COVARIATIONAL REASONING SUPPORTING PRESERVICE TEACHERS' MATHEMATIZATION OF AN ENERGY BALANCE MODEL FOR GLOBAL WARMING

Dario A. Gonzalez
Center for Mathematical Modeling, Universidad de Chile
dario.gonzalez@fulbrightmail.org

I discuss how three preservice mathematics teachers' (PSTs') covariational reasoning supported the mathematization of a simple energy balance model (EBM) for global warming, and how such mathematization shaped PSTs' understanding of the link CO2 pollution and global warming. I use Thompson & Carlson's (2017) levels of covariational reasoning and Thompson, Carlson, Byerley, and Hatfield (2014) descriptions of understanding and meaning to inform the discussion of results. The PST completed the EBM Task during an individual, task-based interview. The analysis revealed that Chunky Continuous Covariation level supports the mathematization of the EBM in terms of a covariation's rapidity of change. The analysis also revealed that particular mathematizations resulted in particular meanings for radiative equilibrium, which in turn have implications for understanding the link between CO2 pollution and global warming.

Keywords: Modeling, Advanced Mathematical Thinking, and Teacher Knowledge

Introduction and Purpose

Global warming refers to an increase in the mean global surface temperature caused by human emissions of greenhouse gases. The planetary scale of this phenomenon makes it difficult for a single person to experience it in its entirety. Mathematical modeling can make global warming visible for people (Barwell & Suurtamm, 2011; Barwell, 2013a, 2013b; Gonzalez, 2016, 2018; Mackenzie, 2007), thus helping them understand it and take action against this new horizon. I consider preservice mathematics teachers an important group to be informed about global warming because they will educate the members and future leaders of this democratic society. Thus, there is a need for studies examining how preservice mathematics teachers can learn the mathematics behind global warming.

Lambert and Bleicher (2013) have found that there are two key concepts from climate sciences that preservice science teachers need to learn about in order to understand global warming: (a) the Earth's energy balance, and (b) the link between carbon dioxide (CO2) pollution and global warming. Extending this premise to mathematics education, preservice mathematics teachers (PSTs) can model global warming by reasoning about these two concepts as dynamic situations involving covariation between quantities. In this paper, I discuss how three PSTs' covariational reasoning supported the mathematization of a simple energy balance model (EBM) for global warming, and how such mathematization shaped PSTs' understanding of the link CO2 pollution and global warming.

Conceptual Framework

Covariational reasoning refers to "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002, p. 354). Thompson and Carlson (2017) have identified six distinctive levels of covariational reasoning (Table 1). Thompson and Carson suggested that "a researcher could use [the levels] to describe a class of behaviors, or she could use it as a characteristic of a person's capacity to reason … covariationally." For the current study, I used Thompson and Carlson's levels as a framework to characterize PSTs' covariational reasoning as they mathematize the EBM. Mathematization refers to the process of translating a real-life, non-mathematical situation into a mathematical problem, and then using mathematical tools and processes to solve it (Freudenthal, 1991).

I also made use of Thompson and colleagues' (Thompson, Carlson, Byerley, & Hatfield, 2014) descriptions of understanding, meaning, and way of thinking, in order to characterize PSTs' understandings of the EBM and meanings for radiative equilibrium. Thompson et al. suggests that a person's understanding is an in-the-moment state of equilibrium resulting from assimilating sensory information to the person's current schemes, or from accommodating those schemes to assimilate the new information. A person's meaning, then, is the space of implications that emerges from the assimilation to or accommodation of the person's current schemes. A person's way of thinking is that person's "pattern of utilizing specific meanings or ways of thinking in reasoning about particular situations" (Thompson et al., 2014, p. 12).

Table 1: Major Levels of Covariational Reasoning

Level	Description
Smooth continuous covariation	The person envisions increases or decreases (hereafter, changes) in one quantity's or variable's value (hereafter, variable) as happening simultaneously with changes in another variable's value, and the person envisions both variables varying smoothly and continuously.
Chunky continuous covariation	The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables changing by intervals of a fixed size (not necessarily of the same size). The person imagines, for example, the variable's value varying from 0 to 1, from 1 to 2, from 2 to 3 (and so on), like laying a ruler. Values between 0 and 1, between 1 and 2, between 2 and 3, and so on, "come along" by virtue of each being part of a chunk—like numbers on a ruler—but the person does not envision that the quantity has these values in the same way it has 0, 1, 2, and so on, as values.
Coordination of values	The person coordinates the values of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y).
Gross coordination of values	The person forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases." The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, nonmultiplicative link between the overall changes in two quantities' values.
Preccordination of values	The person envisions two variables' values varying, but asynchronously—one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects.
No coordination	The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values.

Methodology

This paper is part of a larger study that investigated how PSTs make sense of introductory mathematical models for global warming. Three secondary PSTs —hereafter Jodi, Pam, and Kris— enrolled in a mathematics education program at a large Southeastern university participated in the larger study. These PSTs had completed Calculus I and II and an Intro to Higher Mathematics course and were completing a Math Modeling for Teachers course by the time the larger study took place. The PSTs were asked to complete an original sequence of mathematical tasks while participating in individual, task-based interviews (Goldin, 2000). In this paper, I focus on the PSTs' responses to the EBM task.

The Energy Balance Model (EBM) Task

An *energy balance model* (EBM) describes the continuous heat exchange between the sun, the planet's surface, and the atmosphere (Figure 1a). The sun warms up the planet's surface at an approximately constant rate S. As the surface heats up, it radiates heat to the atmosphere (R), the majority of which (B) is absorbed by *greenhouse gases* (GHG). The atmosphere then re-radiates a fraction of the absorbed heat back to the surface (A), further increasing its temperature. The continuous heat exchange between the surface and the atmosphere is known as the *greenhouse effect*, which is responsible for enhancing the *planet's mean surface temperature* T(t). Changes in the greenhouse effect result in changes in T(t). Let $N(t) = [S + A(t)] - R(t)$ be the *net planetary energy imbalance*, then it is said that the energy balance is in radiative equilibrium when $N(t) = 0$, which implies that T(t) remains constant. There are *forcing agents* that can push the energy balance out of radiative equilibrium, resulting in $N(t) \neq 0$. My study focuses on modeling the impact that an increase in the atmospheric CO_2 concentration has over the Earth's energy balance, and how such impact affects the planet's mean surface temperature. CO_2 pollution is one of the main drivers of global warming (Intergovernmental Panel on Climate Change [IPCC], 2013).

The Energy Balance Model (EBM) Task (Figure 1b) describes a simplified scenario with a unique, instantaneous increase in the *Atmospheric CO_2 Concentration Function*, C(t), at time t = 0. This increase results in an initial positive heat imbalance $N(0) = [S + A(0)] - R(0) > 0$. This initial imbalance is known as *positive forcing by CO_2* and is denoted by $F = N(0)$. The positive forcing results in a surface absorbing heat at a higher rate than that at which it is releasing it. Thus, the surface warms up as time passes, which causes it to radiate heat at an increasing rate R(t). The atmosphere then absorbs more heat from the surface, which causes it to radiate heat back to the surface at an increasing rate A(t). This feedback process continues until radiative equilibrium is restored so that $N(t) \to 0$ and $T(t) \to T_{NE}$ as time increases, where T_{NE} represents a new and higher equilibrium temperature.

The EBM Task thus required PSTs to reasoning about the above process and draw the graphs of the functions N(t) and T(t). The EBM Task has two prompts: (a) Determine how N vary over time t (in years) and sketch the graph of N(t) and (b) Determine how T vary over time t (in years) and sketch the graph of T(t).

Data Collection

PSTs watched a 7-minute long video introducing the concept of EBM, followed by a Q&A session with me. I next gave them the EBM task. PSTs were also given a diagram of the EBM showing initial values for S, R, B, and A. They were expected to sketch the graphs of N(t) and T(t) assisted by the recursive rules $B_i = 0.794 \cdot R_i$, $A_i = \frac{1}{2} \cdot B_i$, $R_{i+1} = S + A_i$, and $N_i = [S + A_i] - R_i$ (for *i* = 0, 1, 2, …). The rules were meant to give PSTs a sense of how the heat flows change after a positive forcing. Each PST completed the task in a 60-minute, semi-structured, task-based interview (Goldin, 2000). The interview was video recorded and transcribed for analysis. PSTs' work on paper was also collected for analysis.

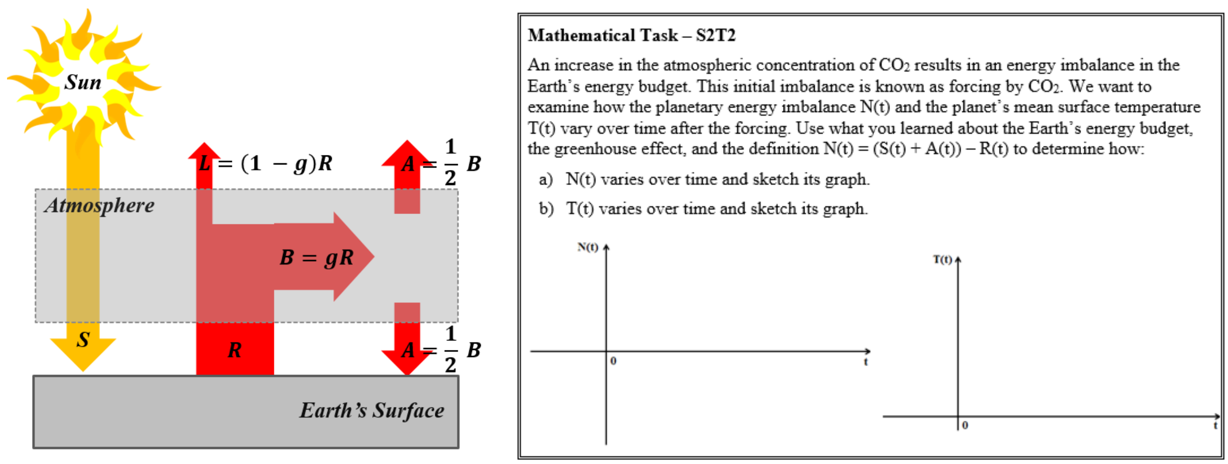

Figure 1: (a) The Earth's Energy Balance (Left) and (b) The EBM Task (Right)

Data Analysis

Videos and transcripts were analyzed through Framework Analysis (FA) method (Ward, Furber, Tierney, & Swallow, 2013). FA has five stages of data analysis: the familiarization with the data, the development of the analytic framework, indexing and pilot charting of the data, summarizing data into the analytic matrix, and the synthetization of the data by mapping and interpreting. Through these stages the research develops an analytic framework and uses it to analyze, reduce, and index data into analytic matrices, FA's distinctive feature.

I watched all interview videos and took notes while doing so. The videos were separated into shorter, more manageable *episodes*. An *episode* showed evidence of PSTs' ways of understanding the EBM or ways of reasoning about covariation. The notes informed my first round of coding. Then, the episodes were sorted according to particular ways of understanding the Earth's energy balance. Looking for patterns in participants' responses, I developed six *energy balance (EB) codes*. I repeated the process with PSTs' ways of envisioning covariation, which resulted in four *covariational reasoning (CR) codes*. These codes represented the *Analytic Framework* for the study.

Using the analytic framework, I indexed all episodes into three analytic matrices, one per participant. I looked for patterns in the distribution of CR codes in relation to EB codes across all three matrices, examining ways in which covariational reasoning supported PSTs' mathematization of the EBM. Then, I compared EB codes across participants in order to identify particular ways of thinking about the EBM. The analysis of such patterns provided the information needed to meet the research goals.

Results

The analysis of PSTs' responses revealed that coordination of values and coordination of change are key to mathematize the EBM for global warming. The analysis also revealed that particular mathematizations resulted in particular meanings for radiative equilibrium.

Covariational Reasoning and Mathematizing the EBM

Coordinating values represented a key step in mathematizing the notion that the energy balance restores radiative equilibrium over time. Coordinating values allowed PSTs to translate radiative equilibrium into a quantity, N(t), that decreases as time increases (direction of change). For instance, Jodi's initial understanding of the EBM did not include an energy balance restoring radiative equilibrium after a positive forcing. For her, the increase in C (atmospheric CO_2 concentration) *pushed* the energy balance out of its *normal* state, and it would remain out of that *normal* state unless C decreased to its original value, which is reflected in the way she described change in N(t) ("[N] wouldn't increase or decrease if CO_2 is kept stable"). Coordinating t-values and N-values using the given recursive rules allowed her to create and plot a collection of pairs (t, N) with which she drew a accurate graph for N(t) (Figure 2a). She interpreted the graph as follows "[the graph means] that we are going back to an equilibrium, or we are not as far from equilibrium as we were." While looking at her graph, she added "each time we are increasing t, we are decreasing N by smaller amounts." Jodi's understanding of the EBM extended to include: (a) the idea of radiative equilibrium being restored over time, and (b) a mathematized representation of such process in terms of a covariation, N(t), that decreases (direction of change) by decreasing amounts of change (rapidity of change) as t increases. Describing rapidity of change of N(t), I would argue, reveals a more sophisticated mathematization of radiative equilibrium than describing direction of change alone; it represents a higher degree of complexity in understanding and describing a covariation.

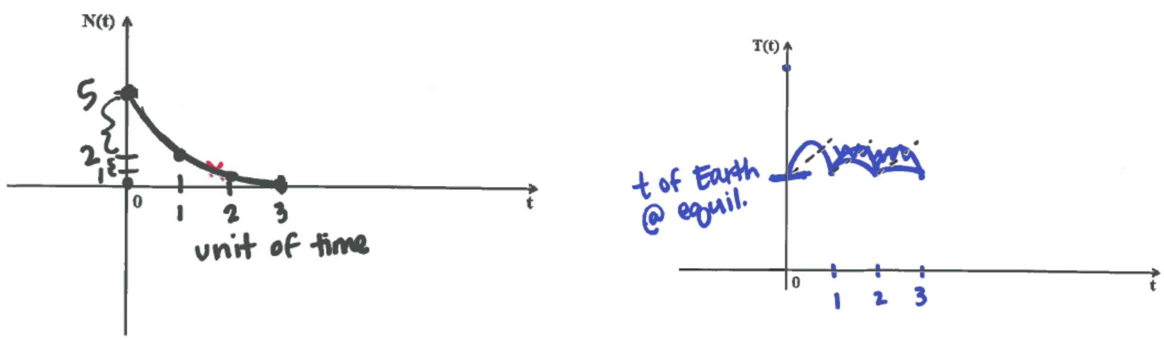

Figure 2: (a) Jodi's Graph of N(t) (Left) and (b) Jodi's Graph of T(t) (Right)

Pam's initial understanding of the EBM included the idea of radiative equilibrium being restored over time; she correctly anticipated N(t) to be a decreasing function of time. Her mathematization of radiative equilibrium, however, was limited to indicating direction of change alone ("[N] was five, and then it would decrease to be zero again") and did not support drawing a graph for N(t). After coordinating t-values and N-values using the given recursive rules, Pam noticed that:

> [N] decreased pretty quickly, like 3 units of $J/s/m^2$. Then, it decreased by about 1 $J/s/m^2$ … I am assuming [N] is going to decrease by a little bit, and a little bit, and a little bit, until it reaches zero again.

Coordinating values allowed Pam to think about changes in t in relation to changes in N. This, in turn, helped her draw an accurate graph for N(t) (Figure 3a). Pam's mathematization of radiative equilibrium extended from describing *direction of change* alone ("[N] was 5, and then it would decrease to be zero again") to describing *rapidity of change* ("[N] is going to decrease by a little bit, and a little bit, and a little bit, until it reaches zero again"). Pam reasoned about rapidity of change by coordinating two sequences indicating change in N: the sequence of values of N and the sequence of amounts of change $\Delta_i N$. The coordination of these two sequences allowed her to mathematize radiative equilibrium as a covariation, N(t), that decreases by decreasing amounts of change as t increases.

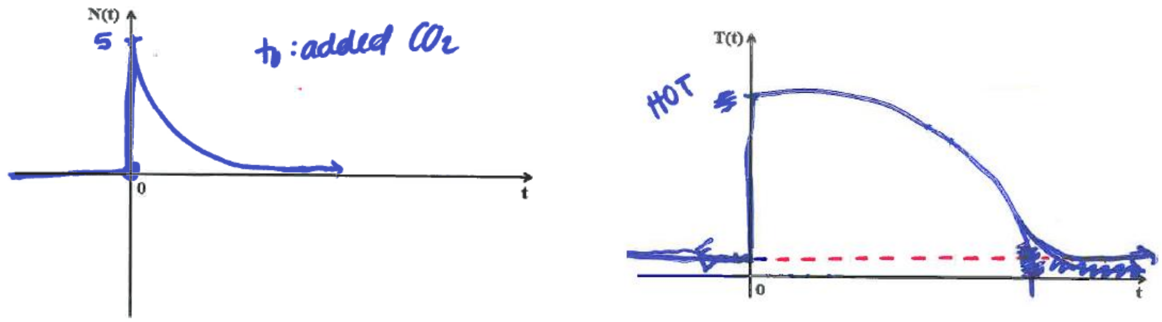

Figure 3. (a) Pam's Graph of N(t) (Left) and (b) Pam's Graph of T(t) (Right)

Kris's initial understanding of the EBM included the idea of radiative equilibrium being restored over time. She, however, struggled to see that idea reflected in the covariation N(t). Since the heat flows R and A were increasing as t increased, Kris thought that N(t) would be increasing too given that $N(t) = [S + A(t)] - R(t)$. She, however, expressed surprise about her conclusion "So, as R increases, A increases [*points at R and A in* $N = (S + A) - R$] … [N] can't just keep increasing!" Coordinating values by using the given recursive rules helped Kris clarify her confusion. She noticed that the heat flow B(t) was increasing by decreasing amounts of change

> Well, this difference right here, between 320 and 328 ($\Delta_1 B = 328 - 320 = 8$), is greater than the difference between these two values, the 328 and 331.5 ($\Delta_2 B = 331.5 - 328 = 3.5$), and the difference between these two ($\Delta_3 B = 332.9 - 331.5 = 1.4$) is less than those [*points at 328 and 331.5*], which is less than that [*points at 320 and 328*]. That tells me that there is eventually going to be a limit … Yeah, it is going to reach a new equilibrium point somewhere

The coordination of values helped Kris reconcile (and mathematize) radiative equilibrium with (and in terms of) the covariation N(t). Her analysis of the rapidity of change of B(t) supported drawing an accurate graph for N(t) (Figure 4a). Like Pam, Kris's extended her mathematization of radiative equilibrium from direction of change to rapidity of change by coordinating two sequences indicating change in B: the sequence of values of B and the sequence of amounts of change $\Delta_i B$. The coordination of these two sequences allowed her to mathematize radiative equilibrium as a covariation, N(t), that decreases by decreasing amounts of change as t increases.

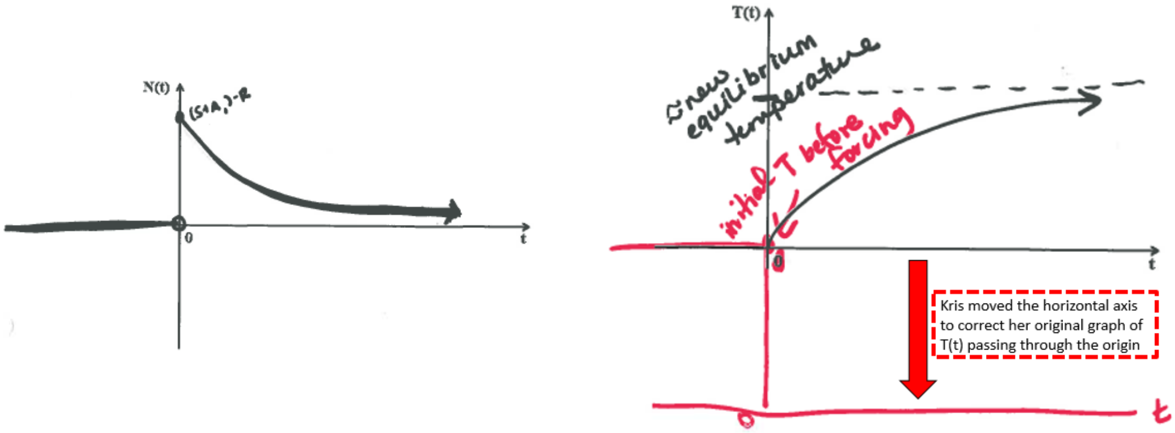

Figure 4. (a) Kris's Graph of N(t) (Left) and (b) Kris's Graph of T(t) (Right)

PSTs use the recursive rules to coordinate values of an independent variable *t* and a dependent variable *y*. Coordination allowed them to mathematize radiative equilibrium in terms of the direction of change of a covariation N(t). Coordination also made sequences of values and graphs available to PSTs. They used such objects to coordinate changes in *t* with changes in *y*. The coordination of changes, in addition to the coordination of values, helped them extend their mathematization of radiative equilibrium from direction of change (N(t) decreases as t increases) to rapidity of change (N(t) decreases by decreasing amounts of change as t increases). The later represents a higher level of complexity in describing covariation.

Mathematization and Understanding Global Warming

PSTs' ways of thinking about their mathematization of EBM resulted in two different meanings for radiative equilibrium: *Single Equilibrium Meaning* (SEM) and *Multiple Equilibrium Meaning* (MEM). The distinctive feature between the two meanings for radiative equilibrium was the PSTs' conception of *what is measured* by N(t) in the EBM.

I asked PSTs to interpret their graphs of N(t) in terms of whether the energy balance was losing heat (cooling down) or gaining heat (warming up) as time increased. Jodi and Pam concluded that the energy balance was losing heat or cooling down as time increased. Jodi stated that "the line (the graph of N(t)) is going in the negative direction, and we know that as N decreases, the surface is losing energy," while Pam stated that "we are losing because if we have gained energy [the surface] would get hotter, but it is not getting hotter because N is smaller so it's cooling off." It seems that Jodi and Pam arrived to such conclusion because they conceived N(t) as a measure of *how much heat* needs to be lost for the energy balance to *return* to radiative equilibrium (e.g., Jodi: "when the input is greater than the output, then we need to decrease N [*writes* $-N$ *on the right side of* $(S + A) = R$] so we can get back to equilibrium"). Jodi and Pam saw N(t) as an amount of excess heat in the energy balance. Therefore, if N(t) is decreasing, then the energy balance must be losing heat. Notice that seeing N(t) as excess heat also involves envisioning the energy balance *returning* to radiative equilibrium. I use the word *returning* to indicate that Jodi and Pam thought of radiative equilibrium as a unique state, the original state before the forcing by CO_2. For her, the energy balance is *returning* to its original equilibrium because the excess heat, caused by the initial forcing by CO_2, is decreasing. I called this meaning of radiative equilibrium Single Equilibrium Meaning (SEM).

SEM led Jodi and Pam to draw graphs of T(t) showing an overall decline in temperature as time increased (Figure 2b and Figure 3b, respectively). The particular shape of Jodi and Pam's

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

graphs for T(t) were rooted in the way they reasoned about graphs and covariation, particularly in terms of the rapidity of change. A discussion of Jodi's case is presented elsewhere (Author, 2018). As for Pam, a discussion of the particular shape of her graph is beyond the scope of this paper. For now, I am drawing attention to the fact that both, Jodi and Pam, drew a graph of T(t) showing an overall decline in temperature. Their graphs indicate that the planet's surface warms up *solely* when C increases. Thus, if C stops changing, then the energy balance returns to its original radiative equilibrium and the planet's surface cools down as time increases.

When I asked Kris to interpret her graph of N(t) in terms of whether the energy balance was losing heat or gaining heat, she concluded that the energy balance was gaining heat.

> Well, [the surface] keeps in taking. I think it is warming up because once we added more CO_2, that is less of the emitted energy that is getting just like shut out passed the atmosphere, leaks from it. So then, more [radiation] is going to be absorbed by the atmosphere … Whatever is absorbed by the atmosphere [*points at B*] is going to be absorbed back into the [*points at the planet's surface*] … which is going to keep increasing.

Kris did not demonstrate any conflict between the energy balance gaining heat and N(t) decreasing as t increased. This suggests that Kris saw N(t) as another representation of heat gain in the EBM. A possible explanation is that Kris conceived N(t) as measuring *how much heat* needed to be gained for the energy balance to reach a *new* radiative equilibrium. When Kris drew the graph of the Planet's Mean Surface Temperature T(t) (Figure 4b), she wanted to show that T(t) was increasing but stabilizing at a certain value. Such graph implied that: (a) the energy balance gains heat to reach radiative equilibrium, and (b) radiative equilibrium is not unique and can occur at higher levels of heat. These are distinctive characteristics of a *Multiple Equilibrium Meaning* (MEM) for radiative equilibrium. This meaning includes attention to the increasing values of R, B, and A, and their implications in the context of the EBM. It also includes understanding the feedback of heat between the atmosphere and the surface: a fraction of the heat released by the surface is reabsorbed by it, enhancing its temperature.

Conclusion

I use Thompson and Carlson's (2017) levels of covariation to characterize PSTs' covariational reasoning. PSTs needed to reason about covariation at the Coordination of Values Level in order to mathematize radiative equilibrium in terms of *direction of change* of the Net Planetary Energy Imbalance N(t). Through coordinating values, PSTs can notice that N(t) decreases as time increases, which indicates that the Earth's energy balance is restoring radiative equilibrium. When PSTs coordinated changes $\Delta_i y$ with equal changes Δx, in addition to coordination of y-values with x-values, they were able to mathematize radiative equilibrium in terms of the *rapidity of change* of N(t). In other words, PSTs envisioned changes in t and N as occurring simultaneously and by intervals of fixed size. This suggests that covariational reasoning at the Chunky Continuous Level supports the mathematization of radiative equilibrium in terms of rapidity of change.

PSTs' mathematizations of the EBM resulted in two different meanings for radiative equilibrium; these meanings have different implications for understanding the link between CO_2 pollution and global warming. In particular, PSTs' conception of what is measured by N(t) led to two meanings for radiative equilibrium. SEM includes conceiving N(t) as an amount of *excess heat* that must be lost for the energy balance to *return* to radiative equilibrium. SEM also includes understanding radiative equilibrium as a unique state, the original equilibrium. SEM leads to the idea that the planet's surface cools down after a positive forcing by CO_2. This way of thinking about global warming is unproductive because CO_2 pollution has long-term and long-lasting effects on the planets mean surface temperature (IPCC, 2013). SEM may also open the door for misconceptions regarding global warming (e.g., "we can stop global warming at the moment we stop CO_2 emissions" or "as long as we maintain our current level of CO_2 emissions, the planet wouldn't get any hotter"). MEM includes conceiving N(t) as an amount of *heat to be gained* for the energy balance to *reach* radiative equilibrium. MEM also includes understanding that there are radiative equilibriums at higher levels of heat. MEM leads to the idea that the planet's surface warms up after a positive forcing by CO_2, even after the atmospheric CO_2 concentration is stabilized. This way of thinking about the link between CO_2 pollution and global warming is a productive one. It supports an understanding of the real impact of CO_2 emission in the climate, as well as their long-lasting effect in the planet's mean surface temperature.

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