Ellis, 2011) and Structured Interview Probes

Data Analysis Techniques

We used cyclic methods of coding (Miles, Huberman, & Saldana, 2013). In the first cycle of coding, we analyzed participants' representational fluency and functional thinking. In the second cycle of coding we employed axial coding (Strauss, 1987) by analyzing instances of coding-cooccurrences (i.e., where in a code for FT and RF was applied in the same data segment). In this round of coding we aimed to identify relationships between participants' functional thinking and representational fluency. The cycles of coding were an iterative process of questioning and reflecting. We compared and contrasted the data related to PTs' functional thinking with preexisting data and situated it in existing literature (Baxter, & Jack, 2008).

Coding for Representational Fluency

In this study, we employed Dreyfus' (1991) theory of learning with multiple representations, to inform decisions about students' creation, interpretation, and connection of representations. We also employed a variant of Lesh, Behr, and Post's "rule of five" or "web" of representation types to inform distinctions of when the participant was drawing on one or more of symbolic, numeric, graphic, or diagram representations. From this conceptualization or theoretical stance on the use and connection of one or more representations, we employed Fonger's (2019) analytic framework for representational fluency (Table 1) to characterize the nature of the PTs discursive activity in creating, interpreting, and connecting *across* multiple representations in solving a task. The framework developed by Fonger (2019) builds on a structure of observed learning outcome taxonomy (Biggs & Collis, 1982) and employs an actor-oriented approach (Lobato, 2012) to problem solving. Each method or approach to a problem was analyzed as a unit of analysis to discern meaningfulness in representational fluency according to eleven levels (8 are below).

Table 1: An Excerpt of Fonger's (2019) Analytic Framework for Representational Fluency

Lesser Meaningfulness	*Greater Meaningfulness*
Student expresses a canonically inappropriate or incomplete creation, interpretation, or connection.	*Student expresses canonically appropriate and complete creations, interpretations, and connections.*
Move Between. Moves between more than one representation type, does not correctly create *and* interpret the meaning of the representations to solve the problem.	*Uni-Directional Translation.* Perform a translation from one representation type to another representation type. *Bi-Directional Translation.* Perform translation and complementary translation processes. *Multi-Directional Translation.* More than two representation types are related by translation processes.
Multi-Representationally Grounded. Moves between more than one representation type, giving an incorrect creation *or* interpretation of their meaning.	*Uni-Directional Connection.* Create and give a correct interpretation of an invariant feature across multiple representations or types. *Bi-Directional Connection.* Create and interpret a source representation with respect to a target representation and vise-versa, and recognize invariant features across the two representation types. *Multi-Directional Connection.* More than two representation types are related by translation processes and involve a correct interpretation of an invariant feature.

Lower levels include: *Pre-structural* students create and interpret a representation with incomplete understanding and *Multi-structural* students interpret, create, or connect more than one representation type without making sense of a mathematical object. Higher levels include: *Unistructural* students create and interpret a representation without making a connection to more than one representation, and *Relational* students create, interpret, and connect multiple representations with sophisticated understanding of a mathematical object.

Coding for Functional Thinking

We employed Thompson's (2011) theory of quantitative reasoning to inform our conceptualization of participants' engagement with one or more quantities and relationships between them. From this orientation toward students' quantitative reasoning and ways of thinking about function, we employed an analytic framework on functional thinking as articulated in the literature (e.g., Confrey & Smith, 1991; Stephens et al., 2017a; Thompson & Carlson, 2017; Fonger, Ellis, Dogan, 2016) as an a priori lens. We attended to three types of functional thinking: recursive, correspondence, and coordinated change.

Table 2: A Quantitative Lens on Modes of Functional Thinking

Functional Thinking	includes students' abilities to generalize and represent functional relationships
Coordinated change	students' reason with change in more than one quantity together by comparing or linking change across these different quantities (e.g., change in x and change in y) with explicit quantification of the magnitude of both changes (e.g., change of 1 in x and 3 in y) (Fonger, Ellis, Dogan, 2016).
Correspondence	students determine output values (range) related to input values (domain) as a direct mapping or dependency relation
Recursive	"Recursive patterns describe variation in a single sequence of values, indicating how to obtain a number in a sequence given the previous number or numbers" (Stephens et al., 2017, p. 145)

Coding Co-occurrences

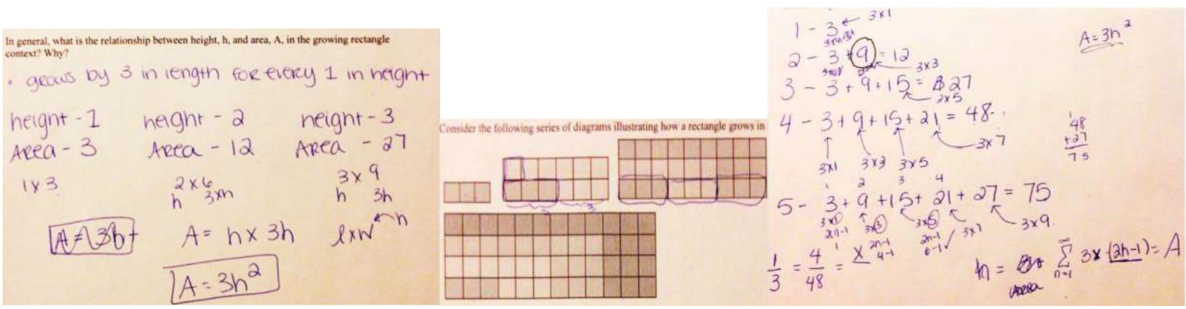
In the second round of analysis we re-analyzed instances of code co-occurrences for themes. This allowed us to identify relationships between the PTs' representational activity and their ways of thinking about function. In essence, this thematic analysis was concerned with answering the question of "what counts" as meaningful understanding when a coordination of analytic lenses it taken together?

Characterizing Meaningful Understanding in Problem Solving

In this section we highlight Vignettes to elaborate how the coordination of lenses enriches researcher sensitivity to characterizing meaning in students' problem solving activity.

Meaning May Emerge from Representational Disfluencies

In Vignette 1 (Figure 2a, b, lines 1-5) we show how meaning may emerge in problem solving.



(a) (b) (c)

Figure 2: Emy's Problem Solving Activity in Words, Numbers, Symbols, Diagrams

1 Emy: The question is like what is the relationship between height and area and the growing rectangle? So the relationship between the height and length is that it grows a 1 box in height for every 3 boxes in length, which makes it grow. So if the height is 1, the area is 3, if the height is 2 to the area is 12, then 1, 2, 3; 1, 2, 3; 4, 5, 6 times. Height is 3, an area is 27. So it's relation between height and area and the growing rectangle context. So it's 3, 3 times 1. Yeah, 2 times 6, 3 times 9, 3 times 9. So, I would say that the area equals 3, 3 times. Yeah. Three times the height (see left side of Figure 2a)

2 Nicole: How did you get a equals three h [A=3h]?

3 Emy: how we got the area, which would be the height was type 1 times 3 and then I looked in the box and it was 2 for the second one. … No, hold on. … This may be wrong. Yeah. Area equals the height times 3 times of height. So, this is what we are equals 3 height squared because if you look at the height in each one of these … So each of these is just starting out and then this is the high times 3, height times 3 times 3.

4 Nicole: So what was this, like 3h, like how did you get, how did you know to change your rule?

5 Emy: Well, because if you try to apply this to this one, it doesn't work. So area equals 3h would work for a 3 times 1, but it doesn't work for the height of two because 2 times 3 to 6. … if you try to look at that pattern and the next one, … you'll have to get 3h squared.

In interpreting this vignette from our networked lenses, we notice that Emy initially attended to coordinated change in height and length of the growing rectangle diagram by stating “the relationship between the height and length is that it grows a 1 box in height for every 3 boxes in length.” Emy created a table of values and interpreted a correspondence between height and area stating “So if the height is 1, the area is 3, if the height is 2 to the area is 12” and concluding “So I would say that the area equals … 3 times the height”, yet came to an incorrect generalization of A=3h (lines 1-2, *multi-representationally grounded*, *coordinated change*, *correspondence*).

Prompted to elaborate (line 2), Emy initially expressed uncertainty about her generalization A=3h, “No, hold on. … This may be wrong” drawing on a correspondence perspective of this function rule, Emy articulated independent and dependent quantities that did and did not match her rule, leading her to express a correct generalization and symbolic rule “Area equals the height times three times of height” (lines 3, *bidirectional translation*, *correspondence*). Asked to explain (line 4), Emy interpreted the symbolic rules A=3h and $A=3h^2$, and discerned that the rules give the same (height, area) only when h=1, while $A=3h^2$ accurately generalizes heights 1, 2, and 3 (line 5, *unidirectional connection*, *correspondence*).

From this vignette we learn that Emy’s representational fluency changed from lesser to greater sophistication in her problem solving approach (*multistructural* to *relational*), and her engagement in functional thinking (*coordinated change* and *correspondence* reasoning) may have supported that shift as she interpreted the diagram, and created and interpreted a table of values to lead to her generalization in words and symbols. In this case, evidence of Emy’s expressions of functional thinking give insight into her meaningful understanding of the growing rectangle situation as represented in diagrams, values, and a symbolic rule.

Flexibility in Meanings of Functions Engenders Representational Fluency

In Vignette 2 (see Figure 2c above, and lines 6-8 below), Emy was asked to solve the same problem in a different solution approach. Emy engaged in recursive thinking with greater sophistication in representational fluency across a diagram, table, and symbols.

6 Emy: Because you're adding a 3 on the bottom to the original 3 and then 2, 3 is on the top. So you're always going to be adding an odd number of 3s … to the original one from the past. … It's the length that's being added. So, you can group them by 3s is what I'm trying to make sense of … So you're adding 3 times. One, 2, 3 times 3 instead of 3 times 2. Um, and then you're adding 3 times 5, but we're skipping 3 times 4 and you're adding 3 times 7 but no 3 times 6 if that's like a pattern. But, um, I saw something like that. I don't know how to write that.

7 Nicole: Does it make sense why that's occurring?

8 Emy: Yes. Because you're adding a 3 on the bottom to the original 3 and then 2, 3 is on the top. So you're always going to be adding an odd number of 3s because I'm here. You're adding two 3s to bottom and then an even number 1, 2, 3, 4, 5, 6, 7, 8. This is an odd number 3 is from before and the third one? … You're adding an odd number of 3s to the original one from the past. … It's a sequence of adding groups of 3s. I don't know how to write the odd groups of 3s. I guess this could be like 3 times 2 N minus 1. Would that work?... This would be the formula [$A=\sum 3*(2h-1)$] for the area and I guess n would actually be h, it would be the height that would be a formula that relates height to area.

From a networking of lenses, we learn about Emy’s meaningful understanding of this quadratic growth situation from a recursive perspective. Emy engaged in recursive reasoning, moving fluently between the diagram and the table she created to explain her generalization of adding an odd number of groups of three to the previous area in both the diagram and the table (line 6, *bidirectional connection*, *recursive*). Prompted for sense-making (line 7), Emy elaborated her recursive thinking as a "sequence of adding groups of 3s," as strongly connected to her interpretation of the diagram, again concluding with a correct symbolization of a generalization (*multidirectional translation*, *recursive*).

In this vignette we learn that in her solution approach Emy demonstrated greater sophistication in representational fluency at a *relational* level, and in moving among representations her ability to generalize the functional relationship was interpreted as meaningful from a recursive perspective. Taken together with the first vignette, we see evidence of Emy's representational fluency as going hand in hand with her functional thinking. For instance, if we illustrate Emy's activity from a "web" diagram perspective, with each node taken as a representation type, and Emy's creation, interpretation, and connection across representations as a form of functional thinking, we observe important connections. For instance, Emy began with coordinated change on a diagram (Figure 3), and side bar chart (Figure 2c). Then she built correspondence reasoning on coordinated change by shifting into the summation formula. Emy used a correspondence approach to check if the summation formula she created is meaningful for her by plugging in values of height and area for each step through correspondence reasoning.

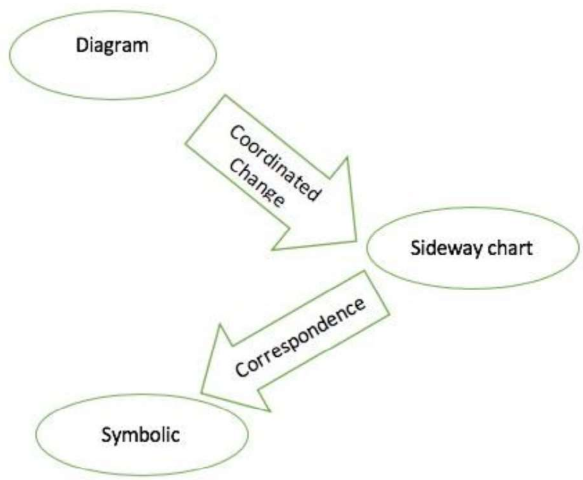


Figure 3: Emy's Solution Web with Representations and Functional Thinking

Discussion and Conclusion Characterize Meaningful Understanding

The networking of theories in this study was borne out of a need to address an enduring challenge in the field of mathematics education: characterizing students' meaningful mathematics learning. In this paper we elaborated how a networking of theories of multiple representations and quantitative reasoning is one productive approach for characterizing meaningful learning of mathematics. The theoretical groundings and assumptions guiding this study span constructivist and semiotic lenses on cognition. In particular, we drew on theories of learning that posit that students learn by creating, interpreting, and connecting multiple representations in doing and communicating about mathematics (Dreyfus, 1991), with covariation, correspondence, and recursive reasoning about relationships among linked quantities as supports for students' meaningful understandings of functions (e.g., Confrey & Smith,1994, 1995; Ellis, 2011).

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

From this networked stance, we examined how students' representational fluency and quantitative reasoning about relationships between varying and dependent quantities seemed to support and constrain one another. Results of this study highlight (a) how lesser meaningfulness in representational fluency may serve as a productive starting point for more sophisticated thinking and mathematical meaning to follow, and (b) how flexibility in thinking across multiple meanings of function (i.e., correspondence, covariation, and recursive modes) go hand in hand with representational fluency. We see these results as contributing evidence for the productiveness of networking theories to advance tools for characterizing meaningful learning.

Lessons Learned on Networking Theories

We have also learned that the activity of networking theories has contributed to our sensitivities as researchers to the importance of communicating explicit assumptions in background theories, theoretical constructs, and analytic tools. As we reflect on how we've learned new nuances in characterizing meaningful learning, we are left asking how other theoretical backgrounds and assumptions about learning might be brought to bear to paint a different picture of students' meaningful learning. For example, tracing the bodies of work that researchers draw on in defining "meaning" and "meaningful learning" brings us to other theories such as Voigt's (1994) articulation that personal meanings are negotiated through social interaction. Such social interaction and proactive role of the teacher-researcher (e.g., lines 2, 4, and 7) needs to be accounted for in analyses of situations, even interviews that would otherwise be thought of as "purely" psychological in orientation (see also Cobb & Yackel, 1996).

Next Steps

Next steps of this research program are to further investigate how the networking of theories can inform not only our ability to characterize more nuanced models of students' meaningful learning, but also to elaborate design principles as the basis for instructional supports aimed at engendering meaning-making. Future studies can build on this networking across the design, enactment, analysis, and communication of research.

We found the recommendations of the international working group (Bikner et al., 2019) to be a helpful grounding for continuing the work of theory development vis-à-vis networking theories in mathematics education. We encourage others to engage with and extend these recommendations for theory networking to advance theory building in mathematics education toward aims of addressing enduring challenges related to understanding and supporting students' meaningful mathematics learning.

References

Baxter, P., & Jack, S. (2008). The Qualitative Report Qualitative Case Study Methodology: Study Design and Implementation for Novice Researchers. *The Qualitative Report*, 13(4), 544–559.

Biggs, J. B., & Collis, K. F. (1982). *The quality of learning: The SOLO taxonomy (Structure of the Observed Learning Outcome)*. New York: Academic Press.

Bikner-Ashbahs, A., Bakker, A., Johnson, H. L., Chan, E. (2019). Theoretical perspectives and approaches in mathematics education research. *Congress of European Research in Mathematics Education Thematic Working Group 17*. Utrecht University: The Netherlands.

Bikner-Ahsbahs, A., & Prediger, S. (2010). Networking of Theories—An Approach for Exploiting the Diversity of Theoretical Approaches. In B. Sriraman & L. English (Eds.), *Theories of Mathematics Education: Seeking New Frontiers* (pp. 483–506). Berlin, Heidelberg: Springer Berlin Heidelberg.

Brownell, W. A. (1947). The place of meaning in the teaching of arithmetic. *The Elementary school journal, 47*(5), 256-265.

Cai, J., Bie, B., & Moyer, J. S. (2010). The teaching of equation solving: Approaches in Standards-based and traditional curricula in the United States. *Pedagogies: An International Journal*, 5(3), 170-186.

Cobb, P. (2007). Putting philosophy to work: Coping with multiple theoretical perspectives. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 3–38). Charlotte, NC: Information Age.
Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, 31(3/4), 175-190.
Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. In *Learning Mathematics* (pp. 31-60). Springer, Dordrecht
Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.). *Advanced mathematical thinking (pp. 25–41). Netherlands: Springer.*
Drijvers, P. (2019). Embodied instrumentation: combining different views on using digital technology in mathematics education. *Congress of European Research in Mathematics Education* (CERME11): Utrecht University: The Netherlands.
Ellis, A. (2011). Generalizing promoting actions: How classroom collaborations can support students' genearlizations. *Journal for Research in Mathematics Education*, *42*(4), 308-345.
Fonger, N. L. (2019). Meaningfulness in representational fluency: An analytic lens for students' creations, interpretations, and connections. Journal of Mathematical Behavior.
doi: https://doi.org/10.1016/j.jmathb.2018.10.003
Fonger, N. L., Ellis, A., & Dogan, M. F. (2016). Students' conceptions supporting their symbolizations and meanings of function rules. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.
Glasersfeld, E. v. (1995). Radical constructivism: A way of knowing and learning. Washington, D.C.: Falmer Press. Silverman, J. (2011). Supporting the development of mathematical knowledge for teaching through online asynchronous collaboration. *Journal of Computers in Mathematics and Science Teaching*, *30*(1), 61-78.
Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). New York: Mcmillan.
Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-27). Hillsdale, NJ: Erlbaum.
Huntley, M. A., Marcus, R., Kahan, J., & Miller, J. L. (2007). Investigating high-school students' reasoning strategies when they solve linear equations. *The Journal of Mathematical Behavior*, *26*(2), 115-139.
Johnson, H. L., & McClintock, E. (2018). A link between students' discernment of variation in unidirectional change and their use of quantitative variational reasoning. *Educational Studies in Mathematics*, *97*, 299-315. doi: https://doi.org/10.1007/s10649-017-9799-7
Johnson, H. L., McClintock, E., & Gardiner, A. M. (2019). Locally integrating theories to investigate students' transfer of mathematical reasoning. Paper presented at the *Congress of European Research in Mathematics Education* (CERME11), The Netherlands.
Kaput, J. J., Blanton, M., & Moreno, L. (2008). Algebra from a symbolization point of view. In J. J. Kaput, D.
Carraher, & M. Blanton (Eds.). *Algebra in the early grades*. New York: Lawrence Erlbaum Associates.
Kutzler, B. (2010). Technology and the yin and yang of teaching and learning mathematics. In Z. Usiskin, K. Anderson, & N. Zotto (Eds.), *Future Curricular Trends in School Algebra and Geometry: Proceedings of a Conference* (pp. 73-91). Charlotte: Information Age Publishing.
Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representations in the teaching and learning of mathematics* (pp. 33-40). Hillsdale, NJ: Lawrence Erlbaum.
Lobato, J. (2012) The Actor-Oriented Transfer Perspective and Its Contributions to Educational Research and Practice, *Educational Psychologist*, 47:3, 232-247.
Miles, M. B., Huberman, A. M., & Saldana, J. (2013). *Qualitative data analysis*. Sage.
Moore, K. C. (2012). Coherence, quantitative reasoning, and the trigonometry of students. In R. Mayes and L. L. Hatfield, & S. Belbase (Eds.), *Quantitative reasoning and mathematical Modeling: A driver for STEM integrated education and teaching in context* (pp. 75-92). Laramie, WY: University of Wyoming.
Moore, K. C. (2014). Signals, symbols, and representational activity. Paper presented at the WISDOMe, University of Georgia.
Moore, K. C., Paoletti, T., & Musgrave, S. (2013). Covariational reasoning and invariance among coordinate systems. *Journal of Mathematical Behavior*, *32*(3), 461–473.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.
National Council of Teachers of Mathematics. (2014). *Principles to Actions: Ensuring Mathematical Success for All*. Reston, VA: National Council of Teachers of Mathematics.
National Governors Association Center for Best Practices, Council of Chief State School Officers (2010). *Common Core State Standards for Mathematics*. Washington D.C.: Authors.
Piaget, J. (Ed.). (2001). *Studies in reflecting abstraction*. Philadelphia: Taylor Francis.
Selling, S. K. (2016). Learning to represent, representing to learn. *Journal of Mathematical Behavior*, *41,* 191-209. doi: http://dx.doi.org/10.1016/j.jmathb.2015.10.003
Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher*, *27*(2), 4-13.
Skemp, R. (1976). Instrumental understanding and relational understanding. *Mathematics Teaching*, *77*, 20-26.
Steffe, L., & Olive, J. (2010). *Children's fractional knowledge.* New York: Springer.
Stake, R. E. (1995). *The art of case study research*. Thousand Oaks, CA: Sage.
Strauss, A. L. (1987). *Qualitative analysis for social scientists*. New York, NY: Cambridge University Press.
Stephens, A., Fonger, N. L., Strachota, S., Isler, I., Blanton, M., Knuth, E., & Gardiner, A. M. (2017a). A learning progression for elementary students' functional thinking. *Mathematical Thinking and Learning*, *19*(3), 143-166.
Stephens, A. C., Ellis, A. B., Blanton, M., & Brizuela, B. M. (2017b). Algebraic thinking in the elementary and middle grades. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 386-420). Reston, VA: National Council of Teachers of Mathematics.
Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation and functions: Foundational ways of mathematical thinking. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). Reston, VA: National Council of Teachers of Mathematics.
Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. A. Chamberlain & L. L. Hatfield (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for wisdom^e* (Vol. 1, pp. 33–56). Laramie, WY: University of Wyoming College of Education.
Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation and functions: Foundational ways of mathematical thinking. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). Reston, VA: National Council of Teachers of Mathematics.
Thompson, P. W., & Saldanha, L. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, G. Martin, & D.
Schifter (Eds.), *Research companion to the Principles and Standards for School Mathematics* (pp. 95–114). Reston,VA: National Council of Teachers of Mathematics.
Voigt, 1. (1994). Negotiation of mathematical meaning and learning mathematics. *Educational Studies*, *26*(2/3). 273-298.