BOB'S ADDITIVE REASONING: IMPLICATIONS FOR KNOWING FRACTIONS AS QUANTITIES

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Fractions are one of the most difficult areas of mathematics for all students and especially for students with learning disabilities (LD). An incomplete understanding of fractions during the elementary and middle school may be why some students view fractions as "really small" or "less than 1" compared to others who view them as multiplicative magnitudes. We analyzed how one student with LD worked on tasks involving both whole numbers and fractions using the framework of units coordination. Through our analyses of 14 teaching experiment sessions, we determined that he was operating on two levels of units with whole numbers, but only one level of units with fractions. We argue that this limited his ability to operate multiplicatively with fractions. However, his identified LD did not appear to impact his work with the fractions tasks he was given. Implications for future research and instruction are shared.

Keywords: (Dis)ability Studies, Rational Numbers

Background and Study Objectives

Fractions are one of the most relentless areas of difficulty in mathematics for all students (Siegler et al., 2010) and are especially for students with learning disabilities (LD) (Cawley & Miller, 1989). When asked to place fractions in order from least to greatest on two separate assessments, middle school students with LD answered only 47% and 1% of questions correctly, compared to 85% and 60% by students without LD (Mazzocco & Devlin, 2008). Similarly, researchers report fourth and fifth grade elementary students with LD begin their study of fractions with a diminished conceptual understanding compared to their peers (Geary et al., 2008) and show significantly less improvement in their ability to solve problems, estimate, and apply computational procedures with fractions over time (Hecht & Vagi, 2010). An incomplete understanding of improper fractions during the late elementary and middle school years may be the driving force behind why some students view fractions as "really small" or "less than 1" versus other who view them as multiplicative magnitudes (Resnick et al., 2016).

In this study, we present the whole number and fractional reasoning of one student with LD across 14 teaching experiment sessions. We utilize a three phase qualitative analysis to illustrate how this student evidenced his understanding of fractions as quantities through his interactions with varied tasks. Through our analyses of these data, we raise questions about the child's thinking and what his apparent knowing and learning were relying upon. The research questions addressed are (1) How does one child with LD conceive of whole number and fractions as quantities across varying task types? and (2) What persistent ways of reasoning were apparent across the child's activity and how did these contribute towards this child's knowledge of fractions?

Conceptual Framework

We think of students' understanding of fractions as quantities as a result of their propensity to multiplicatively coordinate units(s). Units coordination refers to how students create units and maintain relationships with other units (Norton, Boyce, Ulrich, & Phillips, 2015). It explains transitions children make from additive operations to multiplicative operations. Norton et al. (2015) explain that a student uses one level of unit when she conceives of situations such as five iterations of four by counting on from the first or second set by ones and double-counting the number of fours to reach a stop value (e.g., 4, 8, 12, 13-14-15-16, then 17-18-19-20). A student who uses two levels of units might conceive of five units of four as two units of four plus three units of four (e.g., three 4s is 12; 13-14-15-16; 17-18-19-20). This student breaks apart the composite unit of five into three and two and uses each of those parts to arrive at the solution. This student also sees "20" as both 20 ones and 5 fours all at once, so finding six units of four would not require reconstructing the first five units of four- the student would count-on from 20. Using three levels of units involves three related or coordinated units: (a) one unit of 20 that contains (b) five units of four, each of which contains, for instance, (c) four units of one. Students who anticipate three levels of units demonstrate flexible, strategic reasoning with each of the units.

Fractional units can also be coordinated. Steffe (2002) hypothesized that students would reorganize whole number unit coordinating to develop similar understandings with fractional units. Put differently, students use their whole number units coordination to conceptualize fractional units as a result of equi-partitioning whole units. The size of each fractional unit is one that when iterated *n* times would result in the size of one (Olive, 1999). To understand a non-unit fraction (*m/n*) as a number, students must conceptualize *m/n* as equivalent to *m* x *1/n*, *n* of which are the same value as 1. When the number of iterations of *1/n* exceeds *n/n*, students must accommodate their part-whole reasoning from thinking of *1/n* as one out of *n* total pieces to thinking of *1/n* as an amount iterated more than *n* times without changing its multiplicative relationship with the size of 1. Thus, a multiplicative conception of fractions as quantities involves coordinating three levels of nested units: 10/4 as 10 times (1/4) *and* 1 is the same value as 4 times (1/4). Ten-fourths involves both a unit of 1 and a unit of 1/4 coordinated with 1. If students name the size of proper fractions by counting the number of parts of size *1/n* within the bounds of a whole of *n/n*, they do not yet iterate *1/n* (Tzur, 1999). This limits students' propensity to understand unit fractions beyond the size of the whole in a multiplicative manner.

Methods

"Bob"

The data utilized in this study illustrates the case of one fifth grade student who we refer to as "Bob." Eleven years old at the time of the study, Bob was an outgoing child who often imagined creative ways to solve problems and was eager to interact with problematic situations. Inclusion criteria for the study were as follows: (a) individualized education program goals in mathematics, (b) a cognitively-defined label of learning disability (LD) with working memory as the dominant cognitive factor, (c) identification through clinical interview data that the child had constructed at least a parts within the whole concept of unit and non-unit fractions, and (d) identification by the classroom teacher as 'non-responsive' to supplemental, small group intervention from a textbook or supplemental curriculum in advanced fraction concepts.

For Bob, school-based instruction included opportunities to shade pre-partitioned wholes to reflect a designated number of parts. Bob extended this activity to represent various fractional quantities using area, set, and linear models. These experiences were designed to help Bob see the relationship between one of the equal parts and the whole. Area, set, and linear models were also used during school-based instruction to represent equivalent fractions as well as to identify given fraction representations as equivalent. Procedures for fraction operations followed.

Furthermore, although Bob was able to construct unit fractions and identify shaded parts of wholes (e.g., identify one part shared within a partitioned item as one-seventh; determine the share of 4 objects among 3 people as one whole and one third); he could not yet draw parts outside of and relative to the whole. He did not yet conceive of non unit fractions as a summation of unit fractions (¾ = ¼ + ¼ + ¼) or a multiplicative relation with respect to a unit fraction (e.g., ¾ as three iterations of ¼). Instead, ¾ was three of four parts.

Study Design and Data Sources

We report on five sessions conducted as part of a larger individualized teaching experiment of 14 sessions (Steffe & Thompson, 2000) because these sessions provided evidence of how Bob coordinated units with whole numbers and fractions. We also conducted two formative clinical interviews that serve as pre-and post-measures (Ginsburg, 1997). Each interview and instructional session lasted about 30 minutes. The second author was the researcher-teacher. The first and second authors both engaged in the retrospective analysis after data collection concluded. The instruction was conducted in a one-on-one setting in the school library. The classroom was equipped with one circular table, unifix cubes, pens, paper, and chart paper. Sessions took place after school and were in addition to Bob's regular math class time. We collected three sources of data: video recordings, written work, and field notes.

Data Analysis

Data analysis occurred in three phases. The first involved ongoing analysis of video data and written work immediately following each teaching session. The ongoing analysis led to planning the following teaching episode and consistent modification of hypothesis regarding the child's mathematics and goals in activity. The focus was on generating (and documenting) initial hypotheses as to what conceptions could underlie the child's apparent problem-solving strategies during these critical events. These hypotheses led to (a) designing subsequent teaching episode(s) and (b) guiding retrospective analysis.

The second phase of data analysis included a retrospective identification of broad indicators, or stages, of conceptual growth after the sessions concluded (Leech & Onwuegbuzie, 2007). We conducted a line by line examination of the transcribed data, student work, and notes with a side by side of video data to consider Bob's thinking within each session. Analysis of each teaching episode focused on expanding our model of Bob's thinking. We coded segments within the session for indications of Bob's units coordination and operations at the time, using prior research as a framework. Bob's gestures and explanations (e.g., from post-episode field notes) were incorporated into the analysis to provide a thick depiction (Geertz, 1988) of his current operations— served as further triangulation. The third phase involved fine-grained analysis to closely consider how Bob advanced his thinking from one stage to the next and the difficulties he experienced (Tzur, 1999).

Results

"I Just Added Two"

During the first instructional session, Bob was presented with the task of making "double" of several whole numbers of squares. He doubled one and two squares to make two and four squares, yet when asked to double four squares, he made six squares. His explanation for doubling both two and four was, "I just added two."

<u>Session 1: Make double of four squares</u>

Instructor: Could you make double that [points to line of four squares]?

Student: Yeah.
Instructor: Maybe – [student traces one square on each end of line of four squares] What'd you do there?
Student: I just added two.
Instructor: You added two. Hmm, gotcha. So, what's this whole length called? [runs finger along line of six squares]
Student: [tapping pencil on desk] One or one, two, three, four, five, six, six. [pointing at squares, but not counting one to one]

We questioned whether Bob was coordinating whole numbers multiplicatively. Moreover, it was unclear to us whether Bob connected "doubling" with two levels of units (e.g., viewing four as two units of two). This wondering would permeate throughout the teaching sessions.

"One, Two, Three, Four"

The next several sessions continued to provide contradictions in how Bob was coordinating units. For instance, in the transcript below, Bob shares three French fries among four people. At first glance, it seems as if Bob is utilizing two levels of units multiplicatively to consider the resulting quantity. Yet, as the transcript shows, Bob coordinated units with whole numbers and fractional units utilizing additive reasoning.

<u>Session 5: Share three fries among four people</u>
I: Could they each get a whole one? Could they get less than a whole? Remember these are really big fries.
Bob: So one of them you could cut in half [makes a chopping gesture]... two of them could be cut in half.
I: Two of them need to be cut in half, okay.
Bob: And then we need to have one more French fry.
I: What did you do to that one?
Bob: And then now we'll have to split that one up… into… fourths so if I did one [uses finger to mark first line and second line]– that's…
I: Are you showing me the lines?
Bob: So that's 1, 2, 3, 4 [uses fingers to show lines] and there and there [marks lines].
I: Like that? Nice. There you are [draws a smiley face on paper]. How much do you get?
Bob: [draws a smiley face on one of the halves and one of the fourths]
I: So you get this part and this part? [points to pieces labeled with smiley faces] How much is that? Bob: ½ [points to the one-half].
I: What's this part called?
Bob: 1/4. [points to the one-fourth]
I: So would it be fair to say that you get one fourth and one half of a fry?
Bob: [measures ¼ piece and adds it to the end of "his" half] This is how much I will get.

Bob coordinated four sharers across two wholes by partitioning each whole into two parts. At the time, we viewed this as evidence that Bob utilized two levels of units to create four from two twos multiplicatively. However, to coordinate fourths with respect to one whole, Bob seemed to count by ones as opposed to partitioning into halves and then fourths (Confrey, Maloney, Nguyen, & Rupp, 2014). Furthermore, although Bob was able to draw the two fractional pieces together on one strip, he did not yet see the parts as a continuous quantity (i.e., as three-fourths).

Thus, Bob coordinated whole numbers on two levels additively and fractions on one level additively.

"Two Out of Five"

In Session Six, Bob displayed part-whole knowledge of fractions to consider two-fifths as two parts out of five parts. He was able to use his part whole knowledge to create one-fifth using the computer program JavaBars and iterate the one-fifth to make seven-fifths. Yet, his language shows that he views the created quantity as seven as opposed to seven fifths.

> <u>Session 6: This bar is two-fifths. Can you make seven-fifths?</u>
> Bob: Two-fifths [splits bar into two], so it's two… it's two out of five, so maybe if I… That's two – every one out of five, two out of five…
> I: Mm-hmm.
> Bob: There's two now and, well, uh, and… and then that one. [pulls out one fifth]
> I: What's that one? [points to pulled out one-fifth]
> Bob: One-fifth.
> I: It's one-fifth.
> Bob: And it –
> I: We want seven-fifths.
> Bob: Seven fifths… one out of five, seven-fifths, so we want… and then that is… [repeats 1/5 four times, pauses to count]. Okay, you just do that. [repeats two more fifths to make seven fifths] One, two, three, four, five, six, mm, six, seventh. But this is how big it is.
> I: That's how big it is?
> Bob: Yeah.
> I: That's how long? [student accidently repeats an additional one-fifth] Whoops.
> Bob: Uh… one, two, three – eight-fifths. Eight-fifths, whoa.
> I: Yeah, a question for you. See this eight-fifths? How many times bigger is that than the two-fifths?
> Bob: Mm… three.
> I: Three times bigger?
> Bob: Because ... these two come together [points to two one-fifth sized parts]. So, these two come together, and these two come together, and these two come together [points to third and fourth, then fifth and sixth, then seventh and eighth one-fifth parts]. And that; two, four, six – and then those two.

In this excerpt, Bob interprets two-fifths as two pieces out of five pieces, with each piece representing one-fifth. This thinking promotes him to pull out one fifth from the two-fifths displayed; he then repeats the one-fifth he removed from two-fifths four times to make five fifths. He pauses, then repeats the part two more times to create seven fifths. We interpreted Bob's reasoning as additive; he adds one more to achieve seven one fifths. Bob's justification of the relationship of eight fifths to two fifths was *three more times,* or three extra pairs of two, as opposed to four times as large. Furthermore, we argue that, for Bob, the parts are not fifths, but *ones* that are part of a whole comprised of five units. Evidence for this claim rests in his consistent numbering and counting of the parts as whole numbers and only naming the parts with fractional language when prompted.

"Six Times Six is 12"

Because of his persistent additive reasoning to this point, in Session 11 Bob plays "Please Go and Bring for Me" (Tzur et al., 2013), a game designed to promote multiplicative units coordinating with whole numbers. Below, Bob considers how to coordinate five iterations of a unit of six.

<u>Session 11: Make five towers of six</u>
I: Okay. Five towers of six. Okay. So, um, how many towers did you bring?
Bob: Um, five towers of six. [brings towers back to table (2 purple, 1 green, 2 pink)]
I: Okay, so five towers, and how many in each tower?
Bob: Six.
I: So how many cubes did you bring in all?
Bob: Six [slides over 1 tower], 12 [slides over 1 tower], 24 [slides over 2 towers], 24…, 25, 26, 25, 26, 27, 28, 29, 30 [counts on fingers], 30.
I: Thirty cubes. Can you tell me how you figured that out?
Bob: I just showed – so, six times six is 12, and then 12 plus 12 is 24, and here and I already have a six, which is 30 [moves towers as he talks about them].

Bob conceived of five towers of six as two, two, and one tower with visible cubes. Interestingly, he builds the towers using three different colors of cubes, suggesting that the groups of two created in his towers belong together. This seems to support his counting of the cubes as a "double" (i.e., "Six, 12, 24"). Bob seemed to conceptualize doubling as *adding the same group*, supporting his breaking apart of the composite five into two, two, and one. Thus, he could access a two level structure additively, utilizing his way of doubling for a few iterations. His way of doubling breaks down with larger numbers. Other sessions continued to show sustained use of additive reasoning to solve problems.

Later in this session, the researcher-teacher confirmed understanding of how Bob utilized doubling in both whole number and fractional situations. For this task, Bob again used the computer program JavaBars. Below, Bob considers how to coordinate fractional units beyond a whole by employing his conception of doubling.

<u>Session 11: This ribbon is one-eighth of a yard long. What does twice as much ribbon look like?</u>
I: I want you to pretend this is a piece of ribbon, and you own a store where you sell ribbon in pieces. And the pieces that you sell are one-eighth of a yard long.
Bob: Okay.
I: I'm thinking about two times as much as that.
Bob: [repeats the bar]
I: Okay. What's the name that you give that?
Bob: One-eighth. I mean two-eighths.
I: Two-eighths. How do you know it's two-eighths?
Bob: 'Cause it's two out of eight, and I had just added one, so it's two out of eight.
I: Now, I'm thinking about an, an amount that's two times as much as this [points to Bob's two-eighths].
Bob: Okay.
I: What would that look like?
Bob: That big. [repeats one-eighth piece twice; adds on to two-eighths]

I: Okay. That big. Um, how much of a whole yard is that?
Bob: Four fourths – I mean, four-eighths.
I: Four fourths, four-eighths, which one?
Bob: I think four-eighths.
I: Four-eighths? How do you know it's four-eighths of a yard?
Bob: Um, 'cause it's four pieces out of eight.
I: Four pieces out of eight?
Bob: Uh-huh.
I: Okay, um, tell me more about that. How do you know it's four pieces out of eight?
Bob: 'Cause there's four, and there's supposed to be eight.
I: How do you know there's supposed to be – ?
Bob: 'Cause that's how big a yard is, and I know, and I'm the manager.

Previous tasks had shown Bob using his reasoning about doubling, but in this task, he explicates his notion of "double" as "add one more group." Furthermore, he displayed his persistent part whole concept of units (e.g., four pieces out of eight; two parts out of eight parts). Bob's activity confirmed our hypothesis that his additive view of whole numbers and fractions was a salient and persistent way of reasoning.

"Then it Would Be a Ninth"

In the post assessment, Bob shows the limitations of his whole number, additive reasoning. He was able to split three-eighths to create a one-eighth but continued to name fractional units as whole number units. The limitations of utilizing the whole number reasoning and equal groups reasoning becomes evident in Bob's conceptualization of one whole. When asked to create one and one-eighth (as opposed to an improper fraction name such as nine-eighths), Bob loses the unit structure.

<u>Post Assessment: This is three-eighths. Make one and one-eighth.</u>
I: This is three-eighths. [gives new piece]
Bob: Three. This is three.
I: That's three-eights.
Bob: Three-eighths. It's three-eighths.
I: What are you doing there?
Bob: Making three parts [folds a strip of paper].
I: You mean three in there? Okay. And then you're gonna use one of them?
Bob: So then, then it's one-eighth now. [has piece folded into thirds] One.
Bob: Two. Three. Four. Five. Six. Seven. Eight [traces eight of the one-eighths in a line].
I: I want you to make one and one-eighth.
Bob: One and one-eighth [traces one more of the one-eighths separately from the line of eight].
I: Oh. So this is one? [runs finger along line of eight] And that's one-eighth? [points to one piece]
Bob: Mm-hmm.
I: Okay. Would it be the same thing if you had made this one-eighth down here? [points to end of line of eight pieces] Would it be the same or different?
Bob: No. Different.
I: That'd be different.

Bob: Then it would be a ninth.
I: Would it be a ninth? How can it be a ninth?
Bob: Because it would at one. [pause] No and the nine it wouldn't. Or it wouldn't work. 'Cause, 'cause this is one entire length [points at line].
I: Can you circle the one whole?
Bob: There [circles one-eighth].
I: Can you circle the one whole?
Bob: This one? [circles the line of eight one-eighths]
I: Where's the one whole that you made? So if I tacked on another eighth ... I couldn't do that? [motions to one-eighth and puts it at end of line again]
Bob: Nope. You'd have to make these smaller.

Discussion

Throughout the sessions, researchers worked to reorganize Bob's whole number unit coordination to support understanding of fractional units. We hypothesized that Bob would use his whole number units coordination to consider fractional units as multiplicative quantities. To that end, we designed various situations to elicit multiplicative reasoning, both in whole number and in fractions. Two key findings emerged from the analysis related to Bob's reasoning.

First, Bob did utilize his whole number reasoning across situations, whether the situations elicited fractions or not. Yet, because his reasoning was largely additive in nature, this placed limitations on how Bob conceived of fractions as numbers. Namely, Bob's pervasive use of a part whole notions of fractions limited his understanding of fractions as quantities. Second, we saw little to no evidence of Bob's cognitive difference (i.e., working memory) apparent across the sessions. For instance, Bob did not seem to have a need to re-run his activity or re-verbalize his thought processes in the midst of learning. For Bob, the single disabling factor in his reasoning seemed to rest in his pervasive part whole notion of fractions and overreliance on additive reasoning.

Implications from the work include the need to provide students with LDs rich experiences to develop multiplicative reasoning in both whole number and fractions, including tasks that utilize a three level structure in early learning experiences. This student with LDs seemed to "need" more experiences developing multiplicative conceptions as opposed directive forms of instruction that emphasized facts and procedures.

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Equity Statement

Our research is based in the notion that students with disabilities can and do advance their notions of multiplicative reasoning in whole number and fractions. Addressing conceptual advancement for this underserved population addressed equity because historically these students have been denied access to opportunities to grow their conceptual knowledge of mathematics, limiting school and life outcomes.

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