VARIOUS MEANINGS A STUDENT USES FOR QUANTIFIED VARIABLES IN CALCULUS STATEMENTS: THE CASE OF ZACK

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This study investigates one Calculus student's meanings for quantifiers in Calculus statements involving multiple quantifiers. The student was asked in a two-hour long clinical interview to evaluate and interpret the Intermediate Value Theorem (IVT) and three other statements whose logical structure was similar to the IVT except for the order of both the quantifiers and their attached variables. Four different meanings for variables attached to universal and existential quantifiers emerged from his responses at various moments of the interview. In this paper we detail these four meanings with empirical evidence and discuss implications of our findings to research and teaching of quantified variables in Calculus statements.

Keywords: Reasoning and Proof, Postsecondary Education, Cognition, Advanced Mathematical Thinking

The purpose of this study is to investigate students' mental processes associated with quantified variables. In particular, we focus on a Calculus student, Zack, and his quantifications for variables in the Intermediate Value Theorem (IVT) and similar statements as we answer the following research question: *What are the meanings for quantified variables that a student uses when interpreting statements from Calculus contexts?*

Literature Review

Quantifiers such as "for all" (∀) and "there exists" (∃) may be used to state important definitions and theorems in Calculus. For example, the Intermediate Value Theorem (IVT), the Mean Value Theorem (MVT), and definitions of limits and continuous functions may be stated with multiple quantifiers (Bartle & Sherbert, 2000; Stewart, 2003). Several studies have reported various student tendencies to mistreat quantifiers when analyzing quantified statements involving either the universal or existential quantifier (Barkai, Tsamir, Tirosh, & Dreyfus, 2002; Epp, 1999). For example, Barkai et al. (2002) reported that some participants of their study tended to suggest that a few examples are sufficient to prove that a statement involving a universal quantifier, in the form 'For all x, P(x),' where P(x) is a statement about x, is true. In the case of a statement involving an existential quantifier, in the form 'there exists x such that P(x),' students also rejected the notion that one example would suffice for proving such a statement (Tirosh & Vinner, 2004). One such explanation for why students tend to have these particular tendencies in their interpretations of quantified statements is that they may confound colloquial language with mathematical language (Epp, 1999). For example, colloquially we may state that "Every book on the bookshelf is French," and we would assume that there is at least one French book on the specified bookshelf. However, in mathematics, we could consider the case where the statement is vacuously true. Colloquial language may explain some student difficulties with quantification, but there may be other reasons why these tendencies exist that have yet to be noted in the literature that may be explained by classifying students' meanings for quantified variables.

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Theoretical Perspective

In this section, we define some terms that we will use to describe one student's meanings for quantified variables. *Quantifiers* are phrases (e.g., "for all" and "there exists") used to indicate the number of elements, x, in the domain of discourse satisfying a predicate, $P(x)$. For example, consider the mathematical statement, "Every isosceles triangle x has congruent base angles." This statement is a quantified statement where the phrase "every" is a quantifier for the elements x within the domain of discourse, which is the set of isosceles triangles, satisfying the predicate, "x has congruent base angles." We refer to x as a *quantified variable*. We use the term *quantify* to mean that an individual is mentally searching for (or anticipating searching for) a specific number of, or quantity of, values of the variable x in the domain of discourse that satisfy the predicate $P(x)$. By *quantification*, we refer to an individual's mental search processes (or anticipated search processes) for a specific number of elements that satisfy the predicate.

Our definitions for the terms *quantifying* and *student quantification* align with constructivist views of meaning, as each individual constructs and reinforces his own quantifications through his own experiences (Thompson et al., 2014). We situate these definitions from a constructivist perspective partially because students may not share conventional meanings for quantifiers in mathematics (Sellers, Roh, & David, 2017; Dubinsky & Yiparaki, 2000; Epp, 1999). Student quantification can be regarded as a meaning for quantifiers as described by Thompson et al. (2014) because quantification is comprised of an *individual's* mental actions or schemes that are easily triggered as a result of the person's understanding (or assimilation to a scheme). In particular, our definition for student quantification emphasizes that a student quantifies in ways that *he* deems necessary based on *his own* interpretation of a given statement, which also aligns with a constructivist view of meaning. Thus, when we refer to a *student's meanings for a quantified variable*, we refer to a student's own constructed quantification for a specific variable in the given statement from the *student's* perspective.

Methodology

We conducted a two-hour long clinical interview (Clement, 2000) with a Calculus student, whom we call Zack, in the spring of 2016. Zack had completed a first semester Calculus course and was currently enrolled in a second semester Calculus course at the time of the interview. We report Zack's interview because of the various meanings for quantified variables that we found across different moments of his interview and our ability to triangulate his words, gestures, and markings on graphs.

Tasks

We asked Zack to interpret and evaluate several complex mathematical statements, shown in Table 1. We did not present Zack with the symbolic representations of the conclusion of each of the statements in Table 1, but we provide these representations in this paper to display the logical differences in the statements. Statement 2 is the IVT and the only true statement. The other three statements reorder the quantifiers and variables in the IVT, but maintain the hypothesis.

Table 1: Statements Presented in Clinical Interviews

Statements	Symbolic Representations
Statement 1 (S1): Suppose that f is a continuous function on the closed interval $[a, b]$, where $f(a) \neq f(b)$. Then, for all real numbers c in (a, b), there exists a real number N between $f(a)$ and $f(b)$, such that $f(c)=N$.	$\forall c(\exists N\, f(c) = N)$

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Statement 2 (S2): Suppose that f is a continuous function on the closed interval $[a, b]$ where $f(a) \neq f(b)$. Then, for all real numbers N between $f(a)$ and $f(b)$, there exists a real number c in (a, b), such that $f(c)=N$.	$\forall N(\exists c\, f(c) = N)$
Statement 3 (S3): Suppose that f is a continuous function on the closed interval $[a, b]$, where $f(a) \neq f(b)$. Then, there exists a real number N between $f(a)$ and $f(b)$, such that for all real numbers c in (a, b), $f(c)=N$.	$\exists N(\forall c\, f(c) = N)$
Statement 4 (S4): Suppose that f is a continuous function on the closed interval $[a, b]$ where $f(a) \neq f(b)$. Then, there exists a real number c in (a, b), such that for all real numbers N between $f(a)$ and $f(b)$, $f(c)=N$.	$\exists c(\forall N\, f(c) = N)$

We first presented each of the statements shown in Table 1 one at a time and asked Zack to explain in his own words the meaning of each statement and also asked him to determine if each statement was true or false. Zack was asked to justify his evaluations and we allowed Zack to draw his own graphs to explain his thinking about his evaluation of each statement. After Zack evaluated all four statements, we provided him with several graphs and asked him if he could use the graphs to support his evaluations of each of the four statements. We allowed (and often asked Zack to) highlight the variables in a given statement on the given graph after he referenced them. These markings as well as his gestures of sweeping or pointing on the graphs were used in the data analysis as an indicator of characteristics for his meanings for the quantified variables in that statement in a given moment.

Data analysis

Our analysis was conducted in the spirit of grounded theory (Strauss & Corbin, 1998). The use of grounded theory allowed new categories to emerge from our data that have not yet been described in the literature regarding students' meanings for quantifiers. We noticed that Zack's meanings for a particular quantified variable with the same statement changed at different moments in the interview. Thus, we also employed Thompson et al.'s (2014) construct of "meanings in the moment" to analyze Zack's meanings for quantified variables. We marked a new moment when Zack was presented with a new interview prompt or task, when he changed his evaluation of a statement, or if he provided a different meaning for a quantified variable while working with the same statement. Every time we found a new meaning for quantified variables in a moment, we added this new meaning into our overall coding system. We also compared Zack's meanings against other students' meanings we interviewed. From this process, similarities and differences in student meanings for quantified variables were refined into four categories. Finally, using the four meanings for quantified variables, we re-analyzed all student interviews and refined our previous coding for each student moment as necessary to ensure that these categories were reliable for all moments, as well as with other students.

Results

We found Zack used four different meanings for quantified variables, which we call MQ1-MQ3 and NQ. Evidence of these meanings came from moments across different moments of Zack's interview. In the subsections that follow, we explain each of the four meanings for quantified variables and provide examples from different moments of the interview when Zack used each of the four meanings.

MQ1: Checking the predicate holds for at least one element

We found that in some moments, Zack described his imagined process of checking the predicate of the statement for *at least one* element of x. We classified his meaning as MQ1 whenever he appeared to strategically search for at least one value of *x*, within his domain of discourse, that satisfies the predicate.

Although MQ1 is consistent with the mathematical convention for existentially-quantified variables, in the following moment, Zack used MQ1 for a variable attached to a *universal* quantifier in the given statement. Zack quantified the variable N in Statement 2 ($\forall N(\exists c\, f(c) = N)$) as follows: "I have read the first part of the second sentence, 'for all real numbers N between $f(a)$ and $f(b)$.' So that made me think, or realize, that there exists a number N between my output variables $f(a)$ and $f(b)$ on this curve." Although Statement 2 contained the phrase "for all N," Zack's interpretation indicates that he was looking for (at least) one N-value as he stated that, "there *exists* a number N."

MQ2: Checking the predicate holds for exactly one element

In some other moments, Zack emphasized that he was looking for *exactly one* value satisfying the predicate. We use MQ2 to refer to Zack's meaning when he appeared to mentally search (or suggested he imagined searching) through every element in the domain of discourse to ensure that *exactly one* value satisfies the predicate. MQ2 follows the mathematical convention for variables attached to "there exists a unique." However, as we see in the moment below, Zack used MQ2 despite the absence of the word "unique" in the provided statements.

We highlight one moment with Zack where he appeared to use MQ2 for the variable N in Statement 4 ($\exists c(\forall N\, f(c) = N)$). Zack first stated that the graph in Figure 1 was "still a function because [Figure 1 is] still in a parabola shape." He then labeled $f(a)$ and $f(b)$ on the graph, and then proceeded to chose a specific value of c, drew the vertical line shown on the graph, and wrote $f(c)=N$ at two different points on the curve. Next, he concluded that Statement 4 was false for this graph and explained his reasoning in the following transcript.

Zack: I don't know if I would use this graph to prove this statement […] I'm getting two output variables from the *c*. So if it was […] a regular parabola, I would say […] If I choose *c* in between *a* and *b*, I am only gonna get one *N*. […] But […] I get two outputs for a *c*.

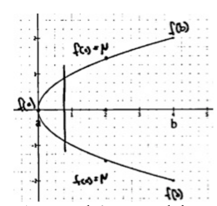

Figure 1. Zack's markings on provided graph for S4.

Zack's explanation above suggests that he anticipated finding one value of N (Lines 3-5). Yet, he found two specific N-values that satisfied the predicate for his chosen c-value (Lines 2, 5), which led him to complete his search process for *exactly one* N-value that satisfies the predicate. Zack evaluated Statement 4 as false because *more than one* element of N satisfied the predicate for the given graph. Zack ultimately determined that this statement was false for this graph because there was more than one N-value for his chosen c, and thus, we claim that Zack utilized MQ2 for N in Statement 4 in the moment above.

MQ3: Checking the predicate holds for all elements

In contrast to the moments in which Zack employed MQ1 or MQ2, we found some moments when he described his imagined process of checking the predicate $f(c)=N$ for *all* values of either c or N. We classified this type of student quantification as MQ3.

We classified Zack's meaning for a quantified variable as MQ3 whenever we observed the following behaviors: (1) he chose a value or values from his own identified domain of discourse and determined whether or not his chosen value(s) satisfy the predicate and (2) he repeated (or imagined repeating) checking the predicate for all elements within his domain of discourse.

MQ3 is akin to the mathematical convention for universally-quantified variables. However, students may use MQ3 for existentially-quantified variables, as shown in the moment with Zack below. In this moment, we focus on Zack's quantification for the variable *c* in Statement 4 ($\exists c(\forall N\, f(c) = N)$. Even though *c* is an existentially-quantified variable in the given statement, Zack's quantification for *c* does not follow convention in this moment.

Zack: If I were to input *c*, whatever number *c* may be, and that's just arbitrary. By choosing that number, I know that I am gonna get *N* […] So, in this case *c* equals $^{-}1$ (*marks c=*$^{-}1$ *on x-axis*) […]
Int: Is there any particular reason why you picked this *c* […]?
Zack: No. I could have […] represented *c* as 2 (*points to the number 2 on the x-axis*) […] I would say *c* only represents one input-output relation at one time. […] I think what I am trying to say is I can choose […] any number between *a* and *b* for an input variable, and that could be *c*. So yeah if I choose again if I choose *c* to be -1 then I know that my *N* would be this number right here (*points to N on the curve*). If I represent any other *x* number to be *c,* then that output would have to be unique to that input.

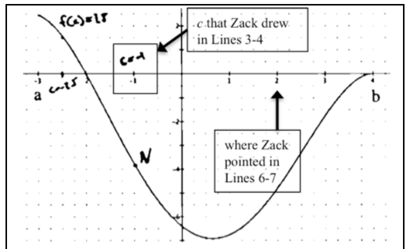

Figure 2. Zack's markings on provided graph for S4.

The transcript above indicates that Zack's domain of discourse was the interval (*a*, *b*) (Lines 9-10) and he checked the predicate for a specific value of *c* in his domain of discourse (Lines 2-3). Initially, he checked the predicate for just this value (Lines 2-3), but also referred to his choice of *c* as *arbitrary* (Line 2). Although Zack chose to check one specific value of *c*, he did not claim that this was the *only* value for *c* that could have been chosen. Instead, he accepted multiple values of *c* in his consideration of the predicate (Line 6). Furthermore, Zack also used words such as *whatever number c* (Line 1) and *any number between a and b* (Line 9), which indicate that Zack considered not only multiple values of *c*, but *all* the values of *c* within his chosen domain of discourse, (*a*, *b*). He also discussed his stipulations for checking the predicate for *any* value for *c* (Line 8). Since Zack explained the satisfaction of the predicate for *any other* values of *c* (Line 11), we conclude that his language suggests an imagined search through all values of *c*, and thus, we conclude that he quantified *c* with MQ3.

NQ: No quantification

Thus far, we have detailed several different ways that Zack quantified variables. In some other moments, Zack did not quantify a variable in the given statements (see Table 1). Regardless of the presence of the quantifier words in a statement, in these moments Zack focused on certain attributes of *x other than the quantity* and often attended to properties of a variable without attending to the number of elements in the domain of discourse. Indeed, Zack did *not* search for a specific number of elements of *x* that satisfied the predicate in these moments. We refer to this type of student quantification (no quantification) as NQ. We illustrate characteristics of NQ from a moment in which Zack first analyzed Statement 3 ($\exists N(\forall c\, f(c) = N)$), without being given any graphs. In this moment, we highlight Zack's quantification for both variables *c* and *N*.

Zack: If I draw this line…I am now going to write *c* in between these two [*a* and *b*] such that for a real number *c* in interval (*a*, *b*), if I put […] *c* into the function, […] then I get my number *N*. So, right here (*marks N on the line*). […] I will get my output variable *N*.
Int: […] In your own words how would you explain this? […]
Zack: So I know that I can only work between my *x* values of *a* and *b* on a closed interval. […] I know that *c* lies in the interval between *a* and *b*. So I am just gonna put it right in the middle. If I put this *c*-value into the function, I get *f*(*c*), which is equal to my number *N* in between *f*(*a*) and *f*(*b*).

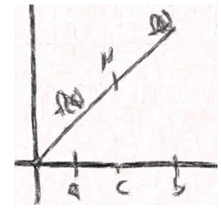

Figure 3. Zack's hand-drawn graph for S3.

Zack stated that he knew that *c* has to be in the interval (*a*, *b*) and also stated that *N* was between *f*(*a*) and *f*(*b*) (Lines 2, 9). Thus, he recognized a domain of discourse for *c* and *N*, respectively. However, Zack does not refer a specific number of *c*-values or *N*-values that he is checking to ensure that *f*(*c*)=*N*. Rather, he stated that he was drawing his graph in such a way that his *c would* yield *N* (Lines 1-4). We take Zack's words as indicative that he is *interpreting* the predicate and drawing his graph in such a way to ensure that he gets *N*-values that satisfy the predicate, *f*(*c*)=*N*. Thus, we conclude that Zack used NQ for both *c* and *N* in this moment.

Conclusion

Zack used four different meanings for quantified variables throughout his interview, which we refer to as MQ1-MQ3 and NQ. MQ1-MQ3 are akin to mathematical conventional uses of the existential, existential unique, and universal quantifier meanings, respectively. Zack also exhibited one other meaning for quantified variables, which we categorized as "No quantification" (NQ). NQ is not characteristic of any conventional mathematical meaning for quantified variables, and we conjecture that in moments where Zack used NQ, he lacked a mental search for a search for a number of elements of either variable *c* or *N* that satisfied the predicate. In these moments where he used NQ, he appeared to only interpret the meaning of the predicate instead of checking the validity of the predicate.

The four categories of meaning are highlighted in Table 2. In this table, we provide descriptions of evidence that we considered in our classifications and the mental actions, which we theorize comprise each meaning for quantified variables. We utilize *x* as an arbitrary quantified variable and X as a generic domain of discourse in Table 2, as these meanings applied to either variable *c* or *N*. We categorized Zack's meaning for a quantified variable based on crucial observable behaviors that distinguished his meaning from another meaning, even if he did not exhibit all observable behaviors listed. These crucial behaviors are italicized in Table 2.

Table 2: Student Meanings for Quantified Variables

Meaning	*Mental Actions*	*Observable Evidence*
MQ1: Checking the predicate holds for at least one *x* in X	1. Identify the domain of discourse, X. 2. Choose (or imagine choosing) one value (x_0) for *x* from X, then check if the predicate is satisfied by x_0 (i.e. $P(x_0)$ is true). 3. Repeat (or imagine repeating) this mental action until at least one value of *x* in X is found that satisfies the predicate. May complete without exhausting all values of *x*.	• Marks off the domain of discourse, X. • Marks one value of *x*, x_0, in X & explains or illustrates whether or not this value of *x* satisfies the predicate. • May mark more values of *x* & may explain or illustrate that at least one value of *x* in X satisfies the predicate. • *Uses phrases such as there is, some, or at least one to refer to the values of x in X that satisfy the predicate.*

MQ2: Checking the predicate holds for exactly one *x* in X	1. Identify the domain of discourse, X. 2. Choose (or imagine choosing) one value (x_0) for *x* from X, then check to determine if the predicate *P*(*x*) is satisfied by this value of *x*. 3. Repeat (or imagine repeating) step 2 until all the elements of *x* in X are exhausted to ensure that exactly one value of *x* in X satisfies the predicate.	• Marks off the domain of discourse, X. • Marks one value of *x*, x_0, that satisfies predicate. • *Claims that the value, x_0, is the only value that satisfies the predicate.* Explains or illustrates that other elements of *x* do not satisfy the predicate. • Uses phrases such as there is exactly one to refer to the values of *x* in X that satisfy the predicate.
MQ3: Checking the predicate holds for all *x* in X	1. Identify a domain of discourse, X. 2. Choose one value (x_0) for *x* from X, then check if the predicate is satisfied by x_0 (i.e. $P(x_0)$ is true). 3. Repeat (or imagine repeating) step 2 until all the values of *x* in X are exhausted.	• Marks off the domain of discourse, X. • Marks one value of *x*, x_0, in X & explains or illustrates whether or not this value of *x* satisfies the predicate. • *Explains or illustrates that the predicate holds for every x.* One possible illustration may be sweeping along X. • Does not use the phrases there is, there exists, or for some, but may use words all, every, each, any or arbitrary to refer to values of *x* in X that satisfy the predicate.
NQ: No quantification	1. Identify the domain of discourse, X. 2. Choose (or imagine choosing) one value (x_0) for *x* from X & interpret predicate *P*(*x*) using the chosen x_0. (i.e. $P(x_0)$ means…)	• Marks off the domain of discourse, X. • May mark a value of *x* in X. • *Interprets the given predicate P(x) in their own words, but does not explain how many values of x satisfy P(x).*

For all four categories, including MQ1-MQ3, Zack often used meanings for quantified variables in unconventional ways, i.e. he used these meanings *regardless of the given quantifier in the given statement*. For example, Zack quantified a variable with a meaning that is more akin to a mathematical meaning for a universally-quantified variable even though the variable is existentially quantified in the given statement and vice versa in different moments.

Discussion

All four meanings for quantified variables that emerged from our study are applicable to analyze students' meanings for quantified variables in other mathematical contexts. Our discovered four meanings may explain findings in previous research, and they have potential to guide teaching in many content domains.

The four categories of meaning that emerged in this study have explanatory power for previous research findings. Students' reasoning from a given statement may not be perceived as erroneous if that reasoning is based on a different interpretation of the variable given in the statement. As previously mentioned, prior research has found that students may determine that a few examples are sufficient to prove a universally-quantified statement (Barkai et al., 2002). This behavior could be explained if the student interpreted the given universal statement with MQ1, checking the predicate for at least one value of *x* in X. If the student perceives that the statement is implying that there should be at least one *x* that satisfies the predicate, then this meaning explains their acceptance of few examples to prove the statement true. As another example, other studies have noted that some students state that one example is insufficient for an existentially-quantified statement (Tirosh & Vinner, 2004). This type of reasoning is not an erroneous argument if a student's current meaning for x is MQ3, searching through all elements of x to ensure that all values of x satisfy a given predicate. Thus, if a part of a quantified statement leads a student to believe that a variable should be quantified in a way different than intended, then students' arguments may also deviate from convention.

These categories for student meanings for quantified variables may also be used by a variety of undergraduate mathematics teachers to aid them in characterizing their own students' meanings. All undergraduate mathematics courses involve quantified statements, and as such, all undergraduate mathematics instructors should be attuned to students' types of quantification. Beyond having a variety of meanings, different parts of a mathematical statement may cause students to quantify in unconventional ways. Students may even skip over quantifier words altogether and interpret pieces of these statements rather than quantifying variables. Thus, we view one of the primary uses of our findings as a tool for educators to identify students' meanings and address their current meanings through questioning and activities that promote reflection of differences in types of quantified variables. Our findings suggest that quantification as a mental process for students is much more complicated than researchers and teachers may expect, and Calculus students in particular could benefit from learning opportunities that would help them confront their various meanings for quantified variables. We hope that teachers and curriculum writers will consider thoughtful questions and activities that will give students an opportunity to become aware of their meanings and address inconsistencies in their meanings for quantified variables.

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