PERSPECTIVES ON THE NATURE OF MATHEMATICS

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As mathematics educators, we teach and research a particular form of knowledge. However, in reacting to Platonic views of mathematics, we often overlook its unique characteristics. This paper presents a Kantian and Piagetian perspective that defines mathematics as a product of psychology. This perspective, based in human activity, unites mathematical objects, such as shape and number, while explaining what makes mathematics unique. In so doing, it not only privileges mathematics as a powerful form of knowledge but also empowers students to own its objects as their own constructions. Examples and interdisciplinary research findings (e.g., neuroscience) are provided to elucidate and support the perspective.

Keywords: embodied cognition, mathematical epistemology, neuroscience, radical constructivism, reflective abstraction

Mathematics has been described as both a science and a language. Most sources define it as a collection of abstract sciences, but its objects of study are so varied that, according to Wikipedia, "mathematics has no generally accepted definition" ("Definitions of mathematics," n.d.) Consider the following attempts:

- "the abstract science of number, quantity, and space" ("Mathematics," n.d.)
- "the science of numbers and their operations, interrelations, combinations, generalizations, and abstractions and of space configurations and their structure, measurement, transformations, and generalizations. Algebra, arithmetic, calculus, geometry, and trigonometry are branches of mathematics" ("Mathematics," n.d.)
- "a group of related sciences, including algebra, geometry, and calculus, concerned with the study of number, quantity, shape, and space and their inter-relationships by using a specialized notation" ("Mathematics," n.d.)

What do number, quantity, space, and the various branches of mathematics have in common?

Mathematics is a unique body of knowledge owing to its apparent infallibility. Across millennia, continents, and cultures, mathematics has produced stubborn facts, so much so that we confidently assume that any alien life form, if intelligent enough, would recognize the prime numbers (Sagan, 1975). Students often appreciate the way that mathematics builds on itself, such as the way real numbers build on rational numbers, which build on integers. Scientists marvel at the "unreasonable effectiveness of mathematics in the natural sciences" (Wigner, 1960), such as when mathematical models predicted the existence and location of Neptune before it was discovered (see Norton, 2015). No wonder Platonism still holds sway in society and scientific communities alike.

As a mathematics education community, we often confront Platonist ideals, which position mathematics as something that lies beyond human experience. We understand the cultural influences and psychological roots of mathematical development and mathematics itself. We challenge mathematical myths but rarely acknowledge their persistent epistemological basis. For example, we cite Kline's (!982) "Loss of Certainty" to break down perceptions of mathematics as a collection of immutable truths because such perceptions disinvite students to participate in

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creating mathematics (e.g., Chazan, 1990). However, we overlook the apparent certainty of mathematics as a feature that garners students' interests to begin.

Popular characterizations of mathematics do have a valid basis. There is a sense in which mathematics is infallible and builds upon itself, and mathematics holds a privileged position of predictive power among the sciences. However, these characterizations require psychological explanation rather than a Platonic dodge. Moreover, we need a definition that presents mathematics as a unified field of study rather than a collection of abstract sciences. What unifies mathematics? What are its objects of study? What is the basis for its reliability, utility, and ubiquity?

This paper presents a Kantian/Piagetian response—one grounded in cognitive psychology and buttressed by recent findings from neuroscience. Kline (1982) summarized the Kantian position as follows: "mathematics is not something independent of and applied to phenomena taking place in an external world but rather an element in our way of conceiving the phenomena" (p. 341). Piaget (1942), with Inhelder (1967), built upon this position by specifying children's development of mathematical structures used to organize the world, such as space and number. These structures depend on operations that, at once, demonstrate the unity and power of mathematics.

Mathematical Objects

Mathematical objects arise from our own activity within the worlds we experience. This is a view espoused by social constructivists, radical constructivists, and embodied cognitionists alike (Núñez, Edwards, & Matos, 1999; Vygotsky, 1986). The distinguishing feature of the Piagetian perspective concerns the role of abstraction, particularly *reflective abstraction*, in constructing those objects. Reflective abstraction is a psychological process that is notoriously difficult to grasp. As Chomsky lamented during a debate with Piaget, "my uneasiness with reflective abstraction is … that I do not know what the phrase means, to what processes it refers, or what are its principles" (Piattelli-Palmarini, 1980, p. 323). Here, we will attempt to specify the process of reflective abstractions and its principles, as well as its role in constructing mathematical objects.

We find Piaget's plainest description of reflective abstraction in *Genetic Epistemology* (1970). There, he describes the sensorimotor basis for logic and mathematics: "the roots of logical thought are not to be found in language alone, even though language coordinations are important, but are to be found more generally in the coordination of actions, which are the basis of reflective abstraction" (p. 19). He goes on to describe how actions become coordinated with one another, through reflective abstraction; but reflective abstraction does not apply to any and all actions—only those that are reversible.

Reversibility is another distinguishing feature of mathematics. Addition-subtraction, greater than-less than, integration-differentiation all form inverse pairs. However, Piaget (1970) refers primarily to reversibility of the mental actions that constitute these formalized operations, rather than the formalized operations themselves. For example, a student might know the sum of 10 and 5 but not know the sum of 9 and 6, even if she also knows that 9 is 1 less than 10 and 6 is 1 more than 6. In other words, she has not yet coordinated the actions of iterating (repeatedly integrating) units of 1 and disembedding them (separating units of 1 within the whole). Such a coordination relies upon organizing the actions of iterating and disembedding within a structure for composing and reversing them (compared to 5, 6 has an extra iterated unit of 1, which can be disembedded and composed with 9 to make 10). As educators, we might think about these as strategies, but through reflective abstraction, strategically coordinated actions become structures

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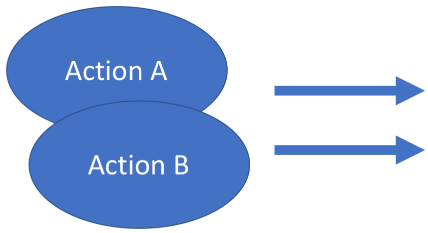
for assimilating mathematical situations so that the two sums become the same mathematical object, 15. Of course, numbers do not exist in isolation, so the coordination of mental actions like iterating and disembedding reorganize the child's entire number sequence, including 5, 6, 9, 10, and 15 (see Steffe, 1992).

Figure 1 illustrates the process of reflective abstraction. Several mathematics educators have recognized this process, or similar processes, as essential for mathematical development: Sfard (1992) referred to it as reification; Dubinski (2002) as interiorization; and Tall as encapsulation (Tall, Thomas, Davis, Gray, & Simpson, 2000). They all describe a process by which existing mental actions become coordinated to constitute a new mathematical object.



	i	A	B
i	i	A	B
A	A	B	i
B	B	i	A

Figure 1. Reflective abstractions and the construction of mathematical objects.

The table on the right side of Figure 1 represents the organization of actions within a mathematical group. This is a researcher's model for describing how the students' mental actions become coordinated with one another as composible and reversible operations; it does not imply that students are aware of a group structure (Piaget, 1970). Figure 1 illustrates the simplest example of coordinating mental actions, wherein two actions (A and B) are coordinated as inverses of one another and where i represents the identity element of the group (note that Piaget also referred to group-like structures that do not satisfy all conditions of a mathematical group but nonetheless model the coordination of reversible mental actions). The two arrows in Figure 1 represent the two aspects of objects noted by Piaget and Garcia (1986): "First of all, it is 'what can be done with them' either physically or mentally… (2) The meaning of object is also 'what it is made of,' or how it is composed. Here again, objects are subordinate to actions" (pp. 65-66).

The coordination of mental actions within group-like structures explains many of the unique features of mathematics. In particular, the reversibility of mental actions within the structure explains the reliability of mathematics. In the sciences, reliability is repeatability. The natural sciences never attain perfect repeatability because the initial conditions of a situation cannot be precisely reproduced. However, in mathematics, reversing one action with another action (e.g., A and B compose to form i, in Figure 1) returns one to the same exact starting point every time.

Coordinations of action also explain the ubiquity of mathematics because they become structures for organizing experience. For example, when I see seven cars in a parking lot, nothing in the parking lot imposes 7 upon me. Instead, I assimilate my perceptual experience by coordinating mental actions of unitizing (separating out each perceived car and treating it as a unit identical to the others) and iterating resulting units in one-to-one correspondence with my number sequence.

Furthermore, coordinations of action explain how mathematics builds upon itself, because the process of reflective abstraction does not end with the construction of the first structures. Rather, those structures, as objects, become material for further operating. For example, I can consider any multiplicity of 7 by acting upon one copy of my number sequence with another copy of it (Steffe, 1992). Such structures explain the subjectivity of mathematical experience when, for example, I see three rows of seven cars and assimilate them as three 7s, whereas a young child might see a spatial pattern but not the numerical structure of 3 times 7.

Evidence from Neuroscience

As noted in the introduction, definitions of mathematics generally refer to the study of a collection of objects, usually including number and space. As mathematics educators, the construction of number may seem more familiar, but Piaget and Inhelder (1967) used space (along with number; Piaget, 1942) as a primary example of a mathematical object. They demonstrated that space does not exist as an innate construct, as Kant had assumed, but that children construct it through the coordination of displacements within a group for composing and reversing them. In this section, we will see tight connections between space and number as psychological and neurological phenomena that depend on coordinated actions, beginning with sensorimotor activity.

One early connection concerns object permanence and the onset of self-locomotion (crawling). Developmental psychologists take object permanence as a critical marker in early child development, whereby children learn that objects persist in space even when removed from the child's perceptual field. Bell and Fox (1996, 1997) conducted studies on 76 eight-month-old infants, separated into four groups: pre-crawlers, crawlers with 1-4 weeks of experience, 5-8 weeks of experience, and 9 or more weeks of experience. Greater experience in crawling was associated with the development of object permanence.

Piaget and Inhelder (1967) had tied object permanence to children's construction of sensorimotor space, wherein objects would have residence. More recently, psychologists have associated object permanence with "spatial working memory", wherein children coordinate spatial transformations, such as displacements and rotations (e.g., Bell, 2001). Together with the findings from Bell and Fox (1996, 1997), the collective literature suggests that crawling provides sensorimotor experience that is critical to the construction of space as a coordination of displacements. After all, crawling enables children to transform their perceptual fields through voluntary movement, which (from the child's perspective) amounts to a displacement of space itself, similar to the transformations of space described by a vector field (or the group of vectors, under addition).

We find similar connections between embodied/sensorimotor experience and the child's construction of number. In particular, manual and numerical digits go hand-in-hand, in a manner that transcends etymology (see Norton, Ulrich, Bell, & Cate, 2018). For example, as mathematics educators, we know that children generally learn to count with the aid of their fingers as manipulatives, but recent neuroimaging studies indicate that the connection persists into adulthood. Specifically, neural substrates for finger recognition and finger use (e.g. pointing) overlap with those for counting and arithmetic, even among adults (Soylu, Lester, & Newman, 2018)—so much so that researchers now hypothesize that areas of the brain that evolved for manual dexterity (e.g., tool use) have been re-purposed to support mathematical development (Penner-Wilger & Anderson, 2013). In considering these neural substrates, the intraparietal sulcus (IPS) stands out.

Figure 2 presents a diagram of the neo-cortex—the outer layer of the human brain—and a few of its main regions. The frontal lobe lies above the eyes and plays the leading role in executive function (working memory, inhibitory control, and decision making). The parietal lobe rests toward the back of the brain and is generally associated with spatial reasoning, including hand-eye coordination. Between those two lobes sits the sensorimotor cortex, which initiates voluntary movement. The IPS aligns with the sensorimotor area associated with hand movement and runs between the upper and lower halves of the parietal lobe. This positioning would suggest that the IPS plays an important role in the manipulation of objects in space, which it does.

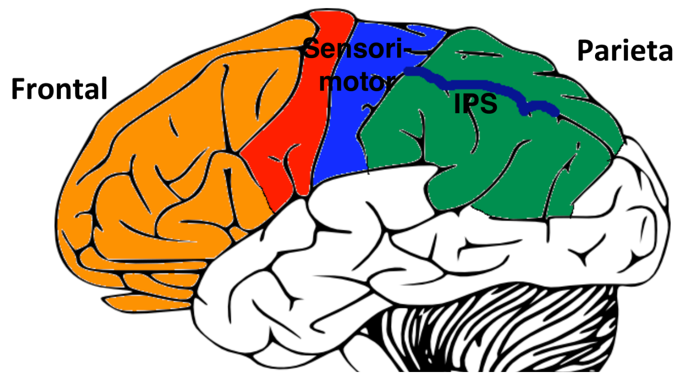


Figure 2. Neo-cortex.

In addition to its role in tool use and other coordinated actions involving the hand (e.g., Mruczek, von Loga, & Kastner, 2013), the IPS is implicated in virtually every neuroimaging study of numerical and spatial reasoning (e.g., Dehaene, 1997; Kucian et al., 2007). In and around the IPS we find the common neural substrate for the two primary objects of the mathematical sciences: space and number. There we also find their common link to coordinated sensorimotor activity, especially involving the hands.

The IPS exists as part of a network that includes the frontal lobe and the angular gyrus—an area in the lower part of the parietal lobe associated with memorized tables of information (e.g., multiplication tables). Studies of mathematical development generally show a shift, from frontal to parietal areas of the network, as children learn: "Solving a new multiplication problem involves the IPS bilaterally and also the frontal lobes, while dealing with the same problem a second time shifts the focus of activity to the angular gyrus in the left parietal lobe" (Butterworth & Walsh, 2011, pp. 19-20). Other studies (e.g., Ansari, 2008), show a similar frontal-parietal shift associated with age.

As we have mentioned, executive function is a primary role of the frontal lobe. It directs limited working memory resources (including spatial working memory) to solve novel problems. As children learn—either rotely through memorizing multiplication tables or through the development of conceptual structures—working memory is offloaded so that the same task becomes less demanding. We posit that areas in and around the IPS serve as the neural substrate for spatial-numerical structures. This view, too, is supported by neuroimaging studies (Hubbard, Piazza, Pinel, & Dehaene, 2005) and implies that the IPS is heavily involved in assimilating mathematical experiences. Resources from the frontal lobe are recruited when the assimilated experience becomes problematic. As such, frontal-parietal coherence (areas within the two lobes working in tandem, as indicated by brain wave frequencies) would be the neural correlate of mathematical development.

Returning to Bell and Fox's (1996) study of crawling, infants with 1-4 weeks of crawling experience exhibited greater frontal-parietal coherence than pre-crawling infants and infants with more crawling experience. Thus, self-locomotion appears to provide a sensorimotor foundation for the development of object permanence and the construction of space—the play space for subsequent geometric construction. In the next section, we consider the case of Euclidean geometry.

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Euclidean Objects

Euclid was the first mathematician to formalize mathematics on an axiomatic basis. The purpose of the Elements, Book I, was to prove the Pythagorean (sic) theorem from common notions and postulates (axioms). This is evidenced by the appearance of the Pythagorean theorem and its converse as the final two propositions in the book (Propositions 47 & 48). But what was the basis for these axioms and formal arguments? Evidence appears in the axioms and arguments themselves. The first three axioms (Postulates 1-3) are Plato's rules for straight edge and compass constructions, indicating their sensorimotor basis within Greek culture. The chain of propositions leading from those axioms to the Pythagorean theorem indicates the kinds of mental actions behind Euclid's intuitions. Here, we demonstrate how coordinations of spatial transformations, like sweeping, shearing, and rotating, form the psychological basis for geometric objects.

Figure 3 illustrates the diagram Euclid used to support his arguments for the Pythagorean theorem. Essentially, he argued that the areas of the yellow and blue squares were equivalent to the areas of the yellow and blue rectangles, respectively (Proposition 47). The argument depended on previous propositions demonstrating that shearing triangles and parallelograms does not affect their areas (Propositions 35-38). In Figure 3, triangle DAC is the result of shearing triangle DAG (half of the blue rectangle) along segment FC. Likewise, triangle ABE is the result of shearing triangle ACE (half of the blue square) along segment HB. Because these triangles are congruent (Euclid relied on Proposition 4—a side-angle-side argument, which he demonstrated through Common Notion 4, displacing those elements from one triangle onto another and showing that the remaining sides must also coincide), the areas of the blue rectangle and blue square (each having twice the area) are equivalent. The same argument works for the yellow regions, thus proving the Pythagorean theorem.

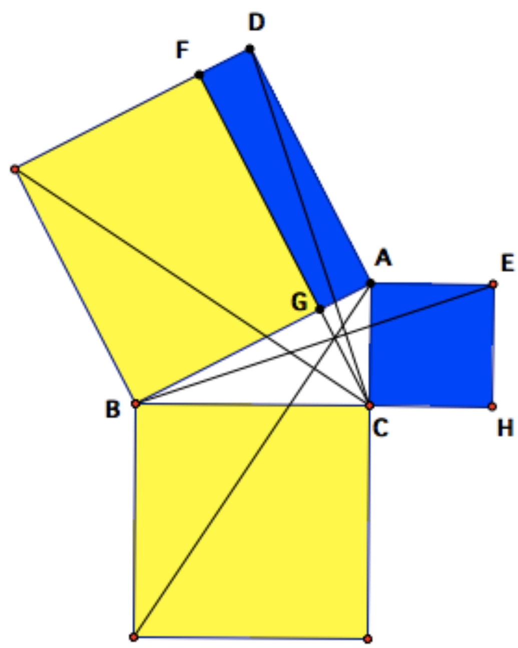

Figure 3. Euclid's proof of the Pythagorean theorem.

In sum, Euclid proved the Pythagorean theorem by transforming mathematical objects (e.g., squares) through mental actions of bisecting, displacing, and shearing. The mathematical objects being transformed are themselves the result of mental actions, such as sweeping (sweeping a

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point to make a line segment and sweeping a line segment to make a square, as in Proposition 46). Constructing and transforming mathematical objects in this way fits Piaget's descriptions of mathematical objects as coordinations of mental actions, as indicated by the two arrows in Figure 1: (1) mathematical objects arise through the coordination of actions and (2) can be subsequently transformed through further action. Thus, mathematical objects are characterized by both the actions that constitute them and the manner in which actions transform them, particularly aspects of the objects that remain invariant under transformation (e.g., the area of a parallelogram as invariant under the transformation of shearing).

Consider the simpler example of angle sums within a triangle (Proposition 32). Like number, children have to construct triangles: "children are able to recognize and especially to represent, only those shapes which they can actually reconstruct through their own actions" (Piaget & Inhelder, 1967, p. 43). Understanding triangles as mathematical objects requires children to move beyond the figurative material that represents or symbolizes them and to focus on the underlying mental actions that constitute them. The perfect triangle does not exist as a Platonic ideal, but rather as a coordination of actions, including sweeping and rotating.

To demonstrate that the angles in a triangle sum to a straight angle (pi, or 180 degrees), consider the construction of the triangle itself. It begins with a segment (side) swept from one vertex to another. Each pair of adjacent sides forms an angle, which measures the degree of openness, or rotation, between them (Moore, 2013). Figure 4 illustrates the three rotations (A, B, and C) that occur between pairs of adjacent sides. Each rotation is a transformation of one side onto the adjacent side, preserving the property of being a straight segment (a sweep from one vertex to another) but transforming its length and direction. After three such transformations, the original segment has been transformed back onto itself but in the reverse direction. In other words, the combined effect of composing the three angle rotations is a rotation of 180 degrees.

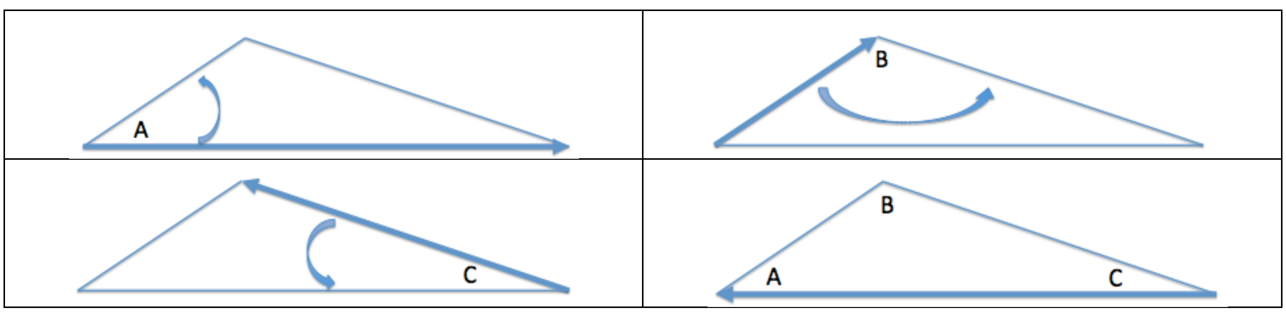

Figure 4. Sum of angles in a triangle.

What we see in the *Elements* is the historical trace of Euclidean geometry from sensorimotor activity all the way up to the first axiomatic system. Thus, we can trace formal mathematical objects, such as right triangles with all of their properties, all the way back to their psychological roots. Like numbers, shapes and their properties (e.g., the Pythagorean theorem) depend upon the coordination of mental actions. For the Greeks, those mental actions were derived from the sensorimotor activity of playing in the sand with compass and straight-edge. Reflective abstraction provides the mechanism for moving from each stage to the next: from sensorimotor activity to mental actions, to the construction and transformation of triangles, to the formal demonstration of the Pythagorean theorem.

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Summary

From the Kantian/Piagetian perspective described here, mathematics can be defined as the study of reversible mental actions and the structures that organize them (Piaget, 1970). This unifying definition applies to shape as well as number, both of which arise from the coordination of actions that have a sensorimotor basis (Piaget, 1942; Piaget & Inhelder, 1967). The definition also explains unique features of mathematics while empowering students to construct, transform, and study mathematical objects on the basis of their own activity. The infallibility of students' constructions owes to the reversibility of the mental actions that undergird them (Piaget, 1970). For example, if a child has defined a triangle on the basis of its three planar angles (rotations), composing those rotations with one another inevitably leads to the conclusion that they form a straight angle—a single 180-degree rotation that can be partitioned into the three angles that constitute it. The trick is to find a way to compose all three rotations without appealing to the drawn figure itself but rather to the organization it represents. This process of organizing rotations within a structure for composing and reversing them is the process of reflective abstraction.

When we focus on students' available mental actions and their engagement in sensorimotor activity, we are valuing students' mathematics as they construct new mathematical objects—objects that empower students to model and structure the worlds they experience. Thus, the appeal to students' mental actions is an appeal for equity in mathematics education. Building from the work of Noddings (1999), Hackenberg (2010) has described the appeal in terms of *mathematical caring relations*, wherein the teacher builds models of the students' available mental actions and engages the student with tasks likely to foster new coordinations.

Although Kant and Piaget set the stage for investigating students' mathematical constructions, researchers have just begun the work of describing those constructions as coordinated mental actions. The task before us is compounded when we consider the entire body of formal mathematics, ultimately entailing an account of the sensorimotor basis of the mental actions that undergird it. For example, can we account for the development of geometry from the onset of crawling to the Pythagorean theorem? This work too is mathematical because it requires us to explicitly identify the structures that organize reversible mental actions. Only then will we fully understand mathematics as a human construction rather than a Platonic ideal.

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