FROM COMPUTATIONAL STRATEGIES TO A KIND OF RELATIONAL THINKING BASED ON STRUCTURE SENSE

Cesar Martínez-Hernández
Universidad de Colima
cmartinez7@ucol.mx

Carolyn Kieran
Université du Québec à Montréal
kieran.carolyn@uqam.ca

This paper provides evidence on how elementary school students from a Mexican public school move from using an operational sense, expressed in computational strategies, to a kind of relational thinking based on structure sense ideas and expressed by number decomposition. Even though the results are somewhat preliminary, they illustrate how students can leave behind their computational strategies and develop a more sophisticated mathematical reasoning regarding equivalence of numerical expressions and equalities; however the strategies they developed tended to be based on "ad hoc decomposition of one side and comparison with the initial form of the other side," rather than on compensation.

Keywords: Algebra and Algebraic Thinking

Background

Among the different approaches to developing algebraic thinking in the early grades, recent studies have begun to distinguish more clearly the notion of structure sense in arithmetic (Kieran, 2018b) and to stress its importance as a key feature of algebraic thinking that should be promoted from an early age onward. Equivalence tasks involving equalities such as $a+b=c+d$ have been shown to be particularly important (Martínez & Kieran, 2018). Among the studies dealing with structure sense, Asghari and Khosroshahi (2016) have described how certain tasks in the elementary grades that use the equal sign can provoke a kind of mathematical thinking involving both operational and structural conceptions.

Pang and Kim (2018), as well as Schifter (2018), in studies on equalities of the form $a+b=c+d$, have emphasized the importance of engaging students in discussions on the structural properties of such equalities. In particular, Schifter states that if students focus on the structure of equalities such as 57+89=56+90 in order to determine their veracity, they are thereby indicating a relational thinking with respect to equivalence tasks, which is considered a fundamental aspect of algebraic thinking (Carpenter, Franke, & Levi, 2003).

Based on Schifter (2018), and the theoretical perspective addressed in Kieran (2018b), Martínez and Kieran (2018) demonstrated that students at early ages accept equalities such as $a+b=c+d$ and are able to rewrite them in various ways when asked to identify the equivalence of both sides of the equality. However, their strategy for rewriting an equality to show equivalence (e.g., 172+18=182+8 as 100+72+10+8=100+82+8) indicates that they have a natural inclination to rewrite the equality in a manner that clearly relates to the total of each side. That is to say that they do not decompose and recompose the numbers with the aim of producing a written form where both sides look alike, for example, rewriting 172+18=182+8 as, among others, 100+72+10+8=100+72+10+8, or as 182+8=182+8. In other words, even when they are able to decompose the numbers, they do not establish a clear relation between both sides of the equality, but rather with respect to the total of each side. Therefore, according to Martínez and Kieran (2018), for students to determine the veracity of an equality (e.g., 172+18=182+8) and its various rewritten forms (e.g., 100+72+10+8=100+82+8), they resort to a computational strategy:

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

calculate the total of each side of the equality. Such a strategy and the form they use to rewrite the equality does not allow them to reason about the expressions of the equality based on structural aspects.

However, the results obtained by Martínez and Kieran (2018), especially the fact that students accept equalities of the form $a+b=c+d$, and that they are able to rewrite them, served as a basis for the current study. This study, which is reported herein, involved designing an equivalence task on numeric equalities where students were to be explicitly instructed not to calculate the total of both sides in order to determine the veracity of the equality. This design was considered to be a pivot for developing in students a sense of structure with respect to equivalence, since accepting numerical sentences as bona fide numerical objects and being able to operate on and with them as objects is viewed as fundamental in looking for and expressing structure in arithmetic (Martínez & Kieran, 2018, p. 169). Hence, the question that guided the research of this study is the following: *What is the nature of the strategies students use to determine the equality of the right and left sides of a numerical equation when they are explicitly asked to not calculate the total of each side?*

Theoretical Framework

Different theoretical viewpoints underpin discussions on the development of algebraic thinking at early ages (see Cai & Knuth, 2011; Kieran, 2018a). One such perspective focuses on seeking, using, and expressing structure in numbers and numerical operations (Kieran, 2018b).

Structure in Numbers and Numerical Operations

The main idea in this perspective lies in the notion of structure sense in arithmetic. As Kieran (2018b) points out, there are several researchers interested in the notion of structure in algebra. Linchevski and Livneh (1999) and Mason, Stephens, and Watson (2009), among others, emphasize structure as a feature of algebraic thinking. On the one hand, Linchevski and Livneh relate the development of structure in arithmetic with the development of structure in algebra. Mason et al., on the other hand, maintain that students need to develop *structural thinking*, that is, a focus on relations rather than on procedures.

According to Kieran (2018b, p. 80), the notion of structure is one of the central ideas in mathematics; however, not only are there different perspectives on it, but also it is often treated as if it were an undefined term. Nevertheless, it is strongly related to generalization in the literature on algebraic thinking. On this base, generalization involves identifying the structural, and vice versa, the structural involves identifying the general.

In contrast, Kieran (2018b) discusses aspects related to structure that might expand this notion and that, at the same time, involve more than just the basic properties of arithmetic. She proposes a focus on the notion of structure in arithmetic that draws on Freudenthal (1983, 1991, quoted in Kieran, 2018b). According to Freudenthal, the whole numbers constitute an order structure. This order structure leads to the addition structure, as well as to the multiplication structure. Following Freudenthal, Kieran points out that structure in numbers and numerical operations involves different *means of structuring*, according to, for example, factors, multiples, powers of 10, evens and odds, decomposition of primes, etc. This perspective is embodied in the idea that developing a sense of structure entails seeing through mathematical objects and drawing out relevant structural decompositions.

Based on Freudenthal´s ideas, and the various perspectives found in the literature regarding the notion of arithmetical structure, Kieran suggests promoting various experiences with equivalence that involve decomposition, recomposition, and substitution. Such experiences would allow students to become aware of different means of structuring numbers and numerical operations. The research literature suggests that students think in a structural way about equivalence when they transfer part of a number to another. For example, Carpenter et al. (2003) have shown how students indicate a relational thinking about equivalence when they implicitly use the associative property to decide about the equality of the two sides. Without writing it as such (e.g., for the numerical equality 56+47=54+49), students express that 56+47=(54+2)+47=54+(2+47)=54+49. Britt and Irwin (2011) have referred to this approach as the compensation strategy, which they express alternatively as follows: 56+47=56–2+2+47=54+49. However, the results from our pilot study (see Martínez & Kieran, 2018) suggested that the compensation strategy is not the one that underpins the approach used by the students when they determined the equivalence of the left and right sides of an equality. Consequently, we had to extend our research.

Method

Participants

For this second stage of the project, three students (S1, S2 and S3) from a Mexican public school participated, ages between 11 and 12 years. When data were collected, the students had just finished elementary school. Those three students were selected because they had taken part in the first stage of the project (see Martínez & Kieran, 2018).

Task Design

The proposed task in this second stage of the project is a redesign of Task 3, reported in Martínez and Kieran (2018). Hence, the task involves the equal sign to show the equivalence of expressions in an equality (e.g., 10+7=5+12). The task has four sections, the first three having the following structure:

- A true equality (numerical equation) is presented.
- Students are asked to show, <u>without calculating the total on both sides</u>, that the equality is true. (The task explicitly asks for them not to calculate the total.)
- They are asked for an explanation of their reasoning.

Section four asks for a generalization of the students' strategy for showing the veracity of the equalities as given in the three previous sections.

Data Collection

Data collection was carried out in one single session, eight months after the initial study (reported in Martínez & Kieran, 2018). The technique used in data collection was that of the group interview, conducted by one of the researchers (subsequently referred to as "I"). The aim of the group interview was to allow students to verbalize their reasoning. During the interview, each participant was given the printed task sheet in order to have a written record of his/her work. The 40-minute session was also video-recorded.

Results and Discussion

Due to space constraints, this report concerns only the work developed by the three students during the first part of the task, which involved the equality 10+7=5+12, within the setting described in the Task Design section above (where the equality 10+7=5+12 is presented; students are asked to show, <u>without calculating the total on both sides,</u> that the equality is true, along with providing an explanation of their reasoning). Data analysis is based on each student's written work sheets, as well as their verbalizations as evidenced by the video-recordings, and the researcher's field notes.

Spontaneous Exhibition of Operational Sense

The first item in the initial section from the task displays the expression 10+7=5+12 and explicitly asks for showing, without calculating the total on both sides, that this equality is true. Despite the explicit request, students spontaneously referred to the total on both sides of the given equality after individually reading and answering the question. The following verbatim, translated from the original Spanish transcriptions, reflects this moment of the interview:

> I: Would there be a way to show that this equality [is true]? [referring to 10+7=5+12, the interviewer does not complete the question because S3 interrupts]
> S3: By adding [referring to the total; the other students agree]

As can be seen, in spite of the instructions on the work sheet to not calculate the total, S3 answers spontaneously that one needs to add up each of the sides. S1 and S2 explain in a similar manner. Fig. 1 shows S1's work, where her initial written answer is observed ("It is true because if we add the two sides, the sums that are the results are the same.").

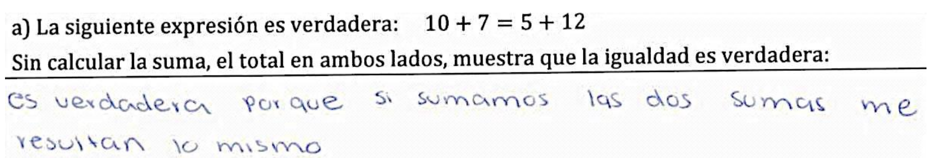

Figure 1: S1's Initial Answer

Based on these answers, the interviewer asks for the possibility of using a particular strategy that the students already know and exhibited in the earlier study (Martínez & Kieran, 2018) – as described in the Background section above, that is, to rewrite the given equality in another equivalent form:

> I: [The researcher writes on the board the equality 10+7=5+12] Is there another way to rewrite this equality?
> S2: As a subtraction.
> S3: As a subtraction [inaudible].
> I: All right, as a subtraction, according to S2 […] For example S2?
> S1: Twenty minus three [note that S1 answers]
> S2: Equals eighteen minus one [S2 completes the equality 20-3=18-1]

As we observe from the verbatim, the strategy followed by S1 and S2 is to search for one or more numbers in such a way as to preserve the total value for each side. These initial results, as reflected in the report of the pilot study, show a strong operational sense in students. They manage each side of the equality in an independent way, guided by the total.

Transition to a Kind of Relational Thinking

From the students' initial responses, the interviewer tries once again to see if the students would apply their earlier strategy for rewriting a given equality (Martínez & Kieran, 2018):

I: Would there be a way to write also [referring to rewriting the given equality] but using, say, these same numbers? [points to the board to the equality 10+7=5+12; see Fig. 2]
S3: Yes.
I: […] For instance, could this [points to the left side, 10+7, but S3 interrupts]?
S3: 5+5+5+2 [verbalizes the expression].
I: Ok, S3 says that this [referring to the left side of the equality 10+7=5+12] could be written as 5+5+5+2 [writes on the blackboard as expressed by S3] Is this OK?
S1 and S2: Yes [both at once].
S3: Is equal to [inaudible].
I: […] This [referring to the right side of 10+7=5+12], in which other way? Look, this [referring to 5+5+5+2] already has a form of, I mean, it [10+7] can be re-expressed in this way [Referring to 5+5+5+2] […]. Ok, Can this [the right side 5+12] be rewritten identically to this? [referring to 5+5+5+2].
S3: Yes.
I: Why?
S2: It gives the same
S1: Because it gives the same.
I: How would you rewrite it?
S3: 5+5+5+2 [verbalizes the expression].
I: Ok, 5+5+5+2 [writes the equality 5+5+5+2=5+5+5+2 as proposed by S3, on the blackboard, see Fig. 2]

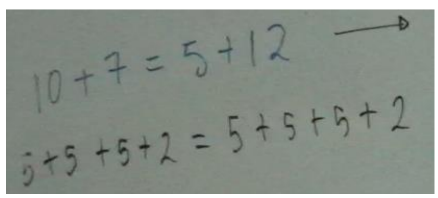

Figure 2: Equality Proposed by S3

As can be observed from the preceding verbatim and from Fig. 2, S3 is able to propose an equivalent form (5+5+5+2=5+5+5+2) of the equality 10+7=5+12*,* in such a way that both sides of the equality have the same form. This, however, does not necessarily mean that a kind of relational thinking is at play, since the students are working each side separately. Furthermore, S1 and S2 justify based on the preserved total. This is seen when S1 answers: *"Because it gives the same"*.

S2 and S3 are, however, showing that their initial strategies are evolving. The form proposed by S3 is in fact a decomposition of each side of the initial equality. This suggests that S3 is leaving behind the idea of looking for two or more numbers that preserve the total. The evidence for this is derived from the following verbatim:

I: Does this 5 [points to the first number 5 on the left side, 5+5+5+2, of the equality seen in Fig. 2] come from any part here [points to the initial equality 10+7=5+12]? […] Or, are you looking for two or more numbers in order to get 17 [the total]?
S3: Yes, I rewrote the two 5s [the first two numbers of 5+5+5+2] from the 10 [referring to the 10 in the expression 10+7], and from the 12.

S2: From the 7! [S2 interrupts, referring to the 7 in the expression 10+7].
S3: From the 7, we had 2 left, that is why I wrote the plus 2 [S3 explains that the 7 is rewritten as 5+2]
I: And, how would you justify these? [referring to the right side of the equality 5+5+5+2=5+5+5+2, seen in Fig. 2]
S2: The 5 comes from the 5 [of the right side of the initial equality 10+7=5+12], the two 5s and the 2 come from the 12.

As suggested by the verbatim, S2 and S3 are able to decompose each side of the given equality 10+7=5+12 in a common form. They explain, through decomposition, where each number comes from in 5+5+5+2=5+5+5+2. However, the right side in this last equality might be an *ad hoc* writing, in order to coincide with the right side (5+12) of the initial equality – a notion that is synthesized in Fig. 3 and which is discussed more fully in the paragraphs that follow.

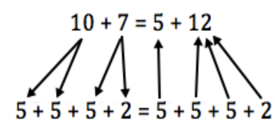

Figure 3: Rewritten Equality Through Decomposition

Based on this new strategy of rewriting each side (or at least one side) through decomposition of numbers and getting both sides of the rewritten equality to have the same form, the interviewer returns to S3's equality to inquire into the need to calculate the total:

I: Seeing it in this way [points to the equality 5+5+5+2=5+5+5+2; see Fig. 2], is it necessary to add up in order to decide if the equality is true? Would you still add [calculate the total] or is the addition no longer necessary?
S3: It is no longer necessary for me.
I: Why not, S3?
S3: Because it is easy to see what will be the result.
I: Ok [….] What is this and this expression like? [pointing to both sides of 5+5+5+2=5+5+5+2]
S2: The same
S3: The same
I: Then, is it necessary to add?
S1: Oh, no!
S2: No, because [inaudible]
S3: But to know the result of each one? [referring to the total of each side]
S2: No, but if it is the same [in the same form], obviously it will give the same [the total will be the same]. If the expression is the same, it will be equal, it will give the same.

As can be seen from the above verbatim extracts, S3 relates the decomposed equality to both sides of the initial equality, but still thinks about the total as a means to validate it. S2 and S1 support the idea that the decomposed equality of S3 is all that is needed for validation, due to the common form in which both sides could be rewritten. It is important to note how S2's speech and thinking have changed. At the beginning of the task, S2 states that the total *"is" the same*, which means that S2 has calculated the total. But now S2 justifies with another tense: *"will be" the same*, which means that S2 is aware that she will get the same total, but that it is not necessary to calculate it. It is enough to observe that both sides of the equality are written in the same form.

As of this moment, the students apply the following strategy to generate an equality equivalent to a given equality and to justify its veracity: rewrite one of the sides of the given equality based on numerical decomposition, and use this to guide the decomposition of the other side; if this decomposition fits the numbers on the other side of the initial equality, then it is not necessary to calculate the total of each side of the newly obtained equality to prove that it is true.

The above described structural approach is clearly observed when the interviewer subsequently asks the students to rewrite in another different form (different from 5+5+5+2=5+5+5+2) the initial equality 10+7=5+12, in such a way that both sides look alike. Fig. 4 shows S1's work during the group interview. However, S1 genuinely decomposes only the left side (10+7) of the initial equality to obtain 2+2+2+2+2+2+2+3, and copies this expression onto the right side. This is made obvious because S1 could not relate the expression 2+2+2+2+2+2+2+3 (i.e., the right side of the equality in Fig. 4) with the right side (5+12) of the initial equality. After a few futile attempts to relate the right side of the equality in Fig. 4, with the right side of the initial equality, S1 starts anew. She proposes a different expression for the right side, 2+2+1+5+5+2, and then guided by this expression rewrites the left side of the initial equality in the same way (see Fig. 5).

Figure 4: Rewritten Equality by S1

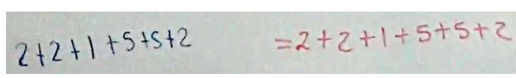

Figure 5: Second Rewritten Equality Proposed by S1

Now, the students are able to rewrite an equality by means of a decomposition that clearly shows equivalence; however, the truth of the equality refers to the rewritten form, not necessarily to the initial equality – perhaps because from the beginning they are told that the given equality is true. This is seen, for example, in the explanation offered by S1 (Fig. 6): *"Because if I look and compare each number, they are the same and obviously it is the same"*

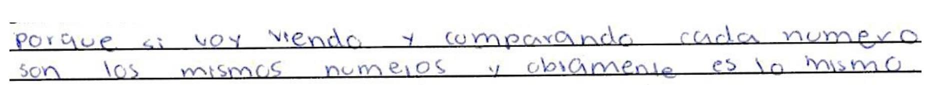

Figure 6: S1's Explanation of the Veracity of the Rewritten Equality

Summarizing, the students' strategy consists of the following: decompose the left (or right) side; copy it to the right (or left side accordingly); compare both sides of the rewritten equality with their corresponding expressions in the initial equality; if they are able to establish the equivalence relation (by pointing to where each number of the rewritten equality comes from with respect to the given equality), then they can conclude that both sides are equal (and that it is not necessary to refer to the total of each side). As previously noted, we refer to this structural approach for generating equivalent numerical equalities as “ad hoc decomposition of one side and comparison with the initial form of the other side”.

Interaction Between Participants and Task Design

As could be discerned from the verbatim extracts, the participants’ interactions, the task design, and, not least of all, the interviewer’s interaction with the students were all crucial for the students to evolve in their mathematical reasoning.

Firstly, it was of the utmost importance to state explicitly to the students that they were not to calculate the total for each side. However, as discussed above, they spontaneously referred to the total as their initial approach for comparing right and left sides of an equality. Thus, the interviewer’s role, based on the participants’ impulse to calculate, was also crucial in promoting other approaches. For example, a key question that led the students to relate the initial equality with the numerical decomposition they were proposing for their equivalent equality was to have them explain where the numbers came from in their rewritten equality. This helped them focus on the given equality and its rewriting (for at least one side of the equality), based on the involved numbers (see Fig. 3). Lastly, the contribution each student provided in the first part of the task – especially the initial decomposition of the left side proposed by S3 – was crucial for the eventual consolidation of the new strategy that grew out of the students’ interactions. The analyzed interview episodes illustrate clearly the significance of a sociocultural environment in the construction of mathematical knowledge.

Conclusions

The first set of results emanating from the designed task of this study support the results and conjectures put forward in Martínez and Kieran (2018). Students show a strong operational sense that leads them to use computational strategies, as a first idea, in equivalence tasks. However, based on their acceptance of equalities such as $a+b=c+d$, students are able to develop a structural sense that is anchored in their arithmetic.

If we compare, on the one hand, the strategy that the students in this study developed with, on the other hand, the compensation strategy reported in the literature, we note several differences. Applying the compensation strategy to the equality 10+7=5+12 involves a simultaneous comparison of the relation between the numbers on the left side (10+7) with those on the right side (5+12), an observation that our participating students never explicitly mentioned. The compensation strategy involves noting that the 10 on the left side is 2 less than the 12 on the right side, and the 7 on the left side is 2 more than the 5 on the right side; so everything is balanced. According to the compensation strategy, this could be expressed as 10+7=10+2-2+7=12+5=5+12. Alternatively, this kind of thinking could be expressed in written form by decomposing the left side into 10+2+5 and the right side into 5+2+10, leading to the equivalent equality, 10+2+5=5+2+10. However, our data did not provide evidence of the use of such strategies. In contrast, the students developed a different structural strategy for generating equivalent equalities – a strategy that has not been reported in the literature up to now.

Such preliminary results lead to further questions regarding whether or not students apply the same strategy for generating equivalent equalities when presented with initial equalities involving much larger numbers where they cannot immediately observe the total for each side or with equalities where they are not told at the outset that the equalities are true. Analyses bearing on such questions are being prepared for follow-up reports.

Finally, even though the students showed a noticeable evolution in the development of their mathematical thinking, we want to emphasize the importance of the role played by the teacher in this development – a role that involved helping the students to move from an operational to a structural kind of thinking in order to determine the equivalence of numerical equalities.

Acknowledgments

We express our appreciation to the Secretaría de Educación Pública (SEP) of Mexico for the financing of the project of the research group “UCOL-CA-102 Educación Matemática” within the program PRODEP 2017. We are also grateful to the authorities, teacher, and students of the participating school for their collaboration.

References

Asghari, A., & Khosroshahi, L. (2016). Making associativity operational. *International Journal of Science and Mathematics Education, 15*(8), 1559-1577.

Britt, M. S., & Irwin, K. C. (2011). Algebraic thinking with and without algebraic representation: A pathway to learning. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 137-160). New York: Springer.

Cai, J., & Knuth E. (Eds.). (2011). *Early algebraization: A global dialogue from multiple perspectives*. New York: Springer.

Carpenter, T., Franke, M., & Levi, L. (2003). *Thinking mathematically. Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.

Kieran, C. (Ed.). (2018a). *Teaching and learning algebraic thinking with 5- to 12-year-olds: The global evolution of an emerging field of research and practice*. New York: Springer.

Kieran, C. (2018b). Seeking, using, and expressing structure in numbers and numerical operations: A fundamental path to developing early algebraic thinking. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds: The global evolution of an emerging field of research and practice* (pp. 79-105). New York: Springer.

Linchevski, L., & Livneh, D. (1999). Structure sense: The relationship between algebraic and numerical contexts. *Educational Studies in Mathematics*, *40*(2), 173–196.

Martínez, C., & Kieran, C. (2018). Strategies used by Mexican students in seeking structure on equivalence tasks. In T. E. Hodges, G. J. Roy, & A. M. Tyminski (Eds.), *Proceedings of the 40th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 163-170). Greenville, SC: University of South Carolina & Clemson University.

Mason, J., Stephens, M., & Watson, A. (2009). Appreciating mathematical structure for all. *Mathematics Education Research Journal, 21*(2), 10-32.

Pang, J., & Kim, J. (2018). Characteristics of Korean students’ early algebraic thinking: A generalized arithmetic perspective. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds: The global evolution of an emerging field of research and practice* (pp. 141-166). New York: Springer.

Schifter, D. (2018). Early algebra as analysis of structure: A focus on operations. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds: The global evolution of an emerging field of research and practice* (pp. 309-328). New York: Springer.

DE ESTRATEGIAS DE CÓMPUTO HACIA UN PENSAMIENTO DE TIPO RELACIONAL BASADO EN UN SENTIDO DE ESTRUCTURA

Este reporte muestra evidencia de cómo alumnos de primaria, de una escuela pública de México, transitan de un sentido operativo, manifestado en estrategias de cómputo en tareas de equivalencia, hacia un pensamiento de tipo relacional basados en cierto sentido de estructura y manifestado en la descomposición de números. Aunque los resultados son preliminares, estos muestran que los alumnos son capaces de desprenderse de estrategias de cómputo y desarrollar un razonamiento matemático más sofisticado en torno a la equivalencia de expresiones e

igualdades numéricas; sin embargo, la estrategia que desarrollan se sustentan en una "descomposición ad hoc de un lado y comparación con la forma inicial del otro lado de la igualdad", más que en la compensación.