USING NUMBER SEQUENCES TO MODEL MIDDLE-GRADES STUDENTS' ALGEBRIC REPRESENTATIONS OF MULTIPLICATIVE RELATIONSHIPS

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The number sequences describe a hierarchy of students' concepts of number. This research uses two defining cognitive structures of the number sequences—units coordination and the splitting operation—to model middle-grades students' abilities to write linear equations representing the multiplicative relationship between two unknowns. Results indicate that students who have constructed a tacitly nested number sequence (TNS), second in the hierarchy, do not represent multiplicative relationships algebraically. Students who have constructed an advanced tacitly nested number sequence (aTNS), third in the hierarchy, do so inconsistently, and students who have constructed an explicitly nested number sequence (ENS), fourth in the hierarchy, do so consistently. Coordinating three levels of units in activity advantages aTNS and ENS students, while splitting further advantages ENS students' algebraic reasoning.

Keywords: Algebra and Algebraic Thinking; Number Concepts and Operations

Literature Review

Algebra is persistently characterized as difficult for students to master (e.g., Kieran, 2007), and is a staple in school mathematics. Preceding a formal algebra course, algebra can be conceptualized as an extension of arithmetic. Russell, Schifter, and Bastable (2011), for example, conclude that arithmetic ideas such as understanding operations, generalizing and justifying, extending the number system, and using symbolic notation contribute to early algebraic reasoning. With these ideas in mind, algebraic reasoning is defined by Hackenberg (2013) as "generalizing and abstracting arithmetical and quantitative relationships, and systematically representing those generalizations in some way… [and] learning to reason with algebraic notation in lieu of quantities" (pp. 541–542).

In addition to defining algebraic reasoning, it is important to consider what concepts support algebraic reasoning. Students' concept of the equal sign is one such concept (Carpenter, Franke, & Levi, 2003; Matthews, Rittle-Johnson, McEldoon, & Taylor, 2012). Students can have two distinct notions of equality (Kieran, 1981): relational and operational. A relational concept indicates an understanding of the equal sign as indicating a balance, or equality of expressions on both sides of the equation. This is the more sophisticated of the two notions, but many children conceive of the equal sign operationally (Baroody & Ginsberg, 1983), indicating a conception of the equal sign as an operator. This is detrimental to students' ability to solve equations (Carpenter et al., 2003), and many mistakes in high school mathematics can be attributed to incorrect uses of the equal sign (Kieran, 1981). Furthermore, a relational concept of the equal sign compliments, rather than replaces, an operational concept (Matthews et al., 2012), implying that students who have constructed a relational concept may also apply an operational concept.

The present research study will examine the ability of middle-grades students to write linear equations representing the multiplicative relationship between two unknowns. Students' concept of the equal sign will be considered as it supports or limits their algebraic reasoning. Finally,

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cognitive structures that define students' number sequences (Steffe & Cobb, 1988) will be used to model students' algebraic reasoning.

Theoretical Framework

The number sequences (Steffe & Cobb, 1988) were originally developed in research with elementary students, and described a hierarchy of four concepts of number: the initial number sequence (INS), the tacitly nested number sequence (TNS), the explicitly nested number sequence (ENS), and the generalized number sequence (GNS). Ulrich (2016b) subsequently identified a student who had constructed an advanced tacitly nested number sequence (aTNS), and Ulrich and Wilkins (2017) found 36% of sixth-grade students to be operating with only an aTNS. The aTNS falls in the number sequence trajectory between the TNS and the ENS (Ulrich, 2016b), making the trajectory of number sequences for middle-grades students INS, TNS, aTNS, ENS, GNS. The present research will focus on the algebraic reasoning of middle-grades students who have constructed an aTNS, and how their algebraic reasoning compares to that of students who have constructed a TNS and an ENS.

Units Coordination

The number sequences are defined by cognitive structures, such as units coordination and construction (Ulrich, 2015, 2016a). Figure 1 is a visual representation of the levels of units students can coordinate, organized by number sequence. Students with a TNS assimilate tasks with one level of units and can coordinate or construct a second level of units in activity (Figure 1, row 1). Thus, TNS students can immediately conceptualize a number such as seven as seven individual units of one. In mental activity, TNS students can chunk those seven individual units into a single composite unit containing seven individual units (Ulrich, 2015). A composite unit provides economy in reasoning because students must only maintain one unit of seven, rather than seven units of one. However, because the composite unit is constructed in activity, it decays following mental activity (Ulrich, 2015), leaving only seven units of one (i.e., the assimilatory structure) available for reflection.

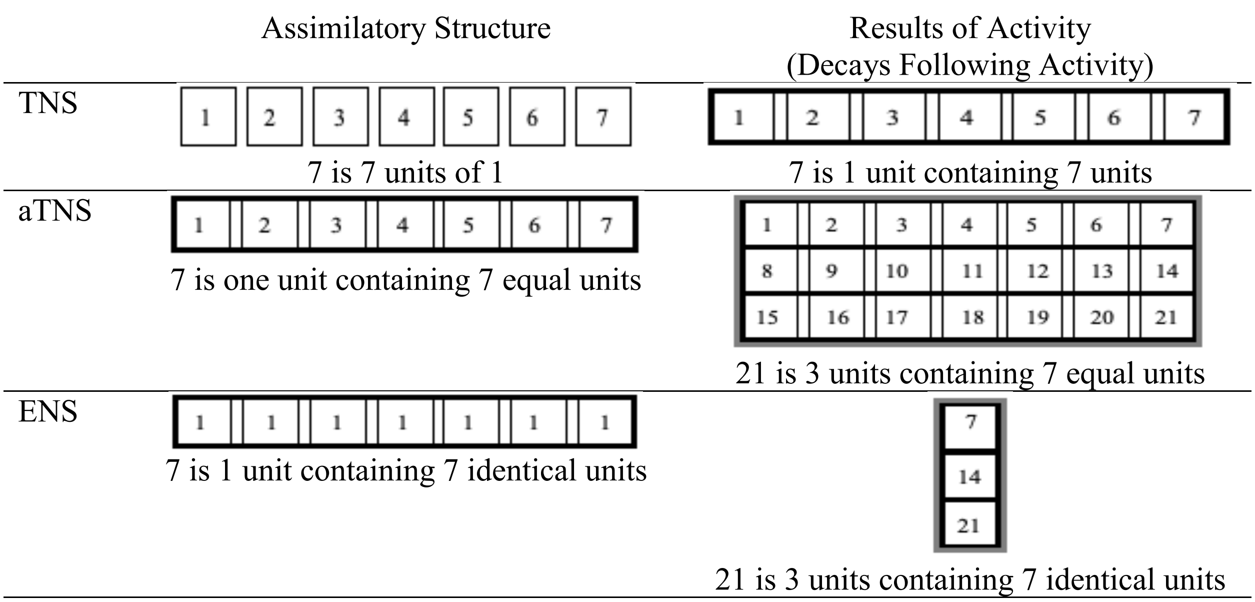

Figure 1: Depictions of 1-, 2-, and 3-levels of Units, Organized by Number Sequence

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Students who have constructed the next number sequence, an aTNS, assimilate with two levels of units, or a composite unit, and construct a third level of units in activity (Ulrich, 2016b; Figure 1, row 2). aTNS students can conceptualize seven as a single composite unit containing seven units, and can operate on a composite unit to conceptualize a three-level unit structure in activity. This allows aTNS students to conceptualize 21, for instance, as three groups of seven. But, following activity, the third-level of units decays leaving aTNS students to reflect only on a composite unit.

Similar to aTNS students, ENS students assimilate tasks with a composite unit and construct a third-level of units in activity (Ulrich, 2016a; Figure 1, row 3). However, ENS students have also constructed an iterable unit of one and a disembedding operation (Steffe, 2010a; Ulrich, 2016a). These operations advantage their reasoning over aTNS students (Ulrich, 2016b) by supporting a conception of seven individual units of one as identical rather than equal (Steffe, 2010a). ENS students can reflect on the relationships between the units contained within the composite and the composite as a whole, without destroying either quantity (Steffe, 2010a). These operations support multiplicative reasoning because ENS students conceive of the composite unit as seven times the size of each individual unit (Ulrich, 2016a).

Splitting

A splitting operation can also be understood in terms of students' number sequences. The splitting operation, defined as simultaneously partitioning and iterating (Steffe, 2010b), is within the zone of potential construction (Norton & D'Ambrosio, 2008) of students who have constructed an ENS. That is to say, ENS students can learn to split; this is not true for TNS students (Ulrich, 2016b). aTNS students, on the other hand, have been found to *solve* splitting tasks, although Ulrich (2016b) concludes that they do so by *sequentially*, rather than simultaneously, partitioning and iterating. Accordingly, this research will examine how solving splitting tasks by sequentially partitioning and iterating advantages the algebraic reasoning of aTNS students over that of TNS students, and how the simultaneity of splitting advantages the algebraic reasoning of ENS students over aTNS students.

Algebraic Reasoning

Hackenberg et al. (Hackenberg, 2013; Hackenberg, Jones, Eker, & Creager, 2017; Hackenberg & Lee, 2015) use the multiplicative concepts to model students' algebraic reasoning. The multiplicative concepts are based on students' levels of units coordination (Hackenberg & Tillema, 2009). Students who assimilated with three levels of units demonstrated "swift" equation writing when representing the multiplicative relationship between two unknowns (Hackenberg & Lee, 2015, p. 219). On the other hand, students who assimilated with two levels of units demonstrated "effortful" equation writing, and even with considerable effort, only four out of six of these students were able to write a correct algebraic representation of the multiplicative relationship between two unknowns (Hackenberg & Lee, 2015, p. 214). Only two out of six students who assimilated with one level of units represented the relationship between two unknowns algebraically (Hackenberg, 2013). Hackenberg et al. (2017) find that an unknown quantity constitutes a composite unit. This directly ties students' abilities to operate on composite units to their algebraic reasoning, and can be used to explain, at least in part, the difficulty students may have in representing the multiplicative relationship between two unknowns algebraically. The implications of these findings are that students must have constructed an assimilatory composite unit in order to operate on unknown quantities, and that students who assimilate with composite units may or may not represent the multiplicative relationship between two unknowns algebraically.

Existing research uses students' multiplicative concepts to model their algebraic reasoning. The present study expands on existing literature by using the cognitive structures that define students' number sequences to model their algebraic reasoning. The number sequence framework allows this research to distinguish between the algebraic reasoning of two groups of students – aTNS and ENS – both of whom assimilate with composite units, but only one of whom (ENS students) reason multiplicatively and split.

Methods

This report draws from the qualitative results of a larger, mixed methods study. In phase one, 326 students in grades six through nine were given a survey (Ulrich & Wilkins, 2017), the purpose of which was to attribute to each student a number sequence. In phase two, students were selected to participate in qualitative interviews based on the results of the survey. In total, 18 students were interviewed across two days, for approximately 45 minutes each day. Interviews were audio and video recorded for retrospective analysis, and students' written work was collected. The interviews included confirmation of students' number sequence attribution from the survey, characterization of their concept of the equal sign as either operational or relational, and characterization of their algebraic reasoning on 11 tasks.

Results are from the responses of 16 participants who had constructed either a TNS, an aTNS, or an ENS (Table 1). Students were assigned pseudonyms that begin with the same letter as their number sequence attribution (e.g., Ann has constructed an aTNS). The focus of the analysis is on one of the algebra problems, the phone cords problem. Students were told: Steven's cord is five times the length of Rebecca's cord. Draw a picture to represent the two cord lengths, and use algebra to represent the relationship between the two cord lengths (Hackenberg, 2013). The first part of the problem, drawing a picture to represent the cord lengths, will be referred to as the splitting task. The splitting task was considered correct if students drew a picture in which one cord length was about five times the length of the other, and the student explained that Steven's cord was the longer. The algebraic portion of the problem was considered correct if students wrote an equation equivalent to $y = 5x$ and explained that y represented the length of Steven's cord and x represented the length of Rebecca's cord.

Table 1: Participants by Grade and Number Sequence

	6^{th} Grade	7^{th} Grade	8^{th} Grade	9^{th} Grade
TNS	Tabitha	--	--	Travis
aTNS	Ann	Ava	Amanda	Alex
	Aaron	Alyssa		
	Abby	Andy		
ENS	Elle	--	Erin	Elizabeth
	Evan		Emily	Emma

Results

Concept of the Equal Sign

Tabitha and Ann were determined to only reason with a relational concept of the equal sign. On a question that asked if the equation $37 + 29 - 5 = 48 + 14$ was true or false, they both responded false. Although their response was correct, their reasoning indicated that they conceive of the equal sign as an operator rather than a relation. For example, Tabitha said, "I added 37 plus 29 and I got 66. So I subtracted 66 by 5 and I got 61. And on here, it said that 37

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plus 29 minus 5 equals 48 but actually it equals 61." Ann's response was similar. On the same question, all other students explained that the equation as false by calculating the value of each expression and indicating that they were not equal ($61 \neq 62$). This indicates a relational concept of the equal sign. Therefore, only an operational concept of the equal sign was attributed to Tabitha and Ann. A relational concept was attributed to all other students.

The Splitting Task

Neither TNS student solved the splitting task within the phone cords problem. Tabitha drew a seemingly arbitrary length to represent Steven's cord. She said, "Maybe this long?" but expressed no comprehension of a relationship between the lengths nor did the lengths have a 1:5 relationship. Travis, on the other hand, stated one correct numerical example for Steven's and Rebecca's cords, but did not generate any other correct numerical examples and did not generate a picture of the cords even at the interviewer's request. This is not surprising, given that TNS students have not been shown in previous research to split (Steffe, 2010b). Four aTNS students out of eight solved the splitting task, presumably by sequentially partitioning and iterating (Ulrich, 2016b). All ENS students solved the splitting task, which is again, not surprising, given that splitting is within the ZPC of ENS students (Steffe, 2010b).

The Phone Cords Problem

Neither TNS student solved the splitting task or represented the problem algebraically (Table 2). Both TNS students were prompted to use a numerical example to try to make sense of the relationship. Travis determined that if Rebecca's cord was five feet Steven's would be 25, but that was the extent of his progress on the problem. Tabitha did not generate multiplicatively related cord lengths. She indicated that if Rebecca's cord length was two then Steven's would be seven because, "I went up from two, and then I went up to five, and I counted. Like, 2; 3, 4, 5, 6, 7." When counting from two to seven, Tabitha raised two fingers on her left hand, and then raised one additional finger on her right hand each time that she uttered a number word. On the phone cords problem, Tabitha inappropriately applied additive reasoning to a numerical example of a multiplicative relationship.

Table 2: Number of Students by Number Sequence Who Solved the Splitting Task and Represented the Phone Cords Problem Algebraically

	Solution to the Splitting Task		
Algebraic Representation	Correct	Incorrect	Total
Correct	**7 Total**	**2 Total**	**9 Total**
	0 TNS	0 TNS	0 TNS
	1 aTNS	2 aTNS	3 aTNS
	6 ENS	0 ENS	6 ENS
Incorrect	**3 Total**	**4 Total**	**7 Total**
	0 TNS	2 TNS	2 TNS
	3 aTNS	2 aTNS	5 aTNS
	0 ENS	0 ENS	0 ENS
Total	**10 Total**	**6 Total**	**16**

Results of the ENS students will be presented next so that aTNS students' responses can be situated between those of TNS and ENS students. All six ENS students represented the phone cords problem algebraically by writing an equation equivalent to $y = 5x$, and explained the meaning of each variable (Table 2). Five of the six ENS students first wrote an incorrect equation, but used numerical examples to generate a correct equation. Elizabeth, for instance, first wrote two expressions: $5n$ and $n + 4$. She reasoned that if Rebecca's cord was three feet long then "his [Steven's] consists of, like, five little sections of that, then his would be fifteen feet long. So then five times three does equal 15… yeah, it [$5n$] works." In this excerpt, Elizabeth built on the numerical example $5 \cdot 3 = 15$ to determine the correct expression, and subsequently generated a correct equation $5n = S$.

Erin used a numerical example to decide between an equation ($5R = x$) and an inequality ($x + 5 > R$), despite being specifically asked to write an equation. She explained the inequality saying that "we don't know exactly how long they [the cords] are," so she used an inequality to indicate that Steven's (x) was longer. It was not until Erin worked through a numerical example that she determined the equation $5R = x$ was correct. Comparably, Emily used a numerical example to correct a reversal in her equation, and Evan began with a numerical example and then substituted variables for Steven's and Rebecca's cord lengths into the structure of that example.

aTNS students' solutions to the phone cords problem were inconsistent. Three aTNS students algebraically represented the phone cords problem, and there were two commonalities among these solutions. First, the successful aTNS students all began with numerical examples and substituted variables into the structures of those examples. Second, they all reverted to an operational concept of the equal sign, despite having demonstrated a relational concept earlier in the interview. Both behaviors can be observed in Alyssa's work. Alyssa initially wrote $x5 - y$.

> Alyssa: So Steven's would be *x* and Rebecca's would be *y*. And Steven's is 5 times as long as hers so you do this [multiply *x* by 5] and subtract hers [*y*] and you get the answer, I guess.
> Interviewer: OK. And what would the answer be?
> Alyssa: I don't know because there's no numbers. It just has 5.
> Interviewer: That's OK, but what would it, what would the answer represent? …
> Alyssa: The amount of cord that was timesed onto it. …
> Interviewer: So what if we say, for example, that Rebecca's cord is 3 feet long. …
> Alyssa: His would be 15, cause hers, his is 5 times as long as hers, so 3 times 5 is 15.
> Interviewer: Oh, very good. So if Rebecca's is 3 feet, Steven's is 15 feet. Does that work in our equation? …
> Alyssa: I don't think it would work with that equation [$x5 - y$] because you can't get these two [3 and 5] times each other. So it would probably be *x* equals 5 times 3. And then you would get 15 and that'd be his. … Yeah. *x* equals five times *y*.

It was not until Alyssa considered a numerical example that she rewrote her equation. Prior to that, she tried to multiply five by the length of Steven's cord, which she represented as $x5$ because the problem states that his cord is five times the length of Rebecca's. After determining numerically that Rebecca's cord should be multiplied by five, she was able to generate the equation "*x* equals five times three" and then "*x* equals five times *y*." The use of numerical examples was common among the three successful aTNS students.

The second behavior that can be observed in this excerpt is that Alyssa reasoned with an operational concept of the equal sign when she indicated that the result of $x5 - y$ is "the answer, I guess," and explained that you do not know the answer because *x* and *y* are not assigned numerical values. Alyssa was focused on finding a numerical result, and not on the equality of the expressions. Of the three successful aTNS students, all three reverted to an operational concept of the equal sign, as did Alyssa, at some point in their reasoning.

Interestingly, in Table 2 all TNS and ENS students are represented on the main diagonal—they either solved both portions of the phone cords problem (ENS students) or they did not (TNS students). The reasoning of aTNS students was less consistent. Three aTNS students represented the problem algebraically, and five did not. Of the three who correctly represented the problem algebraically, only one also solved the splitting task. Conversely, of the five aTNS students who did not represent the problem algebraically, three solved the splitting task. The five aTNS students who solved only one portion of the task are on the off-diagonal of Table 2; all students on the off-diagonal are aTNS students. Thus, despite both aTNS and ENS students assimilating with composite units, aTNS students were less successful and the results of the splitting task were disconnected from the algebraic representation.

Discussion

TNS students made very limited progress on the phone cords problem. Tabitha applied additive reasoning, Travis only generated one multiplicative numerical example, and neither student represented the relationship algebraically. All ENS students correctly represented the relationship algebraically. These results are comparable to those of Hackenberg et al., who found that four out of six students who assimilate with composite units (Hackenberg & Lee, 2015) and two out of six students who assimilate with one level of units (Hackenberg, 2013) represented the multiplicative relationship between two unknowns algebraically.

Also, five of the six ENS students interviewed used numerical examples to facilitate their equation writing. This is also consistent with Hackenberg et al.'s (2017) result that students who assimilate with composite units may build this type of equation using numerical examples. Therefore, an unknown quantity is assimilated by ENS students with a composite unit, in which Rebecca's cord length, for example, is a quantity containing an unknown number of units of one. This allows ENS students to operate on the unknown (i.e., the composite unit) in activity to form a three-level unit structure constituting the relationship between Steven's and Rebecca's cords.

Erin was the only ENS student who wrote an inequality to represent the relationship between the cords. This is similar to a student Hackenberg et al. (2017) identified, Tim, who like Erin, could assimilate with composite units. On a problem that is parallel to the phone cords problem, Tim insisted that a five times multiplicative relationship between two unknowns was "approximate" because "we don't know it [the exact lengths]" (Hackenberg et al., 2017, p. 46). Like Tim, Erin expressed a conception that while Steven's cord was longer than Rebecca's, the exact multiplicative relationship was unknown as long as the lengths were unknown. Consistent with Hackenberg et al.'s (2017) analysis, the use of an inequality to represent a multiplicative relationship between two unknowns can be explained as the manifestation of Erin's need to reduce the complexity of the unit structure on which she was operating.

aTNS students did not consistently solve the splitting task or the algebraic portion of the phone cords problem. Three aTNS students represented the problem algebraically, only one of whom solved the splitting task. Three aTNS students solved the splitting task but did not represent the problem algebraically. These results are evidence that although aTNS students may solve splitting tasks by sequentially partitioning and iterating, the sequential nature of that activity is not sufficient to support their algebraic representation of the resulting relationships. The sequential partitioning and iterating behavior of aTNS students is qualitatively distinct from

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the splitting in which ENS students are likely to engage. This provided ENS students with an advantage in their algebraic reasoning.

Both aTNS and ENS students used numerical examples to build an algebraic equation on the phone cords problem. However, ENS students tended to only work through one numerical example prior to writing a correct equation, while aTNS students tended to work through several. Also, whereas building an equation using numerical examples was a productive behavior for all five of the ENS students who took that approach, it was only productive for three out of eight aTNS students. Furthermore, all three aTNS students who represented the phone cords problem algebraically reverted to an operational concept of the equal sign at some point during their reasoning, despite demonstrating the ability to reason with a relational concept earlier in the interview. In contrast, no ENS students demonstrated reasoning consistent with an operational concept of the equal sign on the phone cords problem. Hackenberg and Lee (2015) conclude that students must assimilate with three levels of units, as would GNS students, before they are likely to represent multiplicative relationships between two unknowns without relying on numerical examples. This explains why all successful aTNS and ENS students, who only assimilate with composite units, relied on numerical examples to some extent. Additionally, this demonstrates a range of ways in which two groups of students who assimilate with composite units may rely on numerical examples to support their algebraic reasoning.

Conclusions

While both aTNS and ENS students assimilate with and operate on composite units, there are qualitative differences in their algebraic representations of multiplicative relationships between two unknowns. Consistent with the analysis of Hackenberg et al. (2017), both aTNS and ENS students are likely to have assimilated the phone cords problem with a composite unit consisting of Rebecca's cord, which contains an unknown number of units of one. Then, in activity, both aTNS and ENS students are assumed to operate on said composite unit to generate the relationship between Rebecca's and Steven's cord lengths; this facilitates their equation writing. Following activity, however, one level of units decays leaving both aTNS and ENS students to reflect only upon Rebecca's cord as a unit containing an unknown number of units of one.

For the three aTNS students who wrote the correct equation, the decay of Steven's cord from the unit structure resulted in their applying an operational concept of the equal sign. In other words, because the relationship between Steven's and Rebecca's cords was lost, the variable that had previously been representative of Steven's cord length was now simply "the answer." ENS students have the same limitation following activity; the third level of units decays and the relationship between the cord lengths is lost. In contrast to aTNS students, however, ENS students did not reason operationally about the equal sign. This is because ENS students are advantaged by the ability to reflect on the results of the splitting operation.

For ENS students, the results of splitting (i.e., Steven's cord being five times the length of Rebecca's) is maintained following activity. Therefore, despite the decay of the third level of units, the splitting operation allowed ENS students to maintain the relationship between Steven's and Rebecca's cords and reason normatively about their equation. In constrast, aTNS students presumably solve splitting tasks by sequentially partitioning and iterating, if they solve them at all, making the results of the splitting task unavailable for reflection following activity. As such, when the third level of units decayed following aTNS students' equation writing, they must recreate the results of the splitting task if they are to interpret the equation normatively.

Reverting to an operational concept of the equal sign has been noted in previous literature (Matthews et al., 2012), and this research provides a theoretical rationale for such a regression in the situation of writing an equation to represent the multiplicative relationship between two unknowns. aTNS students reverted to an operational concept of the equal sign to compensate for the inability to reflect on the results of the splitting task when the third level of units decayed. Therefore, although both aTNS and ENS students assimilate with and operate on composite units, the splitting operation advantaged ENS students' ability to represent the multiplicative relationships between two unknowns algebraically.

References

Baroody, A. J., & Ginsberg, H. P. (1983). The effects of instruction on children's understanding of the "equals" sign. *Elementary School Journal, 84*, 199–212.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school.* Portsmouth, NH: Heinemann.

Common Core State Standards for Mathematics. (2010). *Common Core State Standards Initiative.* Retrieved from http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

Hackenberg, A. J. (2013). The fractional knowledge and algebraic reasoning of students with the first multiplicative concept. *The Journal of Mathematical Behavior, 32*, 538–563.

Hackenberg, A. J., Jones, R., Eker, A., & Creager, M. (2017). "Approximate" multiplicative relationships between quantitative unknowns. *Journal of Mathematical Behavior, 48*, 38–61.

Hackenberg, A. J., & Lee, M. Y. (2015). Relationships between students' fractional knowledge and equation writing. *Journal for Research in Mathematics Education, 46*(2), 196–243.

Hackenberg, A. J. & Tillema, E. S. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fraction composition schemes. *The Journal of Mathematical Behavior, 28*, 1–18.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*, 317–326.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. K. Lester Jr. (Eds.), *Second handbook of research on mathematics teaching and learning* (pp. 707–762). Charlotte, NC: New Age Publishing; Reston, VA: National Council of Teachers of Mathematics.

Matthews, P., Rittle-Johnson, B., McEldoon, K., & Taylor, R. (2012). Measure for measure: What combining diverse measures reveals about children's understanding of the equal sign as an indicator of mathematical equality. *Journal for Research in Mathematics Education, 43*, 316–350.

Norton, A., & D'Ambrosio, B. S. (2008). ZPC and ZPD: Zones of teaching and learning. *Journal for Research in Mathematics Education, 39*(3), 220–246.

Russell, S. J., Schifter, D., & Bastable, V. (2011). Developing algebraic thinking in the context of arithmetic. In J. Cai & E. Knuth (Eds.) *Early algebraization: A global dialogue from multiple perspectives* (pp. 43–70). New York, NY: Springer.

Steffe, L. P. (2010a). Operations that produce numerical counting schemes. In L. P. Steffe & J. Olive (Eds.), *Children's fractional knowledge* (pp. 27–48). New York, NY: Springer.

Steffe, L. P. (2010b). The partitive and part-whole schemes. In L. P. Steffe & J. Olive (Eds.), *Children's fractional knowledge* (pp. 75–122). New York, NY: Springer.

Steffe, L. P., & Cobb, P. (1988). *Construction of arithmetical meanings and strategies.* New York, NY: Springer.

Ulrich, C. (2015). Stages in constructing and coordinating units additively and multiplicatively (part 1). *For the Learning of Mathematics, 35*(3), 2–7.

Ulrich, C. (2016a). Stages in constructing and coordinating units additively and multiplicatively (part 2). *For the Learning of Mathematics, 36*(1), 34–39.

Ulrich, C. (2016b). The tacitly nested number sequence in sixth grade: The case of Adam. *The Journal of Mathematical Behavior, 43*, 1–19.

Ulrich, C., & Wilkins, J. L. M. (2017). Using written work to investigate stages in sixth-grade students' construction and coordination of units. *International Journal of STEM Education, 23*(4). Retrieved from https://stemeducationjournal.springeropen.com/articles/10.1186/s40594-017-0085-0