INEQUALITIES AND SYSTEMS OF RELATIONSHIPS: REASONING COVARIATIONALLY TO DEVELOP PRODUCTIVE MEANINGS

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Systems of equations are an important topic in school mathematics. However, there is limited research examining productive ways of supporting students' understandings of systems of equations. In this paper, we first present a conceptual analysis of potential ways students may leverage their quantitative and covariational reasoning to graph systems of relationships. We then describe results from a design experiment in which we examined the potential of supporting middle-grades students in reasoning in ways compatible with this conceptual analysis. We highlight two different ways of reasoning students engaged in as they compared the relative magnitudes of two quantities with respect to a third quantity and leveraged this reasoning to graphically represent two relationships on the same coordinate system. We draw implications from these results for the teaching and learning of systems of equations.

Keywords: Algebra and Algebraic Reasoning, Cognition, Middle School Education

Systems of equations are an important topic as indicated by its presence in both U.S. and international mathematics curricula (Bergeron, 2015). Despite this, there is a dearth of research examining productive ways of supporting students' meanings for systems of equations (see Häggström, 2008). Several researchers (Olive & Caglayan, 2008; van Reeuwijk, 2001) have noted students can leverage thinking about experientially real contexts to determine solutions for systems of equations involving discrete quantities. For instance, Olive and Caglayan (2008) examined how students solved a system of equations by leveraging their quantitative reasoning (Thompson, 1995) to coordinate several quantitative units to simplify a system of two or three equations into one equation with one variable. In this report, we extend Olive and Caglayan's (2008) work by examining ways to support middle-grades students in reasoning quantitatively and covariationally to construct, graph, and determine solutions to systems of relationships. We first present a conceptual analysis (Thompson, 2008) in which we explain ways of reasoning we conjectured could be productive for students graphing systems of relationships. We then present results from a design experiment in which we highlight students reasoning in ways compatible with this conceptual analysis. Specifically, we highlight that students were able to reason quantitatively and covariationally to construct and reason about two relationships and leverage this reasoning to accurately graph both relationships on the same coordinate system. We discuss implications of these findings for school mathematics teaching and curriculum.

Reasoning Quantitatively and Covariationally to Represent Systems of Relationships

In this report, we adopt Thompson's (1993, 1995, 2008) theory of quantitative reasoning. Specifically, Thompson (1993) noted:

> A quantity is not the same as a number. A person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the possibility of measuring it... Quantities, when measured, have numerical value, but we need not measure them or know their measures to reason about them. You can think of your height, another person's height,

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> and the amount by which one of you is taller than the other without having to know the actual values. (p. 165)

We highlight two aspects of quantities as defined by Thompson. First, and consistent with others (Glasersfeld, 1995; Steffe, 1991), Thompson emphasizes that quantities are conceptual entities constructed by an individual in order to make sense of their experiential world. As such, understandings of a quantity can and will differ from individual to individual and it is critical for teachers and researchers to attend to *students'* conceptions of quantities. Second, quantitative reasoning may entail reasoning about numeric values, but such reasoning does not *require* numeric values (Johnson, 2012; Thompson, 2011). Instead a student may reason about the magnitude, or 'amount-ness,' of a quantity without using numeric values.

Leveraging Thompson's characterization of quantitative reasoning, Carlson et al. (2002) defined covariational reasoning as entailing a student coordinating two varying quantities while attending to the ways the quantities change together. They described five mental actions students engage in that allow for fine-grained analysis of students' activity. The mental actions include coordinating *direction of change* (surface area increases as height increases; MA2), *amounts of change* (the *change* in surface area *increases* as height *increases in successive equal amounts*; MA3), and *rates of change* (surface area *increases at an increasing rate* with respect to height; MA4-5). Although researchers (Johnson, 2012; Moore, 2014; Paoletti & Moore, 2017) have described productive ways high school and post-secondary students engage in reasoning compatible with the mental actions, there is a dearth of research examining middle-grades students' reasoning in such ways. In this study, we pay particular attention to MA3 as critical to students representing relationships between covarying quantities.

A Conceptual Analysis with an Example

In this section, we leverage one use of conceptual analysis described by Thompson (2008), namely "describing ways of knowing that might be propitious for students' mathematical learning" (p. 46). We present a conceptual analysis of ways students may leverage their quantitative and covariational reasoning in order to describe and graphically represent a system of relationships. Specifically, we conjectured we might be able to support students in reasoning about and graphically representing a system of relationships by having the students engage in a series of activities. We intended to support students to (a) construct and reason about a relationship between Quantity A and Quantity B, (b) construct and reason about a relationship between Quantity A and Quantity C, (c) compare the relative magnitudes of Quantity B and Quantity C, including when Quantity B is greater than, less than, or equal to Quantity C for all possible values of Quantity A, and (d) leverage the reasoning from (a)-(c) to graph a system of relationships and interpret the resulting graph in terms of the inequalities determined in (c).

We use the Cone/Cylinder Task, which we designed with this conceptual analysis in mind, to describe steps (a)-(d). Prior to the session addressing the Cone/Cylinder task, we spent two to three teaching episodes engaging students with the Cone Task, which entails a dynamic cone with a varying height (the same cone is seen in Figure 1). We ask students to describe quantities they could imagine in the situation to support their reasoning quantitatively. We then task students with describing how either the surface area or volume of the cone varies, without providing numeric values, as the height varies (for more on the evolution of this task see Paoletti, Greenstein, Vishnubhotla, & Mohamed, accepted). We provide students with a sheet of paper showing the cone at five equal changes in height and prompt them to describe how the surface area (or volume) of the cone changes for successive equal changes in height to support their engaging in the mental actions described by Carlson et al. (2002). After coordinating the quantities situationally, each group was able to represent the relationship they conceived between surface area (or volume) and height of the cone in a normatively correct graphical representation.

After addressing the Cone Task, we began the next session with an applet showing the same cone with a growing cylinder next to it (Figure 1). Prior to asking the students to graphically represent the surface area (or volume) of each 3D shape with respect to height, we first asked students to engage in two other activities: (1) we asked students to describe how the surface area (or volume) of the cylinder grows for equal changes in height attempting to promote their reasoning in ways described by Carlson et al. (2002), and (2) we prompted them to describe height values (as defined by the slider, *a*) such that they conceived the surface area (or volume) of the cone is greater than, less than, or equal to the surface area (or volume) of the cylinder. Consistent with quantities being conceptual entities, we were not concerned if the students' approximation for an *a*-value such that the two quantities are equal is accurate. Instead, we intended to explore how the students described the relative magnitudes of the surface areas (or volumes) before and after their determined numeric height value.

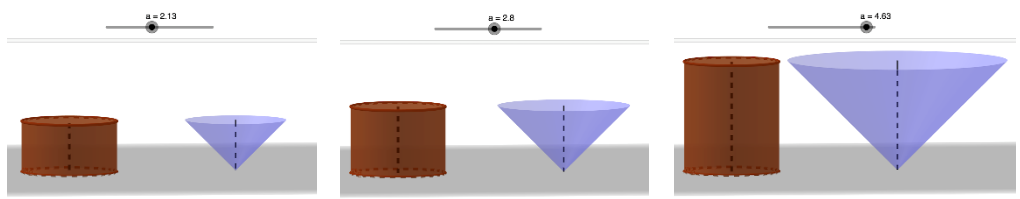


Figure 5: Several Screenshots of Cone/Cylinder Task

After addressing the two aforementioned activities, we asked students to graphically represent the relationships between surface area (or volume) and height of the two shapes on the same coordinate system. We note that since the Cone Task, involving only the cone, required several teaching episodes, each group had already constructed at least one graph representing the relationship between surface area (or volume) and height of the cone but had not yet constructed a graph representing the surface area (or volume) and height of the cylinder.

Methods, Participants, and Analysis

We conducted a design study (Cobb, et al., 2003) consisting of five small-group (one to three students) teaching experiments (Steffe & Thompson, 2000) to examine ways to support middle-grades students' (age 10-13) quantitative and covariational reasoning including the possibility of supporting their leveraging such reasoning to represent systems of relationships. We opted to engage middle-grades students as such students had not yet taken or completed Algebra I and did not have experience graphing systems of equations. The teaching experiments occurred in an underperforming school (as measured by standardized testing), which hosts a diverse student population (over 75% students of color and qualify for free or reduced-price lunch), in the Northeastern U.S. We recruited students through teacher recommendations and we engaged all students who returned consent forms.

To analyze the data, we leveraged a second use of conceptual analysis, "building models of what students actually know at some specific time and what they comprehend in specific situations" (Thompson, 2008, p. 60). Specifically, we generated, tested, and adjusted models of student's mathematics so these models provided viable explanations of each student's activity.

We used open (generative) and axial (convergent) approaches (Strauss & Corbin, 1998) to analyze the data. First, we watched videos identifying instances that provided insights into each student's meanings. We used these instances to generate tentative models of each student's mathematics, which we compared to notes taken during on-going analysis. We tested these models by searching for supporting or contradicting instances in other activities. When evidence contradicted our models, we revised our models and returned to prior data with these new hypotheses in mind to modify previous models. This process resulted in viable models of each student's mathematics. After this, we used cross-case analysis (Yin, 2003) to compare the different students' activity to identify themes in the students' reasoning and meanings.

Results

Across the five groups, we saw two different approaches students used to graphically represent the two relationships on the same coordinate system after comparing the relative magnitudes of the surface area (or volume) of the shapes with respect to height. Four groups initially reasoned covariationally as they accurately represented each relationship on the same coordinate system without attending to the inequalities they determined (e.g., drew a linear graph to represent the relationship for the cylinder and a concave up curve to represent the relationship for cone). Two of these groups later adapted their curves in order to maintain the relationships represented by the inequalities they determined earlier in the session. The third group (a single student) reasoned about two researcher drawn curves on a coordinate system as consistent with her conceived inequalities. Due to time constraints, we were unable to probe the fifth group (a single student) in regards to how she could adjust her curves or interpret curves on a new coordinate system in relation to the inequalities she determined. To exemplify these students' activities, below we present the activities of one pair of students, Zion and Reggie.

In contrast to the other students' activities, one group, Candice and Amber, reasoned about the *difference* between the surface area of the cone and cylinder throughout their activity. By focusing on the difference in the two quantities magnitudes, the pair constructed curves that maintained the relationships they inferred regarding height values such that one surface area was greater than, less than, or equal to the other but did not accurately represent the individual covariational relationships between each shape's surface area and height. The pair adjusted their curves while continuing to reason about the difference between magnitudes to construct normatively accurate graphs.

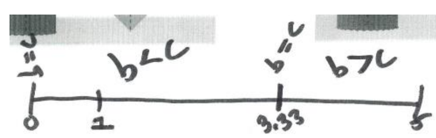
Reasoning Covariationally About Each Relationship then Adjusting: Zion and Reggie

After the sessions in which Zion and Reggie reasoned about the directional and amounts of change of surface area with respect to height of the cone (MA 2-3) and graphically represented this relationship, we presented them with the Cone/Cylinder applet and asked them to determine intervals on which the surface area of the cone (which they defined with the variable b) was greater than, less than, or equal to the surface area of the cylinder (which they defined with c). Moving the slider on the applet, they determined the two surface areas were equal at a height value of 3.33, and that for height values less than 3.33 the surface area of the cone was less than the surface area of the cylinder ($b < c$) and vice versa for values greater than 3.33 (Figure 2*a*).

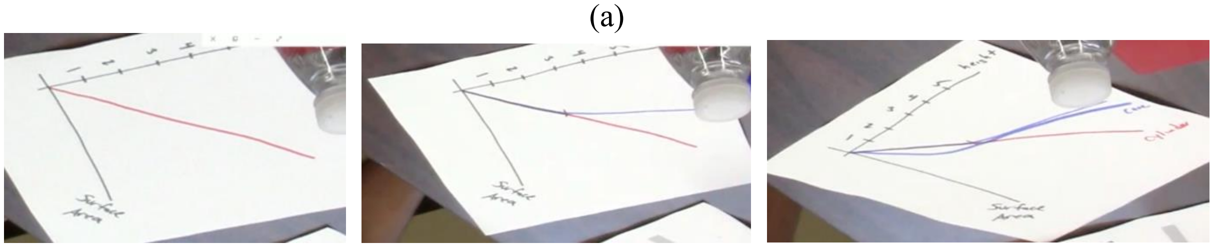
Having compared the two surface areas for all height values, the teacher-researcher (TR) asked the pair to describe how the surface areas of the cone and cylinder were increasing. Based on their activity in the Cone Task, the students spontaneously shaded in areas representing amounts of change of surface area of the cone for successive equal changes in height to argue the surface area of the cone was "growing by more and more each time." Then, the pair worked to represent via color-coordination, amounts of change in surface area of the cylinder for equal changes in height. Specifically, on a paper showing five instantiations of the cylinder and cone, the TR shaded in blue the outer surface area of the cylinder with a height of 1. In the cylinder representing the second change in height (from 1 to 2), Zion shaded in blue the same area shown in the first cylinder (with a height of 1) then shaded in brown area representing the amount the surface area changed by from a height of 1 to 2.

> TR: So that you were trying to make this brown part [*pointing to the cylinder showing the second change in height*] the same as what?
> Reggie: [*Zion and Reggie both point to the surface area representing the cylinder with a height of 1*] Same as this.
> TR: As that one?
> Reggie: 'Cause that's how much it grows every time [*pointing to the surface area of the cylinder with a height of 1*].

Shortly after this, Zion described that the surface area of the cylinder "grows constantly." We infer Reggie and Zion each understood the surface area of the cylinder increased by the same amount for equal changes of height (MA 2-3). Further, Zion sketched a linear graph representing this relationship (Figure 2*b*). We note that although the graph had the horizontal axis labeled "Surface Area," throughout their activity each student made statements that implied surface area and height were represented on the vertical and horizontal axes, respectively, and later in their activity relabeled the horizontal axis "height" when they noticed this label.



(a)



(b) (c) (d)

Figure 2: (a) Zion and Reggie's Segment Representing Height Values and (b)-(d) Several Screenshots of Their Work Graphing Each Relationship on the Same Coordinate System

Immediately after the interaction above, Reggie drew a second curve in blue to represent "the cone" (Figure 2*c*). The TR probed the students to examine if they intended to represent the surface area of the cone and cylinder as being equal for small height values:

> TR: Are they growing from 0, 0 to 3.33 [*marks a point on the curves where the blue and red curve diverge*]… are they the same, are the surface areas the same for that, from 0 to 3.33?
> Reggie: Yeah [*using his fingers to indicate the segment between (0,0) and the point on the*

graph where the curves diverge], uh-huh.
TR: Should they be?
Zion: [*crosstalk*] I mean this one would be a little bit lower I guess [*tracing an imaginary concave up curve from (0,0) to where the curves diverge as if drawing a new blue curve*] because it's not…until we said until 3.33 the cylinder is going to be bigger. [*Zion draws a new curve in blue on their coordinate system, Figure 2d, then the TR asks him to explain why he changed the graph.*] What we said is that the, that from this point [*motioning over the interval from 0 to 3.33 on the segment in Figure 2a then pointing to b > c*] that the cu-cylinder is gonna be bigger than the cone and from this point on [*pointing to 3.33 then to 5 on the segment in Figure 2a*] the cone is gonna be bigger than the cylinder [*pointing to b < c*].
TR: How does this graph [*pointing to the graph in Figure 2d*] show you that… the surface area of the cone is less than the surface area of the cylinder?
Reggie: The cone [*tracing over the blue curve from the origin to the point of intersection*] is lower than the cylinder [*tracing over the red curve from the origin to the point of intersection*].

We highlight two aspects of this interaction. First, we infer Reggie argued the two surface areas were equivalent from 0 to 3.33 by interpreting what it would mean for the two curves to overlap on this interval (i.e., using his fingers to indicate the gap between (0, 0) and the point where the curves diverged on the graph), rather than on a conception of the relationship between the quantities' magnitudes. This inference is supported by him quickly assimilating Zion's argument regarding the need to represent the cone's surface area as being below the cylinder's surface area for height values between 0 and 3.33. Second, we highlight how Zion leveraged the inequalities they had determined earlier (i.e. explicitly referencing the inequalities they had determined in Figure 2a) to adjust their curve to create a representation that reflected their stated inequalities; Zion coordinated his activity reasoning about inequalities regarding the two surface areas in relation to height as he constructed two curves representing each surface area-height relationship.

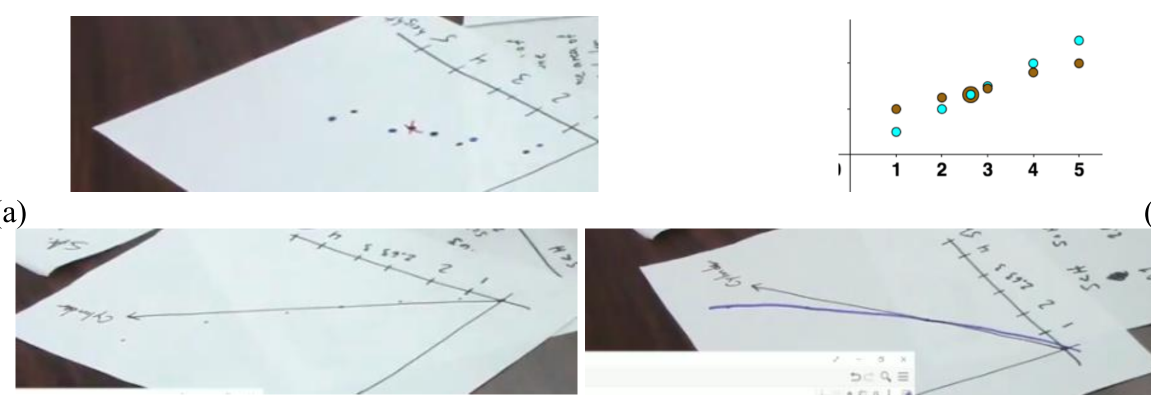
Reasoning About the Difference of Two Quantities to Represent a System of Relationships

Like Zion and Reggie, at the outset of the Cone/Cylinder Task session, Candice and Amber described the surface area of the cone increased by more for equal changes in height (MA 2-3) and the surface area of the cylinder increased by equal amounts for equal changes in height (MA 2-3). When prompted to determine a height value such that the two surface areas were equal, they approximated a height value of 2.63 and indicated for height values between 0 and 2.63 the surface area of the cone was less than that of the cylinder and vice versa for height values between 2.63 until 5. Each student had sketched a linear function to represent the relationship between surface area and height of the cylinder but had labeled their axes differently (Candice labeled the vertical axis "height" and Amber labeled the horizontal axis "height").

Attempting to ensure the students would discuss the same quantities when discussing their graphs representing both relationships, the TR created new axes with height values from 1 to 5 on the horizontal axis and asked the students to color coordinate points (blue for the cone and brown for the cylinder) representing the height and surface areas of the two shapes. Addressing this, each student considered the difference in the two surface areas for corresponding height values as they plotted points. Specifically, after plotting the brown points above the blue points at height values of 1 and 2 (Figure 3*a*/*b*), the TR asked how they had chosen to space the points. Candice maintained the points above the height value of 2 were closer than the points at a height value of 1 and Amber stated the points above the height of 1 should have been spaced farther apart. We infer each student understood that the surface area of the cylinder was bigger than the surface area of the cone and that the difference between surface areas was decreasing until 2.63.

After plotting these points, the TR moved the applet to a height value of 2.63 and each student indicated the surface areas “are going to be the same.” Candice then placed a blue point above a height value of 3 and Amber placed a brown point on top of this blue point. Noticing the horizontal placement of the points, the TR motioned to the value on the axis below their points to examine if the students intended to plot points above a height value of 3. Amber quickly indicated their points should be above a point between two and three on the height axis, putting an X on top of their points (Figure 3*a*). Candice then placed a tick on the height axis representing a value of 2.63 and the students plotted overlapping points above this tick. We infer the students understood the two shapes had the same surface area at 2.63, which they understood meant the two curves would overlap at this value.

The students continued plotting points after 2.63 such that the blue point was above the brown point and the distance between points was “getting farther [apart]. Because [the difference between surface areas is] getting bigger.” Throughout their activity the students focused on the difference in surface areas as they plotted points. However, they were not explicitly focusing on the locations of the points in relation to an individual shape’s surface areas, which the TR conjectured could be problematic. For instance, both he and Amber perceived that the blue points (Figure 3*a*/*b*), representing the surface area and height of the cone, fell in a straight line that did not accurately represent the relationship. Hence, the TR drew new coordinate axes and a linear curve representing the surface area and height of the cylinder, which each student had produced earlier. He then asked the pair to sketch the curve representing the surface area and height of the cone. Consistent with reasoning about the difference between surface areas, Candice plotted points above and below the red line that maintained the difference in surface areas they had described previously. For instance, the spacing between their plotted points and points on the line for height values greater than 2.63 was increasing (Figure 3*c*). Candice then connected these points with a curve (Figure 3*d*). Compatible with attending to difference in surface areas rather than the individual surface areas, when Candice drew the curve, Amber exclaimed “Oh my god! It looks like the other line.” Amber then explained the blue curve was similar to the curve they had constructed to represent the surface area and height of the cone in previous sessions.



(a) (b)

(c) (d)

Figure 3: Several Screenshots (and a Recreation) of Candice and Amber’s Work

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Throughout their activity, we infer Amber and Candice focused on the relationships they inferred creating inequality statements (i.e. a difference) rather than on the covariational relationships they had inferred regarding each individual surface area-height relationship. After plotting points that reflected how the difference between quantities was changing but not the quantities themselves, they were able to use the difference between surface areas to accurately sketch a curve representing the relationship between surface area and height of the cone on the same coordinate system with a line representing the surface area and height of the cylinder.

Discussion

In this paper, we presented a conceptual analysis of four interrelated activities, (see steps (a) – (d) in Conceptual Analysis with an Example), we conjectured could be productive for students graphing systems of relationships. While both pairs of students presented eventually constructed curves representing a system of relationships based on the relationships they inferred in the situation (step (d)), they did so in different ways. Specifically, Amber and Candice's reasoning focused on the *difference* in surface areas' magnitudes which they inferred from comparing the two surface areas (step (c)) as they initially plotted points. They continued to reason about the difference in surface areas to plot points representing the surface area and height of the cone after the TR provided a new coordinate system with a linear graph representing the surface area and height of the cylinder (which they had previously constructed addressing step (b)). This process resulted, to Amber's surprise, in a curve similar to the one they obtained when addressing step (a). In contrast, Zion and Reggie, as well as the three other groups, initially produced a graph of each individual relationship that reflected the covariational relationship they inferred from the situation (e.g. the relationships they inferred in steps (a) and (b)). During this activity, they did not attend to how the two curves may intersect. After considering the inequalities they deduced when comparing the relative magnitudes of the two surface areas (step (c)), two of these groups were able to sketch a graph accurately representing the two relationships they conceived.

The results we presented provide novel insights into students developing understandings of systems of relationships and also add to the literature on students' quantitative and covariational reasoning in two important ways. First, we highlight that the students were never presented with, nor created, values for the surface area of either shape. Rather they reasoned about quantities' magnitudes (Thompson, 1995) when reasoning about the direction and amounts of change of surface area as well as when comparing the relative size of the two shape's surface areas. The students later used height values to describe and graphically represent these varying magnitudes. Hence, the results presented here highlight ways in which middle-grade students can productively reason quantitatively and covariationally about both magnitudes and values to graphically represent relationships.

Second, the results provide empirical examples that, at least some, middle-grades students are capable of leveraging the mental actions describe by Carlson et al. (2002) to reason about and graphically represent relationships between covarying quantities. We conjecture supporting middle-grade students in developing such reasoning as a way of thinking (Thompson, 2016) could serve as the foundation for their developing notions of linear and non-linear relationships as well as rate of change more generally. Future researchers may be interested in examining ways to support middle-grades students engaging in such reasoning to develop such notions as well as to develop other relation classes (e.g., exponential, quadratic relations).

Finally, we note that our focus was broader than determining a solution to a system of equations, which is often a goal of school mathematics (e.g., Häggström, 2008). We were interested in exploring ways students can reason about systems of relationships, including how two quantities are represented when one is greater than, less than, or equal to the other with respect to a third quantity. The students in our study rarely experienced difficulties comparing two quantities magnitudes situationally or graphically. Hence, we encourage future researchers and curriculum designers to examine the possibility of supporting students explicitly comparing two quantities magnitudes or values prior to tasking them with determining unique solutions to systems of equations. That is, we conjecture reasoning about systems of relationships can serve as a foundation for students' developing productive meanings for systems of equations.

Acknowledgments

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