USES OF COORDINATE SYSTEMS: A CONCEPTUAL ANALYSIS WITH PEDAGOGICAL IMPLICATIONS

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Conventional coordinate systems are often considered representational tools for reasoning about mathematical concepts. However, researchers have shown that students experience persistent difficulties as they engage in graphing activity. Using examples from research and textbooks, we present a framework based on a conceptual analysis of the use of coordinate systems. We discuss the implications of the framework for student learning, curriculum design, and teaching.

Keywords: Cognition, Curriculum Analysis, Modeling

Conventional coordinate systems, such as the Cartesian and polar coordinate systems, are often considered representational tools for reasoning about mathematical concepts such as number systems, geometric figures, ratios and proportional relationships, and equations and functions (e.g., Common Core State Standards for Mathematics). Not only are coordinate systems used in the learning, teaching, and doing of mathematics, they are also commonly used in other fields like science, technology, and engineering as a means to communicate information (Paoletti et al., 2016; Roth, Bowen, & McGinn, 1999; Rybarczyk, 2011).

Despite a widespread use of coordinate systems, researchers have shown that students experience persistent difficulties as they engage in graphing activity—constructing and interpreting graphs—in their mathematics (e.g., Leinhardt, Zaslavsky, & Stein, 1990) and science courses (Potgieter, Harding, & Engelbrecht, 2008). For instance, when constructing graphs, students struggle with establishing axes and scales and transitioning from discrete graphs to continuous graphs (Herscovics, 1989), connect points without considering what happens between points (Yavuz, 2010), and adhere to familiar forms of regularity like linearity (Leinhardt et al., 1990). When interpreting graphs, students show common difficulties such as treating graphs as literal representations of a situation (e.g., interpreting a graph of a biker's speed vs. time as the biker's traveled path) (Clement, 1989; Oehrtman et al., 2008), and focusing on one quantity while tacitly overlooking the other quantity when prompted to interpret relationships between two quantities (Leinhardt et al., 1990; Oehrtman et al., 2008).

A frequently overlooked aspect accounting for students' persistent difficulties in graphing activity is students' understanding of underlying coordinate systems. Researchers and curriculum developers often take coordinate systems for granted as a primitive structure. Little focus is given to students' construction of coordinate systems or their understanding of what is being represented within coordinate systems. For example, most prior research on students' understandings of graphs presumes the Cartesian plane as a given structure in students' reasoning without exploring what meanings students hold for the underlying coordinate system (e.g., Levenberg, 2015). In this report, we outline a framework distinguishing two uses of coordinate systems: situational coordination and quantitative coordination. This explicit distinction has received little, if any, attention in extant mathematics education research and curricula. By bringing attention to the underlying coordinate systems on which students are asked to reason, we intend to provide an explanatory construct as to why students may have difficulty constructing or interpreting graphs represented on coordinate systems. We discuss implications our framework could have and pose future research directions.

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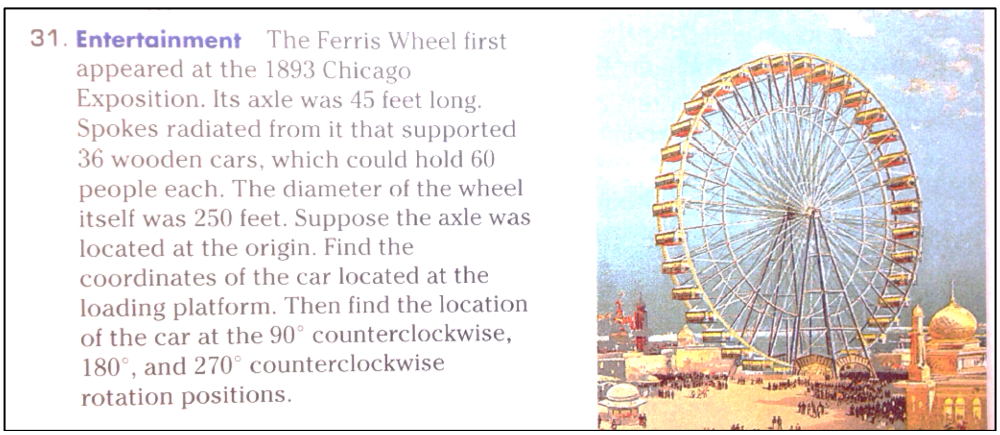
Conceptual Analysis

The framework we propose is based on our conceptual analysis (Thompson, 2008) of students' potential uses of coordinate systems as representational tools. Our conceptual analysis is informed by research on spatial cognition (e.g., Levinson, 2003; Tversky, 2003), Thompson's (2011) theory of quantitative reasoning, research examining students' construction and use of coordinate systems (e.g., Lee, 2016; Lee & Hardison, 2016; Lee, 2017), research examining students' graphing understandings (e.g., Moore, Paoletti, & Musgrave, 2014; Joshua, Hatfield, Musgrave, & Thompson, 2015), and our experiences working with students.

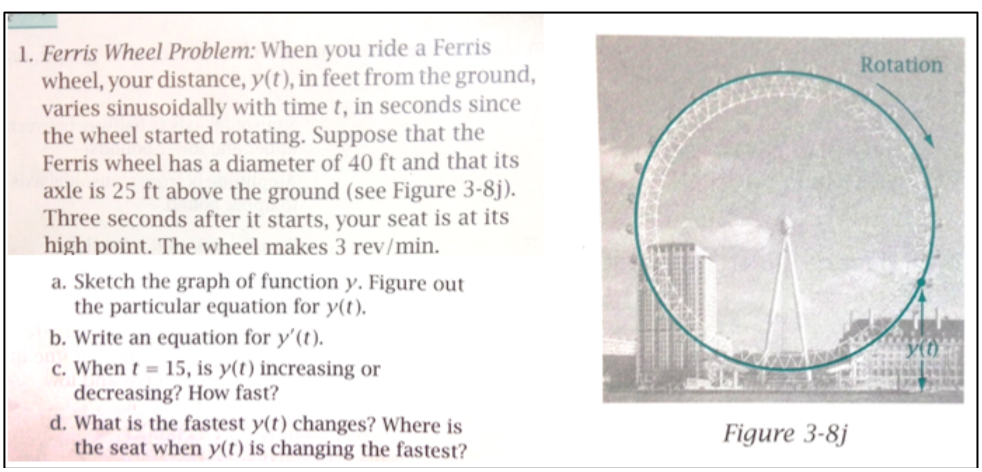
Two Contrasting Examples

To illustrate our distinction between two kinds of coordinate systems, we present two tasks from textbooks (Figures 1 and 2). Both tasks involve a Ferris Wheel context and, from our perspective, both tasks involve establishing a coordinate system. Task A (Figure 1) prompts students to find the coordinates of the car located at the loading platform and in other positions, with the axle of the Ferris Wheel defined to be at the origin. In this case, students are asked to coordinate the location of each car within the space in which the Ferris Wheel is *situated*. In other words, the coordinate system in this task is used for spatially organizing the location of each car in reference to the position of the axle of the wheel.



31. **Entertainment** The Ferris Wheel first appeared at the 1893 Chicago Exposition. Its axle was 45 feet long. Spokes radiated from it that supported 36 wooden cars, which could hold 60 people each. The diameter of the wheel itself was 250 feet. Suppose the axle was located at the origin. Find the coordinates of the car located at the loading platform. Then find the location of the car at the 90° counterclockwise, 180°, and 270° counterclockwise rotation positions.

Figure 1. Task A (Holliday, Cuevas, McClure, Carter, & Marks, 2006, p. 95).



1. *Ferris Wheel Problem:* When you ride a Ferris wheel, your distance, $y(t)$, in feet from the ground, varies sinusoidally with time t, in seconds since the wheel started rotating. Suppose that the Ferris wheel has a diameter of 40 ft and that its axle is 25 ft above the ground (see Figure 3-8j). Three seconds after it starts, your seat is at its high point. The wheel makes 3 rev/min.
   a. Sketch the graph of function y. Figure out the particular equation for $y(t)$.
   b. Write an equation for $y'(t)$.
   c. When $t = 15$, is $y(t)$ increasing or decreasing? How fast?
   d. What is the fastest $y(t)$ changes? Where is the seat when $y(t)$ is changing the fastest?

Figure 2. Task B (Foerster, 2005, p. 112).

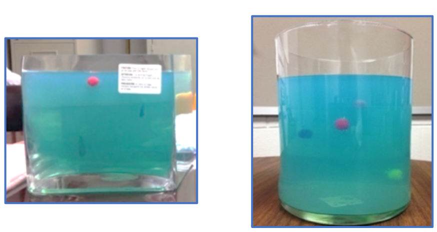
Now consider Task B (Figure 2). Rather than asking students to use a coordinate system to spatially represent the situation, Task B prompts students to graph the relationship between the time elapsed since the wheel started rotating and a rider's distance from the ground. To solve this task, students must extract two quantities (Thompson, 2011), time and distance, from the Ferris Wheel situation and coordinate them in a new space in order to produce a graph. This new space does not entail the spatial situation from which the quantities were extracted. In other words, the Ferris Wheel situation is *not* contained in the graph students are asked to produce or in the coordinate system containing this graph.

Two Uses of Coordinate Systems

As demonstrated through the contrasting examples above, coordinate systems can serve different purposes, which potentially lead to different graphing activities. We propose a framework to distinguish between two uses of coordinate systems, *situational coordination* and *quantitative coordination*. We emphasize we are distinguishing uses of *coordinate systems*, which are different from distinctions others have made about students' *graphing* activities (e.g., Moore & Thompson, 2015); coordinate systems can, but need not, contain graphs. We describe each use of coordinate systems with examples from our research and from textbooks; textbook examples are used to illustrate our interpretations of the textbook author's intended use of coordinate systems, which do not necessarily coincide with how students might perceive of the coordinate system. Through hypothetical examples, we also illustrate how a student might engage in graphing activity in each case.

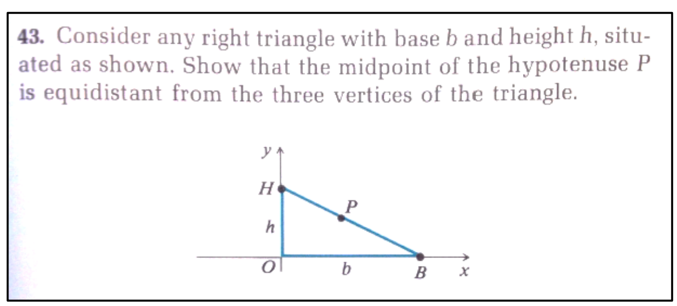
Situational Coordination

Situational coordination refers to an individual using a coordinate system to represent or mathematize a space or physical phenomena, as in Task A (Figure 1). Constructing a coordinate system for situational coordination (i.e., constructing a *situational coordinate system*), involves establishing frames of reference (e.g., Levinson, 2003) to gauge extents of various attributes of objects (e.g., relative location or orientation of an object) within the space or phenomena. When constructing a situational coordinate system, an individual can produce quantities by measuring attributes of the space/situation using their frames of reference and coordinate such measurements to represent attributes of objects *in the space or situation*. An everyday example of a situational coordinate system is used is a map. Graphs constructed on situational coordinate systems can be viewed as projections or traces of physical objects or phenomena onto the *same space* containing the original objects or phenomena (e.g., the movement of a car shown on a GPS).



Locate the four fish in the fish tank models.

Task C: Fish Tank Task (Lee, 2017)



43. Consider any right triangle with base b and height h, situated as shown. Show that the midpoint of the hypotenuse P is equidistant from the three vertices of the triangle.

Task D: (Keedy & Bittinger, 1981, p. 159)

Figure 3. Two Examples of Coordinate Systems Used for Situational Coordination.

As an example of a task designed for students to construct a situational coordinate system, consider Task C (Figure 3) from the first author's research involving students' organization of

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space. In this task, the researcher asked students to locate four fish figures in three-dimensional tanks. The students' responses to this task (see Lee, 2017) provide an example of how a student might construct a coordinate system to represent objects in space by establishing a frame of reference and locating points within the space using coordinated measurements. As a second example, consider Task D from a college algebra text (Keedy & Bittinger, 1981, p. 159), which asks students to reason about a geometric figure and its characteristics situated in a coordinate system. Here, the *x*-axis, *y*-axis, and origin suggest a frame of reference used to locate the points or line segments within the geometrical figure (i.e., triangle). Tasks C and D exemplify situational coordinate systems as the coordinate system is (or can be) used to locate or mathematize objects (e.g., the fish in Task C or right triangle in Task D) in a given space.

The examples above highlight how situational coordinate systems can be used to mathematize a single moment in time (i.e., a snapshot). Situational coordinate systems can be used when representing dynamic situations as well. In other words, a student may think of overlaying a frame of reference on an imagined movie. For example, in Task A, one can observe or imagine a rider's position on the Ferris Wheel changing over time within a situational coordinate system (e.g., Williams, 2018). Similarly, in Task C, one can observe or imagine a fish's position in the fish tank changing over time within a situational coordinate system.

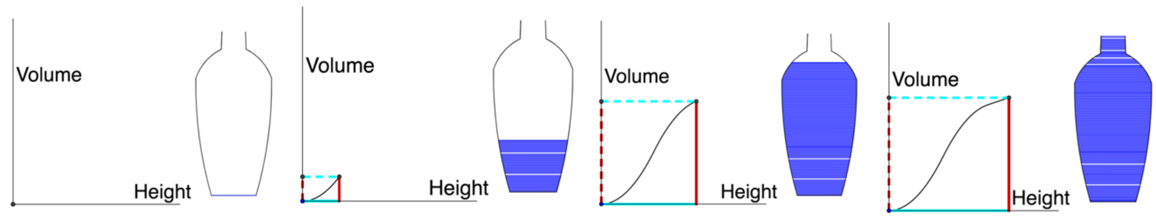
When constructing a situational coordinate system, one can establish and insert a multitude of quantities onto the situation. We have found it helpful to imagine a situational coordinate system as mentally tagging an object with an *n*-tuple and imagining the values within the *n*-tuple changing over different situational instances. For example, for a Ferris Wheel car's location in Task A, one can mentally tag a *4*-tuple such as (θ, h, a, t), where θ represents the angle through which a car on the Ferris wheel has rotated, h represents the height of the car, a represents the arc length the car has traveled, and t represents the time the car has been in motion. The quantities one inserts onto the situation need not be purely geometric or temporal in nature, as in Tasks A, C, and D. For example, an individual might also consider inserting the blood pressure of the rider in the car into the situation and therefore the tuple. The critical distinguishing feature of a situational coordinate system is that frames of reference are established and used to construct and tag quantities *onto* the situational space or physical phenomena.

Quantitative Coordination

Quantitative coordination refers to an individual using a coordinate system to coordinate sets of quantities by obtaining a geometrical representation of the product of measure spaces, such as in Task B above. To establish a quantitative coordinate system, an individual must establish quantities within the given space/situation, disembed (Steffe & Olive, 2010) these quantities (i.e., extract them from the situation while maintaining an awareness of the quantities within the situation), and project them onto some new space, which is different from the space in which the quantities were originally conceived. Graphs as emergent traces (Moore & Thompson, 2015) representing relationships between disembedded quantities can be constructed within a quantitative coordinate system. These graphs are not projections of physical objects or phenomena from the same space containing the original objects or phenomena.

Consider Task E (Paoletti & Moore, 2017), a variation of Swan's (1985) bottle problem, which has been used to examine students' graphing activity (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). In this problem, students are asked to imagine a bottle being filled with liquid and sketch a graph relating the volume of liquid in the bottle and height of liquid in the bottle. To solve this task, a student must first conceive of two quantities, liquid volume and liquid height. The student may then consider how the two quantities vary as liquid is poured into the bottle.

Next, the student may conceive of two number lines, one for liquid height and one for liquid volume, and use these number lines to represent the varying magnitudes or values of each quantity. The student can then construct a Cartesian plane with the two number lines—the horizontal axis representing liquid height and the vertical axis representing liquid volume—and imagine a point in this space as simultaneously representing the liquid height and liquid volume as water is poured into the bottle (Figure 4a-d). The two-dimensional space that is made by the product of the two axes form a new space (a {height×volume} plane), which is different from, but related to, the space containing the bottle itself. The infinitude of traced points (height, volume) in the graph (Figure 4d) do not depict the physical bottle being filled but represents the covarying quantities.



(a) (b) (c) (d)

Figure 4. Task E: Representing the height and volume of liquid in a bottle.

For an individual to construct a quantitative coordinate system she must first establish a situational coordinate system in terms of the relevant quantities in the imagined situation before disembedding these quantities to create a quantitative coordinate system (Lee & Hardison, 2016). For example, in Task B, in order for a student to coordinate the time traveled and vertical distance of car from the ground, she will first need to establish a situational coordinate system on the Ferris Wheel situation (as in Task A) through which she could gauge the time traveled and vertical distance from the ground before disembedding those quantities into a new space. Similarly, in Task E, a student will first need to establish a situational coordinate system on the bottle situation through which she could gauge the liquid height and liquid volume as the water fills the bottle.

Implications of the Framework and Future Research Directions

We encourage researchers to focus their attention on the two uses of coordinate systems when studying students' construction of coordinate systems and also to use this framework as an analytical tool in future studies examining students' graphing activity. Findings from such studies can support curriculum development and classroom instruction related to coordinate systems and students' graphing activities, which we further discuss in the sections below.

Using the Framework to Better Understand Students' Graphing Activities

The proposed framework may help explain some of the students' challenges in constructing or interpreting graphs represented on coordinate systems as reported in the literature. That is, identifying whether a student is aware of the purpose of the coordinate system he has constructed and/or is reasoning upon can provide insight for his graphing activity. For example, when a student constructs axes, examining whether he is (a) attempting to quantitatively organize a particular situation, (b) representing magnitudes of quantities abstracted from an outside situation, or (c) attempting to reproduce an image from their prior classroom instruction can be informing. Similarly, for a student who draws graphs by connecting points without considering what happens between points (Yavuz, 2010), it could be insightful to investigate whether the student understands points as representing the coupling of two quantities (Saldanha & Thompson, 1998) disembedded from the situation.

The lack of explicit attention to the two uses of coordinate systems can become problematic when a student interprets what a teacher or researcher intends to be a quantitative coordinate system as a situational coordinate system. In such a case, the student will be constrained, at best, to reasoning about points and lengths in graphs as opposed to reasoning about the contextualized quantities outside of the perceptually available graph. Such an interpretation may explain why researchers report students treating graphs as literal representations of a situation (e.g., interpreting a time-speed graph of a biker as the biker's traveled path) (Clement, 1989) or why some students have difficulty "distinguishing between visual attributes of a physical situation and similar perceptual attributes of the graph of a function that models the situation" (Oehrtman et al., 2008, pp. 153).

Looking forward, we intend for our conceptual analysis to be a tool for empirical research examining students' construction of coordinate systems and graphing activities. Furthermore, because a quantitative coordinate system presupposes some situational coordinate system, we hypothesize that many of students' difficulties interpreting and constructing graphs within quantitative coordinate systems can be explained by students' challenges in constructing situational coordinate systems. Even if an individual is capable of constructing a situational coordinate system, transitioning to a quantitative coordinate system is nontrivial. If an individual has constructed a situational coordinate system in the bottle problem, for example, establishing a quantitative coordinate system requires transforming a volume in the situational coordinate system to a directed length on a number line; this transformation cannot be taken for granted when teaching and researching students' graphing activity. As such, we argue students need experiences constructing situational coordinate systems prior to constructing quantitative coordinate systems and we echo others' calls (e.g., Thompson & Carlson, 2017) for more research on how students map quantities onto number lines.

Using the Framework to Reflect on Curriculum and Teaching

Our framework can be used to analyze mathematical tasks. For example, students' thinking of a graph as a picture of a physical situation may be unsurprising when students are, at times, presented with graphs that seem to conflate the two uses of coordinate systems. In Task F (Figure 5), we see a quantitative coordinate system used for relating time and distance. Yet, the use of arrows and labels in the text is problematic from our perspective. The stoplight, school, and school zone are *not* represented physically on this graph as the arrows seem to indicate. These objects exist in a space different from the coordinate plane established by the quantities time and distance. We hypothesize that conflations between quantitative and situational coordinate systems in mathematical tasks can lead to students' difficulties in graphing activity. In analyzing tasks, we also noticed that many graphing exercises in textbooks did not explicitly include quantitative referents for axes, which is consistent with the analysis of pre-calculus and calculus textbooks reported by Paoletti et al. (2016). They found that popular pre-calculus and calculus textbooks almost exclusively used graphs to represent decontextualized functions of x and y. Absent of any quantitative referents, students are likely left with little choice but to interpret these decontextualized graphing tasks in terms of situational coordinate systems. This is problematic as a common way STEM fields communicate information is through graphical representations of two (or more) covarying contextualized quantities.

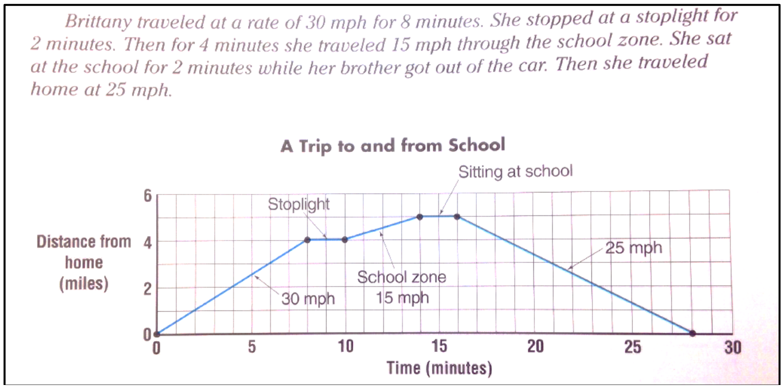


Figure 5. Task F: (Holliday, Cuevas, McClure, Carter, & Marks, 2006, p. 95)

Also relevant to our distinction, Paoletti et al. (2016) found that although many STEM fields (e.g., Biology, Chemistry, medicine) often use quantitative coordinate systems to represent two or more covarying quantities in their practitioner journals and textbooks, other fields (e.g., engineering, physics) commonly employ situational coordinate systems to mathematize a situation or phenomena. This difference between the use of coordinate systems in pre-calculus and calculus textbooks and other STEM fields may help explain why students often do not see connections between the mathematics they learn in mathematics classes and the mathematics they use in STEM courses (Britton et al, 2005). We encourage mathematics educators to provide students with opportunities to develop a balanced understanding of both uses of coordinate systems because it is important for their mathematical development as well as potential future STEM courses and careers.

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