USING STRIP DIAGRAMS TO SUPPORT EXPLANATIONS FOR KEEP-CHANGE-FLIP FOR FRACTION DIVISION

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Using memorized rules and algorithms without coherence and understanding is a perennial problem for teachers and students especially in the teaching and learning of fraction operations. I present data in which prospective middle school teachers explain a commonly used rule for fraction division—keep-change-flip. I argue that using both strip diagrams and a single, quantitative definition for multiplication support prospective teachers when explaining why the rule works. The results of the study provide impetus for both mathematics teachers and mathematics teacher educators to teach with coherence across multiple mathematical topics.

Keywords: Teacher Education-Preservice, Rational Numbers, Middle School Education

Curricular documents have highlighted the crucial role for representations in the learning and teaching of mathematics (National Council of Teachers of Mathematics [NCTM], 2000) and the development of well-prepared beginning teachers (Association of Mathematics Teacher Educators, 2017). Although there are many representations used in mathematics education, I focus on representations as models of problem situations, specifically strip diagrams. The importance of using representations in educational contexts is not new (Ball, Thames, & Phelps, 2008; NCTM, 2000), and researchers' work on teachers' use of representations has helped to identify related issues. First, teachers' conceptions of representations and their role in problem solving are relegated to the periphery of mathematical activity—representations are not "real" mathematics (Stylianou, 2010). Teachers prefer to prioritize abstract, procedural rules over productive representations (Eisenhart et al., 1993). Second, teachers' content knowledge constrains their pedagogical purposes for using representations (Izsák, Tillema, & Tunç-Pekkan, 2008), and teachers' mathematical knowledge has been shown to be primarily procedural without a strong grasp of the mathematical underpinnings (Mewborn, 2003). Prospective middle school teachers, in particular, frequently use the keep-change-flip procedure (KCF; i.e., $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$) to explain fraction division but have difficulty explaining why the procedure works (Li & Kulm, 2008). This procedural knowledge of mathematics could explain the reluctance of teachers to use representations when teaching (see Eisenhart et al., 1993). One viable avenue is to provide teachers with opportunities to learn with representations which influences both teachers' knowledge and use of representations (Jacobson & Izsák, 2015). In this report, I present such an opportunity. I examine how prospective middle school teachers use strip diagrams to solve fraction division problems in a content course and how leveraging the diagram supports productive explanations for the keep-change-flip procedure.

Theoretical Framework

Definition of Representations

Researchers who have studied representation use in class (e.g., Izsák, 2003; Saxe, 2012) have generally agreed to distinguish what is being represented and what is "doing" the representing (cf. von Glasersfeld, 1987). In this study, I refer to representations as observable geometric inscriptions that can be referred or pointed to as the object of discussion (Goldin, 2002). It is this

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indexical and communicative nature of representations allowing students to explain their thinking and for others to engage in another's way of reasoning. When students create a display to represent their thinking, they also have a communicative aspect. In other words, they tailor their display with an audience in mind (Saxe, 2012) and thus students select salient features to highlight and point when creating and talking about representations. Additionally, I frame representations as culturally and historically rooted. A representation's cultural and historical meaning grow out of how communities have interacted with an inscription over time (Blumer, 1986). For example, a class can ascribe the meaning to the inscription "=" as "execute the arithmetic to the left" if they are continually asked to solve result-unknown problems over time.

Strip diagrams. The Common Core authors recommended strip diagrams when reasoning about ratios and rates in 6th grade (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). A strip diagram is usually drawn as a rectangle that can be partitioned into different sized parts where each part may refer to a quantity. Although researchers have identified the strip diagram as a feature of mathematics instruction in high performing countries such as Japan and Singapore, research on strip diagrams themselves is sparse especially in the United States (Murata, 2008; Ng & Lee, 2009).

Form-Function Relationships

Saxe's account (2012) of cultural forms and functions accounts for how historically-rooted artifacts change over historical time. *Forms* are socially-rooted systems of artifacts perceivable by members of the community such as the base-27 system of the Oksapmin peoples (Saxe, 2012) or a number line in a math class (Saxe, de Kirby, Le, Sitabkhan, & Kang, 2015). *Functions* are how the forms are used to achieve goals. To characterize cultural forms and their functions, the researcher must investigate the creation of a "common ground" or a taken-as-shared ways of talking and doing (Saxe et al., 2015). Saxe and colleagues identified three strands describing how individuals contribute to common ground. In this repor, I account for two of these strands:

1. *Microgenesis.* This process shows how individuals contribute to a common ground, often using a form in public, by describing how forms serve certain functions. For example, if a student wants to show how 3/2 is equivalent to 6/4, a student may create a strip diagram partitioned into three parts and partition each part into two in order to show six sub-partitions using a different color to show the relationship between the two partitions. The student is contributing to common ground by producing a particular form (strips with partitioned partitions) to describe fraction equivalency.
2. *Ontogenesis.* This process shows the continuity and discontinuity of forms to serve new functions. In some instances, if a new function is necessitated, some previously used forms may be employed (continuity) or new forms (discontinuity) may emerge to serve the new function and accomplish the goal.

Data and Analysis

Context of the Study

I analyzed four days of instruction from the second course of a sequence of two mathematics content courses for prospective middle school teachers (PSMTs) enrolled in a teacher education program. The same teacher taught both courses. The program was geared towards certification to teach mathematics in Grades 4–8. The objective of the course was to strengthen the students' mathematical understanding of middle school topics such as base-10, fractions, and ratios. The 13 PSMTs enrolled in the course were predominantly white women. Two class norms were developed in class by the time of the classes selected for analysis. First, students were expected

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to use a multiplicand-multiplier definition for multiplication, notated by equation $N \cdot M = P$ (Beckmann & Izsák, 2015). In this equation, N denotes the number of base units in one group (the multiplicand), M denotes the number of groups (the multiplier), and P denotes the total number of units in M groups. With this definition, the order of the factors matters e.g., 12 boxes with six donuts in each box ($12 \cdot 6$) is different from six boxes with 12 donuts in each box ($6 \cdot 12$) even if they are numerically the same number of donuts in total. The class was also expected to use the Common Core definition of fraction (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) where the fraction a/b referred to the quantity formed by a parts of size $1/b$. Lastly, students were expected to explain their thinking with drawings rather than memorized algorithms or symbol manipulation. Specifically, students were expected to use strip diagrams and double number lines.

The teacher usually began class by orienting the PSMTs to the mathematical topic of the day. She gave the class a problem to solve and the PSMTs worked at their table with two to five other PSMTs. As they worked, the teacher walked around class, supporting or pressing the PSMTs. The teacher would redirect the PSMT if they breached any of the two sociomathematical norms. PSMTs were given the option of using iPads. After a period of time, PSMTs presented the strategies in whole-class discussion. To present their strategies, they could recreate their strategies on the mounted whiteboards or project their iPad screen on one of four mounted screens. Some students used the iPad's camera to project written work. The whole-class discussion focused on students' strategies and connections between different strategies.

Data Collection and Analytical Techniques

The main data corpus for this study was video and audio-recorded lessons from class. One stationary camera was set at the back of the class and captured the whole class within one frame. The other camera was also stationary during whole-class discussion but followed the teacher during small group discussions. Two microphones mounted on the ceiling captured audio during whole-class discussion while four flat microphones captured audio at each table. In post-production, all video and audio data were condensed into a single file. The two videos were synchronized and combined into a single frame. The file contained all the audio feeds such that I could select any audio and listen to one audio source.

The primary analytical techniques were modified from Saxe et al., 2015 and focused on identifying forms and functions of the representations to characterize the microgenetic process. To identify forms, I located PSMTs' inscriptions in the classroom data. I relied on both discursive and gestural indicators. As the PSMTs talked about their drawings, I noted when they physically gestured to an inscription or used pointing language such as "This is…" or "I drew…" Two grain sizes for forms emerged from this analysis. A *micro form* was a single geometric inscription as fine as a line or rectangle and a *coarse form* was a group of microforms used to address a larger goal such as an entire strip diagram with its annotations to solve a multiplication problem. To identify a function, I found pointing language referring to what a particular form represents e.g., "This is a cup" and annotations on the drawings. To address the ontogenetic strand, I searched for moments between problems wherein forms changed. I identified the difference between the question or problem that was asked before and after the change.

Results

Over the course of the lessons, the PSMTs used strip diagrams to explain their thinking. The coarse form changed when the problem type changed. In Table 1, I summarize and illustrate the coarse and micro forms used. As the lessons progressed, the function of a partition of a strip changed to accommodate multiple functions which allowed students to explain KCF.

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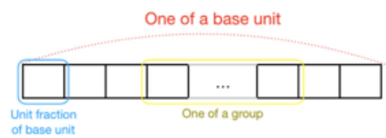
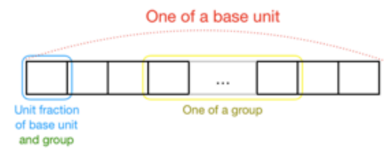
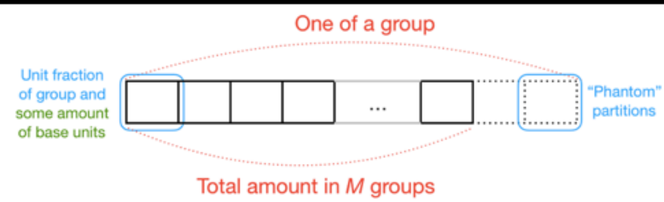
Table 1: Coarse Forms and Functions of the Micro Forms

		Measurement Division Forms: First coarse form	Measurement Division Forms: Second coarse form
			
Micro forms	Whole strip	One of a base unit	One of a base unit
	Set of partitions	One of a group (*N*)	One of a group (*N*)
	One of a partition	Unit fractional amount of a base unit	Unit fractional amount of a base unit and a group

		Partitive Division Form
		
Micro forms	Whole strip	One of a group (*N*)
	Set of partition	Total amount in *M* groups (*P*)
	One of a partition	Unit fractional amount of a group and some amount of base units
	"Phantom" partitions	Partitions intended to complete one of a group

Note: *M, N,* and *P* refer to quantities in the multiplication definition in Beckmann & Izsák (2015)

Measurement Division Forms

The first two lessons centered on solving measurement or how-many-groups division. The two coarse forms resembled the definition of a fraction where each strip stood for one of a base unit e.g., one cup and the partitions referred to a unit fraction of the base unit e.g., $1/n$ of a cup. The PSMTs indicated one of a group by highlighting a set number of parts (such as the different colors in Figure 1a). Using the set number of parts, they counted how many groups were in the total amount of base units. The second form resembles the first coarse form; however, the function of a partition changed to holding two referents, the base unit and the group.

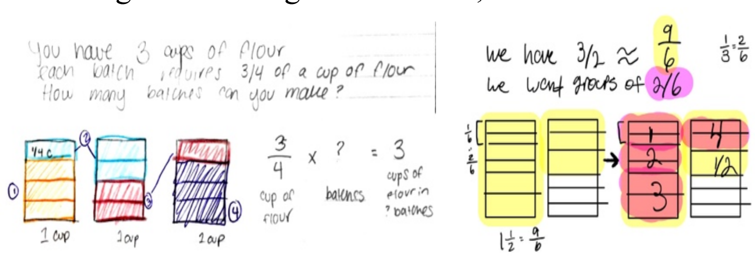


Figure 1. (a) Sophie's Diagram for 3 ÷ 3/4 = ? (b) Jack's Diagram for 1 ½ ÷ 1/3 = ?

In the first lesson, students created a word problem for the measurement division problem 3 ÷ 3/4 = ?. They solved the problem using strip diagrams and in some cases double number lines. All the strip diagrams produced resembled Sophie's diagram in Figure 1a. Sophie created the word problem "You have three cups of flour. Each batch requires three-fourths cup of flour.

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How many batches can you make?” During whole-class discussion, she explained, “I first drew three full cups each of those is a cup.” Based on the annotations at the bottom of her strip and her equation, Sophie created a whole strip as a cup. Because the class was annotating equations with one definition of multiplication and Sophie’s placement of cups as the multiplicand in her equation, I inferred she assigned cups as her base unit. She explained, “each of those is made of four parts, each of size one-fourth of a cup and then I colored three of them which is three fourths of a cup each makes a batch” thus a set of three partitions was one group or batch in her context. In the next lesson, PSMTs were also asked to create a measurement division problem for 1½ ÷ 1/3 = ? and solve the problem. Jack wrote the problem “You have 1½ liters of apple juice. You want to pour this apple juice into glasses, which can hold 1/3 liters each. How many glasses can you fill with your apple juice?” Similar to Sophie’s strategy, he created a whole strip indicating one cup of apple juice or one of a base unit. He shaded one and a half strip indicating the total amount he needed to count (see yellow partitions in Figure 1b). During whole-class discussion, Jack explained he anticipated the size of the partitions he wanted “[because] halves and thirds don’t mix perfectly…so I needed to put them in same sizes so the easiest one was sixths ‘coz that’s our common denominator–least common denominators.” Thus, he chose partitions that were 1/6 of the strip. He counted sets of two partitions showing one of a group or 2/6 of a liter. Finally, he commented on the left-over partition and described it simultaneously as one sixth of a liter and half of a glass. This signaled describing a partition with respect to both the group and base unit as a new function for a partition. The ontogenesis of this new function for the partition could perhaps be explained by the number choice of the problem in the first problem where the total number of partitions is divisible by the numerator of the divisor.

Partitive Division Form

The next two lessons centered on solving partitive or how-many-in-1-group division problems. The PSMTs initially worked on partitive division problems where the dividend was a whole number and the divisor was a fraction between zero and one. One coarse form emerged from solving these problems. They created whole strips referring to one of a group e.g., one serving and the partitions were a unit fraction of the group. They also referred to the partition as some amount of base units. Describing the partition with respect to both groups and base units is perhaps a function rooted in the function emerging towards the end of the measurement division lessons. Finally, students also distinguished some partitions. Jack called these partitions that “aren’t really there” as “phantom” partitions. The phantom partitions were drawn to complete one of a group because the sizes of the group that given in the problems were less than one.

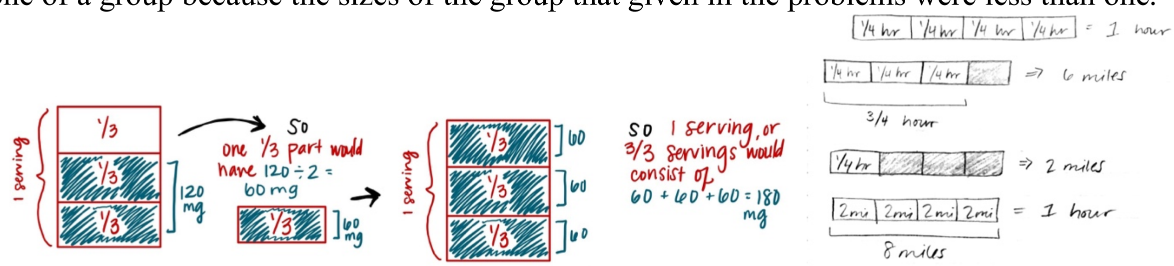

Figure 2. (a) Elizabeth’s Diagram for 120 ÷ 2/3 = ? (b) Catherine’s Diagram for 6 ÷ 3/4 = ?

First, the class worked on the problem “2/3 of a serving of noodles contains 120 mg of sodium. How much sodium is in one bowl of noodles?” Afterwards, they worked on the problem “Running at a steady pace, Anna ran 6 miles in 3/4 of an hour. At that pace, how far will Anna run in one hour?” Elizabeth and Catherine’s strip diagrams in Figure 2 are both exemplars of the coarse form PSMTs used when solving partitive division problems. In both diagrams, each

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whole strip denoted one of a group and assigned a subset of the partitions in a strip to denote the total amount of base units in *M* groups. In Elizabeth's first strip diagram, the set of blue partitions referred to 120mg of sodium and 2/3 serving. Catherine similarly drew her second strip with the total amount of miles in 3/4 hours. They then considered just one of these partitions and described the partition in both the quantity of the group and base unit similar to Jack's partition in the measurement division lesson. Elizabeth and Catherine also assigned equal amounts of the base unit in each partition. For Elizabeth, she annotated this in the middle of her drawing (Figure 2a) by labelling one of her partitions as both one-third of a serving and 60mg of sodium. Catherine showed this function for a partition in her third strip (Figure 2b) where she annotated one of her partitions as one-fourth of an hour and two miles. They both iterated this partition to complete the whole strip and counted the amount of base units in the whole strip. Although the functions for the whole strip and the partition shifted i.e., measurement division problems showed one of a base unit while partitive division problems showed one of a group, the function of the partition as both an amount of a group and base unit was present in both problem types.

Explaining Keep-Change-Flip

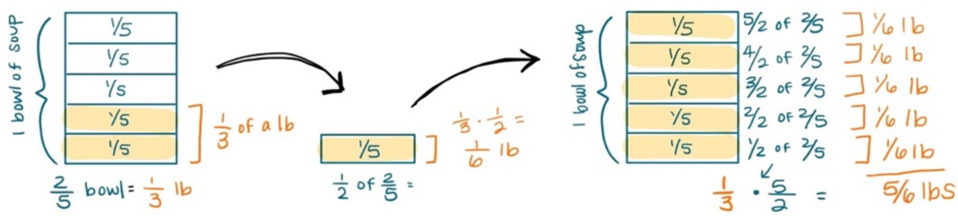

Figure 3: Elizabeth's Strip Diagram for 1/3 ÷ 2/5 = 1/3 · 5/2

PSMTs explained KCF with strip diagrams and the definition of multiplication when explaining partitive division problems when prompted by the instructor. The instructor asked the PSMTs to find ways to explain 1/3 ÷ 2/5 = 1/3 · 5/2. The PSMTs leveraged the function of a partition to describe both the base unit and the group. To explain the rule, PSMTs used a third function for a partition—describing the partition with respect to the size of the group of the total amount of base units. Consider Elizabeth's explanation for 1/3 ÷ 2/5 = 1/3 · 5/2.

Elizabeth drew the strip diagram in Figure 3 to show her thinking for 1/3 ÷ 2/5 = ? and explained her thinking in whole-class discussion. She created a partitive division word problem "A third of a pound of chicken is enough for 2/5 of a bowl of chicken soup. How many pounds of chicken is in 1 whole bowl of chicken soup?" Elizabeth's drew a strip diagram with the coarse partitive division form. First, she drew the strip on the left functioning as one of a group or one bowl of soup. She partitioned the strip into five parts and annotated her parts as 1/5 of the bowl and "colored in two of the fifths and called that 1/3 of a pound." The set of partitions referred to the size of the group, i.e., 2/5 of the bowl, but also the corresponding quantity in base units i.e., 1/3 pound of chicken. Using the function of the partition, she described one partition in two ways, as 1/5 of the bowl and 1/6 of a pound as seen in the middle of Figure 3. She iterated this part to build the whole bowl of soup and kept track of both quantities simultaneously to get 5/6 pounds of chicken in the whole bowl similar to the previous partitive division coarse forms.

To explain the equivalence 1/3 ÷ 2/5 = 1/3 · 5/2, Elizabeth described the situation considering two groups—the original group of one bowl and a new group of 2/5 of a bowl. Considering this new group, she explained the partition is also one-half of two-fifths of the bowl. This activity indicated a new function for a partition in addition to denoting a unit fractional amount of a group and base unit. She used the partition as a unit fractional amount of the size of the group of the total amount of base units in addition to one-fifth of the bowl and one-sixth of

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the pound. In other words, one partition refers to 1/6 pound of chicken, 1/5 of the bowl, and one-half of two-fifths of a bowl. She counted the five partitions in the whole strip and used the new function to get one bowl as five halves of two-fifths of the bowl.

Using the new group, Elizabeth created the expression $1/3 \cdot 5/2$ following the definition of multiplication used in class. In the annotation, she explained there is one third pound of chicken in one of the new group, two-fifths bowl (amount in one group, *N*) and there are five-halves of the new group in the whole bowl of soup (number of groups, *M*). In summary, Elizabeth's group changed from one bowl to two-fifths of a bowl when asked to explain KCF. Because of her new group, she added a new function to one partition. By using the class definition of multiplication, she annotated her thinking when she considered the new group to obtain the expression $1/3 \cdot 5/2$.

Two reasons could explain the ontogenesis of the new function when solving partitive division problems. First, the instructor's prompt of asking students to explain how to see $1/3 \cdot 5/2$ in their drawing seemed to initiate the creation of the function. If the PSMTs were to simply solve the partitive division problem, the third function for the partition may not have emerged. Second, the nature of partitive division problems may afford a flexible choice of groups. Elizabeth retained 1/3 in base units and flexibly chose her group as either the whole group or 2/5 of her group. In the problem Jack was solving, 1 1/2 and 1/3 both referred to base units and could potentially restrict a group to solely 1/3 of a base unit.

Discussion and Conclusion

The results of this study provide a characterization of how functions of inscriptions evolve over time. In this case, an evolving function for a partition in a strip allowed students to explain KCF. PSMTs' explanations of the algorithm were rooted in two practices—using strip diagrams and a definition of multiplication. Strip diagrams were not templates with predetermined rules and meanings. In fact, most of the forms were rooted in the activity of the class and previous uses of the strip diagram. Initially, PSMTs used the strip diagram to solve measurement division problems. By changing the number choices within the problem, a new function of the partition of a strip diagram emerged which proved to be useful in subsequent solutions. The strip diagrams remained relatively similar when solving partitive division problems. When prompted to explain KCF, the students used the partition in a new way not previously used in solving problems.

The results I report here provide future steps for both researchers and teachers. When analyzing inscriptions, researchers must attend and be explicit about the grain size of the inscription. By capturing two grain sizes, I was able to determine new uses for smaller inscriptions embedded in larger ones. Additionally, an analysis of inscription use in classrooms provides researchers with continuities and discontinuities between points in time in order to characterize teaching opportunities for new forms and functions to emerge. For teachers and teacher educators working with strip diagrams, activities wherein opportunities are provided to assign new functions to diagrams and its elements can equip students with ways of reasoning to be used in future tasks. Additionally, this report provides a case for using representations and how keeping coherent activity across multiple lessons provides students and prospective teachers with a powerful opportunity to learn mathematics.

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