

REPRESENTATIONAL SAMENESS AND DERIVATIVE

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This study focuses on students' understanding of multiple representations of functions. It examines student responses to a task in which calculus students are asked to evaluate the derivative at a point of the cubing function when represented piecewise. Results suggest that attending to the graph of the piecewise function does not improve students' ability to differentiate it. Results also suggest that students tended to view a piecewise-defined function not as a singular function, but as a set of instructions for which function to use.

Keywords: function, derivative, multiple representations

Introduction and Literature Review

Multiple representations of functions play an important role in mathematics and mathematics education. There is a body of literature addressing college students' difficulties linking multiple representations of functions, and some studies suggest that post-secondary students struggle translating between different representations (Even, 1998; Gagatsis and Shiakalli, 2004; Chinnappan and Thomas, 2001). The literature on multiple representations tends to focus on translation between multiple *types* of representations (e.g., graphical, analytical, and verbal), rather than multiple representations of the same type. However, working with multiple representations of the same type is also a crucial part of mathematics. Although there is no body of literature specifically devoted to within-representation-type translation, some authors have highlighted the importance of this for specific types of representations. For example, Moore and Thompson (2015) make the point that math students should be able to move flexibly between different coordinate systems, being able to recognize when the same graph has two different visual representations.

I argue in Mirin (2017) that students' understandings of sameness of representation of function, by which I mean student assessments of which function representations represent the same function, are inextricably linked to their concept of function. We can see how one's concept of sameness-of-representation-of-function and the function concept itself are interlinked when we consider how a student might view derivative. If a student views a derivative as operating on a function, then his concept of function is inextricably tied to his concept of derivative. For example, his criteria for determining whether two function representations share a derivative might be influenced by his criteria for determining whether those representations refer to the same function. This leads us to the following research question: What are students' understandings of multiple analytic representations of a single function as it relates to derivative?

We follow Thompson's (1982) constructivist approach of being sensitive to student understanding by asking, "What is the problem that this student is solving, given that I have attempted to communicate to him the problem I have in mind?" (p.153). This is akin to Harel, Gold, and Simon's (2009) description of the "interpreting" mental act; in analyzing students' responses to a task, we, as researchers, pay careful attention to how students interpret a task. Thompson makes the point that, when referring to representations *of* something, we ought to be clear about *to whom these are representations* of whatever "something" is (Thompson & Sfard, 1994). In the case of this study, there is a representation of the cubing function *to us* (as mathematicians), but to students, it may not be. So, we ought to be sensitive to the fact that a

student might agree with the assertion that two representations of the same function share a derivative, but these students might have non-standard understandings of what “same function” is. In fact, this is precisely the sort of reasoning a particular student used to determine that sharing a graph was not sufficient for sameness of functions; she concluded that two particular representations of functions share the same graph but do *not* share a derivative, leading her to conclude that, to be the same function, having the same ordered pairs on the graph is not sufficient (Mirin, 2017).

Task Design and Methodology

This study arose from an anecdote that Harel and Kaput (1991) share in which calculus students, when prompted to differentiate the function g defined piecewise by $g(x) = \sin x$ if $x \neq 0$ and $g(x) = 1$ if $x = 0$, answered with $g'(x) = \cos x$ if $x \neq 0$ and $g'(x) = 0$ if $x = 0$, appearing to use the constant rule. To these students, the only aspect of the representation as relevant for determining the value of $g'(0)$ is the second line of the piecewise function definition. It seems reasonable to believe that, if the definition of g were modified to instead have $g(x) = 0$ if $x = 0$ (resulting in a nonstandard representation of the sine function) students would answer identically. However, given the anecdotal nature of Harel and Kaput’s claim, there is no data available to substantiate how common such errors are or why they occur.

This paper undertakes the task of studying this phenomenon more systematically. I designed the following task to address this issue:

Let f be the function defined by

$$f(x) = \begin{cases} x^3 & \text{if } x \neq 2 \\ 8 & \text{if } x = 2 \end{cases}$$

Evaluate $f'(2)$, and provide an explanation of your answer.

Figure 1. The Task on which This Study is Based (Quiz A)

Henceforth, the task of evaluating $f'(2)$ for f defined piecewise as above will be referred to as “The Task”. Notice that the function f is simply the cubing function, but represented in a non-standard way. Whether students see it that way is part of the investigation.

Subjects and Methods

Initially, The Task, exactly as pictured in Figure 1, was given to 240 introductory calculus students during the last week of the semester at Anonymous State University (ASU). Referred to as “Quiz A”, it was administered in an exam environment by course instructors as part of the course, where students were required to work silently and independently. I conducted follow-up interviews of eight individual students. I collected and coded their responses before performing the interviews, and, informed by the interviews, re-coded the responses to reflect students’ rationales as suggested by the interviews.

The interviews each lasted 60-90 minutes. The interviews operated according to clinical interview methodology (a la Clement, 2000) and served as establishing students’ rationale for their responses to The Task. Additionally, students were given similar problems, as well as asked to graph the function f and asked to make sense of their answer to The Task with their graph of f .

Results, Quiz A vs. Quiz B

Results of Quiz A suggest that Harel and Kaput (1991)'s anecdote is indeed indicative of a wider phenomenon, as only 18.3% of students gave the correct answer. Like in Harel and Kaput's story, the majority (53.6%) of students claimed that the answer was "0". Further, many students (41.2% of total) explicitly cited the constant rule.

Now that we have established that there is a larger phenomenon, the next natural question to ask is, "why"? It is possible that some students erred due to inattention or carelessness, rather than a major misconception. That is, they might have simply seen the "8" and applied the constant rule out of habit or simply not realized that 2^3 is 8 and that the given function is in fact continuous. This would explain why some students answered "undefined," and it is also consistent with some of the graphs that students volunteered (graphs with removable discontinuities). Further, it might not have occurred to students to compare the graph of f with that of the cubing function - as discussed earlier, the piecewise-defined f is a representation of the cubing function to *us*, but perhaps not to students.

Suppose f is the function defined by $f(x) = x^3$ if $x \neq 2$, $f(x) = 8$ if $x = 2$.

1. Calculate 2^3
2. Below to the right is a graph of $y = x^3$. On the blank axes provided, graph $y = f(x)$ for the function f defined above.

3. Evaluate $f'(2)$ for f defined above. Please provide a 1-3 sentence explanation of your answer. (a)0, (b)8 (c)12, (d)undefined, (e)0 if $x = 2$, and 12 if $x \neq 2$, (f) 0 if $x = 2$, and undefined if $x \neq 2$.
4. Suppose g is a function and h is a function. What does it mean for g and h to be the same function?

Figure 2. Quiz B (Visually Condensed)

Accordingly, informed by Quiz A results, Quiz B (Figure 2, above) was created to test this possibility that inattention accounts for student responses. Quiz B involves The Task (multiple choice form), except, prior to attempting The Task, students are prompted to calculate 2^3 and to graph $y = f(x)$ aside a provided graph of $y = x^3$. Also included on Quiz B is a task asking students to state their definition of when a function g is the same function as a function f . If inattention accounts for student responses, then the following hypotheses should hold:

- (1) Overall, students will perform significantly better on The Task in Quiz B than Quiz A, leaving open the possibility that inattention or carelessness could account for students' tendency to do poorly on The Task in isolation. Students might, because of the prompting, be more likely to compare f to that of the cubing function.
- (2) Quiz B students who answered "12" would be more likely than Quiz A students who answered "12" to provide a justification involving the comparison of f with the cubing function.
- (3) Students who provided a mathematically normative definition of function sameness (Problem 4) would be more likely to answer "12" than students' who did not.

As discussed earlier, a student's criteria for determining whether two function representations share a derivative might be influenced by his or her criteria for determining whether those representations refer to the same function. If a student believes that having the same set of

ordered pairs is sufficient for function h to be the same as function g , then it seems she is more likely (than a student who does not believe this) to conclude that because g and h are the same function, their respective derivatives are the same function. These hypotheses center on the idea that if students are positioned to compare the ordered pairs on $y = f(x)$ to that of $y = x^3$, they are more prone to answer The Task correctly.

Table 1: Student Answers to Quiz A and Quiz B

	0	8	12	Undefined	Multiple answers	Other	Blank	Total
Quiz A %, n	56.3%, 135	4.6%, 11	18.3%, 44	5.8%, 14	(8.3%, 20)	5.4%, 13	1.3%, 3	100%, 240
					0&12: (5.8%, 14)			
					0&undef: (2.5%, 6)			
Quiz B %, n	40.2%, 41	7.8%, 8	23.5 %, 24	9.8%, 10	(18.7%, 19)	N/A	0%, 0	100%, 102
					e: (12.8%, 13)			
					f: (5.9%, 6)			

The data reveal no evidence to support that inattention could account for student responses. Although there was a slight improvement in correctness rate from Quiz A to Quiz B (see Table 1), this improvement was not statistically significant ($\chi^2=1.21$, $p>.05$), contrary to (1). In other words, prompting students to compare the graph of $y = f(x)$ to that of $y = x^3$ did not appear to cause improvement, suggesting that students did not err simply due to inattention to the function's graph. Moreover, the Quiz B students who answered "12" were no more likely than the Quiz A students who answered "12" to draw an explicit comparison between f and the cubing function (4.2% of Quiz A students who answered 12 did, whereas only 2.9% of Quiz B students did so), contrary to (2). Also, the students who provided a mathematically normative definition of function sameness were no more likely to answer "12" than those who did not, contrary to (3). These results suggest that, contrary to my hypotheses, prompting students to compare f to the cubing function did not appear to encourage them to infer that f and the cubing function share a derivative at 2. This naturally led to the emergent question: if inattention to the graph of f does not account for students' tendency to answer incorrectly, then why are students answering the way they are answering?

Phenomena and Student Rationale

To answer this question, we turn to the student graphs together with the student interviews. Normatively, two graphs (of functions) are the same if and only if they consist of the same ordered pairs. It seems reasonable to believe that some students might *not* have this criterion for sameness of graph; indeed, interviews suggested that some students viewed a graph of x^3 with an extra "dot" placed at (2,8) as different from a graph of x^3 without one. Some students referred to the point (2,8) as "separate". Accordingly, a sub-category (category B) of "correct" was created: mathematically normative graphs that highlighted (2,8) in the sense that they had a dot on (2,8) that was more prominent than any other dots. The remaining "correct" graphs - those that were correct but indicated nothing special about (2,8) - were grouped together as category. The remaining graphs were classified as follows: those with a single dot at (2,8) (2.0%), those with just a graph of $y = 8$ (6.9%), those with a removable discontinuity at $x=2$ (6.9%), blank (4.9%), those with graphs of both $y = 8$ and $y = x^3$ (2.9%), and other (8.8%). Among the correct graphs (A and B), graph A students were more likely than graph B students to answer "12" on The Task ($\chi^2=3.932$, $p<.05$), suggesting some sort of difference (in the graph B

students' minds) between f and the cubing function. How students understand their graphs in relation to The Task will be further elaborated below.

Now we turn to the qualitative data: the student interviews, which shed light on why students answer the way they do. Musgrave and Thompson's (2014) construct of "function notation as idiom" was useful in accounting for student responses. A student views function notation idiomatically when he or she views " $f(x)$ " in its entirety as a name for a function (Musgrave & Thompson, 2014). Such students might view " $f(x)$ " as no more than another name for " y " (Thompson, 2013). It appeared that many students used this way of thinking when evaluating $f'(2)$, as students seemed to view " 8 " and " x^3 " as names of functions with " $f(x)$ " referring to both of these functions. Another common theme, appearing both on the written quizzes and in the interviews, was the viewpoint that the "if $x \neq 2$ " served as a *restriction* on the domain for students, rather than as a condition.

Since there is not space to discuss every student in detail, we provide insight from the interviews that is consistent with various student responses. The following subsections should be viewed as *descriptions and illustrations* of student thinking that explain students' answers, rather than rigorous evidence of such phenomena. Moreover, we discuss only the parts of the interviews that explain why students answered the way they did originally, rather than elaborating on the in-depth portions that were more exploratory. Each subsection begins with a direct, written quote from a student, which provides a concise summary of the way of thinking described in the subsection. We also discuss how, for the students, the point (2,8) was special and the way students made sense of their graphs. Additionally, we discuss how students' ways of thinking are reflected in their responses to the interview prompt to find $h'(5)$ for the function h defined by $h(x) = x^3$ if $x \neq 5$, $h(x) = x^2 + 100$ if $x = 5$.

Students who answered "0"

"When the graph is at the point $x=2$, the function is determined by the piecewise part ' 8 '. So, $f(x)$ itself equals 8. When 8 is derived, it becomes 0" [Pete, Quiz B student (emphasis added)]. The rationale summarized by Pete appears to exemplify a common way of thinking amongst students who answered "0". For these students, the " $f'(2)$ " tells them that they are in the situation " $x = 2$," which serves as an instruction to use the function " 8 ". Here, the " 8 " serves as a name of a function rather than a particular output, suggesting an idiomatic conception. Many of these students provided a category B graph of f (graph of $y = x^3$ but a special dot at (2,8)) and found no issue with the fact that they couldn't "see" that $f'(2) = 0$ in their graph; when asked to explain graphically, they would provide a graph of $y = 8$ and explained why its derivative at 2 is 0.

Interviewed students extended this way of thinking to evaluating $h'(5)$ for the function h defined by $h(x) = x^3$ if $x \neq 5$, $h(x) = x^2 + 100$ if $x = 5$. It was common for students to answer "10" by evaluating the derivative of $x^2 + 100$ as $2x$ and substituting $x=5$ to result in 10, with the rationale that "Um I used this part, the part that makes the parabola [$y = x^2 + 100$]. Because we're interested in the time when x equals 5. And that's kind of the rule here, when x equals 5 to use the parabola" [Jennifer, Quiz A student]. She elaborated: "The derivative of h when x equals 5 is gonna be $2x$ um....if x were to equal some number other than 5, you would use this (underlines x^3) function up here, but because x is 5 we use this one." Jennifer's rationale exemplifies the way of thinking that led students to answer " $f'(2) = 0$ ": viewing the conditions on a piecewise-defined function as instructions for which function to use, and a piecewise-defined function involving two different functions.

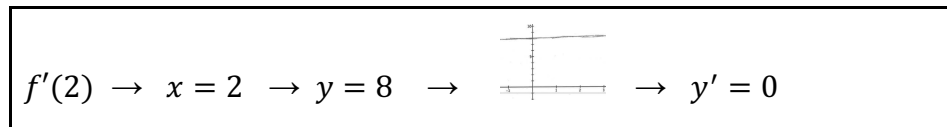


Figure 3. Student rationale for answering “0”

Many of these students (Pete included) provided a graph of f that was like $y = x^3$ but with a dot at (2,8) (category B graph). It seems that students viewed the dot at (2,8) as separate or independent from the rest of the graph. For example, one student recreated his graph during the interview, explaining his reasoning as follows: “At the point (2,8) I draw a circle to show there is an opening there, there’s a gap. I’m excluding that point from what it is we are talking about in this point in time.” He elaborated: “So the *two*...they’re existing on the same coordinate system but *existing independent of each other*” (emphasis added).

Students who answered both “12” and “0”

“If $f(x)$ does not equal 2, the function is x^3 . The derivative of x^3 equals $3x^2$, then substitute 2 for x , $3(2)^2=12$. However, if x is allowed to be 2, then the derivative of $8=0$ ” [Carlos, Quiz A student]. The case of Carlos illustrates how a student can reason idiomatically to get the answers 0 and 12. In the interview he reiterated his reasoning: “If x isn’t 2 then the function is x^3 . The derivative of x^3 is $3x^2$. Then substitute 2 for x here and you get 12. However, if x is allowed to be 2, then the derivative of 8 is 0”. For Carlos, the “if $x=2$ ” condition told him that he was in the case in which “the function” is the function “ $f(x) = 8$,” and that the “if $x \neq 2$ ” condition told him he was in the case in which “the function” is x^3 . Carlos did not even make the connection that the “2” in “ $f(2)$ ” told him he was in the case where “ $x = 2$ ”; for him, the “ $f(x)$ ” was just a shorthand for “ y ”. When prompted to graph f , he provided a graph of (what he thought was) $y=8$ as well as a graph of $y = x^3$, indicating that he viewed himself as graphing two separate functions. When asked how $f'(2)$ can be 12 while he had said prior that it was 0, he explained: “this is an *entirely different function*”, indicating that the conditions on the piecewise function were instructions about which function to use.

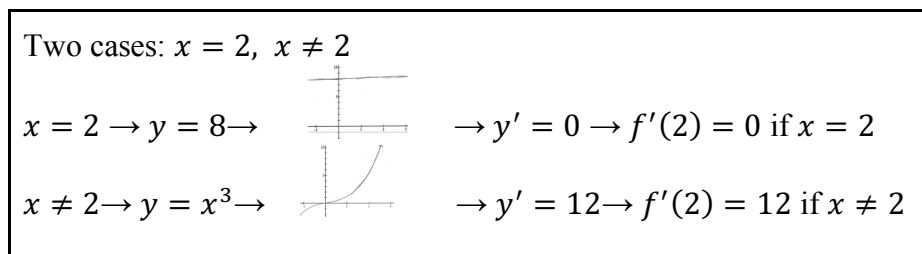


Figure 4. Student Rationale for answering “0 if $x = 2$, 12 if $x \neq 2$ ”

Carlos’ way of thinking was confirmed when he was asked to calculate $h'(5)$ when h is defined by $h(x) = x^3$ if $x \neq 5$, $h(x) = x^2 + 100$ if $x=5$. He graphed $y = x^3$ and $y = x^2 + 100$ on the same axes (Fig.4). When prompted to find the value of $h'(5)$, he differentiated x^3 and plugged in 5 to get 75, and then he differentiated $x^2 + 100$ and plugged in 5 to get 10. When asked which was the value of $h'(5)$, he exclaimed confidently, “both! 75 and 10!”.

Students who answered both “0” and “undefined”

“If just looking at $f(x) = 8$, the derivative of a constant would make $f'(2) = 0$. If just looking at $f(x) = x^3$, the derivative would be undefined because $f(2)$ is not on the graph of x^3 .”

There is a hole at $x=2$ " [Eric, Quiz B]. Eric's reasoning exemplifies how students could have come to select choice "F" in Quiz B. A different student, Sarah, explained her reasoning in detail in the interview. Sarah initially answered that "both" are undefined, but during the interview, she revealed that she interprets "0" to mean the same thing as "undefined" (which was a common trend in student responses). Like Carlos, she viewed two functions as being involved, which was again confirmed when she was asked about the piecewise-defined function " h ". She appeared to reason about two different functions, and calculated $f'(2)$ by treating the first function as " $y = x^3, x \neq 2$ ", and the second function as " $y = 8, x = 2$ ". She interpreted the " $x \neq 2$ " as a restriction on the *first* function, and the " $x = 2$ " a clarification that such a restriction did not exist on the *second* function. Thus, for the first function, $f'(2)$ is undefined, and for the second function, $f'(2)$ equals 12.

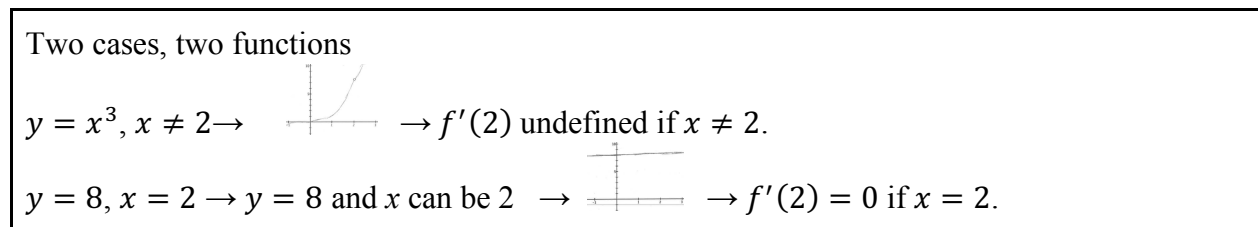


Figure 5. Student Rationale for answering "0 if $x = 2$, undefined if $x \neq 2$ "

Conclusions and Discussion

The results of Quiz A showed us that Harel and Kaput's (1991) anecdote is indeed indicative of a larger phenomenon: many students appeared to differentiate a piecewise function formally by differentiating each expression as a separate function. By comparing the results of Quiz A to Quiz B, we confirmed that this phenomenon cannot be attributed merely to inattention. The interviews, together with the Quiz B results, suggest that a non-normative understanding of piecewise function notation, stemming from a view of function notation as idiom and the conditions on the domain as either instructions or as restrictions, accounts for many students' responses.

This study shows that students do not view the same function, represented in two different analytic ways, as sharing a derivative at a particular value. However, this last sentence was ambiguous; when we say "a function", we are not being clear if students view these function representations as referring to the *same* function. Students might, for example, consider it possible for two *distinct* functions to share a graph, and we can ask: do students believe that same graph implies same derivative? The answer to this appears to be "no," as many Quiz B students provided normative graphs of f yet did not evaluate $f'(2)$ normatively. Yet, we run into another ambiguity: what students view as "same graph" might not be consistent with the normative notion of "same graph," as suggested by students' insistence that the point (2,8) being highlighted. This means that, although it is tempting to conceptualize this study as one about students' understanding of derivative, its results highlight how students think about function notation. To illustrate this point consider the way of thinking that accounted for students answering "0." It arose from a misconception of function notation: no matter how strong of a meaning the student has of "derivative", the student was still reasoning with the graph of " $y = 8$ ", leading to an answer of "0".

As discussed earlier, it seemed reasonable to hypothesize that students who provided normative definitions of what it means for functions g and h to be the same (Problem 4, Quiz B)

would be more likely to correctly evaluate $f'(2)$; this is because it seems these students would be more likely to assess piecewise-defined f and the cubing function as “the same,” positioning them to infer that f and the cubing function share a derivative. In light of the interviews and students’ ways of thinking, the counter-intuitive result – that this hypothesis did not hold – makes sense. This is because, to students, f was not a function in the same way that the cubing function is; instead, f was *two functions*. Having a strong criteria for sameness of functions did not help students evaluate $f'(2)$ because f was not in the category of “functions” to which sameness can apply!

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