RELATIONSHIPS BETWEEN UNITS COORDINATION AND SUBITIZING

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This proposal explores relationships between young children's unit development/coordination and young children's subitizing. In particular, this theoretical commentary considers students' degrees of abstraction, students' development of actions on units, and students' operations with units when subitizing. As a result of this commentary, this author offers questions regarding how subitizing may elicit actions on units and perceptual/figurative material in a different order. These questions indicate possible alternative means in which comprehensive operations with natural numbers may develop. Possible educational implications and future research around these questions are also discussed.

Keywords: Early Childhood Education, Number Concepts and Operations

Steffe's (2017) plenary for PME-NA outlined comprehensive means in which radical constructivist learning theory has explained children's construction of mathematical concepts that have children's construction and coordination of units as a foundation. Steffe also explained that on average about 40% of first graders do not yet use figurative material to stand in for perceptual material when counting and unitizing. By third grade, this figure remains at about 5-8% of students. This finding suggests differences in children's number abstractions in early elementary grade levels that may also indicate distinct differences in their unitizing activity later. This theoretical commentary explores how young children's unitizing may relate to *subitizing* activity (individuals' quick apprehension towards the numerosity of a small set of items), which may explain differences in children's number abstractions in early grade levels.

Freeman (1912) first proposed that subitizing requires an individual's attention towards units of units when encoding number. MacDonald and Wilkins (under review) found that one child's subitizing activity may relate to her composite unit development. However, these findings are still exploratory and provide more questions than answers. In particular, if young children are developing units and/or acting on these units through their subitizing activity, how might this affect the development of their actions and operations? For instance, Steffe and Cobb (1988) found that young children develop a singular unit through their counting activity, which they iterate before engaging with partitioning. However, subitizing activity requires students partition patterned sets of items to associate with number before they iterate singular units.

If children order their actions on material differently, how might this affect their ability to coordinate actions and form operations? If both counting and subitizing promote children's coordination of actions earlier, which promotes early operation development, how would this affect children's number development? Therefore, when considering subitizing activity in relation to students' composite unit development, there seems to be new perspectives when investigating learning trajectories in number. Through this theoretical commentary, I will discuss how actions and operations may develop differently when young children subitize versus count. Resulting from this commentary are future research directions and educational implications.

The purpose of this commentary is to consider alternative learning trajectories that take on theoretical aspects in the Neo-Piagetian literature and may explain how subitizing relates to young children's development and their coordination of units. To discuss the intersect between subitizing and children's development of units, I will (a) provide a theoretical framework, (b)

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define unit construction, (c) describe how subitizing activity may relate to unit construction, and (d) propose future research that should consider how the two types of activity may relate.

Theoretical Framework

This theoretical commentary is grounded in the radical constructivist paradigm and more specifically, a Neo-Piagetian perspective. Essentially, in adopting a radical constructivist paradigm, I acknowledge that children learn through active engagement and reflection of their perceived reality. By adopting this paradigm, I also acknowledge that each individual constructs a unique mathematical reality that can be partially understood by others developing a second-order model of his or her mathematics.

Abstractions

With this paradigm, Piaget explains changes in mathematical realities in varying means through degrees of abstraction children rely on when engaging in mathematics. Piaget (1977/2001) described abstractions students rely as beginning as a reliance on *empirical abstractions* (abstractions of actions on perceived objects) towards *reflective abstractions* (abstractions on projected operations). Glasersfeld (1995) explains that individuals rely on two types of empirical abstractions (empirical abstractions and pseudo-empirical abstractions). *Empirical abstractions* explain children's attention towards rules and patterns when acting on perceived objects (e.g., counting manipulatives and knowing the last number word signifies the total). *Pseudo-empirical abstractions* are defined as children's ability to coordinate figurative material, which is indicative when students represent perceived objects with figurative material (e.g., fingers, tapping, verbal utterances), when solving a task. Students transitioning from empirical abstractions towards pseudo-empirical transitions internalize mathematical patterns and rules while coordinating their unitizing, regardless of material presented to them. As children internalize their actions and further step away from perceived material presented to them their degree of abstraction transitions from empirical abstractions towards reflective abstractions.

Piaget (1968/1970) first defined reflective abstractions as simply "coordinated actions" (p. 18). Glasersfeld (1995) further interpreted Piaget's (1977/2001) reflective abstraction delineations by describing two types of reflective abstractions (reflective abstraction and reflected abstraction) that children rely on when interiorizing mathematical patterns to form logical structures. *Reflective abstraction* (first subset of reflective abstraction) explains an individual's projection and reorganization of his or her coordinated actions or operations at another conceptual level (1995). *Reflected abstraction* (second subset of reflective abstraction) explains this same activity, but also explains that an individual is also aware of his or her projection and reorganization (1995).

Actions Versus Operations

As children rely on different degrees of abstractions, they develop operational fluency with number operations, as structures for number become interiorized. Boyce (2014) explained how these structures develop in weak forms versus relatively stronger forms. Essentially, Boyce distinguishes between children's development and coordination of actions versus their development and coordination of operations to explain different forms of reflective abstractions. For instance, if a child is capable of coordinating his or her <u>actions</u> to create a goal and develop a means in which to take a unit and iterate it, then the child has created goal-activity and an iterable unit (an abstract unit capable of iteration). The unit has become abstracted, but the operational structure has not been developed to allow student anticipation of his or her actions on the unit and in coordination with his or her other actions. This is an example of a lower form of reflective abstraction because the unit is acted upon in activity. An operation to anticipate this

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activity is not created. This would explain why a child may be transitioning away from a reliance on perceptual material towards figurative material, but still struggles to interiorize structures for natural numbers. Comparatively, if a child is capable of developing an iterating operation, then the operation is one that can be acted upon, not the unit. This allows for anticipated activity (c.f., Tzur & Simon, 2004; Steffe, 1992; Ulrich & Wilkins, 2017), which allows a child's natural number schema to become interiorized.

In this commentary, I will not focus on the operations, which are beyond the capability of a young child in the early childhood years, but on how early perceptual actions may explain later conceptual operations. For instance, Piaget (1968/1970) posits, "in developmental psychology … there is never an absolute beginning" (p. 19). What Piaget seems to be referring to as the "absolute beginning" in this argument is the beginning of logical structures. When the coordination of actions begin framing our discussion, Piaget posits that the coordination of actions can go back to biological or organic coordination of actions.

I posit that many of these early roots of biological coordination do not directly relate to a child's development of his or her mathematical structures for number, but may explain the root of his or her early perceptions and coordination of activity when explaining latter operation development and unitizing (Glasersfeld, 1981). Therefore, I want to focus in on the roots of early development and coordination of actions to determine how early forms of operations may explain unitizing that young children engage in. Coupling this focus with subitizing development may explain differences in young children's unit development. Thus, this proposal will further discuss how particular actions with subitizing may be important for young children and how the coordination of these actions with their counting actions may relate to earlier forms of operations.

Unit Construction and Unit Coordination

The term *unit* has become polysemous in the Neo-Piagetian field, as a unit symbolizes a variety of means in which unitizing develops relative to the context the proposed unit is set in. Ulrich (2015) cited Glasersfeld (1981) when defining unitizing as the "generalized and generative process of abstracting out the 'one'-ness from some aspect of experience" (as cited by Ulrich, 2015, p. 3). Frege (1884/1974) and Husserl (1887/1970) found in their work that children engaged in conceptual activity when required to cut "discrete items out of the flow of experience," which were found to promote the construction of "unitary wholes and ultimately of countable units" (as cited by Steffe & Cobb, 1988, p. 3). Steffe and Cobb posited that children younger than two years of age are capable of this activity, yet it is rare to find studies investigating or even theorizing this development or its nature in the early childhood years. For instance, Clements (1999) first proposed that subitizing relates to young children's number development and MacDonald and Wilkins (Under Review) found that one preschool student's subitizing activity related to her conceptual forms of units that she used when solving a counting task. However, it has not been determined how subitizing activity may relate to unitizing and how this development may relate to latter counting development.

A Unit

Boyce (2014) defines a unit as "something that has been unitized or set apart for further action" (p. 3) and characterizes a unit as "an object that can be transformed" (p. 4) and something that can be iterated (p. 24). A very different definition comes from Ulrich and Wilkins's (2017) study where they define a unit as "interiorized counting acts, so they can be used to enumerate the size of the sets of visible or invisible items and can themselves be counted" (p. 2). Finally, a third (very different) definition from Ulrich (2015) where a unit is

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defined as that, which "allows [students] to measure the number of items in a collection" (p. 3). These definitions are distinctly different because each researcher investigated different aspects of students' mathematics learning. Thus, the term unit becomes polysemous because this one word has multiple meanings that are related, but not alike. As described earlier, when the context becomes more sophisticated, the word, unit represents that which is the basis for measuring what the children are developing and coordinating within their respective mathematical structures. To provide insight into how a unit and unitizing action might relate or not relate to early childhood subitizing and number development, I will adopt Boyce's definition in hopes to explain early forms of empirical and pseudo-empirical abstraction development.

Actions and Operations on Units

The coordination of actions upon a unit explain how students develop operations. Boyce (2014) explains that a coordination of operations allow anticipatory frameworks (see Tzur & Simon, 2004) to develop, as operations are acted upon and allow for anticipation. This iterative cycle explains the nature of development and learning in mathematics. For instance, Norton and Boyce (2015) explain that children produce and coordinate four actions (unitizing, iterating, partitioning, and disembedding) when developing operations (e.g., distributing operation) that promote unit coordination. When students are able to operate flexibly and anticipate appropriate actions and results within a particular mathematics domain, Norton and Boyce posit they are able to do so because they have produced and coordinated levels of units. Coordination of these units results from students' development and coordination of the aforementioned actions.

Steffe and Cobb (1988) also found that young children evoked counting actions to develop early forms of units described as prenumerical singular units. These prenumercial singular units were described as evidence of young children's reliance on actions with perceptual material, figural patterns, motor patterns, verbal utterances, and abstract numbers (1988). As children distance themselves from perceptual material towards more abstract material, Steffe and Cobb found that children unitized the perceptual and figurative material before developing actions upon singular abstract units. Once actions upon abstract singular units are developed, children are able to iterate these units to construct numerical sequences that they segment. Through their segmenting actions, children develop composite abstract units. Once children iterate and partition composite units, they are capable of disembedding parts from whole sets. These actions promoted more operational understandings so that they are able to be aware of and work within mathematical structures for number. Steffe and Cobb found that when children could (1) count-on, (2) double count (i.e., 1, 2, 3, … one three; 4, 5, 6, … two threes), and (3) count by multiples (i.e., 3, 6, 9, 12) to solve problems, they were using all four actions to develop a nesting of sums for multiplicative and fractional reasoning.

With this learning theory in place, mathematics educational researchers have been able to explain nuanced development of children's counting, fractions, and multiplication. However, it is still not clear how subitizing activity may relate or not relate to children's unit development and coordination. Thus, these unit development and coordination learning theories will be discussed as related to different types of perceptual and conceptual subitizing.

Perceptual and Conceptual Subitizing Related to Composite Unit Development

Subitizing, initially defined in the psychology field (Kaufman, Lord, Reese, & Volkmann, 1949), describes individuals' quick attention to the numerosity of a small set of items. Unitizing and unit coordination have not been described in this research. Historically, subitizing has been described in the psychology field as a quantification encoding process and visual information processor where the numerosity of a small sets of items (ranging 1 to 5) are identified (Klahr,

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1973). More recently in the mathematics education field, Sarama and Clements (2009) delineated a hypothetical trajectory of subitizing activity and explained subitizing as relying on either perceptual or conceptual activity when developing early number understanding. More specifically, Sarama and Clements explain that individuals *perceptual subitizing* use an encoding process for small numerosities, but also draw from attentional resources when associating a number word with the numerosity of the set. Individuals engaging with *conceptual subitizing* use relatively more advanced number understandings to make sense of larger sets of items (≥ 5) (Sarama & Clements, 2009). However, unit development and coordination may explain, in more nuanced ways, how conceptual subitizing develops in relation to number understanding.

Perceptual Subitizing and Unit Development

Freeman (1912) initially proposed that subitizing may introduce children to the perception of "units of units" when encoding number. To consider early forms of unit development with subitizing activity, I consider actions students use to determine how "units of units" may be produced through subitizing activity. Clements (1999) first described *perceptual subitizing* and *conceptual subitizing* activity as relying on actions that promote early unit development. Perceptual subitizing was defined as students' ability to intentionally quantify a set of items through their subitizing activity yet be unable to be aware of any mathematical processes. To be capable of engaging in perceptual subitizing, Sarama and Clements (2009) suggest that students would need to be capable of cutting away a set of items to determine these as a unit. Sarama and Clements defines conceptual subitizing as children's ability to be aware of units of units when quickly associating sets of items with number words. However, it is not yet clear what type of units students may be developing and coordinating when subitizing.

MacDonald and Wilkins (2016) found in an exploratory study that preschool children engage in unitizing that may relate to early forms of composite unit development. For instance, children were found to engage in *Initial Perceptual Subitizing* where simple associations were made between shapes or motion when intentionally naming a number word. I argue this is very similar to Sarama and Clements (2009) description for perceptual subitizing and explains early forms of figural or motor singular unit item development. This form of unit development would explain how students rely on patterns shown to them visually with figurative patterns and how they may even represent them rhythmically with motor patterns. Through this unit development, young children may begin acting upon the rules of the patterns instead of simply acting on the actions related to the patterns. This would be indicative of a child pointing to his or her paper to show the pattern or shape when justifying why he or she knows she saw "three."

MacDonald and Wilkins (2016) also found that young children could subitizing two or more subgroups of items before they were capable of composing these subgroups. This type of perceptual subitizing activity was described as *Perceptual Subgroup Subitizing*. Quite often, the orientations shown to the students were clustered and regular. For instance, an orientation shown to a child may have two dots in a column on the right-hand side of the mat and three dots in a triangular orientation on the left-hand side of the mat. These orientations afforded students the opportunity to unitize more than one subgroup without requiring them to partition or iterate.

MacDonald, Boyce, Xu, and Wilkins (2015) also found that when students were shown orientations with regular patterns that were symmetrical (i.e., four dots in a rectangular orientation) in nature, they were capable of unitizing and iterating subgroups. For instance, when Frank, a four year-old student, was shown four dots in a rectangular orientation he initially said he saw, "T … four" (p. x). This suggests that he iterated two to build up towards four. This also

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suggests that he was capable of using a unit of two and partitioning a unit of four. Through these actions, MacDonald et al. posited that Frank was capable of coordinating actions to produce a unit for four. The symmetrical aspects of the orientation may have also afforded Frank the opportunity to partition because the line of reflection fell upon these same partitioning lines. Empson and Turner (2006) found that early forms of partitioning that began with paper folding were foundational for students' construction of functional relationships. Thus, this activity may provide foundations for more sophisticated coordination of units later.

Perceptual Subitizing and Actions on Units

If students are capable of (1) subitizing and then composing units after subitizing – *Perceptual Ascending Subitizing*, or (2) composing and subitizing and then decomposing units – *Perceptual Descending Subitizing*, then MacDonald and Wilkins (2016) found students were building necessary activity for conceptual subitizing. The distinction between this activity and conceptual subitizing is that students are shown items that are clustered and patterned, which does not require partitioning. Thus, this Perceptual Ascending Subitizing and Perceptual Descending Subitizing can allow students to unitize and act on the units developed. It is not clear how these operations develop and what type of composite unit (perceptual or figurative) students are developing or coordinating. These early operations may develop through a coordination of counting actions, through a segmenting of numerical sequences, or through students' development of patterned or figurative patterns. Regardless, this transition from students' development of actions to their development of operations is key, as students are now capable of being aware of number structures that afford them units of units perspectives.

Furthermore, when children engage in Perceptual Descending Subitizing, MacDonald and Wilkins (2016) argue that children are engaging in bi-directional activity. For instance, students who subitize two clustered subgroups may state that they saw "two and three." Once asked, "how many is that altogether?" they may then compose these units and state "five." MacDonald and Wilkins explain that this is Perceptual Ascending Subitizing because the student is ascending from the groups to the total set. However, students who subitize and compose two clustered subgroups may state that they saw "five." When asked, "how do you know there are five there?" they would then need to reflect on their actions and decompose the total set. Steffe, Glasersfeld, Richards, and Cobb (1983) described this bi-directional activity when children reversed their counting actions and found that this activity provided foundation actions for reversible counting later. Steffe explained that the distinction between bi-directional counting and reversible counting was that in bi-directional counting children would rely on prenumerical units and in reversible counting, children would rely on abstract units. This distinction leaves a lot to educators to determine the nuances of the development between bi-directional counting and reversible counting. Thus, in Perceptual Descending Subitizing, I argue that children are not relying on abstract composite units and therefore require the activity and other means to represent these units when reversing their activity. Also, questions arise from some this development in subitizing. For instance, how could different types of perceptual subitizing activity describe different types of prenumercial composite units?

Conceptual Subitizing and Unit Action Coordination

Once children are capable of partitioning and unitizing subgroups from a total set of items, MacDonald and Wilkins (2016) found that children were capable of conceptual subitizing. When children conceptually subitize, it seems that are capable of partitioning and unitizing before they iterate units. Children inverse these actions when they are counting. For instance, when counting, Steffe and Cobb found that children iterated units before they were capable of partitioning or

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segmenting numerical sequences. As students develop operations, they begin coordinating actions, which I posit occurs when students reverse the order of these actions. For instance, if children only engage in counting, then they will develop schemes that promote a particular order of actions. Children will first unitize to develop units and then iterate their actions in accordance with these units in a 1:1 correspondence. Here the actions are still in the material presented to the children and number is also present in their actions on these objects. However, through a distancing of these, children's reliance upon particular items shown to them (e.g., through use of fingers, through use of motor actions) children's units become more abstract until they are capable of operating on a series of abstract units. These early operations involve children segmenting numerical sequences that they built through their unitizing and iterating of abstract units. Thus, when counting children unitize, iterate, and then partition.

Comparatively, if children only engage in subitizing, then they may develop schemes that promote a different order of actions. Children will first unitize to develop units (perceptual subitizing) and then partition these units to develop a unit of units understanding (conceptual subitizing). Only then, will they begin to iterate units (e.g., five and five make ten) to (de)compose multiplicative units. Thus, when subitizing children unitize, partition, and then iterate. I posit that when children engage in counting and subitizing, their actions become operations because they are now asked to change the order of their actions and coordinate them in operational structures that are more comprehensive. This allows for more sophisticated reflective abstractions where children can use when anticipating actions and solutions in mathematics (Boyce, 2014).

In closing, by investigating relationships between subitizing and composite unit development, mathematics educational researchers may be able explain differences in children's composite unit development and provide alternative trajectories in learning mathematics.

Future Research and Possible Educational Implications

This theoretical commentary provided insight into how subitizing may or may not relate to unit development and coordination. From this discussion, it seems evident that when students engaging in counting activity they unitize and iterate before partitioning their developed number sequences. This allows students opportunities to develop actions and abstract these actions on their developed units. However, students engaging in subitizing activity may be unitizing and partitioning before iterating their developed spatial patterns. This allows students opportunities to develop these same actions on their developed units, but in a different order. However, if students develop more than one order of actions, would this provide a more comprehensive set of operations and allow students operation development earlier? By leveraging this development, I posit that gaps that Steffe (2017) and others (e.g., Clements & Sarama, 2011; Siegler & Ramani, 2008) have found in early elementary school may shrink. Thus, future research should focus on how different forms of activity in early elementary years may provide students alternative means to develop operation and units in which to operate on. Findings from studies like this might serve educators alternative actions that would serve children's disembedding actions and distributive operations. Also, when engaging in trajectories with different ordered actions, would children be capable of simultaneously coordinating these actions to develop operations similar to a splitting operation (see Wilkins & Norton, 2011)? Finally, how might alternative trajectories be used to leverage development for students who require different curricula or educational support (e.g., students identified as having a learning disability)? Thus, future research in how subitizing and counting relate to unit development and coordination could serve theoretical frameworks and educational curricula and should be explored further.

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