BUSINESS CALCULUS STUDENTS' INTERPRETATIONS OF MARGINAL CHANGE IN ECONOMIC CONTEXTS

Thembinkosi P. Mkhatshwa
Miami University
mkhatstp@miamioh.edu

Despite the overwhelming amount of research on students' interpretations of rates of change in real-world physical science contexts, similar studies in real-world economic contexts are sparse. Contributing towards addressing this gap in knowledge, this study reports on how 12 pairs of business calculus students interpreted marginal change (marginal cost and marginal revenue) in different economic contexts and function representations while solving two optimization tasks. Analysis of task-based interviews conducted with the students revealed that a majority of the students interpreted marginal change as an amount (the difference) and not as a rate (the difference quotient). A few students interpreted marginal change as the derivative. Implications for instruction are discussed.

Keywords: rates of change, marginal change, business calculus, optimization problems

Much research has reported on students' difficulties with interpreting rates of change (average rate of change and instantaneous rate of change) in real-world physical science contexts such as in fluid dynamics, kinematics, and thermodynamics (e.g., Carslon, Jacobs, Coe, Larsen, & Hsu, 2002; Prince, Vigeant, & Nottis, 2012). However, research on how students, especially at the undergraduate level, interpret rates of change (herein referred to as marginal change) in real-world economic contexts is limited, which is the motivation for this study. According to Gordon (2008), more than 300,000 students take business calculus in the United States each year. Understanding marginal change in a business context is vital in fields such as marketing, managerial accounting, supply chain management, finance, and economics. This study reports on business calculus students' interpretations of marginal change while reasoning about two optimization tasks (shown in the methods section), which are situated in two different economic contexts that have different function representations. Our study was guided by this research question: How do business calculus students interpret marginal change when solving optimization problems that have different function representations and are situated in different economic contexts?

The term marginal change, as used in this study, refers to marginal cost or marginal revenue. Marginal cost is the cost per additional unit produced and marginal revenue is the revenue generated per additional unit sold. Mathematically, marginal change can be calculated as an average rate of change where the length of the interval of change is one unit. Marginal change can be approximated using the instantaneous rate of change.

Related Literature

Rates of Change in Context

Several studies have reported on students' tendency to conflate the rate of change of a quantity with either the amount of the quantity or the amount of change of the quantity when solving application problems that are situated in real-world physical science contexts (e.g., Lobato, Hohensee, Rhodehamel, & Diamond, 2012; Prince et al.,2012; Rasmussen & Marrongelle, 2006). A majority of the 373 engineering students in Prince et al.'s (2012) study had difficulty distinguishing between the rate of change and the amount of change of a quantity in a thermodynamics context. Rasmussen and Marrongelle (2006) reported on students who failed to make “a conceptual distinction between rate of change in the amount of salt and amount of salt” (p. 408) in a differential equations course. In a remote-controlled airplane task where the amount of change had the same numerical value as the rate of change (as it is in the economic context of marginal change), Lobato et al. (2012) reported that a majority of the 24 students who participated in their study confused the distance traveled by the airplane in the task with the speed of the airplane. The study reported in this paper extends the body of research on students’ reasoning about rates of change from the physical sciences to economic contexts by examining students’ interpretations of marginal change in two different profit maximization contexts.

Rates of Change in Multiple Function Representations

The importance of the concept of function in students’ learning of calculus cannot be overemphasized. Oehrtman, Carlson, and Thompson (2008) argued that “the concept of function is central to undergraduate mathematics, foundational to modern mathematics, and essential in related areas of the sciences” (p. 27). In his review of research literature on students’ understanding of functions, Thompson (1994c) argued that “tables, graphs and expressions might be multiple representations of functions” (p. 23) to instructors and researchers, but that they are not “multiple representations of anything to students” (p. 23). Thompson’s argument is well supported by evidence from research on students’ understanding of rates of change in multiple function representations (e.g., Klymchuk, Zverkova, & Sauerbier, 2010; Villegas et al., 2009). Much of the existing research on students’ understanding of rates of change in multiple representations is limited to graphical, algebraic, and textual representations of functions. This study extends the literature on students’ understanding of rates of change by reporting on students’ interpretations of rates of change in two functional situations, one that is based on a numerical table of values and another that is algebra-based.

Theoretical Perspective

The current study draws on the theory of quantitative reasoning (Thompson, 1993; Thompson, 1994b; Thompson, 2011). Quantitative reasoning is the analysis of a situation in terms of the quantities and relationships among the quantities involved in the situation (Thompson, 1993). Thompson (2011) described three tenets (a quantity, quantification, and a quantitative operation) that are central to the theory of quantitative reasoning. A quantity is a measurable attribute of an object. Thompson (1993) distinguished between a quantity and a numerical value: a quantity has a unit of measurement, and a numerical value does not. Examples of quantities in this study include marginal cost and marginal revenue.

Quantification is the process of assigning numerical values to quantities (Thompson, 1994d). A quantitative operation is the process of forming a new quantity from other quantities (Thompson, 1994b). In economics, for example, comparing (by way of finding the difference) marginal revenue and marginal cost with the intent to find the excess of marginal revenue against marginal cost otherwise known as marginal profit is a quantitative operation known as a quantitative difference. We designed two mathematical tasks (shown in the next section) that provided students with opportunities to reason about quantities and relationships among quantities. The interview protocol used during data collection engaged pairs of students in reasoning about relationships among quantities. We believe that having pairs of students share ideas while solving the tasks helped to reveal students’ interpretations of the concept of marginal change in detail, something that could have been harder to achieve when interviewing individual students. The interviewing of pairs of students further shifted the students’ focus from the researcher to the tasks.

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Methods

Setting, Participants, and Data Collection

The study participants were 24 undergraduate students at a research university in the United States who had recently completed a business calculus course. The students were chosen based on their willingness to participate in the study, their major (business or economics), and their prior exposure to the ideas of marginal cost and marginal revenue through the course textbook and course lectures. The cumulative grade point averages (GPAs) of the 24 students had a mean of 3.43 on a 4.0 scale, a standard deviation of 0.37, and a range of 1.56. All but two of the students had earned at least a B grade in their business calculus course. The students were recruited from five different sections of a business calculus course.

We remark that the concept of marginal change was poorly presented in the course textbook and in course lectures. Specifically, in both the textbook and in course lectures, marginal change was defined as a rate (the difference quotient) and interpreted as an amount (the difference) (Mkhatshwa, 2016). As we will show in the results section, we argue that defining marginal change as a rate and interpreting it as an amount as was done in course lectures and in the textbook had an influence on students' reasoning about the units of marginal change in Task 2 (and likely in the other tasks we gave to the students) that appears in the following section. Part of the difficulty is that the numeric value of the difference is the same as the numeric value of the difference quotient since the denominator is one in the case of marginal change, a difficulty also noted by Lobato et al. (2012) in a kinematics context. In many cases the "per additional unit" may have been held implicitly by the students when giving units of marginal change in dollars instead of dollars per additional unit. We further remark that students' opportunities to reason about marginal change in course lectures and via the course textbook were mainly limited to the use of algebraic tasks.

Data for the study consisted of (1) transcriptions of audio-recordings of task-based interviews (Goldin, 2000) conducted with 12 pairs of students and (2) work written by the 12 pairs of students during each task-based interview session. Each interview lasted for about one hour and fifteen minutes. The interviews covered four economic tasks, and this study reports on two of the tasks (herein referred to as Task 1 and Task 2). Both tasks are situated in a profit maximization context. In addition, while both tasks are textually represented, Task 1 is also algebraic and Task 2 is numeric (table-based). We describe the design of each of the two tasks:

Task 1 (Haeussler, Paul, & Wood, 2011, p. 617)**:** A manufacturer can produce at most 120 units of a certain product each year. The demand equation for the product is $p = q^2 - 100q + 3200$ and the manufacturer's total cost function is $c = \frac{2}{3}q^3 - 40q^2 + 10{,}000$, where q denotes the number of units that are produced and sold. Find the maximum profit.

We designed Task 1 to examine students' reasoning about optimization problems similar to those given in their textbook and in course lectures. We used this task to examine students' interpretation of marginal change in a profit maximization context when given a task that has an algebraic representation. We also designed this task to examine students' reasoning about relationships among four quantities: the number of units produced and sold, total cost, total revenue (which can be obtained by multiplying the demand equation given in the task by q, the number of units that are sold), and profit (which can be obtained by finding the difference between total revenue and total cost).

Task 2: The following table shows the marginal revenue (MR) and marginal cost (MC) at various production and sales levels (q) for SciTech, a company that specializes in producing and selling computer chips. The company knows that total revenue is greater than total cost at all the production and sales levels shown on the table.

q (units)	400	401	402	403	404	405
MR (marginal revenue)	58	56	55	54	53	51
MC (marginal cost)	52	54	55	57	60	62

What advice can you give to the management of the company about when to increase or decrease production and sales of computer chips?

We designed Task 2 to examine students' reasoning about relationships among four quantities: the number of computer chips produced and sold, marginal cost, marginal revenue, and profit. We used this task to examine students' interpretations of marginal change (e.g., the cost of producing the 401st computer chip) in a profit maximization context when given a task that has a numeric (table-based) representation. We also used Task 2 to examine students' reasoning about the units of marginal change, that is, as a rate (the difference quotient) or as an amount (the difference).

Data Analysis

Data analysis was done in two stages. In the first stage, we carefully read through each interview transcript and coded instances where pairs of students interpreted marginal change (marginal cost or marginal revenue) while they reasoned about each of the two tasks. Some of the codes that emerged from the data (through an inductive process) include interpreting marginal change as an amount i.e. the difference (e.g., saying the units of marginal cost in Task 2 are dollars), interpreting marginal change as a rate i.e. the difference quotient (e.g., saying the units of marginal cost in Task 2 are dollars per unit), and interpreting marginal change as the derivative (e.g., saying the derivative of the total cost function in Task 1 is marginal cost).

In the second stage of our analysis, we did a comparison across tasks to see how interpretations of marginal change given by pairs of students changed (or did not change) across the representations and contexts of the tasks they were given. For example, if a pair of students interpreted marginal change as the derivative in Task 1 (a task with an algebraic representation) and marginal change as an amount in Task 2 (a task with a numerical representation), we concluded that the students' interpretations of marginal change varied with the representation and context of the task they were given. The comparison made in the second stage provided answers to our research question: - How do business calculus students interpret marginal change when solving optimization problems that have different function representations and are situated in different economic contexts?

In light of the tenets of the theory of quantitative reasoning, and as we would show in the next section, some of the students mentally performed quantitative operations such as by

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conceptualizing the quantity of marginal cost as the derivative of the total cost function given in Task 1. We state as a remark that a majority of the students in this study performed a quantitative operation by conceptualizing a measure of profit as the amount of excess revenue above cost when creating a formula for the quantity of profit. These students engaged in the process of quantification (assigning a numerical value for the quantity of profit) by evaluating the profit formula at a particular production and sales level to find the amount of profit at that level. In our previous work (Mkhatshwa & Doerr, 2018), we showed that some of the students in this study performed a quantitative operation by conceptualizing a measure of the quantity of marginal profit as the difference between marginal revenue and marginal cost while working on Task 2.

Results

Analysis of interview transcriptions and work written by students revealed three findings. Because of space limitations, this paper reports on two of the findings. First, nearly all the pairs of students interpreted marginal change as an amount (the difference) and not as a rate (the difference quotient). Second, three pairs of students interpreted marginal change as the derivative.

Interpreting Marginal Change as an Amount

With the exception of one pair of students (Yuri and Kyle), all the other pairs of students indicated that the units of the marginal cost (MC) and marginal revenue (MR) values given in Task 2 would be in dollars. When asked to justify why the units would be in dollars, a majority of these students indicated that "since we are in the United States," the units must be in dollars. We interpreted the students' claim that the units would be in dollars to mean that they interpreted marginal change as a change (the difference), and not as a rate of change (the difference quotient). A critical reader may argue that this may have been less of a conceptual issue and more of a language issue. For example, it is common for people to speak of marginal cost in units of dollars while holding the "per unit" implicitly. However, these students consistently referred to marginal cost in terms of units of a change in total cost (i.e. dollars) and never mentioned the "per additional unit" while reasoning about marginal cost in other tasks. While this may have been a language issue, we argue that because of the consistency at which the students referred to units of marginal cost as dollars and not as dollars per additional unit that the students were interpreting marginal cost as an amount (the difference) and not as a rate (the difference quotient). As noted earlier, part of the difficulty may have been that that the numeric value for the difference and the numeric value for the difference quotient are equal since the denominator in the difference quotient is one in the case of marginal change.

Yuri, however, stated that the units of the MC and MR values in the table shown in Task 2 would be in "dollars per unit" suggesting that he interpreted marginal change as a rate (the difference quotient). The following excerpt illustrates Yuri and Kyle's reasoning about the MC and MR values in Task 2.

Researcher: What are the units of these numbers [pointing at the MR and MC values in Task 2]?
Kyle: Dollars
Researcher: Yuri?
Yuri: Dollars per unit
Kyle: or cents
Researcher: Yuri, why did you say dollars per unit?
Yuri: Marginal revenue is additional, extra revenue per unit
Researcher: Tell me more about that

Yuri: [Silence]

Yuri's statement that "marginal revenue is additional, extra revenue per unit" in the above excerpt suggests that Yuri interpreted marginal change as a rate, the rate [R(q+1)-R(q)]/1, for a total revenue function R(q) where q is the number of units sold. Kyle's initial response that the units of the marginal cost and marginal revenue values are "dollars" suggests that he was interpreting marginal cost and marginal revenue as total cost and total revenue respectively. This was confirmed later in the interview when Kyle stated that the company mentioned in Task 2 breaks even (i.e., total cost equals total revenue) at a production and sales level of 402 computer chips in the table shown in the task when, in fact, marginal cost equals marginal revenue. However, when he added "or cents" after Yuri had said that the units of marginal cost and marginal revenue would be "dollars per unit," Kyle was either agreeing with Yuri that the units were dollars per unit or cents per unit, or he was still interpreting marginal cost as total cost. Regardless of the units given by each of the 24 students for marginal cost and marginal revenue, the fact that these students assigned units to the marginal cost and marginal revenue values suggests that they interpreted marginal cost and marginal revenue as quantities and not as numerical values.

Interpreting Marginal Change as the Derivative

In this study, interpreting marginal change as the derivative refers to an understanding of the quantity of marginal cost (or marginal revenue) as the result of differentiating a cost function (or a revenue function) and not as a quantity that has rate-related units such as dollars per unit. Three pairs of students interpreted marginal change as the derivative while reasoning about Task 1 and Task 2. Two of these pairs of students interpreted marginal change as the derivative only in an algebraic representation (Task 1) and the other pair of students interpreted marginal change as the derivative only in a tabular representation (Task 2). None of these three pairs of students consistently interpreted marginal change as the derivative in both tasks (Task 1 and Task 2) or even in the other two tasks reported in the larger study (Mkhatshwa, 2016). Alan and Sarah are representative of the two pairs of students who interpreted marginal change as the derivative while reasoning about how to solve the problem posed in Task 1. The following excerpt, which occurred early in the interview, illustrates how Sarah and Alan reasoned about what they needed to do in order to answer the question posed in Task 1.

Sarah: Take the derivative of the demand equation [$p = q^2 - 100q + 3200$]
Researcher: What do you get when you take the derivative of the demand equation?
Alan: Is it the marginal?
Sarah: That would be the marginal cost
Researcher: What is marginal cost?
Sarah: The derivative of the total cost [function], right?
Alan: Yah yah, you are right

In the above excerpt, Alan wondered if taking the derivative of the demand equation would give them "the marginal" while Sarah stated that by taking the derivative of the demand equation what they will get "would be the marginal cost." Alan's wondering about the derivative of the demand equation being "the marginal" and Sarah's assertion that marginal cost is "the derivative of the total cost" were taken by the researchers to be the students' interpretations of the derivative of the demand equation by Alan (or the total cost function by Sarah) as marginal cost. Alan and Sarah, did not reason any further about the idea of marginal cost in solving the problem posed in Task 1. It would appear that these students associated the act of taking the derivative of an equation (e.g., the total cost function in the case of Sarah) with the term "marginal."

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Joy and Nancy are the only pair of students who interpreted marginal change as the derivative while reasoning about the units of the MC values and MR values in Task 2. The following excerpt, which occurred towards the end of Task 2, illustrates how Joy and Nancy reasoned about the units of the MC and MR values in Task 2.

Researcher: What do you think are the units of these numbers [pointing at the MR and MC values in Task 2]?
Nancy: Oh, dollars.
Joy: Dollars.
Researcher: How do you know it's dollars?
Joy: Because revenue and cost is dealing with money
Researcher: But that's marginal cost and marginal revenue, is it the same thing?
Nancy: Yah
Researcher: Joy?
Joy: I think, if you like take the, like if you take the derivative of the revenue it gives you the marginal revenue…

Joy's statement that "if you take the derivative of the revenue, it gives you the marginal revenue" suggests that she was interpreting the marginal revenue as the derivative of the revenue function. There is, however, no evidence that Nancy also thought the same way even though she did not object to Joy's statement about the derivative of the revenue function being the marginal revenue. Joy and Nancy went on to calculate differences between the marginal revenue values and marginal cost values at each production and sales level shown in the table that appears in Task 2. They referred to these differences as "profit," suggesting that these students also interpreted marginal cost as total cost and marginal revenue as total revenue as profit is generally defined as the difference between total revenue and total cost, and this is how profit was defined in the textbook used by the students and during course lectures.

Discussion and Conclusions

Nearly all the pairs of students who participated in this study interpreted marginal change as an amount (the difference) and not as a rate of change per unit of one (the difference quotient). These students stated that the units of marginal cost and marginal revenue in Task 2 would be dollars instead of dollars per unit. Throughout their reasoning about marginal change in other tasks (including two other tasks not included in this paper), the students consistently referred to marginal change in units of dollars and not in units of dollars per unit. We argue that part of the difficulty in this may have been the fact that in the case of marginal change, the numeric value for the difference and that of the difference quotient are equal as the denominator in the difference quotient is one. Only one student interpreted marginal change as a rate of change per unit of one. This student stated that the units of marginal cost and marginal revenue in Task 2 would be dollars per unit. As noted earlier, similar results were reported by Lobato et al. (2012) in a kinematics context. To some extent, the students' interpretation of marginal change as both a difference and a difference quotient can be attributed to the opportunities in the textbook they had to learn about marginal change. Specifically, marginal change was defined as a rate per unit of one (the difference quotient) and interpreted as an amount (the difference) in the textbook used by the students (Mkhatshwa, 2016). Also, the presentation of marginal change in course lectures closely followed the presentation of marginal change in the textbook (Mkhatshwa, 2016).

A majority of the students' interpretations of marginal change tended to change within and across the tasks they were given. For example, Sarah and Alan interpreted marginal change as

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the derivative in Task 1 but then they switched to interpreting marginal change as an amount of change with units of dollars in Task 2. Given that the tasks had different function representations and were situated in different economic contexts, these results suggests that the students in our study had weak understandings of the idea of marginal change in different situations. Specifically, these results suggest that students' interpretations of marginal change varied in different economic contexts and representations of economic situations. As noted in our previous work (Mkhatshwa, 2016), opportunities for students to interpret marginal change (e.g., marginal cost) in the textbook and in course lectures were limited. We argue that by providing more opportunities for students to interpret marginal change in different economic situations and function representations, business calculus instructors and business calculus textbook authors might be able to support students towards developing a robust understanding of marginal change.

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