QUANTITATIVE (AND NON-QUANTITATIVE) METHODS USED BY FUTURE TEACHERS FOR SOLVING PROBABILITY-BASED PROPORTION PROBLEMS

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We have extended two perspectives of proportional reasoning to solve problems based in probability. Four future middle grade teachers were enrolled in a mathematics content course that emphasized reasoning about multiplication with quantities. The course expected future teachers to generate and explain methods for solving proportions. Probability had not yet been discussed in the course, which gave insight into how reasoning fostered in the course was invoked in novel tasks. All four future teachers could reason about multiplication with quantities to solve probability problems and could use quantitative methods that accurately reflected how empirical probabilities approach theoretical probabilities as the number of trials increase.

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Proportional relationships and ratios form an essential domain in elementary and secondary education (e.g., Kilpatrick, Swafford, & Findell, 2001; National Council of Teachers of Mathematics [NCTM], 2000) but are challenging concepts for students to learn (e.g., Lamon, 2007). Siegler et al. (2010) stressed the importance of teachers helping students conceptually understand proportion problems before utilizing rote cross-multiplication procedures. Similarly, the Common Core State Standards for Mathematics stated students should be able to “understand ratio concepts and use ratio reasoning to solve problems… by reasoning about tables of equivalent ratios, tape [strip] diagrams, double number line diagrams, or equations” (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, p. 42). Yet, studies have shown teachers have difficulties similar to their students when reasoning about proportional relationships (e.g., Orrill & Brown, 2012). Thus, it is important to understand how to better prepare teachers to reason about ratios and proportional relationships.

Additionally, standards documents (e.g., Conference Board of the Mathematical Sciences [CBMS], 2012; NCTM, 2000) emphasize the need for students to see mathematics as a coherent whole by developing connections across topics. To this end, ratios and proportional relationships can be situated within the multiplicative conceptual field (Vergnaud, 1994), which also includes multiplication, division, and fractions. Our project group has been investigating how future teachers (FTs) enrolled in content courses on multiplication develop a coherent perspective of the multiplicative conceptual field and construct viable arguments to solve tasks, especially using the multiple batches and variable parts perspectives on proportional relationships (Beckmann & Izsák, 2015). These two perspectives are explained in the theoretical framework section below.

In this study, we investigated how future middle grade teachers engaged in these two perspectives when generating and explaining methods to solve probability-based proportion problems. Due to the tendency for both students and teachers to solve missing-value proportion problems using the cross-multiplication procedure (Orrill & Brown, 2012), it is important to see if FTs can also reason quantitatively in these situations. Two potential ways FTs can reason quantitatively are through the use of the multiple batches and variable parts perspectives.

We focused on probability-based tasks for two reasons. The first is that we wanted to concentrate on a topic tied to the multiplicative conceptual field that had not yet been covered in the FTs' content courses on multiplication. By choosing a topic not covered in class, we could see if and how FTs utilized quantitative reasoning from instruction (i.e., the multiple batches and variable parts perspectives) in novel situations. Probability fit this criterion.

The second is that the variable parts perspective is rarely the focus of instruction (Beckmann & Izsák, 2015) but gives a more accurate representation of what transpires in probability tasks over a series of trials compared to the multiple batches perspective. For instance, if a spinner has five equal sectors and three of those sectors are red, the probability of the spinner landing on red is three-fifths. Yet, this does not mean as more spins are completed, three of each five consecutive spins will be red (a multiple batches approach). A more accurate representation is one based in the variable parts perspective: there are five equal-sized sectors and as the number of spins increases, each sector gets multiplicatively closer to containing the same fractional amount of the total spins (one-fifth). We mention "multiplicatively closer" because as more spins are completed, the additive difference between the number of spins in any two sectors may be growing. Therefore, as the number of spins grows, the number of spins landing on red will approach three-fifths of the total spins. On the other hand, as the number of spins increase, the fractional amount of the total spins in each of the sectors (the empirical probability) should be approaching the same number, namely the theoretical probability of landing in any sector.

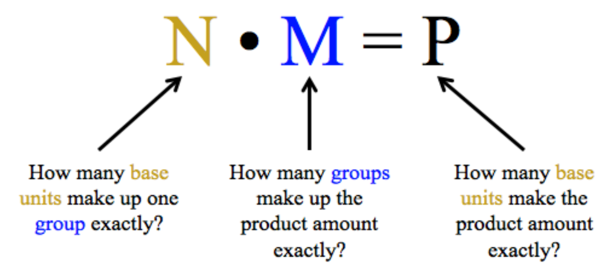
Thus, there is merit in analyzing FTs' methods when solving probability-based proportion tasks. If FTs can access and utilize methods from the multiple batches or variable parts perspectives, it shows FTs can reason quantitatively in tasks that are typically solved using procedures. Further, if FTs can reason using the variable parts perspective, it shows the potential of introducing variable parts-based methods in FTs' mathematics content courses. To investigate this idea, we utilized the following research question: What methods do future middle grade teachers use when solving missing value proportion problems related to probability?

Theoretical Perspectives

In our courses with FTs, we use a quantitative definition of multiplication as a way to provide coherence among topics within the multiplicative conceptual field (Beckmann & Izsák, 2015). The definition of multiplication gives a foothold for FTs to reason about quantities in multiplicative situations and leads to four methods, two in each of the two perspectives (multiple batches and variable parts) on proportional relationships (Beckmann, Izsák, & Ölmez, 2015).

Quantitative Definition of Multiplication

Multiplication can be viewed quantitatively by considering equal-sized groups and units within those groups. We use this idea to produce the definition of multiplication in Figure 1. By consistently ordering the factors as multiplicand (N) times multiplier (M), the definition provides an organizational structure for tasks related to multiplication and allows for differentiation between multiplication, measurement (quotitive) division, and partitive division. An example is also given in Figure 1 to show how the definition of multiplication allows for this differentiation. Based on the choice of group from the situation, the definition of multiplication gives rise to four methods on proportional relationships found in the variable parts and multiple batches perspectives. We will now elaborate on these two perspectives and the four methods.



Multiplication	**N • M = ?**	*If a baker can place 5 cookies on a plate and has 4 plates, how many cookies can the baker put on all of the plates?*	**5 • 4 = ?**
How-Many-Groups Division	**N • ? = P**	*If a baker can place 5 cookies on a plate, how many plates will the baker need to hold 20 cookies?*	**5 • ? = 20**
How-Many-in-1-Group Division	**? • M = P**	*If a baker has 4 plates to put cookies on and has 20 cookies total, how many cookies will the baker need to put on each plate?*	**? • 4 = 20**

Figure 1. A Quantitative Definition of Multiplication with an Example

Four Methods within the Two Perspectives on Proportional Relationships

In this section, we provide an overview of the two perspectives on proportional relationships and the four methods that arise from within these two perspectives. Due to page constraints, this overview is cursory. For a more thorough discussion of the two perspectives, see Beckmann and Izsák (2015). Similarly, for more information on the four methods, see Beckmann, Izsák, and Ölmez (2015).

The two perspectives on proportional relationships are known as *multiple batches* and *variable parts*. In the multiple batches perspective, two quantities measured by fixed units can be joined to form a "batch". A new pair of quantities, measured by the same units as above, is considered proportional to the original batch if the new pair of quantities is a multiple of the original batch. This corresponds to how-many-groups division in the definition of multiplication: the choice of a batch fixes the number of base units in the group (the multiplicand) and the number of groups (the multiplier) varies. In contrast, the variable parts perspective looks at proportionality by fixing the number of groups and varying the number of base units in the groups. This corresponds to how-many-in-1-group division in the meaning of multiplication. Beckmann and Izsák (2015) explained the variable parts perspective in the following way:

> …two quantities are said to be in the ratio A to B if for some-sized part there are A parts of the first quantity and B parts of the second quantity. In contrast to the first perspective [multiple batches], where A and B denote values based on a chosen measurement unit, here A and B specify fixed numbers of parts that can vary in size. (p. 21)

The difference between the two perspectives becomes more distinct when the four methods within these perspectives are shown. The *multiply unit-rate batch* and *multiply one batch* methods are situated within the multiple batches perspective and the *multiply one part* and *multiply whole amount* methods are situated within the variable parts perspective. Each of the four methods differs from the others by the choice of group. To clarify how these group choices affect the corresponding quantitative reasoning, we will consider the following example: *A spinner has five equal sectors with three of those sectors being red. If you spin the spinner 20 times, how many spins should land on red?* To explain how to solve this example, Figure 2 gives a math drawing and the corresponding definition of multiplication for each method. The group can be seen in blue and the number of base units in the group can be seen in red in each drawing.

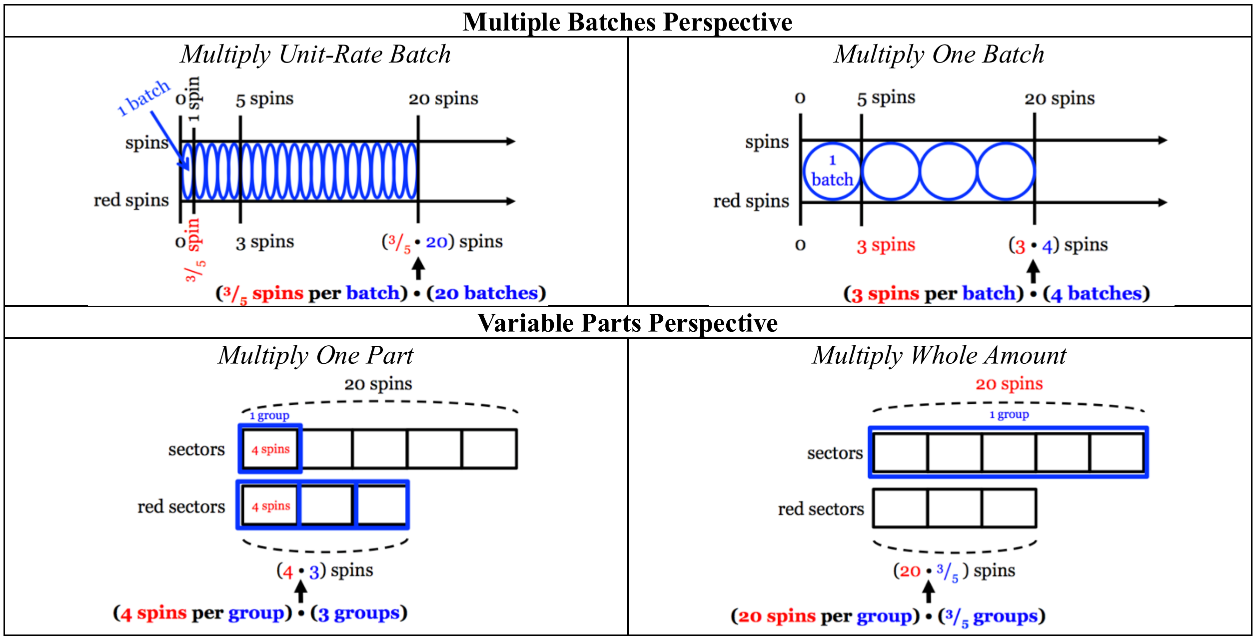


Figure 2. The Four Methods within the Two Perspectives

Data Sources

Data for this report come from a larger study at a public research university aimed at understanding how FTs in middle grade (4-8) and secondary (6-12) math education programs develop understanding of the multiplicative conceptual field. Participants in the larger study took part in six clinical interviews investigating ideas around the multiplicative conceptual field. Two cameras recorded each interview with one focused on a holistic view of the participant and interviewer and one focused on the participant's written work. The data for this report come from the sixth interview with four FTs in the middle grade program who were enrolled in the second content course focused on multiplication taught by the second author. The interview took place near the end of the Spring 2017 semester and focused on probability to see if the FTs could utilize multiplicative reasoning fostered in their content courses with yet-to-be covered topics. Each interview lasted approximately 75 minutes and the tasks used are found in Table 1.

Table 1: Tasks in Interview Involving Missing Value Proportion Problems

Task	Participant(s) Given Task	Wording of Task
1	All	If you were to spin the same spinner [on computer screen with 5 different colored sectors including red] 10,000 times or 100,000 times or 24 hundred thousand times, do you have any expectation about how many times it would land on red? Explain why or why not.
2	Nina Sophie Jack	If you were to spin the same spinner [on computer screen with 5 equal sectors, 3 purple, 2 blue] 10,000 times, 100,000 times or 240,000 times, do you have any expectations about how many times it would land on purple?
3	All	Suppose you want the spinner [5 equal sectors, 3 purple, 2 blue] to land on purple a total of 1500 times, about how many times should you expect to have to spin the spinner?
4	All	Please reconsider the spinner in Task 1. If you were to spin the spinner a bunch of times, say X times, or some specific large number of times, do you have any expectation about how many times it would land on red?

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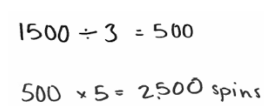
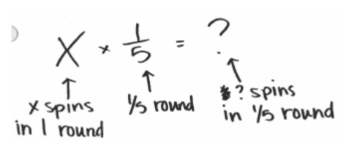
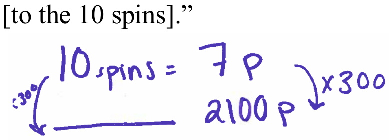
5	Sophie Molly	Please reconsider the spinner in Task 2. If you were to spin the spinner a bunch of times, say X times, or some specific large number of times, do you have any expectation about how many times it would land on red?
6a	Nina	If I wanted to get A spins to be yellow, what total number of spins would you tell me I should use? [Task is referencing a spinner with 10 equal sectors, 7 of which are purple, 3 are yellow]
6b	Sophie	Suppose I wanted to spin the spinner [referencing a spinner with 10 equal sectors, 7 of which are purple, 3 of which are yellow] enough times so that I got 2100 purples, could you give me a recommendation for how many total spins I should make?

Methods

Each interview was transcribed fully and augmented with relevant written work and hand gestures. All authors watched each interview alone and then as a group. The authors then identified all segments related to missing value proportion problems and coded these segments separately. Each author wrote notes and cognitive memos on (a) methods used, (b) concepts discussed related to the multiplicative conceptual field, (c) representations utilized, and (d) quotes that provided insights into methods/reasoning. At least two of the authors then met and discussed all authors' notes, referencing transcripts and videos as needed. Any discrepant interpretations were discussed until resolved. The first and second authors attended all meetings. These meetings led to the creation of the cumulative list of methods used by the participants seen in Table 2. The examples listed in the table do not exhaust all such instances of each method.

Table 2: Methods Used by the Participants During the Interview

Name of Method *Description*	**Example in Data**
Multiply One Part *Variable Parts Perspective* *See Figure 2*	"…we would take 1500 and divide that by 3, so that we know what each of the one-fifth parts are, because there are 3 one-fifth parts in this. And that will give us 500. So one-fifth of the total spins are 500 spins and… so then we'll multiply the 500 by 5 to get a whole, and that would be 2,500 spins."  *–Molly, Task 3*
Multiply Whole Amount *Variable Parts Perspective* *See Figure 2*	"So if you have X spins, so if you're calling like X spins one round [annotates 'X' with 'x spins in 1 round']… then this would be one-fifth of that round [annotates '1/5' with '1/5 round']. So the question mark would be X… or not X. Question mark spins in one-fifth of a round [annotates '?' with '? Spins in 1/5 round']."  *–Molly, Task 4*
Multiply One Batch *Multiple Batches Perspective* *See Figure 2*	"Well, if you know that every 10 times you do it, it'll land on purple 7, that's like the same thing I was doing before where 10 times gives you 7 purple, how many spins gives you 2100 purple? That's multiplying by 300, so you'd have to do the same thing over here [to the 10 spins]."  *–Sophie, Task 6b*

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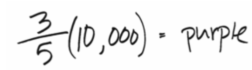
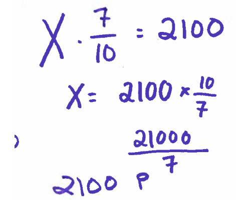
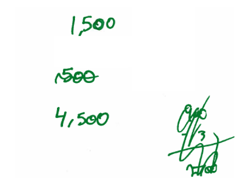
Multiply Unit-Rate Batch *Multiple Batches Perspective* *See Figure 2*	"If the spinner's going to land on a random spot for every spin, I think that for 1 spin, the chance is going to be three-fifths purple. And so, 10,000 spins, each one is also going to be three-fifths chance purple."  *–Nina, Task 2*
Algebraic Manipulation *Writes a multiplication or division equation involving an unknown and uses numeric operations to solve for the unknown with no justification as to what the operations mean.*	Interviewer poses Task 6b, Sophie writes with no verbal justification:  *–Sophie, Task 6b*
Guess and Check *The participant guesses at a potential solution and uses numbers and operations in the context of the problem to check if the potential solution is correct.*	"And we want 1500 purple spins [writes '1500']. We're going to need, what, 3? We're going to need a minimum of 500 spins [writes '500']… other way around [crosses out '500', writes '4,500']. We'd need 45,000 [sic 4,500] spins. [writes '900 •3 = 2700'] Nope, that's not it [crosses out multiplication just written]…"  *–Jack, Task 3*
"For Every" Language Connected with Fractions *Uses language such as "X for every Y" and states this is equivalent to either (a) "X/Y of the total", (b) writing the equation "(Total)•(X/Y)", and/or (c) writing the equation "(X/Y)•(Total)"*	"That if you spin it 10,000 times, 1 in every 5 times… or the red has the chance of being landed on 1 in every 5 times. So every 5 spins it has the chance of being landed on once, so that's the same as 10,000 times one-fifth."  *–Sophie, Task 1*

Results

The results we present focus on the methods used by each FT when solving probability-based missing value proportion problems. Of note, all four FTs were able to engage in sound reasoning regarding the spins and spinners to solve the tasks. Additionally, all four FTs used at least one of the four methods from the two perspectives on proportional relationships during the interview, indicating they could reason about multiplication with quantities to solve probability problems. Lastly, though not always the initial method, all four FTs were able to use methods grounded in the variable parts perspective, showing this approach to proportional relationships is viable even in novel situations.

Sophie

Sophie's (all names are pseudonyms) initial method to solve four of her six interview tasks was not grounded in either of the two perspectives. Rather, Sophie used *"for every" language connected with fractions* to solve the first two tasks and *algebraic manipulation* for two other tasks (Tasks 3, 6b). When prompted for other methods to explain these tasks, Sophie justified her solutions from the multiple batches perspective, successfully explaining the *multiply one batch method* using both strip diagrams and ratio tables. Table 2 shows her typical reasoning for the *multiply one batch method.* For Sophie, using methods from the variable parts perspective only occurred when the questioning from the interviewer cued such reasoning. For instance, Sophie

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explained her equation, *X•1/5 = Red*, through the *multiply whole amount method* when asked to use the definition of multiplication in Task 4 and explained the *multiply one part method* only after being specifically asked to do so in Task 6b.

Jack

With the first two tasks, Jack justified his multiplication equations using *"for every" language connected with fractions*. He consistently discussed the spins landing on red as "1 out of every 5 times", an indicator of thinking about the spins from a multiple batches perspective. He approached Task 3 using *guess and check* (as seen in Table 2) but when that faltered, his language shifted from discussing the purple sectors as "3 out of every 5 times" to "3 parts of 5 that we need" and began reasoning using the *multiply one part method*. Using this method, Jack found the correct solution and provided justification with ease. Jack later revisited Task 3 and articulately explained the *multiply one part* and *multiply whole amount methods* using the definition of multiplication, drawings of spinners, and strip diagrams placed on a Cartesian coordinate system (a representation from the content course). Jack discussed his multiplication equation in Task 4 using *"for every" language connected with fractions* (again, discussing spins landing on red as "1 out of every 5 spins") but when asked about the definition of multiplication, he discussed his equation using the *multiply whole amount method*. Jack used multiple batches language throughout almost the entire interview and tended to justify his multiplication equations using such language. Yet, when these methods did not provide sufficient justification, Jack was able to discuss the two variable parts methods quite easily and with multiple representations.

Nina

Nina's use of the two perspectives was quite similar to Jack's. Nina completed the first two tasks using the *multiply unit-rate batch method* (see Table 2 for her typical explanation) and attempted Task 3 with the *multiply one batch method*. Though Nina found the solution, she was unable to justify it using her multiple batches method. After struggling for several minutes, her language shifted from discussing the purple sectors as "3 times for every 5 times" to "3 of my parts", which seemed to indicate a shift in thinking from a multiple batches perspective to a variable parts perspective. This shift allowed her to justify her answer using the *multiply one part method*. For the remainder of the interview, Nina did not use multiple batches methods again, nor did she use language that would indicate she was approaching the remaining tasks from the multiple batches perspective. Nina explained the *multiply one part method* in Task 4 when asked to use the definition of multiplication and though she used algebraic manipulation to initially solve Task 6a, she discussed the *multiply whole amount method* when asked for other methods.

Molly

Molly was the only FT to strictly use methods within the variable parts perspective throughout the interview. Molly regularly used how-many-in-one-group division to reason through the *multiply one part method* and was also facile in discussing the *multiply whole amount method*. A typical explanation of Molly's *multiply one part method* is in Table 2. When asked for differences between the two variable parts methods, Molly communicated these differences using equations, drawings of spinners, and strip diagrams. Molly needed no prompting to utilize the two methods within the variable parts perspective and used these methods to reason successfully throughout all interview tasks.

Conclusions and Implications

The results we present demonstrate that FTs taking a content course on multiplication can reason through missing value proportion problems using several methods that show conceptual understanding by reasoning with quantities. Thus, an increased focus on reasoning with

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quantities through a structured definition of multiplication could give FTs conceptual ways to approach problems that are frequently solved through rote procedure. Additionally, this study shows that FTs can use ideas from their content courses on multiplication to reason through novel tasks in the multiplicative conceptual field. This is promising because it indicates the FTs are beginning to see cohesiveness across topics within the multiplicative conceptual field and across mathematical ideas, which is a goal of recent standards documents (e.g., CBMS, 2012; NCTM, 2000). Though not typically the initial idea for the FTs, all four FTs could reason from the variable parts perspective and this perspective seemed to be advantageous at times compared to FTs' other methods. This result shows there is merit to giving FTs exposure to the variable parts perspective, a perspective that is typically overlooked (Beckmann & Izsák, 2015).

Future research in probability should explore the variable parts perspective. For example, a study could investigate if FTs can use the variable parts perspective to discuss the law of large numbers informally. Other studies could consider how FTs apply ideas from the multiplicative conceptual field to other mathematical areas, such as linear relationships and trigonometry. More specifically, investigating the accessibility and utilization of the variable parts perspective by FTs in these mathematical areas seems worthwhile. By investigating these topics, the field can gain a better understanding of FTs' abilities to reason quantitatively and the resources needed to support FTs in building connections between areas in the multiplicative conceptual field.

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