COMPLEX CONNECTIONS: REIMAGINING UNITS CONSTRUCTION AND COORDINATION

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Students' construction, coordination, and abstraction of units underlie success across multiple mathematics domains. Structures for coordinating units underscore notions of numbers as composite units (e.g., five is a unit of five and five units of one). In this working group, we seek to facilitate collaboration amongst researchers and educators concerned with units construction and units coordination. The aim for this working group is two-fold: (1) to extend our research around units construction and coordination to new grade levels; and (2) to collaborate with researchers who investigate students with learning differences in school settings to determine diverse students' mathematics learning trajectories.

Keywords: Number Concepts and Operations, Learning Trajectories, Learning Theories

In Steffe's 2017 plenary for PME-NA, he substantiated particular needs for investigating how children develop operations when constructing and coordinating units. The Complex Connections: Reimagining Units Construction and Coordination working group began at PME-NA 2018, with the aim of facilitating collaboration amongst researchers and educators sharing Steffe's concerns about units construction and coordination for all learners. The main goals of the working group are to extend research around units construction and coordination to new grade levels and to facilitate collaborations with researchers who investigate students with learning differences in school settings to determine diverse students' mathematics learning trajectories. We frame this proposal to continue this working group by first providing a background about units construction and coordination. We next describe our progress toward meeting our goals stemming from our inaugural meeting. Lastly, we describe our goals and plans for continuing this work at PME-NA 2019.

Background and Theoretical Perspective: Composite Units – Old and New

We provide a theoretical background for units construction and coordination, focusing on the role of units coordination in students' number sequences (Steffe, 1992), multiplicative concepts (Hackenberg & Tillema, 2009), and fractions schemes (Steffe & Olive, 2010).

Units Construction and Coordination

Unitizing, or setting an object (a unit) aside for further action or activity (Steffe, 1992), is the basis for units construction. Students initially rely on concrete, pictorial, fingers, symbolic numerals, and language to evidence internalized (being able to mentally re-imagine contextual actions) or interiorized (being able to draw on de-contextualized actions) actions. Consider the construction of additive reasoning as an example. When considering how to add eight and seven, students might not yet see the cardinality of eight, counting a set of objects to create eight, then another set of objects to create seven, and finally combining the two sets, beginning at one to quantify the total. Should students see the cardinality of eight, they might use it as an input for solving the problem. They will count on from eight, using objects (cubes, fingers) to keep track of the addition (e.g., 8…9 [raises a finger], 10 [raises another finger]…). The double counting involved in this activity (e.g., 8…, 9 (1), 10 (2), 11 (3), 12 (4), 13 (5), 14 (6), 15 (7)) promotes a coordination of the start value and the stop value. That is, eight, seven, and 15 are taking on some meaning as composites (8 and 7) and a unitized whole (15). Evidence of this meaning includes the breaking apart of one or both of the numbers to arrive at the total (e.g., 8 is 5 and 3; 7 is 5 and 2; 8 + 5 is the same as 5 + 5 + 3 + 2, or 15). This type of units coordination can be explained through the type of numerical sequence students produce and rely upon.

Number Sequence Types (INS, TNS, ENS, GNS)

Steffe and Olive (2010) described four different counting sequences that children may develop and evidence when solving mathematics tasks: (1) Initial Number Sequence (INS), (2) Tacitly-Nested Number Sequence (TNS), (3) Explicitly-Nested Number Sequence (ENS), and (4) Generalized Number Sequence (GNS). Each number sequence can illustrate stages of children's development of units coordination.

Initial number sequence. Steffe (1992) explained that children who segment and interiorize number sequences have developed Initial Number Sequence (INS). Children who develop an INS are characterized by their counting of single units and then their segmenting of a numerical sequence (evidenced through "counting on" activity). When children segment numerical sequences, they are interiorizing patterned templates for counting, which allows them the ability to count on from a composite unit (e.g., developing one composite unit to use when counting on, 4…5-6-7-8). Thus, through counting actions, numerical patterns are developed and become interiorized (evidenced through less reliance on sensory-motor experiences; e.g., verbalizing counts, using fingers, or tapping) before being segmented into a composite unit.

Tacitly-nested number sequence. Once children have developed composite units through their INS activity, they can begin to coordinate these composite units, treating the result of counting activity as both a unit to count on from and one to keep track of when counting. These actions evidence children's development of a Tacitly-Nested Number Sequence (TNS). Steffe (1992) explains that when children have a numerical sequence interiorized and segmented they can use their segmented numerical sequence as material for making new composite units within these numerical sequences. The awareness of one number sequence contained inside another, or double-counting, is an indication of TNS, as is a skip count (i.e., 4, 8, 16,…) to solve early multiplicative kinds of problems, such as how many 3's are contained in 12.

Explicitly-nested number sequence. Children who are described as having part-whole number reasoning in place are capable of disembedding parts from wholes and developing iterable units of one. These children are described as reasoning with an Explicitly-Nested Number Sequence (ENS) (Olive, 1999; Steffe, 1992; Ulrich & Wilkins, 2017). Children capable of multiplicatively understanding a single unit and a composite (whole) unit without disrupting either are said to be operating with an ENS (Ulrich & Wilkins, 2017). The two given composite units (e.g., parts and whole) provide children material to coordinate while constructing a third composite unit; e.g., a unit of units of units (Steffe, 1992). This part-whole reasoning with abstract units provides children multiplicative number structures.

Generalized number sequence. Children capable of developing iterable composite units where units of units of units can be coordinated, are described as operating with a GNS. For example, Olive (1999) explained that when children are asked to find common multiples, they are required to keep track of two series of composite units (e.g., 3, 6, 9, 12; 4, 8, 12; 12. The LCM of 3 and 4 is 12). Children evidencing successful completion of tasks like this are described as reasoning about two iterable composite units (e.g., 3 and 4), while keeping track of the common composite unit in each sequence. At the root of much of this number sequence development, units construction and coordination explain how and why children are capable of transitioning from additive/subtractive operative structures towards multiplicative/division operative structures towards rational number understanding.

Multiplicative Levels of Units Coordination

A student is said to assimilate with one level of units when she conceives of multiplication situations, such as seven iterations of four, by counting on from the first or second set of four by ones and double-counting the number of fours to reach a stop value (e.g., 4, 8,…9- 10-11-12; 13-14-15-16; 17-18-19-20; 21-22-23-24; 25-26-27-28). Here, the child has to model or carry out the situation by using internal (e.g., subvocal counting) or external (e.g., fingers or objects) representations. This is referred to coordinating two levels of units in activity. Units coordination in activity is ephemeral: in a follow-up task, such as how many ones are in eight iterations of four, the student would likely need to repeat a similar process rather than count-on four more from 28.

A student's use of strategic reasoning in such situations may be evidence that she assimilates the situation with two levels of units. For example, a student assimilating with two levels of units might conceive of seven iterations of four as five iterations of four plus two iterations of four (e.g., five 4s is 20; 21-22-23-24; 25-26-27-28). As opposed to modeling the entire coordination, the child can anticipate breaking apart the composite unit of seven into five and two and use each of those parts to solve the problem. For a student assimilating with two levels of units, the result of operating is simultaneously 28 ones and 7 fours; hence a follow-up task of finding the number of 1s in 8 fours would not require building up from 5 fours again.

A student is said to assimilate with three levels of units when she can conceive of a situation such as seven iterations of four as resulting in three distinct yet coordinated units: (a) one unit of 28 that contains (b) seven units of four, each of which contains (c) two units of two. Students assimilating with three levels of units have flexibility to reason strategically with each of the units. For instance, a student assimilating with three levels of units might solve the task, "How many more twos are in 32 than in 28?" by reasoning that 32 is one more 4, which is thus two more 2s. This reasoning involves assimilating three levels of units, multiplicatively.

Norton, Boyce, Ulrich and Phillips (2015) conducted a cross-sectional analysis of 47 sixth-grade students' reasoning in whole number multiplicative settings in clinical interviews. Figure 1 displays descriptors of attributions of students' activities Norton and colleagues identified as corresponding with students' reasoning with one, two, or three levels of units. For instance, they describe students' activity when transitioning from reasoning with one level of units to reasoning with two levels of units with descriptors G-K (Norton et al., 2015, p. 62).

Though there are commonalities with descriptions of students' counting schemes (e.g., descriptors C and J), the focus of the descriptors are more generally about how and whether students are able to flexibly reverse and reflect on their multiplicative reasoning.

Descriptors			
A. Iterates units of 1	E. Works with at least one level of units within a figurative three-level structure	I. Works with at least two levels of units within a figurative three-level structure	M. Assimilates three levels of units within a task
B. Builds composite units	F. Indicates a reciprocal relationship between the size and number of parts in a whole	J. Distributes composite units	N. Reflects on all levels within a three-level structure
C. Iterates composite units	G. Reverses multiplicative relationships	K. Immediacy of response	O. Conflates or confuses units
D. Uses multiplicative relationships	H. Builds a three-level structure	L. Flexibly works with a three-level structure	P. References the structure of units

Figure 1: Descriptors of Sixth-grade Students' Units Coordinating Activity as They Progress from Coordinating 1 Level of Units to Coordinating Three Levels of Units, as Discussed in Norton et al.'s (2015) Findings

These stages of units coordination, as delineated in Norton et al.'s (2015) findings, comprehensively explains transitions children make from additive operations to multiplicative operations. These findings also provide fundamental explanations for fractional unit development, as recursive units coordination in which to act upon and explain how students develop fractions as mental objects.

Fractional Units

Reasoning with fractional units requires unitizing a fractional size, 1/nth. When children first conceptualize fractional units, Steffe (2001) posited that they would reorganize natural number schemes to develop fractional schemes. Olive (1999) and Steffe (2001) argued that they would have to re-interiorize their units coordination operations, to consider a fractional unit as a result of *equi-partitioning* a unit whole into a size that, when iterated *n* times, would result in the size of 1. This re-interiorization of schema requires students to recursively construct and coordinate new composite units relationships (Olive, 1999). Thus, children's production of numerical sequences and their associated units coordination provide children necessary operations for their fractional units coordination.

To conceive of a fraction *m/n* as a number, one must understand m/n as equivalent to *m* 1/*n*ths, n of which are equivalent to 1. In the case of $m > n$, the meaning of 1/*n* must transform from thinking of 1/*n* as one out of n total pieces (*a parts-out-of-wholes scheme*) to thinking of *1/n* as an amount that could be iterated more than *n* times without changing its relationship with the size of 1 (*an iterative fraction scheme*). This measurement conception of fraction (Lamon, 2008) involves coordinating three levels of units of nested units: 8/3 is 8 times (1/3), 1=3/3 is 3 times (1/3), thus an 8/3 unit contains both a unit of 1 and a unit of 1/3 within 1 (Hackenberg, 2010). Students in the intermediate stages of constructing such a measurement conception may reason about the size of proper fractions of form m/n by counting the number of parts of size 1/*n* within a whole of *n/n*. Such students are not yet iterating the amount of 1/*n*, which limits their ability to iterate unit fractions beyond the size of the whole (Tzur, 1999).

Issues Related to the Psychology of Mathematics Education

Steffe's 2017 plenary included both a summary of contributions and important extant problems for mathematics educators pertaining to units coordination. Given these advances in research surrounding K-8 students' units construction and coordination, our mathematics education field is still limited by the context in which students' units construction and coordination develops and how preschool and high school students' units construction and coordination may inform these trajectories. Further, Steffe (2017) proposes that about 40% of first grade students have very different learning trajectories than their peers, suggesting a need to develop alternative means in which units construction and coordination may be developed by children. Finally, Steffe suggests that our field would benefit by investigating children's transitions in scheme development. For instance:

> It is especially crucial to investigate possible changes that indicate *fundamental* transitions between reasoning with two levels of units and three levels of units induced in the construction of quantitative measuring schemes and their use in the construction of multiplicative and additive measuring schemes (Steffe, 2017, p. 46).

One of our intentions for this working group is for researchers from different backgrounds to collaborate to work toward solving such problems. Consider that in their review of research preparing the recent (2016) compendium chapter on quantitative reasoning, Smith and Barrett (2017) note the following:

> [We] found it striking how often the same conceptual principles and associated learning challenges appear in the measurement of different quantities… Despite the clear focus in research on equipartitioning, units and their iteration, units and subunits… curricula (and arguably most classroom teaching) focus students' attention on particular quantities and the correct use of tools, as if each was a new topic and challenge. (p. 377).

Consistent with Arbaugh, Herbel-Eisenmann, Ramirez, Knuth, Kranendonk, and Quander's (2010) call to "develop mathematics proficiency in various school, cultural, and societal contexts" (p. 13), our goal is to connect research programs involving units construction and coordination with research programs that stem from other theoretical perspectives.

Unfortunately, many students with mathematics learning difficulties do not transition from two levels of units to three levels of units at the same pace as their more successful peers. In fact, the construction of ENS is one pervasive mathematical impediment for students with mathematics difficulties (Landerl, Bevan, & Butterworth, 2004). Compared to students without mathematics difficulties, students with mathematics difficulties develop less sophisticated strategies for number computation problems over time, suggesting a lack of a conceptual basis for ENS engagement that actually plays a part in their later disability identification (Butterworth, Varma, & Laurillard, 2011). Therefore, particular research programs with these foci are desperately needed to nurture multiplicative and rational number conceptions (Boyce & Norton, 2017; Grobecker, 1997; Kosko, 2017) and operations (Grobecker, 1997; Grobecker, 2000; Norton & Boyce, 2015) for students who need our support.

Alignment with Conference Theme

By designing interventions with children's mathematics and their units construction and coordination at the center, the field has grown over the years, yet there is much opportunity for improvement. Relations between cognitive factors and test performance may be important, yet these relationships are only one way to conceptualize "cognition." Instead of focusing on aspects of students' working memory or processing, researchers can revolutionize access for students by intervening on the malleable cognitive factors that can be improved upon through students' own development. Research stemming from units coordination and construction is "*against* the new horizon" in the sense that the goal of supporting students' units construction and coordination does not align with goals or initiatives that focus entirely on helping students to meet grade-level, task-based learning objectives. We argue that equitable instruction for all students begins with increased opportunities to adapt their own thinking grounded in a construction within their own mathematical realities. When well-intentioned educators provide children interventions that promote procedures and actions, not only are they not serving their children's mathematics learning needs, they may be preventing them from engaging in learning situations that support the children to adapt their thinking structures and advance their learning.

Research Designs and Methodologies

The primary methodology for investigating units construction and coordination has been the radical constructivist teaching experiment (Steffe & Thompson, 2000). A main role of these teaching experiments is to generate (and refine) epistemic models of students' mathematics – models for how students with common underlying conceptual operations learn within a particular mathematics domain (Steffe & Norton, 2014). Such teaching experiments involve close interactions with a teacher-researcher modeling the dynamics of students' ways of operating longitudinally. Teaching experiment methodology is also used as part of design research (Cobb, Confrey, diSessa, Leher, & Schauble, 2003), to inform instructional approaches or interventions that could be "scaled up" to heterogeneous classroom settings.

Results from analyzing teaching experiments have also informed methods for assessing a child's ways of constructing and coordinating units at a particular moment. In addition to task-based clinical interviews (Clement, 2000), Norton and Wilkins (2009) created written instruments for assessing middle-grades' fractions schemes and operations associated with units coordination. These instruments have been used to validate conjectured learning trajectories for children's construction of schemes for coordinating fractional units (e.g., Norton & Wilkins, 2012). These instruments currently serve as tools for selecting research participants in teaching experiments with middle-grades students (e.g., Hackenberg & Lee, 2015).

Consideration of these methodologies for researching units construction and coordination suggests areas for collaborative work to build our understanding not only of the research programs Steffe (2017) described, but also opportunities and needs regarding related research domains. For instance, Norton and Wilkins' (2009) written instruments have been modified to assess units coordination with fractions with prospective elementary teachers (Lovin, Stevens, Siegfied, Wilkins, & Norton, 2016). Thus, research development with these methodologies are better served in collaborative designs to allow for more perspectives in the design and analyses to more closely determine students' mathematics.

First Aim: Extending Units Construction and Coordination Research

The first aim of this working group proposal is to extend units construction and coordination research to investigations that include both older students and younger students. For instance, questions regarding whether differences in secondary students' units coordination persist beyond eighth grade, and, if so, how these differences manifest in older students' learning, remain underexplored. In interview and teaching experiment settings, Grabhorn, Boyce, and Byerley (2018) have found that students enrolled in university-level calculus do not necessarily coordinate three levels of units. Further expanding understanding of students' units coordination beyond eighth grade would contribute to the development of "coherent frameworks for characterizing the development of student thinking" (Arbaugh et al., 2010, p. 15).

In addition to studies focusing on relationships between units coordination and older students' mathematics, studies of pre-kindergarten children's units construction are also warranted. For instance, Wright (1991) found that students entering kindergarten had a wide variance in their number knowledge, which suggests critical mathematics learning may occur prior to the elementary school experience. With a significant dearth of research studies in the early childhood years (De Smedt, Noel, Gilmore, & Ansari, 2013) we posit that studies of how young children construct their earliest units are a critical area of research. Another potential need regards equity and access in mathematics education.

Researchers might investigate whether analyses of students' units construction and coordination provides insight to the mathematical reasoning of diverse and underserved populations. We envision collaborations around similar conceptual principles and learning challenges, to investigate units construction and coordination of children enrolled in different grade levels and representing different population groups (i.e., special education, early childhood, secondary education, teacher education), which would allow more coherent mathematical learning theory and practical means with which to link research to classrooms.

Steffe (2017) estimated that about 40% of first grade students rely solely on perceptual material when counting all items (Counters of Perceptual Unit Items – CPUI) and that 45% of first grade students are capable of counting figurative unit items (CFUI), a necessary precursor for interiorizing counting actions and developing "counting-on" (e.g., INS) (see Figure 2). Steffe posits that these distinct groups of children require different learning trajectories due to the differences in their construction of units.

With more U.S. students attending preschool programs and an increase of 48% in national funding towards preschool programs, it would be advantageous to develop research that could directly inform early childhood mathematics curricula (Diffey, Parker, & Atchison, 2017; Sarama & Clements, 2009). Also, given the need for these curricula should be coherently aligned with elementary grade mathematics curricula initiatives, it would serve early childhood curriculum designers to bridge research programs around number development in early elementary grades to preschool grade levels. For instance, one author found in a case study that one preschool student may be using subitizing activity to construct prenumerical units (MacDonald & Wilkins, 2019). Investigating how early, perceptual actions may relate to students' actions and operations development around units construction and coordination would serve these foci. Thus, our first aim is to extend research around units construction and coordination to new grade levels.

Grade/N Seq.	CFUI or INS	ENS	GNS
First	≈ 45 Percent	≈ 10 to 15 Percent	≈ 0 to 5 Percent
Second	≈ 30 Percent	≈ 25 to 30 Percent	≈ 0 to 5 Percent
Third	≈ 5 Percent	≈ 45 to 50 Percent	≈ 0 to 10 Percent

Figure 2: Steffe's (2017) Estimated percentage of Students Capable of Counting Figurative Unit Items (CFUI), Engaging with Initial Number Sequences (INS), Engaging with Explicitly-nested Number Sequence (ENS), or Engaging with Generalized Number Sequence (GNS) p.41).

Second Aim: Widening Units Coordination Research

Our second aim is to widen units coordination research by collaborating with researchers who investigate students with learning differences, from diverse cultures, and from low-socioeconomic households to determine diverse students' mathematics learning trajectories. For instance, one author who works closely with students with learning differences in case study research uncovered three important challenges students experience as they work toward more sophisticated coordination of units. First, when engaging in tasks that support the construction of composite units, the use of memorized fact combinations or teacher taught strategies eclipsed the use of one student's natural reasoning (Hunt, MacDonald, & Silva, in press) yet supported a second's (Hunt, Silva, & Lambert, 2017). Both students evidenced initial or tacit reasoning; one reverted to pseudo-empirical abstractions, tricks, or algorithms that she could not explain to solve the tasks. Conversely, the second student leveraged his knowledge of number facts and alternative representations to advance his fractional reasoning and compensate for his perceptual motor differences (2017). For both students, teacher encouragement and support to engage in each student's own ways of reasoning was imperative. Second, Hunt & Silva (in press) found evidence that confirms previous research (Geary, 2010) that one student sometimes lost track of counting during a count on, possibly due to working memory. Hunt & Silva (in press) conjectures that this learning difference interferes with the move from counting on to more sophisticated additive reasoning due to sequential (as opposed coordinated) counting. Yet, the problem was alleviated through opportunities for within-problem reflection through experiences that the child had to construct addends involved in number problems through reprocessed figurative counting (e.g., closing the firsts to recognize ten), sweeping small numbers as lengths, and improving the usability of small composites like 2, 4, 3, 5, and 6. Hunt's research is currently conducting cross-case analysis to discern whether the students' activity is indicative of similar or unique trajectories in number and/or fractional reasoning to students without learning differences.

Specific Goals and Aims

To extend and widen units construction and coordination research this working group intends to accomplish the following: (a) delineate tasks used in various areas of units construction/coordination research/teaching, (b) critically consider student reasoning associated with necessary task features, (c) explore develop means to organize tasks and cross reference with associated study, means for perturbation, materials, etc., (d) discuss webpage revision and creation to house organized tasks, and (e) embark upon collaborations leading to reading groups, research endeavors, and funding opportunities.

Goals and Outcomes from 2018 Working Group

The Complex Connections: Reimagining Units Construction and Coordination Working Group was well attended each day at PME-NA 2018. Including the organizers, there were 16 participants attending all three sessions who also provided their email contact in order to foster continued collaborations.

Session 1: Concept Formation

GOAL: Generation of research questions that are important to the group and/or sub-groups

ACTIVITIES: Introduce focus for the working group by asking "what types of problems would members like to explore?", by viewing/discussing short video clips of students working through various mathematical concepts to better understand the students' thinking, and developing potential research foci (e.g., overall purpose/goals of this working group, ties between composite units, coordination of units, and particular mathematics content). Finally, subgroups will be developed to form research questions that can cross-domains and use questions to form collaborations based on each members' area of interest and expertise.

OUTCOMES: Participants spent a bulk of the session defining terms and discussing students' responses to tasks designed to assess units construction/coordination. Small groups foci included participants' experience in this field with teaching/research.

Session 2: Theoretical Frameworks and Methodologies

GOAL: Explore appropriate research methodologies.

ACTIVITIES: Formulate plans for research and collaboration across group members by examining a variety of methodologies. Means for these examinations would include but not be limited to the following: (a) view videos of work already conducted to highlight possible methodologies for future studies; (b) discuss other potential methodologies not highlighted during the video viewing; (c) discuss how to design robust collaborative studies. Small group would entail: (1) work already done; (2) research agenda development; Large group would entail: (1) sharing of small group discussions; (2) delineate session 3 goals

OUTCOMES: Foci shifted in the session to include task development and website organization.

Session 3: Planning and Writing

GOAL: Embark on collaborations.

ACTIVITIES: Small group will entail: (1) work on written product of research agenda; (2) develop shared conceptual framework and the relationship of our framework to what is currently being done; (3) identify target journals and outlets or grants and funding sources. Large group will entail: (1) share progress and commitments from small group discussion; (2) finalize a plan for individual groups to continue updating progress to the larger group; (3) creation of working group website or blog

OUTCOMES: Collaborative plans around theoretical questions related to particular tasks were developed including task development/organization and webpage development.

Anticipated follow-up activities

Throughout the year, the members of this working group will continue working on research problems of common interest. They will contribute to a common website in which they will update other members of the working group about the progress of the various research collaborations. In the future, this working group will propose a special issue to a leading journal in the field and/or construct a grant proposal to a nationally recognized funder.

Results from 2018 Working Group

Resulting from the 2018 working group were development of three projects (extending to Calculus students, Preservice teachers, early elementary students; widening to special education), five manuscripts, and 11 conference proposals at four (inter)national conferences. Discussions from the 2018 working group further informed nuances in manuscript and project development while also including at least two new members to research initiatives in the units construction/coordination field. The webpage has become further developed to improve organization and include more readings.

Goals and Plan for 2019 Working Group

We primarily plan to build upon the successes of the inaugural Working Group, with two exceptions. We realized during the 2018 working group that the term "unit" needed to be discussed further. This year, we include distinction between research focusing on units construction (to include pre-numerical activity) and research focused on interiorized units coordination. We also plan to spend more time in the first meeting setting up the research paradigm and inviting participants to engage in task-based units coordinating activity themselves (rather than merely watching video excerpts of students' reasoning).

Session 1: Task Organization and Constructivist Groundings

GOAL: Describe main tenets of the Constructivist paradigm; generation/organization of units coordinating tasks set in this paradigm. Introduce focus for the working group by delineating particular assumptions of constructivism:

1. Knowledge is actively created or invented by the child, not passively received from the environment,
2. Children create new mathematical knowledge by reflecting on their physical and mental actions,
3. No one true reality exists, only individual interpretations of the world,
4. Learning is a social process in which children grow into the intellectual life of those around them,
5. When a teacher demands that students use set mathematical methods, the sense-making activity of students is seriously curtailed (Clements & Battista, 2009, p. 6-7).

To draw parallels between these tenets and this working group, members will actively engage with tasks while reflecting on their actions. In particular, they will consider "how did you respond to this task?," "how might students respond to this task?" and "why are these tasks effective for teaching/research purposes?." By breaking into groups to engage with tasks, which exemplify various units construction/coordination, participants can reflect on their own actions to determine how students may develop units and coordinate units through their engagement. Finally, subgroups will come together and discuss possible relationships between student actions and task features when examining students' units construction/coordination.

Session 2: Student Reasoning Relative to Units Construction/Coordination Theories

GOAL: Connect student reasoning to units construction/coordination learning theories.

ACTIVITIES: Participants will discuss students' responses to tasks and explanations relative to appropriate learning theories: (a) documenting anticipated student responses to categorize intended outcomes for tasks; (b) connecting details for student actions to theories in units construction/coordination; (c) documenting learning theories with tasks and references. Small group would entail: (1) task/student actions organization; (2) learning theory discussion to explain importance of particular student actions; large group would entail: (1) sharing of small group discussions; (2) delineate session 3 goals.

Session 3: Webpage Expansion

GOAL: Organize tasks on the webpage. Small group will entail: (1) collaborative webpage expansion by organizing tasks and possible student responses on the webpage; (2) associate each task with intended perturbation/assessment, reference, learning theory, etc. Large group will entail: (1) share progress and commitments from small group discussion; (2) finalize a plan for individual groups to continue updating progress to the larger group; (3) further creation of working group website or blog.

Anticipated Follow-up Activities

Throughout the year, the members of this working group will continue working on research problems of common interest and develop several reading groups. They will contribute to a common website in which they will update other members of the working group about the progress of the various research collaborations and discussion from reading group ideas. In the future, this working group will propose a special issue to a leading journal in the field and/or construct a grant proposal to a nationally recognized funder.

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