SPONTANEOUS GENERALIZATIONS THROUGH EXAMPLE-BASED REASONING IN A COLLABORATIVE SETTING

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This study explores the progression from student justification to generalization in the course of example-based reasoning. Data was collected through group interviews with high school students who were working collaboratively on a task of determining connections between perimeter and area of tile shaped patterns. The task called for making and justifying conjectures regarding patterns of specific number of tiles. Our findings show that the task elicited collaborative example-based reasoning that evoked spontaneous generalizations about patterns of any number of tiles. The findings point to the importance of collaboration in generalizing, as well as to the intuitive nature of generalizations.

Keywords: Reasoning and Proof, High School Education, Middle School Education, Geometry and Geometrical and Spatial Thinking

Background

The study reported in this paper is part of a larger study on the roles of examples in learning to prove (Knuth, Zaslavsky, & Ellis, 2017). The main goals of this study were to examine ways in which the use of examples may facilitate students' reasoning and proving. This approach reflects a shift from focusing on students' overreliance on examples as a stumbling block to learning to prove (Healy & Hoyles, 2000; Harel & Sowder, 2007), to exploring productive ways of using examples for proving. By example-based reasoning we refer to justifications that use examples to convince one's self or others regarding a certain assertion (Rissland, 1991; Zaslavsky & Shir, 2005). The data set of the study consists of transcriptions of videotaped task-based interviews, mostly with individual students and some with groups of two to three students. In this paper we focus on the group interviews and examine the interplay between example-based reasoning and generalization in a collaborative setting.

Conceptual Framing

Generalization is considered to be an important form of algebraic reasoning and an instrumental component of mathematics education reform (Kaput, 1999; National Council of Teachers of Mathematics, 2000; Ellis, 2007; Ellis, 2011). Early theories and frameworks characterized generalization as an individual act (Ellis, 2011), focusing on theories related to the static nature of transfer in which knowledge remains unchanged during the transfer process (Ellis, 2007). However, researchers have more recently considered the dynamic nature of transfer through an actor-oriented perspective in which generalization may be influenced by instructional environments and social interactions (Jurrow, 2004; Ellis, 2007; Ellis, 2011).

Jurrow (2004) and Ellis (2011) show that students can generalize through peer interaction. Jurrow (2004) describes *linking*, the process of creating and applying classification systems, as a way in which students use talk and interaction to generalize. Ellis (2011) identifies seven categories of generalizing-promoting actions within a collaborative setting, noting how student interactions support and shape the generalizing activities. Ellis (2007) also developed a taxonomy for categorizing generalizations in which a distinction is drawn between generalizing actions (*relating, searching, and extending*) and reflection generalizations (a student's ability to identify or use an existing generalization). Vinner (2011) emphasizes that most generalizations made by students are intuitive in that they are immediate, spontaneous, and rely on global impressions, rather than an analytical thought process. He points to this as one reason why generalizations often fail.

Students construct new knowledge through generalizing activities (Ellis et al, 2017). The generalization schema sorts through similarities and differences, forming a generalization through the similar examples (Vinner, 2011). Examples have been identified as the preferred way in which students justify the truth of their conjectures (Stacey, 1989; Ellis, 2007; Vinner, 2011). However, students may not scrutinize the similarities found in examples, leading to false generalizations (Stacey, 1989; Vinner 2011). Justification is an integral aspect of generalization (Kaput, 1999) helping students to establish conviction in their generalization, as well as to prove the generalization (Ellis, 2007); but students attempt to justify through the use of *empirical examples* rather than *generic examples* (Lannin, 2005). *Empirical* example-use is most often characterized by treatment that confirms or contradicts a given conjecture with no consideration of general structure, while *generic* example-use illuminates the general case through a particular case and may convey the main idea(s) of the relevant proof (Mason & Pimm, 1984; Stylianides, 2008; Leron & Zaslavsky, 2013). Ellis et al (2017) developed the Criteria-Affordances-Purposes-Strategies framework to characterize the ways in which students think with examples while investigating and proving conjectures. The CAPS framework identifies *generalization* as one of the affordances of example-use (Ellis et al, 2017).

Although Ellis (2007) contends that the connection between generalization and justification is bidirectional, existing literature is thin with regards to how students progress from example-based justification to develop a generalization. Our study contributes to this area of research.

The Study

In this part of the study we consider the questions: (*i*) How might students' example-based justification of a particular mathematical conjecture promote generalizations? (ii) What roles do peer interactions play in this process?

Research Instrument

The research instrument for this part of the study was "The Tile Task" - a task designed to elicit example-based reasoning in a geometric context. Participants were given the following information: "A design company offers Tile-Patterns in different sizes and shapes. Each tile is a square with a 1-unit length side. Tile-Patterns are constructed by combining some number of tiles so that each tile shares one full side with at least one other tile. No 'holes' are permitted, and no two tiles are allowed to share only part of a side." The following examples of acceptable tile-patterns were provided (Fig. 1):

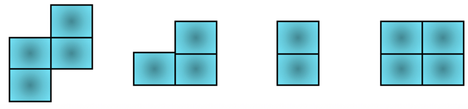

Figure 1. Acceptable tile patterns.

The two parts of the Tile Task that were analyzed for this paper (Question 4 and Question 5) were the same except for the number of tiles used. Question 4 asked: "For a tile-pattern that uses 8 tiles, what would be a shape of the pattern with the smallest perimeter? With the largest perimeter? How do you know?" Question 5 asked the same for a tile-pattern that uses 7 tiles.

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While each of these questions could have been answered by using a previous part of the task in which students determined if the perimeter would increase, decrease, or remain the same by adding one tile, students were not directed to do so.

Data Collection

Students were randomly grouped for participation in four separate one-hour-long semi-structured, task-based clinical interviews. There was one individual interview ('Group 1' – one female), two dyad interviews ('Group 2' – one male one female; 'Group 3' – two females), and one triad interview ('Group 4' – three males). All interviews were recorded and transcribed, and any written work was also collected. Participants used Livescribe pens that captured audio, as well as pen strokes that can be replayed in real time.

Findings

This part of the study revealed that when asked to justify their answer, students attempted to generalize their findings to all tile-patterns regardless of the number of tiles used, with varying degrees of success. Students referred back to the examples they had generated when doing so. Two such instances, described below, occurred after student had identified the shapes of the pattern that would result in the largest and the smallest perimeters for tile-patterns using 8 and 7 tiles, respectively.

Case 1

S1, a 10th grade male, and S2, a 9th grade female, are working as a pair on the Tile Task. After reading Question 4, S1 immediately speaks in general terms about which tile-pattern would make the smallest and largest perimeters. As he speaks he sketches the tile-patterns as in Fig. 1.

S1: Okay. Okay well I guess the largest, the smallest perimeter I guess would be the one that shows the least amount of sides, so...oh. Yeah so so if you have a really long one then that would be the most I think because you're showing as much of the tile as possible for each one and let me see.

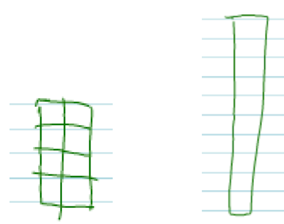

Figure 2. Tile pattern

S2 agrees with this assessment; they count the sides to determine that the smallest perimeter is 12 units and the largest is 16 units (this group incorrectly counted the largest perimeter as 16 while it really is 18 units). Once Question 4 is answered, they move onto Question 5.

S2: I think the largest is still just a long one.

S1: Yeah and this (Fig. 3) comes out to perimeter of twelve like this, so I don't know. I don't-

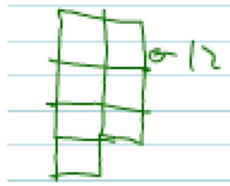

Figure 3. Another tile pattern

S2: I don't think there's anything you can do to make it smaller

S1: Yeah okay so that's smallest and the long one is largest.

Before leaving the Tile-Task, the interviewer asks S1 and S2 to talk about the shapes they used in Questions 4 and 5:

S1: Well the shortest one, or sorry the one with the smallest perimeter is usually in general stout, stouter I don't know how to describe it. Its length, its, the ratio of its length to width was smaller, no larger. I'm not, you put its length over its width it would be closer to one than if you put this length over this width is what I'm saying. So essentially when the closer to a square this gets or the closer to a figure where the ratio of length to width is one, the smaller its perimeter is**.** I think. Does that sound? (refers to Figure 2)

S2: Yeah

S1: So I guess yeah, if we needed to do a formula you could say as l over w approaches 1, the perimeter gets smaller. Or the less the, the less, the more the ratio, or I guess you say the ratio of length over width is inversely proportional to the perimeter. Yeah yeah okay because as the length over width, the ratio itself gets closer to one which is the biggest it can be, the perimeter goes down so yeah.

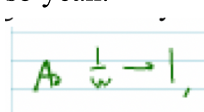

Figure 4. Formula

In describing the shapes used to determine the smallest perimeter, S1 attempts to generalize the relationship between a tile-pattern's length and width with regards to the tile-pattern's perimeter. By doing so, S1 is moving beyond the particular cases concerning tile-patterns that use 7 and 8 tiles into the general structure for all cases, regardless of the number of tiles used. Although S1 explicitly refers to the smallest perimeter, his attempt to generalize also captures all cases for the largest perimeter.

Case 2

The students below are both female, S3 is in 7th grade and S4 is in 9th grade. Each student begins working independently, drawing Figures 5 and 6.

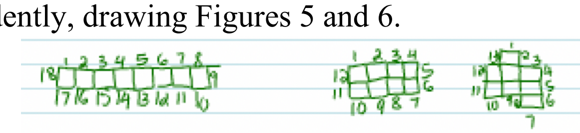

Figure 5. One student's work

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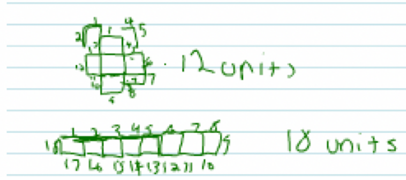

Figure 6. Another student's work

S3: What do you? I said smallest is 12 and longest is 18.
S4: Oh yeah. That's what I got too. But, yeah. I think that, I think that's, is that right? Let me, I'll just try something (Fig. 7).

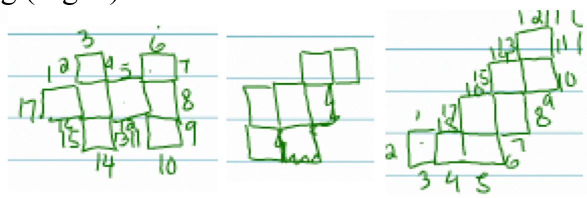

Figure 7. Student's work

S3: Yeah, lemme try a square (Fig. 8).

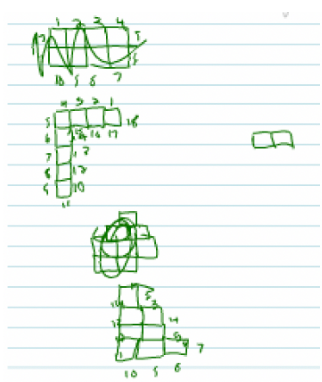

Figure 8. Student's work

S4: Yeah. I think the largest amount is 18.
S3: I'm good with that.
S4: Alright. I think our original answer is justified, um, the largest is 18 units and the smallest is twelve units, right?

The interviewer then asks S3 and S4 how they know that these are the largest and smallest perimeters.

S4: Um, I just personally, I experimented and I found that in um most structures most of the tiles had to only um be exposing to, or, 'cause I kept trying to make it so that each tile was exposing three sides, but um, it didn't work out because you have to plonk the other tile somewhere.

In trying to justify her results for Question 4, S4 talks about structural similarities found in the shapes of the various examples generated. The pair then moved onto Question 5, generating

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several tile-patterns using 7 tiles. After S3 and S4 were in agreement that the smallest perimeter was 12 and the largest 16, the interviewer asked: "How do you know?" The response of S3 is below:

S3: Um. So I tried a different shape than what I did before and it still came in as 12. I feel like if the, um, like if it's larger than I guess 6 maybe, it's going to be, 12 I feel like is the smallest it can go if there's only like 6 tiles, like 5 tiles, the smallest it can go is 12 for the perimeter and then the highest it can go is 2 plus there like, you do 7 plus 7, it's 14 and it's 16, so like the perimeter would be 2 units more than what the number of tiles plus the number of tiles is. (refers to Fig. 9)

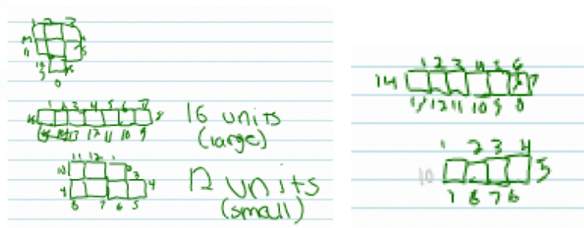

Figure 9. Student tried a different shape

The interviewer probes S3 further, asking if S3 is thinking about the largest perimeter in terms of any number of tiles; S3 replies: "yeah." The following conversation between S3 and S4 ensued:

S4: Then how would you split up the tiles when you're adding them together?
S3: What do you mean, like?
S4: If, if we had 7 tiles, how would you add them together? How would you split them to add them together?
S3: To get a small perimeter or a large perimeter?
S4: A large.
S3: I just did a line and...
S4: But, like applying your rule, how would you do it?
S3: Um, I'm not really sure, but like it's the same thing as the 8 tiles. It came out to 18 for the largest and that's 2 plus 8 plus 8, which is 16, so it's 2 plus, that makes sense.
S4: So it's the number, the number of
S3: Tiles
S4: tiles times two
S3: Yeah.
S4: plus two?
S3: Yeah.
S4: Ok.

Although S3 and S4 did not discuss the shape that would yield the largest perimeter, S3 sought to generalize the numerical value of the largest perimeter, regardless of the number of tiles used. S4 then engaged S3 in a conversation clarifying how to use the "rule" to find the largest perimeter, which further developed the algebraic generalization.

Discussion

The Tile Task was designed with the goal of characterizing students' use of examples in conjecturing and proving when engaged in a task of a geometric nature. As expected, students

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generated their own examples in an effort to answer various questions regarding the size of the perimeter. The spontaneous example-use served to highlight what came naturally to students (Aricha-Metzer & Zaslavsky, 2017) with a goal of building an intuition regarding proving or disproving a conjecture (Zaslavsky, 2017). In addition to this, and as suggested by Vinner (2011), students spontaneously began to investigate the similarities found in their examples in order to make general statements beyond the particular case being considered. Students in each of the four groups, at some point during the Tile Task, engaged in a form of generalization that went beyond the scope of the question asked. This could reflect the intuitive nature of generalizations (Vinner, 2011).

The findings also reveal that students encounter difficulty when shifting from pattern recognition to pattern generalization (Ellis et al, 2017). According to Ellis (2011), students are aided in this shift by sharing ideas with their peers. While individual students initiated the attempt to generalize, the shift from pattern recognition to pattern generalization occurred when students shared their initial ideas with their peers. In case 1, this is evident when both students began to discuss S1's sketches in terms of exposed sides, which lead to an insight of how the length-to-width ratio can be used to determine the shape of a tile-pattern that results in the maximum or minimum perimeter. In case 2, S3 and S4 work individually (e.g. S3: "I tried… S4: "I just personally…") and then share their results. Through this sharing, S3 reveals that she recognizes a pattern between the numbers of tiles used and the largest perimeter of the tile-pattern. S4 propels the generalization forward by asking S3 how she would "split up the tiles" when adding them together. This discussion leads to a general algebraic formula that determines the size of the largest perimeter for a tile-pattern that uses a given number of tiles. Through a collaborative effort, each group volunteered a generalization related to Question 4 and Question 5.

It is also worth noting that while Question 4 and Question 5 discussed in the cases above could have been addressed by using information gained from Question 3, no group drew on this prior knowledge. Question 3 asked if the perimeter of a tile-pattern would increase, decrease, or remain the same following the addition of one tile. All three cases are possible depending on where the additional tile is placed: a) the perimeter decreases when one tile is added to a cavity, b) the perimeter increases when one tile is added to the end of another, and c) the perimeter remains the same when one tile is added to a corner. While only two of the four groups had concluded that the perimeter could decrease, all concluded that the other two cases were possible. However, none of the four groups used the information regarding the increased and constant perimeter when thinking about a shape that would yield the largest and smallest perimeters. The disjointed manner in which students advanced through the Tile Task warrants further exploration.

Pedagogical Implications

The results of this study have pedagogical implications with regards to example-based reasoning and student collaboration. This study shows how both can aid students with generalization. This study also points to the importance of engaging students in rich mathematical tasks.

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