HOW DO UNDERGRADUATE STUDENTS MAKE SENSE OF POINTS ON GRAPHS IN CALCULUS CONTEXTS?

Erika David
Arizona State University
ejdavid@asu.edu

Kyeong Hah Roh
Arizona State University
khroh@asu.edu

Morgan Sellers
Arizona State University
mearly1@asu.edu

The purpose of this study is to examine the characteristics of students' thinking about graphs while evaluating statements from Calculus. We conducted clinical interviews in which undergraduate students evaluated mathematical statements using graphs to explain their reasoning. We report our classification of students' thinking about aspects of graphs in terms of value-thinking and location-thinking, which emerged from our data. These two ways of thinking were rooted in students' attention to different attributes of points on graphs we provided: either the input and output values represented by the points or the location of the points in space. Our findings indicate that students' thinking about aspects of graphs accounts for key differences in their understandings of mathematical statements.

Keywords: Geometry and Geometrical and Spatial Thinking, Post-Secondary Education, Cognition, Reasoning and Proof

The purpose of this study is to characterize students' thinking about aspects of graphs of real-valued functions and to investigate its role in understanding and evaluating statements from Calculus, such as the Intermediate Value Theorem (IVT). Through analyzing students' evaluations and interpretations of these statements using graphs, we seek to address the following research questions: *What are characteristics of undergraduate students' thinking about aspects of graphs related to statements from Calculus contexts?* Specifically,

(1) How do students interpret outputs of a function on a graph, points on a graph, and a graph as a whole?
(2) How do various types of student thinking about graphs of real-valued functions affect students' understanding and evaluation of the Intermediate Value Theorem and similar statements?

Literature Review

Undergraduate Calculus courses, from elementary through advanced Calculus, are comprised of many definitions and theorems about real-valued functions. Often, these statements are accompanied by visual representations in the form of graphs of relevant functions. For example, the Intermediate Value Theorem (IVT) is one such statement commonly associated with a visual representation (e.g., Briggs, Cochran, & Gillett, 2011; Finney, Thomas, Demana, & Waits, 1994; Larson, Hostetler, & Edwards, 1994; Stewart, 2012). Although research has called for the inclusion of such visual representations in mathematics instruction (e.g., Arcavi, 2003; Davis, 1993; Dreyfus, 1991; Hanna & Sidoli, 2007), few empirical studies have been conducted to look at undergraduate students' thinking about graphs of real-valued functions.

While it is hoped that students focus on the details of a provided graph that highlight the intended concept, some students may construe other properties of the given graph rather than the intended ones. For example, Moore and Thompson (2015) found that some students treat graphs as an object itself, and infer details of a situation from the shape of a graph, rather than coordinating the numerical values represented by the points of the graph. If students'

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interpretations of graphs differ from what is intended, students' interpretations of provided graphs might hinder their subsequent mathematical activities, such as rigorous proofs (Alcock & Simpson, 2004). Although several studies have looked at students' understanding of graphs as a whole (Monk, 1992; Moore & Thompson, 2015; Moore, 2016), it is not widely known what meaning students have for various aspects of graphs, such as the input, output, and points on graphs.

Theoretical Perspective

This study, including the data collection, data analysis, and development of the theoretical framework, is grounded in a constructivist perspective. We adopt von Glasersfeld's (1995) view that students' knowledge consists of a set of action schemes that are increasingly viable given their experience. This perspective also implies that we, as researchers, do not have direct access to students' knowledge and can only model their thinking about graphs based upon their observable behaviors.

We adopt components of Arcavi's (2003) definition of visualization for this study, which he states as "The ability, the process, and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper, or with technological tools" (p. 217). While Arcavi's description of visualization is broad, in this paper, we focus on investigating how *students* interpret, use, and think about aspects of graphs of real-valued functions and relations.

Methodology

As part of a larger study, we conducted two-hour clinical interviews (Clement, 2000) with nine undergraduate students from a public southwestern university in the United States. We selected three undergraduates who had just completed one of the following three mathematical courses that may cover the IVT: Calculus I, Introduction to Proof, and Advanced Calculus. During the interview, the interviewer asked students to evaluate each of the four mathematical statements in Table 1 and to provide justification for their evaluations. The second statement, the Intermediate Value Theorem (IVT), was the only true statement we presented. The remaining three statements (1, 3, and 4), all of which are false, were created from the IVT by reordering the quantifiers (for all, there exists) and/or the variables (N, c).

Table 1: Statements Presented to Students

Statement 1	Suppose that f is a continuous function on $[a, b]$ such that $f(a)\neq f(b)$. Then, for all real numbers c in (a, b), there exists a real number N between $f(a)$ and $f(b)$ such that $f(c)=N$.
Statement 2 (IVT)	Suppose that f is a continuous function on $[a, b]$ such that $f(a)\neq f(b)$. Then, for all real numbers N between $f(a)$ and $f(b)$, there exists a real number c in (a, b) such that $f(c)=N$.
Statement 3	Suppose that f is a continuous function on $[a, b]$ such that $f(a)\neq f(b)$. Then, there exists a real number N between $f(a)$ and $f(b)$ such that for all real numbers c in (a, b), $f(c)=N$.
Statement 4	Suppose that f is a continuous function on $[a, b]$ such that $f(a)\neq f(b)$. Then, there exists a real number c in (a, b) such that for all real numbers N between $f(a)$ and $f(b)$, $f(c)=N$.

After students evaluated the four statements, the interviewer presented six graphs. The graphs were chosen to represent a spectrum of possible functions, relations, and relevant counterexamples and included: a polynomial with extrema beyond the endpoints of the displayed function, a vertical line segment, a continuous sinusoidal function, a monotone increasing function, a constant function, and a function that is discontinuous on $[a, b]$. The interviewer asked if the students could use any of these graphs to explain their evaluation of each statement,

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which they could change at any time. Students were also asked to explain how they interpreted various aspects of each graph and to label relevant points and values on the graphs where appropriate.

Our data analysis was consistent with Corbin and Strauss' (2014) description of grounded theory, in which our categories of students' thinking about the graphs emerged from the data analysis. We began preliminary analysis during and immediately following each interview to note relevant findings. After all the interviews were conducted, we employed open coding (Corbin & Strauss, 2014) to document students' interpretations of aspects of the graphs they worked with. We refined these categories and re-coded the video interview data using axial coding (Corbin & Strauss, 2014). Through this process, we finalized two codes, *value-thinking* and *location-thinking*, to broadly characterize student thinking about the graphs.

Results

In our data, we found two distinct ways that students thought about graphs, specifically outputs, points, and the graph as a whole. We observed that some students primarily focused on the *values* represented by the coordinates of points on graphs while others primarily attended to the spatial *location* of these points. To describe these findings, we use the term *value-thinking* to refer to student thinking which focused on the *values* represented by the coordinates of a point. We use the term *location-thinking* to refer to thinking that primarily attends to the spatial *location* of the point. We detail characteristics of both categories of thinking in Table 2 by listing the meanings for aspects of the graph for each category and observable behaviors indicative of these meanings.

Table 2: Comparison of Value-Thinking and Location-Thinking

<table>
<tr><th colspan="2"></th><th colspan="2">Value-Thinking</th><th colspan="2">Location-Thinking</th></tr>
<tr><th colspan="2"></th><th>Interpretations</th><th>Evidence</th><th>Interpretations</th><th>Evidence</th></tr>
<tr><td rowspan="3">Aspects of a Graph</td><td>Output of Function</td><td>The resulting value from inputting a value in the function</td><td>▪ Labels output values on the output axis
▪ Speaks about output values</td><td>The resulting location in the Cartesian plane from inputting a value in the function</td><td rowspan="3">▪ Labels outputs on the graph
▪ Labels points as outputs
▪ Speaks about points as a result of an input into the function (e.g., “an input is mapped to a point on the graph”)</td></tr>
<tr><td>Point on Graph</td><td>The coordinated values of the input and output represented together</td><td rowspan="2">▪ Labels points as ordered pairs
▪ Speaks about points as the result of coordinating input and output values</td><td>A specified spatial location in the Cartesian plane</td></tr>
<tr><td>Graph as a Whole</td><td>A collection of coordinate pairs of values of the input and output</td><td>A collection of spatial locations in the Cartesian plane associated with input values</td></tr>
</table>

Value-Thinking

By *value-thinking*, we mean thinking about graphs that focuses on the input and output values represented by the coordinates of points on the graph of a function. Figure 2 contains two labeled graphs from one of the students, Jay, whom we considered to be engaged in value-thinking. One of the key characteristics of value-thinking is distinguishing between the outputs of a function and the points on a graph. Students who engaged in value-thinking labeled points as

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ordered pairs (see Figure 2, left), and spoke about points as representing both input and output values simultaneously. When considering the output of a function, students who thought in this way tended to label relevant output values on the output axis of the graph of a function (see Figure 2, right), and specifically spoke about the *values* of the output only. Students engaged in value-thinking thus treated graphs as a collection of ordered pairs that relate corresponding input and output values. In terms of the four statements, these students interpreted *N* in the phrase "*N* between *f*(*a*) and *f*(*b*)" as referring to *values* between two values *f*(*a*) and *f*(*b*). They typically indicated the *N* values on the *y*-axis, as shown in Figure 1, right.

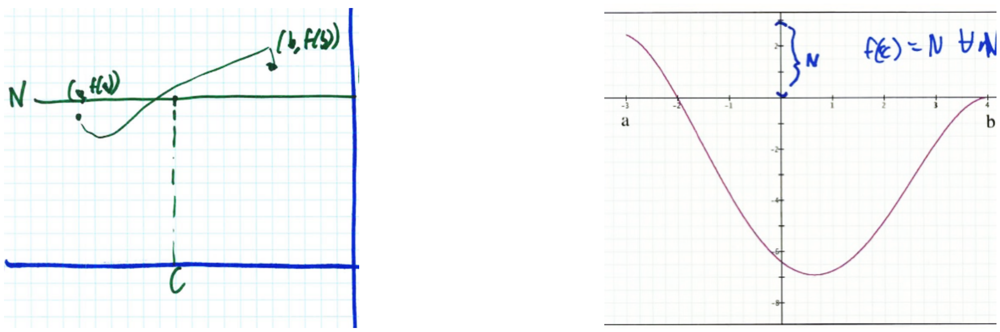

Figure 1. Jay's graph labels, characteristic of value-thinking

Location-Thinking

By *location-thinking,* we mean thinking about graphs that relies on the spatial locations of points in a Cartesian plane. Students who engaged in location-thinking focused on the location of points, while the values of the coordinates for the points were either in the background of their reasoning or absent from it. In contrast with value-thinking, one of the key characteristics of location-thinking we observed was treating the output of the function as indistinguishable from the location of the point. Accordingly, students who engaged in location-thinking often labeled points on the graph as outputs, rather than ordered pairs, and spoke about points in terms of their location in the coordinate plane. While students who engaged in value-thinking labeled outputs on the output axis, students who engaged in location-thinking frequently placed the output label at the location of the point on the graph. Instead of speaking about output *values*, these students speak about *points on the graph* as the result of an input value. Students engaged in location-thinking treated graphs as a collection of locations in space associated with inputs.

Outputs as locations. Zack was one such student who labeled *f*(*a*), *N*, and *f*(*b*) not on the *y*-axis, but on the graphs that he drew, as shown in Figure 3. As a result, Zack did not label points as ordered pairs.

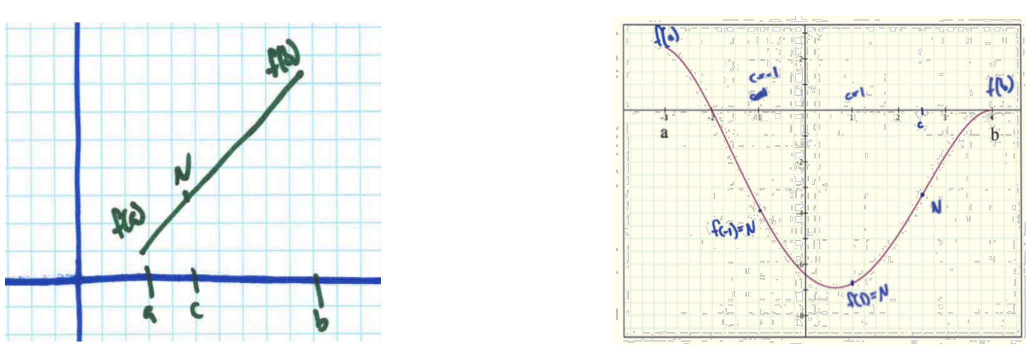

Figure 3. Zack's labels of points as outputs, a common characteristic of location-thinking

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Zack's placement of an output label at a location on the graph, rather than on the y-axis, indicates that he considered these outputs to be locations on the graph, rather than values on the y-axis. Additionally, Zack called the endpoints of the graph in Figure 3, right "f(a)" and "f(b)," again indicating that he conceived of outputs as locations on the graph. Furthermore, when Zack referenced possible N's between f(a) and f(b), he swept along the entire graph of the function, rather than along the y-axis. His gesture along the graph when describing N's between f(a) and f(b) is also consistent with his conception of outputs as locations along the graph. We thus take Zack's label on the graph, words, and gestures as evidence of his consideration of outputs of the function as locations, indicative of location-thinking.

Points as locations. In addition to his graph labels of outputs at points in Figure 3, more striking evidence of Zack's location-thinking was observed when he was working with a constant function and claimed that f(a) is not equal to f(b). When the interviewer presented Zack with the graph of a constant function, Zack confirmed that the function is continuous on the interval [a, b], pointed to the endpoints of the graph, and stated that f(a) is not equal to f(b). Zack read off the remainder of Statement 3, and again pointed to the endpoints of the graph when reading the phrase "N between f(a) and f(b)." Next, he pointed to a spot on the graph, which he explained was an example of N between f(a) and f(b), plotted a dot there, and labeled the dot on the graph as "N." He also labeled the endpoints of the graph as f(a) and f(b), respectively, as shown in Figure 4.

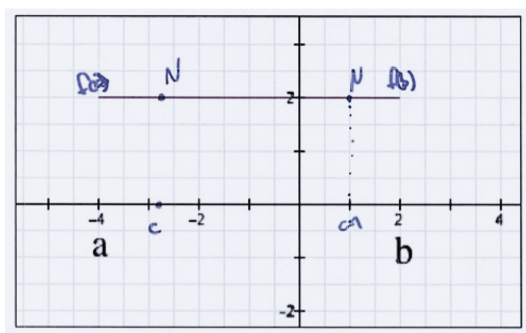

Figure 4. Zack's labeling of *f*(*a*), *f*(*b*), and possible *N*'s when he claimed $f(a) \neq f(b)$.

We take Zack's claim that *f*(*a*) and *f*(*b*) are not equal for the constant function *f* as evidence that he attended to the different spatial *locations* of the endpoints, rather than the pairs of input and output *values* represented at each point. For Zack, the point *N* that he labeled on the graph was in between the *locations* of the endpoints, which he referred to as *f*(*a*) and *f*(*b*). We also note that Zack labeled points as *f*(*a*), *f*(*b*), and *N*, rather than as ordered pairs. In essence, for Zack, there was no difference between outputs of the function and points on the graph, as both referred to spatial locations on the graph. We thus conclude that Zack conceived of *points as locations*, an indication of location-thinking.

***N* as a location between *f*(*a*) and *f*(*b*).** Like Zack and other students who engaged in location-thinking, Nate also considered outputs as locations, rather than values, and points as locations, rather than ordered pairs. We highlight Nate's meaning for the phrase "*N* between *f*(*a*) and *f*(*b*)," which was indicative of location-thinking. When working with one of the provided graphs, Nate first labeled the endpoints of the graph as *f*(*a*) and *f*(*b*), respectively. Then, Nate explained that for every *c* on this axis, he could find an *N* on the curve that *c* maps to. He also motioned from the *x*-axis vertically to the graph when describing that *c*'s mapped to *N*'s *on the*

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graph. Similarly, when describing *N*’s, he swept along the entire graph of the function from what he marked as *f*(*a*) to *f*(*b*). Nate’s graph labels are shown in Figure 5 below.

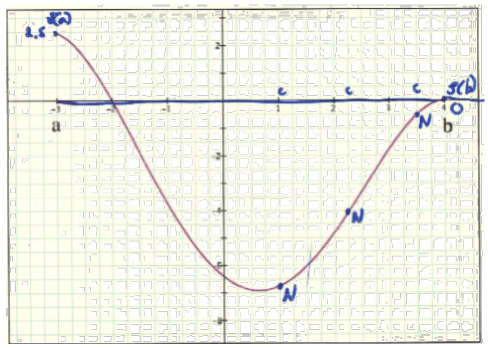

Figure 5. Examples of N’s Nate claimed were between f(a) and f(b)

Nate labeled possible *N*’s on the graph that are, from our perspective, not between *f*(*a*) and *f*(*b*). Noticing Nate’s placement of *N* labels and his sweeping motion along the graph, we infer that Nate interpreted “*N* between *f*(*a*) and *f*(*b*)” to mean *all* the points on the graph between the points that he labeled *f*(*a*) and *f*(*b*).

To further examine Nate’s meaning for this phrase, the interviewer extended the graph to the right and marked a point on this extension of the graph, at approximately (5, 1), whose output value, 1, is between the values of *f*(*a*) and *f*(*b*) (see Figure 6). The interviewer then asked Nate if this output was between *f*(*a*) and *f*(*b*). After thinking about the question briefly, Nate stated that the output was *not* between *f*(*a*) and *f*(*b*).

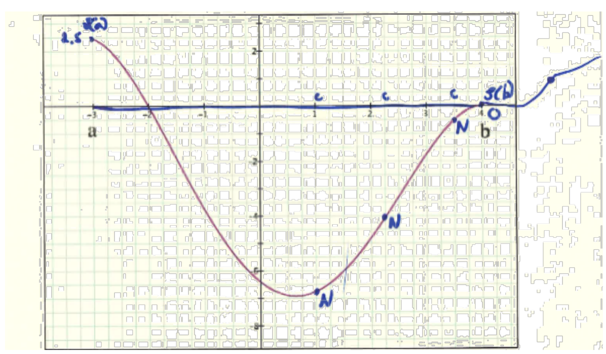

Figure 6. A point at which Nate claimed the output was not between f(a) and f(b)

The interviewer’s prompt allowed Nate to more carefully consider his meaning for “*N* between *f*(*a*) and *f*(*b*).” Nate explained that there are two possible interpretations of this phrase, which he described as a “number interval” and a “function interval.” By “number interval,” Nate referred to the set of all output values between 0 and 2.5. In contrast, Nate used “function interval” to refer to the set of all points on the graph between the endpoints which Nate labeled *f*(*a*) and *f*(*b*). While Nate said the point that the interviewer marked was not between *f*(*a*) and *f*(*b*), he acknowledged that this point was “in the number interval” between 2.5 and 0 (the values of *f*(*a*) and *f*(*b*)). Nate clarified that although the output value of this point was between 0 and 2.5, the point was not between *f*(*a*) and *f*(*b*) because it was not “in the function interval.” As he described the “function interval,” Nate motioned along the entire graph between the points which he had labeled *f*(*a*) and *f*(*b*). Although Nate acknowledged the numerical interval of output

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values, [0, 2.5], he considered his notion of the "function interval" as more relevant for interpreting the phrase "N between $f(a)$ and $f(b)$." We thus take Nate's interpretation of "N between $f(a)$ and $f(b)$" in terms of the *spatial location* of the points, rather than the values of the outputs, as indicative of *location-thinking*.

Value-Thinking, Location-Thinking, & Students' Evaluations of Statements

We found that students' interpretations of aspects of graphs, whether in terms of value-thinking or location-thinking, were related to their evaluations of the four statements we presented. Among the five engaged in value-thinking, three students evaluated all four statements correctly. In contrast, no student engaged in location-thinking evaluated all four statements correctly. Table 3 reports our classification of each student's thinking (Value-Thinking or Location-Thinking), along with each student's mathematical level (Calculus, Introduction to Proof, and Advanced Calculus), and final evaluations of the four statements (True, False, or Sometimes True).

Table 3: Students' Final Evaluations of Statements

Students Observed Engaging in…	Student Name	Math Level	Final Student Evaluations S1(F)	S2(T)	S3(F)	S4(F)
Value-Thinking	Jay	Advanced Calculus	F	T	F	F
	James	Advanced Calculus	F	T	F	F
	Nikki	Introduction to Proof	F	T	F	F
	Ron	Introduction to Proof	T	T	F	F
	Mike	Introduction to Proof	F	F	F	F
Location-Thinking	Zack	Calculus	ST	ST	ST	ST
	Nate	Advanced Calculus	T	T	F	F
	Hannah	Calculus	T	T	T	T
	Marie	Calculus	T	T	T	T

Shaded cells indicate mathematically incorrect evaluations.

Although all students who correctly evaluated the statements engaged in value-thinking, not all the students who engaged in value-thinking evaluated the statements correctly. Ron, a student who engaged in value-thinking, failed to attend to the restriction on the values of N, which led him to evaluate statement 1 as true. Mike, another student who engaged in value-thinking, evaluated all four statements as false due to unconventional meanings for the logical quantifiers (for all, there exists) involved in the statements. Thus, we view value-thinking as necessary, but not sufficient, for correctly evaluating the IVT and similar statements. In contrast, no student who engaged in location-thinking evaluated all four statements correctly. We take this as an indication that location-thinking does not support students in correctly evaluating the statements we presented. Even Nate, a student with a more advanced mathematical background, incorrectly evaluated Statement 1. His location-thinking was the main factor in his incorrect evaluation.

Conclusion & Discussion

Our findings in this study reveal critical distinctions in students' interpretations of aspects of graphs, namely in terms of value-thinking and location-thinking. In our study, some students interpreted and labeled points as pairs of input and output values, while others interpreted points as locations in space and labeled them with output notation. Our results highlight and explain significant aspects of students' interpretations of graphs not previously accounted for by current theories and studies on students' thinking about graphs. Thus, the use of our constructs of value-

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thinking and location-thinking may progress the depth of analysis in the field of students' understanding of graphs.

These two distinct ways in which students interpreted points, and thus graphs, have significant implications for how students understand important mathematical ideas, such as the Intermediate Value Theorem (IVT), as we observed with our students. We acknowledge that, in this study, value-thinking supported students in correctly evaluating the IVT and similar statements. However, in other contexts, such as diagrams in geometric settings, location-thinking may be preferable. Additionally, other contexts, like graphs of parametric curves, may require both location and value-thinking for interpreting various aspects of the same image. Ideally, students should possess the ability to think in both ways, focusing on the values represented at a point and the point's spatial location, along with the ability to discern when it is appropriate to use each way of thinking. To support students in recognizing these two ways of thinking, instructors and curriculum developers may consider providing students with opportunities to think both ways and bring to light this distinction. We hope that our findings increase practitioners' awareness of the subtle yet significant details of students' interpretations of aspects of graphs and may thus inform decisions in curriculum design and instruction.

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