EXPANDING STUDENTS' CONTEXTUAL NEIGHBOURHOODS OF MEASUREMENT THROUGH DYNAMIC MEASUREMENT

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This study is part of a larger project exploring students' thinking of Dynamic Measurement (DYME), an approach to area measurement that engages students in dynamic digital experiences of measuring rectangular surfaces through sweeping lengths. The goal of this study was to evaluate the extent to which students could bridge the mathematical knowledge they gained from these dynamic experiences to other activities that are more static in nature. A classroom of 19 third grade students participated in an 8-period design experiment (DE) centered on DYME. Data to evaluate the bridging were obtained from pre- and post-assessments administered before and after students' participation in the DE. The results suggest that by working with the DYME tasks, students were able develop a conceptual connection between multiplicative reasoning and area measurement that were able to apply to solve static tasks.

Keywords: measurement, technology, situated learning, learning trajectory

> *Researcher*: So, what did you learn from interacting with these tasks?
> *Student*: We learned how to paint inside the lines.
> (*Excerpt from a design experiment after a series of sessions using DYME tasks*)

As researchers, we carefully design tools and tasks to engage students in experiences that would support their learning of mathematics. However, similar to the excerpt above we notice that "unfortunately mere experience is not sufficient for learning, for integrating into one's functioning, or for making, more and more effective actions available to be enacted in the future" (Mason 2015, p. 334). Indeed, students learn how to play with the tool, engage with the task and make some generalizations, which a researcher may describe as powerful for developing advanced mathematical ideas, yet there is "little evidence that students can abstract beyond the modeling context" (Doerr & Pratt 2008, p. 272) or whether students are able to use this knowledge to make sense of other tools or apply it in other contexts.

In fact, many researchers have tried to describe the situated nature of the generalizations that students develop as they engage with contextual problems and digital tools. An example is the work of Hoyles and Noss (1992) on the notion of "situated abstractions" in which they describe the gap between the generalizations that students form in one context but not in others. Many researchers tried to describe this as a failing "transfer" of knowledge. For instance, Broudy (1977) discussed transfer as the ability of students to apply their prior knowledge in order to solve new problems, while diSessa and Wagner (2005) discussed it as re-using the knowledge gained in one situation (or class of situations) to a new situation (or class of situations).
In exploring students' transition from a constructionist learning environment to formal algebra, Geraniou and Mavrikis (2015) raised the issue of what exactly is that "knowledge" being transferred, and chose to focus on "bridging" instead, a metaphor first used by Perkins and Salomon (1988) to describe "a process of abstraction and connection making" (p. 28). Following this notion, Geraniou and Mavrikis (2015) designed a series of "bridging activities" which assisted students in making the connections between the digital tool and the mathematics. These bridging activities included *consolidation tasks* that asked students to reflect on their interactions

with the software, *collaborative tasks* that focused on asking students to justify whether their rules were equivalent or not to each other, *tool-like paper* tasks that looked like the software tasks but were on paper, and finally textbook or exam-like tasks. This variety in the design of the bridging activities can be seen as aiming to expand what Pratt and Noss (2010) refer to as students' *contextual neighborhood*, or the range of contexts and variety of circumstances in which the students' knowledge is made relevant and accessible.

In this paper, we describe our efforts in assisting students to expand their contextual neighborhood of measurement through dynamic measurement (which we describe in the next section), and discuss how our task design may have helped them bridge their dynamic digital experiences with typical static measurement tasks.

Dynamic Measurement (DYME)

DYME draws on research on visualizing area as 'sweeping' (e.g. Lehrer, Slovin, Dougherty, & Zbiek, 2014; Thompson, 2000) to engage students in digital experiences of visualizing the multiplicative relationship of length *times* width that underlies the area formula of a rectangle. To understand this approach, imagine a paint roller of length *a* being swept over a distance of *b* (width) to generate a rectangle of area *ab* (Figure 1). DYME presents area as a continuous dynamic quantity that depends on both the length of the roller and the length of the swipe. Aiming to explore students' DYME reasoning for area, we conducted two cycles of design experiments (DE) (Cobb, Confrey, Lehrer, & Schauble, 2003) and developed an interactive book of tasks on Geogebra [www.montclair.edu/csam/DYME] and a learning trajectory (LT) (Simon, 1995), illustrating how students' thinking of DYME may progress over time.

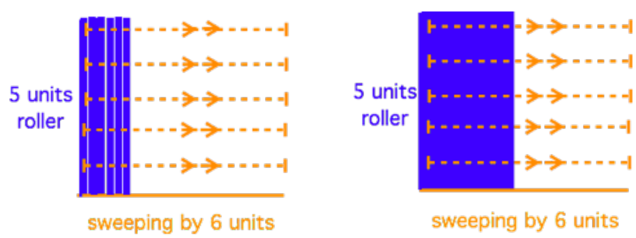
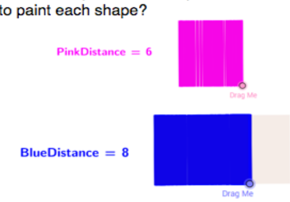
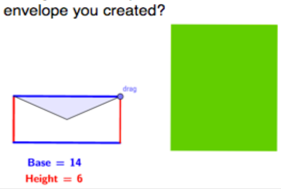
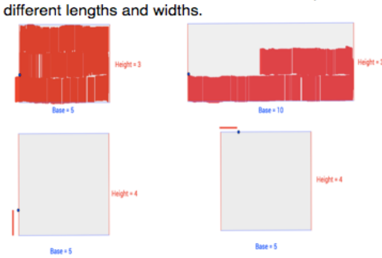
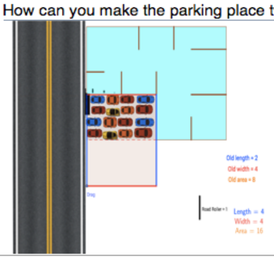
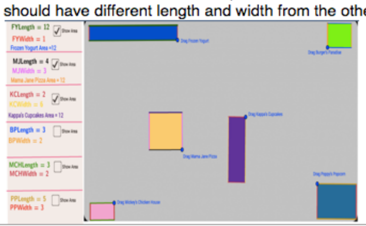
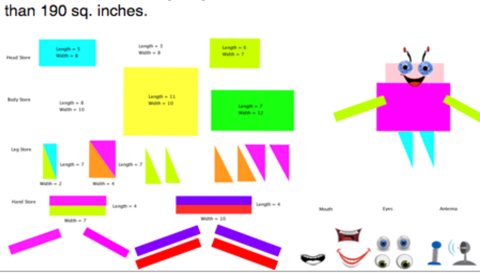


Figure 1. Area as a continuous structure using the 'sweeping' approach (Panorkou, 2017)

Table 1 presents the LT, which consists of a set of DYME constructs from less sophisticated to more sophisticated (levels), a sample Geogebra task for each construct, and a set of observable student generalizations that are more likely to occur at each construct. The goals of Levels 1-3 are for the students to build the idea of 2D space by visualizing area as a continuous structure that can change dynamically through 'sweeping,' recognize that the measurement of a surface requires the coordination of two dimensions (e.g. Reynolds & Wheatley, 1996), and recognize the multiplicative relationship between the two dimensions of a rectangle and its area (Izsak, 2005). To help students identify this relationship, the tasks involve the use of a 1-inch roller to paint shapes of different lengths and widths and constructing a repeating pattern for covering the shape (e.g. Outhred & Mitchelmore, 2000), by considering the distance covered in one swipe with the number of swipes. Subsequently, Levels 4-6 present an exploration of what else is possible to learn after students develop their DYME thinking, therefore they do not follow a specific order (although in this paper we refer to them "levels".) For instance, in Level 4 students use their DYME knowledge of area to identify the effects on the dimensions when the area of a shape is scaled, or in Level 5 students explore dimensions as factors that may give the same area.

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.

Table 1. The hypothetical learning trajectory for DYME

Stage	Example of a task from the Geogebra book	Example of Students' behavior
1. Exploring dimensions and area as continuous attributes: Students visualize area as a continuous structure that can change dynamically.	Drag the paint roller to paint each shape. How far did you drag the paint roller to paint each shape?  	**a.** Students recognize that the dimensions, length of the roller and the distance dragged, define how big or small a shape is, e.g. "*The more I drag the roller, the more space I cover*" *"The longer the roller, the more space it covers."* **b.** They also form relationships between the dynamic action of painting and the dimensions of the shape being painted, e.g. *"The length of the roller needs to be the same as the height of the rectangle"* *"The distance of paint is same as the base of the rectangle."*
2. Coordinating two dimensions to compare area: Students recognize that the measurement of a surface requires the coordination of two dimensions.	Modify the envelope to fit the size of the card! How big is the envelope you created?  What do we need to change? What stays the same?	Students recognize that to compare two shapes, they need to compare both dimensions, e.g. "*The height needs to change, the base needs to stay the same.*"
3. Multiplicative Relationship of length, width and area: Students recognize the multiplicative relationship between the two dimensions of a rectangle and its area.	Students are asked to use a 1-inch roller to paint surfaces of different lengths and widths.  How far did you drag the roller? How many swipes did you need to cover the wall? How much space did you cover?	**a.** Students use the multiplicative 'times' language to find the space covered, e.g. "*This is 30 because the base is 10 and we are going to swipe three times.*" **b.** Students recognize that a roller of size *l* covers the same area as *n* rollers of size *l/n* dragged for the same distance, e.g. "*A 3-inch roller covers the same as three 1-inch rollers. If we cut it into 3 parts and you go across one time it is 10 and then if you go across another time it will be 20 and if you go one more time it will equal 30.*" **c.** Students recognize that the space covered can be found by the length of the roller *times* the base of the rectangle or the distance of swipe, e.g. "*The length of the roller is 3 and the distance is 10, so area is 3 times 10.*" **d.** Students using "length of roller" and "height" interchangeably, and this intuitively leads to *height times base*, e.g. "*The length is 3 and the base is 10. To find area, I did 3 times the base, which is 30.*"
4. Multiplicative Coordination of length and area: Students recognize that length, width and area are coordinated through multiplicative comparisons.	How can you make the parking place twice as big as it is now? 	**a.** Students identify that they can double or triple areas by multiplying only one of the dimensions by the same factor, e.g. *"To double the area, we can double the length or the width."* **b.** Students recognize that in order to split area (fractional thinking), they need to split the length or the width, e.g. students create a cafeteria which is 1/4 of an 8 by 5 inches garden and argue: "*If we split this [the height of the cafeteria] into four parts, then one of the parts will be the cafeteria. It would be 2 inches because the if we use only 1-inch roller it would go 8 times across but if you use 2-inch roller then it would go 1,2,3, and that would go 4 parts.*"
5. Identifying area as a multiple of its dimensions: Students recognize area as a multiple of its dimensions and identify factors that give same area.	Please design the stores in such a way so that each of the food stores will have an area of 12 sq. meters. Also, each food store should have different length and width from the other stores! 	**a.** Students use the commutative property to compare areas, e.g. "*A rectangle with length 4 and width 3 has the same area with a rectangle of length 3 and width 4.*" **b.** Students identify factors that give same area, e.g. "*Length 4 and width 3 is doing 4 swipes of 3. This is same as two swipes of 6, so length 2 and width 6*"
6. Multiplicative Coordination of relative areas: Students recognize that if a shape is 1/n of another shape, then its area is 1/n of the area of that shape.	Make the robot as fancy as you like but its area should be no more than 190 sq. inches. 	Students recognize that if a right triangle is half a rectangle, then the area of that triangle is half the area of the rectangle, e.g. for calculating the area of each blue leg, the students argued: *"You have to do a half of 7 and 2. We do 7 times 2 and a half of that."*

The first two cycles of DE showed DYME's potential for making the multiplicative relationship of the area formula more intuitive and accessible to students (e.g. Panorkou, 2017). However, the nature of DYME tasks is very different than any of the measurement tasks that students encounter in their classrooms and many of the generalizations they make are situated in the specific context of DYME. Therefore, in Cycle 3 we wanted to test out whether students could connect these situated experiences to their other measurement experiences. Our task design and questioning included many tasks that could help them do that. First, we used consolidation tasks that asked students to reflect on their interactions with the software, reflect on their learning in every session and generalize their strategies. An example of the latter was when we asked the students to advise a painter of how to measure the space of any rectangular surface. Second, we used whole class discussions at the end of each session that included collaborative tasks asking students to share different strategies of solving a task and discuss why different strategies generate the same answer. For instance, finding the space of a rectangular surface of 4 inches by 5 inches, by using 4 one-inch swipes of 5 inches or 2 two-inch swipes of 5 inches or using 4 inches *times* 5 inches. Third, in contrast to Geraniou & Mavrikis (2015) that used software-like paper tasks, we used paper-like dynamic tasks, that are similar to what students would encounter on paper but with technological affordances, such as presenting them with a rectangular surface which they can make bigger or smaller by modifying the length and width through dragging (e.g. Level 2 task in Table 1). We conjectured that this kind of design would help students build connections between DYME and other types of measurement.

Aims

This article describes our efforts to investigate how a whole class design experiment (DE) (Cobb et al., 2003) on DYME could help students develop their thinking of area as a multiplicative relationship and whether students are able to transfer knowledge gained from interacting with the DYME tasks to the static traditional area tasks they would encounter in the classroom. More specifically, our goals were to explore:

1. To what extent did the students develop their thinking of area as a multiplicative relationship as a result of the design experiment and the use of the DYME tasks?
2. To do extent did the students bridge the experiences gained from the dynamic environment of DYME tasks to traditional area tasks they encounter in the classroom?

Methods

Nineteen third-grade students from an elementary school in the Northeast participated in the whole class DE (Cobb et al., 2003). The students already had formal instruction on multiplication and area measurement using the common approach of using square units. The DE consisted of six 50-minute periods of instruction using the DYME Geogebra tasks.

Assessment Design

To answer our research questions, we designed and administered an assessment to students at the very beginning of the DE and at the end of the learning experience. The items in both pre- and post-assessments were identical aiming to create an initial and final "profile of strengths and weaknesses" (Huhta, 2008, p. 470) for each student. The assessments were pilot tested with a separate group of students and revised before they were used for the present study. Aiming to examine both the development of students' thinking of area and the extent of bridging, we designed the items based on typical measurement assessments used in the literature (e.g. Battista, 2004) and on questions found in standardized test assessments, such as PARCC. All the items were aligned to a level of the LT and were designed to provoke responses that could indicate how students' reasoning has changed following their interaction with the tasks during the DE.

For example, one of the DYME goals is for students to recognize that *a* swipes of *b* cover the same amount of space as *b* swipes of *a*, what we refer to as commutativity in DYME. To do that, we engage students in dynamic tasks, such as the one presented in Figure 2 (left), where they use paint rollers of different orientations (horizontal/vertical) in order to find area and generalize that 4 swipes of 5 cover the same space as 5 swipes of 4. Instead, typical measurement assessments evaluate the commutative property in static tasks similar to the Question A in Figure 2 (right), which assesses if students recognize that a 3ft by 4ft tabletop has the same area as a 4ft by 3ft tabletop. Question A was used as an item in our pre- and post-assessment. Each assessment included three to four items corresponding to each construct of the LT and these items were evenly distributed throughout the paper so that no two consecutive items were associated with the same LT construct.

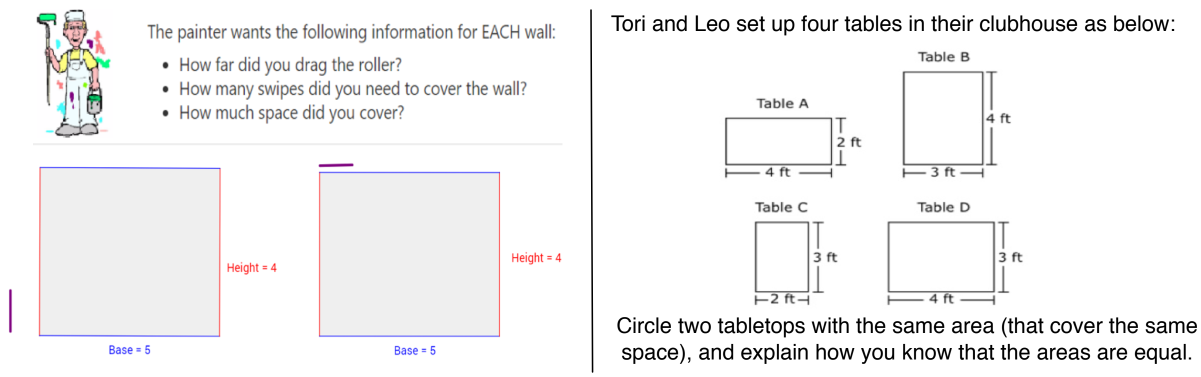


Figure 2. DYME Geogebra task (left); Question A, adopted from the PARCC assessment (right).

Assessment Analysis

For analyzing the students' work, we read every student's response and generated categories to capture the themes in their responses. We then developed a scoring protocol to measure the range of sophistication in their responses. To score the responses, we adopted Norton and Wilkins' (2012) suggestion: 1 for indication of a particular reasoning level and 0 for counter-indication of a particular reasoning level. The levels of reasoning adopted were consistent with the LT levels in Table 1. For instance, a response showing a Level 5 understanding was receiving a point for Level 5, while a response showing a Level 3 understanding was receiving a point for Level 3. The responses were scored by four researchers independently and then negotiated aiming to maximize reliability.

For an illustrative example of the whole process, consider Question A in Figure 2 (right). Table 2 presents the scoring rubric we developed for Question A. In this question, we noticed three different levels of reasoning corresponding to constructs 2, 3 and 5 of the LT. Depending on how the student responded, they would get a point for that construct. For example, if a student used the commutative property to find equal rectangles, they received a point only under level 5.

Results

After each student response was scored, we summed the total scores received by all students under each level (L1 - L6) and color-coded the responses based on these scores in a contingency table (Norton & Wilkins, 2012) (Table 3). As aforementioned, each assessment included three to four items corresponding to each level of the LT. For instance, we included four different items in which students could use level 5 reasoning to respond. If a student received 3 points of level 5 that meant that the student used level 5 reasoning to respond to 75% of the level 5 questions.

Table 2. Rubric for Question A created based on students' responses

Rubric	Example of students' responses	Scoring justification
The student expresses commutative reasoning to find the two rectangles with same area. (1pt for Level 5)	"Table B and table D, because 3x4 = 12, 4x3=12."	Response is aligned to LT level 5. The student exhibits an understanding of the commutative property.
The student multiplies the length and width of the rectangles and compares their area to find rectangles of equal area. (1pt for Level 3)	"I know they are equal because if you multiply them they have the same area."	Response focuses on students' calculation and comparison of area (LT level 3), but does not explicitly state that the commutative property is used.
The student coordinates the length and width of rectangles simultaneously to make judgement about their area. (1pt for Level 2)	"They are same because there is a 4 and a 3."	Response focuses on the numerical values of the sides of the rectangles to make judgements (LT level 2), but does not discuss the order of the numbers.
The student answers correctly but provided vague or no reasoning. (0 points)	"Because when I circle the table. So, I think I should circle the same table."	Response does not provide any description of students' thinking.
The student answers incorrectly. (0 points)	6x5, 3x5, 6x3, 5x6.	Response is illegible.
The student does not respond. (-)		No response.

If a student's overall raw score for a given level was above 60%, then it was inferred that the student reached proficiency in that level and the cell under that level was colored dark grey. If a student's overall score for a particular level was between 30%- 60% the cell was colored grey. For students whose total scores in any level of understanding were less than or equal to 30%, the cells were colored light grey to indicate that students showed some instances of the particular level of understanding. Some students did not answer all of the questions, and thus white cells indicate missing responses. Dotted lines (--) indicate students who did not take the assessment. Table 3 compares students' responses to the pre- and post-assessments showing the extent to which the students' thinking of area as a multiplicative relationship was developed due to their interaction with the DYME tasks. As Table 3 illustrates, out of the 19 third-grade students in the whole class design experiment, only 3 were able to think multiplicatively about area in the pre-assessment (represented by level 3). However, in the post-assessment, 15 students showed an understanding of area as a multiplicative relationship of length *times* width. These results showed that not only students' thinking of area as a multiplicative relationship was developed by engaging with the DYME tasks, but also, that they were able to use their DYME experiences to solve traditional area tasks.

Table 3 also shows that in each level, students understanding was developed from the pre-assessment to the post-assessment (with the exception of level 1 showing that students moved beyond that level.) This is illustrated both by the darker shading, which is more prominent for the post-assessment, and also by the increase in the average score in every level in post-assessment

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compared to pre-assessment. For instance, in level 3, students scored an average of .2 during pre-assessment, which increased to 2.0 during post-assessment. Similar development in students' thinking of area is observed in other levels, especially in level 4, 5, and 6 where the average score of students' responses increased from 0 to 0.6, 0.14 to 1 and 0.07 to 0.6 respectively.

Table 3. Contingency table showing the total scoring of the assessment questions

Student	Pre-assessment						Post-assessment					
	L1	L2	L3	L4	L5	L6	L1	L2	L3	L4	L5	L6
A							--	--	--	--	--	--
B												
C												
D												
E												
F												
G												
H	--	--	--	--	--	--						
I	--	--	--	--	--	--						
J												
K												
L												
M	--	--	--	--	--	--						
N	--	--	--	--	--	--						
O												
P												
Q												
R	--	--	--	--	--	--						
S												
Average	1.35	1.93	0.2	0	0.14	0.07	1.11	1.94	2	0.6	1	0.6

Although students exhibited an overall improvement in their level of understanding in the post-assessment compared to the pre-assessment, we found some exceptions in some students' responses. For instance, though students E and J showed proficiency in coordinating the two sides of a rectangle (Level 2) during pre-assessment, they showed a decrease in their responses in the particular level during the post-assessment. This was most likely because during the post-assessment, the level of understanding of the two students in the certain questions moved beyond level 2 and they showed a higher degree of understanding in level 3 and 5. Another case is student L, who reached level 5 during the pre-assessment but did not proceed beyond level 4 during the post-assessment. On further analysis, we found that student L did not answer the last 9 questions in the post-assessment, which explains the decrease in the level of understanding.

Significance

Researcher: What have you learned all these days?
Student 1: I learned that there are other ways to measure length and width. You can use objects like paint rollers.
Student 2: Yes, and that to double area we need to double one of the measurements.
(*Excerpt from a design experiment after a series of sessions using DYME tasks*)

The findings from the pre- and post-assessment analysis show that students' engagement with the DYME tasks helped them improve their understanding of area. The data analysis also showed that students were able to bridge their DYME experiences and generalizations with typical area tasks. Students extended their contextual neighborhood of measurement to include sweeping-based reasoning as well as reasoning about area as length *times* width. Students not only developed their understanding of area as a multiplicative relationship but they also used this knowledge to respond to more advanced questions in levels 4-6. Among our future goals are to examine the order in the upper LT levels (4-6) and also to further examine the type of activities that assist students in bridging these connections between DYME and other area generalizations.

Acknowledgements

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