TEACHERS' INCOHERENT CALCULATIONS IN PROPORTIONAL TASKS

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We investigated how pre-service teachers (PSTs) interpret their calculations in proportional tasks. A written questionnaire was administrated to 199 PSTs and an inductive content analysis approach used for data analysis. We found that one item that asked PSTs to interpret the meaning of their results had unusually low success; open coding on the responses revealed several common themes. We argue these common errors cannot be dismissed as simple unit or rounding mistakes; they are a reflection on how respondents think about quantities, story problems, and the nature of mathematics itself. We end with suggestions on how to address this problem.

Keywords: Preservice Teacher Education, Teacher Knowledge, Proportional Reasoning, Problem Solving

Introduction

Proportional reasoning has been widely investigated as a key type of reasoning that both students and teachers in K-12 mathematics struggle with (Beckmann, 2015; Byerley, 2017; Langrall, 2000; Son, 2013; Tourniaire, 1985) and is also cited in the Common Core as one of the eleven mathematical domains that span mathematics from elementary to high school (National Governors Association Center for Best Practices, 2010). We surveyed 199 preservice elementary school teachers on 10 tasks that involve proportional reasoning, and the responses revealed that the PSTs struggled to interpret the meanings of their own calculations. This issue was most clearly illustrated in the task that required students to interpret their own calculations in order to give an answer. In this report we will share the major themes we found amongst PSTs that had difficulty answering this task. Our research questions include: (1) How do PSTs solve a proportional reasoning problem including a unit rate? (2) What kind of challenges in reasoning could cause a PST to struggle with answering a proportional reasoning question?

Theoretical Perspective

We approach our research questions and data analysis from a theoretical framework of quantitative reasoning (Smith & Thompson, 2007; P. W. Thompson, 1993, 2011). A quantity is a measurable attribute (such as length, elapsed time, volume, etc.) of an object (such as a car, a person, the Earth, etc.), and a "person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the possibility of measuring it" (P. W. Thompson, 1993). This way of thinking in word or story problems stands in marked contrast to more procedural ways of reasoning such as key-word approaches (replacing "and" with "+" and "less than" with "-", etc.) Quantities as defined by Thompson occur only in the mind of a thinker, who conceptualizes them by making sense of a quantitative situation. Quantitative reasoning, then, is "the analysis of a situation into a quantitative structure--a network of quantities and quantitative relationships" (P. W. Thompson, 1993). A person is reasoning quantitatively when he or she is reasoning about quantities, instead of numbers, undefined variables, or memorized procedures. In the task and responses we present below, the prompt deliberately did not ask for a numerical answer; instead, it asked a yes/no question that required respondents to engage in three steps: a) decide what calculations would be relevant and useful to answering the question, b)

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carry out those calculations accurately, and c) interpret the meaning of their calculated results to answer the prompt. Our analysis focuses mainly, though not exclusively, on the written work that illustrates the third part of this process in our PST population.

Methodology

The proportional reasoning questionnaire was given to 199 elementary preservice teachers over 3 semesters at a large Southwestern university in the U.S. All participants were juniors who enrolled in an elementary mathematics content course covering patterns, functions, and modeling. They were on average 1.5 years away from being becoming full-time elementary school teachers including grades K through 8.

The participants completed a written questionnaire including 10 problems about proportional reasoning and we focused one of the problems (see Figure 1). In the questionnaire, PSTs were explicitly asked to solve all problems using quantitative approach (e.g., using pictures) first rather than using a standard algorithm and to provide their work along with clear justification. The written questionnaire including the problem in Figure 1 was administrated to all PSTs in six sections of a mathematics concept course in the middle of spring 2015, fall 2015, and spring 2016 and it was assigned as take-home homework right after teaching proportional reasoning.

One author came up with a first draft coding strategy for the problems based on the format of the questions. The other author then coded over half the data and looked at how well the coding strategy worked, suggested modifications, and the two authors together agreed on the next draft to the coding strategy. All the data was then coded but several smaller coding strategy changes were also implemented and those sections were recoded. The other author then coded a random subset of responses to verify that the strategy was being implemented uniformly. There was 100% agreement on the coding of 95% of the examples.

Once the initial coding was done we decided that we would like to focus on a specific phenomenon: the cases where preservice teachers either incorrectly interpreted the results of their own calculations, or did not interpret their results at all. We therefore chose to focus on our data on the “Dr. Lee’s Car” item out of the ten problems.

Dr. Lee drove 156 miles and used 6 gallons of gasoline. At this rate, can he drive 561 miles on a full tank of 21 gallons of gasoline? Solve this problem and justify your reasoning.

Figure 1. “Dr. Lee’s Car” Item

Results & Discussion for “Dr. Lee’s Car” Item

We decided that since the prompt asked for a decision of Dr. Lee’s driving ability, and not for a numerical answer, that a correct answer would say that Dr. Lee could not drive 561 miles on 21 gallons of gasoline. 152 preservice teachers made correct final statements such as “No”, “He cannot go 561 miles”, “He can go only 546 miles”, “He has miles left over”, or “He would need more than 21 gallons”; 9 of these teachers still had problems interpreting their own results despite their correct final answer. 36 preservice teachers answered “Yes” or “He can go 561 miles”; only 4 were due solely to arithmetic errors and the other 32 teachers had problems interpreting their own results. 11 preservice teachers gave no final answer, but 5 of them completed all of the necessary calculations yet did not interpret them to answer the question. Overall, 46 out of 199 teachers, or 23.11% of the teachers, demonstrated some problem with interpreting the results of their own calculations.

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Table 1: Summary of all responses

	Correct answer: "He can't make it"	Incorrect answer: "He can make it"	No final answer	TOTAL
Total responses:	152	36	11	199
Responses with interpretation errors:	9	32	5	46

Amongst the 143 teachers that made relevant and accurate computations *and* correctly interpreted those results to answer the prompt, there were three solution techniques: to compute how many miles the car could go with 21 gallons, to compute the gas efficiency needed for both trips, or to compute the gallons needed to complete the second trip. Examples of each strategy are displayed below.

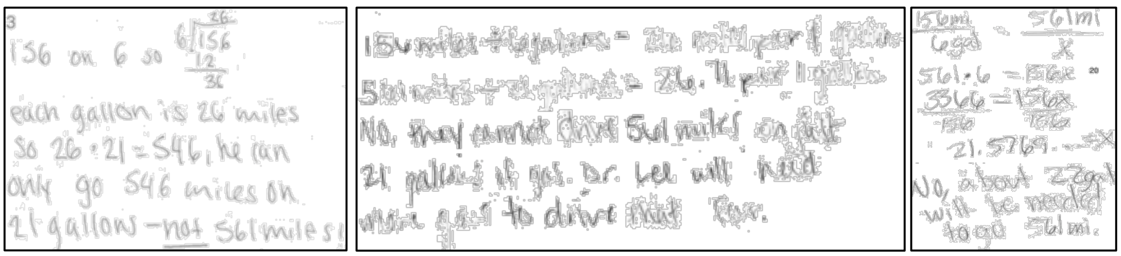

Figure 2. 3 Correct Solution Methods

Among the 46 teachers who had problems interpreting their own calculations, we found several themes.

Whole Number Bias (8.5% of respondents)

By far the most common problematic interpretations revolved around a strong bias towards either only calculating whole numbers, or rounding all values to whole numbers. 17 of the 45 preservice teachers who struggled to interpret their own work did so at least in part because they chose not to reason with decimal numbers. We hypothesize that these teachers looked for similarities in the results of their calculations without attending to the measures that they represented.

5 of the 199 teachers rounded both gas efficiency rates to 26 miles per gallon and concluded that Dr. Lee could make his trip. If a preservice teacher wanted to compare miles per gallon rates to evaluate whether Dr. Lee can make the trip or not, it is clearly relevant that 26.7 mpg is not the same as 26 mpg. Therefore, a teacher that rounded to 26 mph made a problematic interpretive decision when transferring work from calculator to paper, or when deciding to end their long division after finding the whole number part of their response.

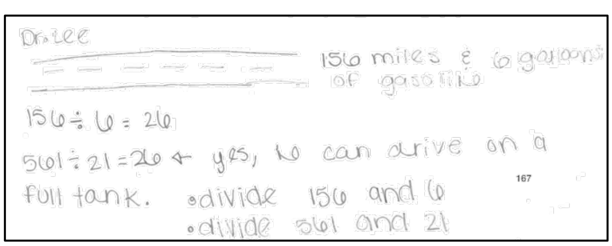

Figure 3. All calculations end in whole numbers

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.
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8 out of the 199 teachers found that the gas efficiency rates of both trips was approximately the same, and concluded that Dr. Lee could complete his trip; 4 teachers qualified their answers by saying that Dr. Lee could just barely make it. In doing so they neglected to keep track of the meaning and significance of their own calculations. The equality of both efficiency rates is irrelevant; rather, it matters whether the needed gas efficiency of the hypothetical trip is *less than or equal to* the known efficiency of the car.

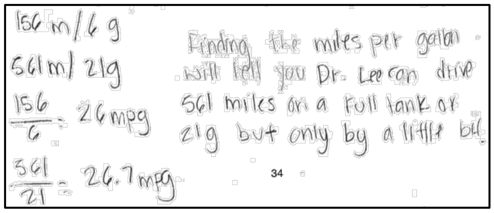

Figure 4. Yes, but only by a little bit [of what?]

5 of the 199 teachers found that it would require approximately 21.5 gallons for Dr. Lee to complete his trip, and concluded that his trip was possible. What matters is whether the gas the trip requires is *less than or equal to* the gas that Dr. Lee has available to him in a full tank, not whether they are approximately equal.

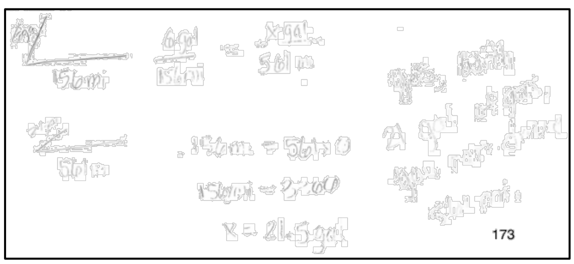

Figure 5. Yes, but only by a little bit [of what?]

Mixed Up Quantities (7.5% of respondents)

The next most common mistake is that preservice teachers did not keep track of the quantitative meaning of their results. These mistakes cannot simply be dismissed as writing down the wrong unit if we start with a presupposition that every calculation should have meaning to the person doing the calculating. We hypothesize that 16 out of the 45 preservice teachers who struggled to interpret their own work did so at least in part because they carried out operations on numbers without following them with operations on quantities.

9 out of 199 teachers calculated values and then ascribed the wrong quantitative meaning to them. For example, they calculated the gallons needed to complete a trip of 561 miles (21.5 gallons) but then wrote down 21.5 miles, calculated the gas efficiency of the shorter trip (26 miles per gallon) but then wrote down 26 gallons, or calculated the relative size of the trips in both gallons and miles (561 miles is 3.5 times as large as 156 miles, and 21 gallons is 3.3 times as large as 6 gallons) and then concluded that the difference meant that “Dr. Lee will be pushing [the car] .2 miles!”.

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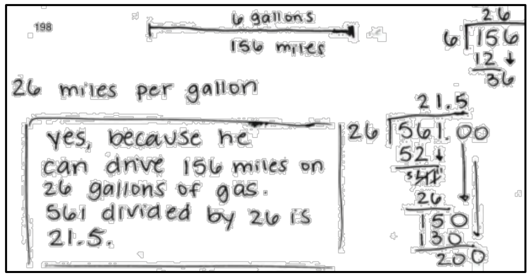

Figure 6. Mixed Up Quantities

6 out of the 199 teachers found values that would enable them to answer the question and interpreted the quantitative meaning of that value correctly, but then did not keep track of the meaning of a difference. 4 teachers correctly set up a proportion to find a value of 546, but then said that this means Dr. Lee *can* make the longer trip; 2 of those teachers interpreted the result of the calculation 561-546=15 to mean that Dr. Lee could drive an extra 15 miles. They did not keep track of the meaning of the difference as the miles that Dr. Lee *could not* drive on a full tank. 2 teachers calculated other differences (in the gas efficiencies of each trip, and gallons of gas used in each trip) and also concluded that Dr. Lee could make the trip.

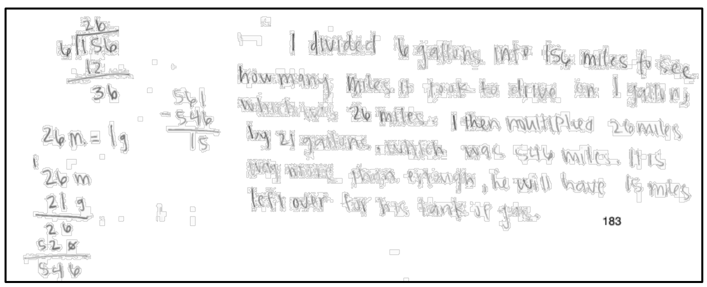

Figure 7. Directionless difference

Values Are Answers, No Meaning (3.0% of respondents)

5 of the 199 teachers reached the values they needed, of either the number of miles Dr. Lee could travel on 21 gallons, or the gas efficiency of both trips, but simply did not answer the question. While this may be an oversight, it may also illustrate a belief that mathematics is about finding numerical answers.

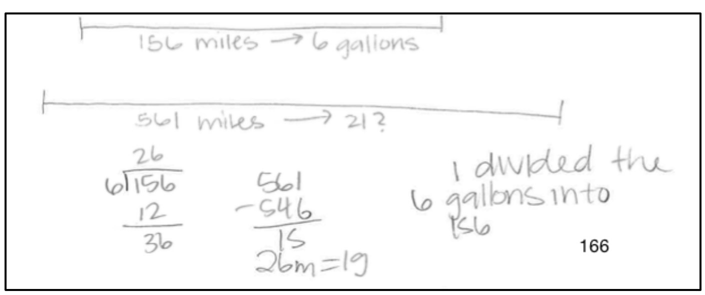

Figure 8. No answer

Chunky Gas (2.0% of respondents)

4 out of 199 teachers reasoned only in chunks of 6 gallons. 3 of the teachers concluded that Dr. Lee could not make the trip, and 1 concluded that he could, but all only calculated the miles

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that 6, 12, 18, and 24 gallons would enable Dr. Lee to drive. We commonly see such chunky thinking in more abstract contexts like linear equations, slope, and accumulation in calculus (Castillo-Garsow, 2013; P. W. Thompson, & Carlson, M. P., 2017), but we were surprised to find it in such a concrete example.

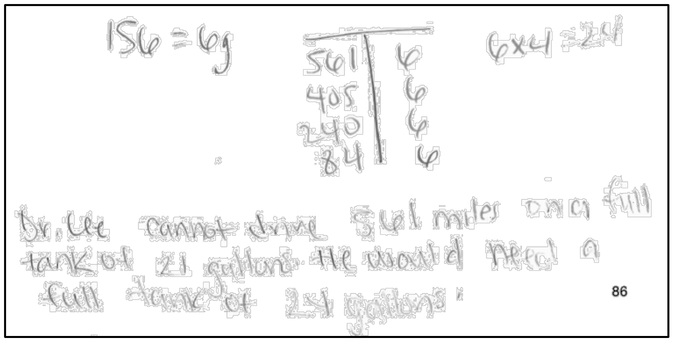

Figure 9. Gas comes in 6 gallon chunks

Equality is Everything (1.5% of respondents)

3 of the 199 teachers used cross-multiplying and looked for the equality of both sides. This strategy is appropriate when checking whether two fractions are equal, but not an appropriate strategy for determining whether one unit rate or gas efficiency is greater than or equal to another. Moreover, the resulting numbers (3276 and 3366) do not have clear quantitative meanings, as can be seen from their incoherent units of "dollar-gallons".

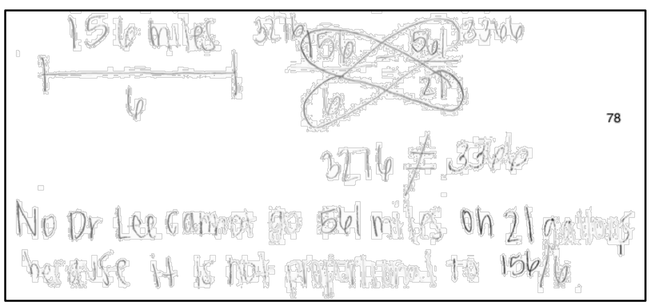

Figure 10. Equality is Everything

Conclusions & Further Thoughts

One of the enduring problems of mathematics education research is how to improve mathematics education in a way that benefits students both inside and outside the classroom. The days when a human calculator was valued and useful are over; now as technology develops, a person's ability to interpret the significance of mathematical outputs has been more emphasized than simple calculation. In the Common Core, the ability to interpret the meaning of one's answer is central to at least five of the eight Mathematical Practices: make sense of problems, reason quantitatively, construct viable arguments, model with mathematics, and attend to precision (National Governors Association Center for Best Practices, 2010). It is deeply concerning that over three semesters at a large university's Teacher's College, almost a fourth of PSTS struggled to interpret the meaning of their own calculations on a sixth-grade level task. There are several implications for students in such a teacher's classroom. Teachers that struggle to assign meaning to their calculations will certainly also struggle to impart that skill to students, but there are also further concerns. Such a teacher is also likely to avoid an area he/she feels

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weak in, which can be reflected in the problems he/she chooses to assign to students or to do with students in class. Additionally, such choices can also convey beliefs to students about what mathematics entails or what kinds of problems should be present in a mathematics class. We also want to note that quantitative reasoning is not only applicable to real world or story problems; for example, students need to reason about the abstract quantities represented by the independent and dependent variables in order to make sense of functions.

The only foreseeable solution to this problem lies in making teachers' *quantitative reasoning* a core focus of both preservice teacher education and inservice professional development. Such a solution would be best implemented by refining the structure and content of current methods courses and professional development opportunities; a separate course might imply that quantitative reasoning is a special and separate topic only applicable to a narrow section of mathematics. The work that has been done on mathematical knowledge for teaching (MKT) shows how crucial it is to student achievement that teachers have a deep understanding of the meaning underlying the procedures they carry out (Hill, 2005; Silverman & Thompson, 2008). We have four specific recommendations for teachers of teachers from our experiences in analyzing this data set: (1) Discontinue the use of the cross-multiplying procedure (where the intermediate results have no clear quantitative meaning and incoherent units such as dollar-gallons) in favor of finding unit rates, or finding the relative size of one pair of measurements and then applying it to the other. (2) Focus explicitly on reasoning with non-integer measurements of quantities. (3) Tasks necessitating proportional reasoning should be given interspersed with tasks that sound similar but do not involve proportional quantities, so that teachers need to genuinely assess whether proportional reasoning is appropriate for each task. Looking back, this was a missing piece in our own data collection. (4) Most significantly, tasks that require teachers to ascribe meaning to their own calculations should be the norm and not the exception. The overall arching problem identified in this data set is that teachers are adept at carrying out calculations without having coherent meanings for their own results. A teacher who cannot make sense of his or her own calculations has no chance of helping students to understand their own.

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